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INTEGRABLE FUNCTIONAL EQUATIONS AND ALGEBRAIC GEOMETRY

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Introduction. The main goal of this paper is to show that certain functional equations can be solved using an appropriate version of the inverse spectral method. (We refer the reader to the books [1]–[5] where the basic ideas of the application of the inverse spectral method to integrable systems of differential equations are explained.) Here we shall demonstrate our method by considering in detail the functional equation

$$\frac{q(x, y)q(y, z)}{q(x, z)} = r(x, y) - r(z, y) + p(x, z). \tag{0.1}$$

Note that, in the case

$$p(x, y) = p(x - y), \qquad q(x, y) = q(x - y), \qquad r(x, y) = r(x - y), \tag{0.2}$$

where $r(z)$ is an odd function, this equation reduces to the more simple functional equation

$$\frac{q(x)q(y)}{q(x + y)} = r(x) + r(y) + p(x + y), \tag{0.3}$$

introduced by Calogero and Bruschi in connection to integrable many-body problems [6].

The usual way to solve functional equations of this type is to derive a differential equation for the involved functions assuming appropriate smoothness. This was done for equation (0.3) in [7]. (The differentiated form of the functional equation (0.3) was also used by I. Krichever [14] in his theory of action-angle variables for Calogero-Moser systems with an elliptic potential.) Here we shall solve the functional equation (0.1) without assuming smoothness or even continuity for the functions p, q, r. We only assume that these functions are Lebesgue measurable.

The equation (0.1) has a rich symmetry group. Indeed, one can transform the functions as follows:

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$$q(x, y) \to \frac{v(x)}{v(y)} q(x, y), \qquad r \to r, \qquad p \to p; \tag{0.4}$$

$$q \to q, \qquad r(x, y) \to r(x, y) + f(y) + g(x), \qquad p(x, y) \to p(x, y) - g(x) + g(y); \tag{0.5}$$

$$p, q, r \to kp, kq, kr. \tag{0.6}$$

Here $v(x)$, $f(x)$, $q(x)$ are arbitrary functions and k is constant. Also, one can change the arguments by an arbitrary function $z(x)$:

$$p(x, y), \qquad q(x, y), \qquad r(x, y) \to p(z(x), z(y)), \qquad q(z(x), z(y)), \qquad r(z(x), z(y)). \tag{0.7}$$

Our main result is that, modulo the ambiguity (0.4)–(0.7), a generic solution of the functional equation has the form (0.2) with

$$p(z) = \zeta(z_0) - \zeta(z + z_0), \qquad q(z) = \frac{\sigma(z - z_0)}{\sigma(z_0)\sigma(z)}, \qquad r(z) = \zeta(z), \tag{0.8}$$

(cf. [7]) where σ and ζ are the Weierstrass elliptic σ- and ζ-functions (see [9]) and z_0 is a complex number.

Our method of solution of the functional equation (0.1) involves first relating (0.1) to the algebraic equation

$$\frac{q_{ij}q_{jk}}{q_{ik}} = r_{ij} - r_{kj} + p_{ik}. \tag{0.9}$$

We then construct a commutation representation ("Lax pair") for equation (0.9). Because of the existence of such a representation, we can call (0.9) an *integrable algebraic equation* and equation (0.1) an *integrable functional equation*. We shall show that the problem of solving the functional equation (0.1) can be reduced to the problem of classifying certain commutative algebras of λ-matrices (λ plays the role of spectral parameter). Using ideas of the algebraic-geometric integration method (see surveys [10]–[12]) we obtain such a classification.

We note that equation (0.9) is not the first algebraic equation to be solved by the inverse spectral method. In fact, Krichever [8] used algebraic-geometric techniques to classify two-component solutions of the Yang-Baxter equation. However, the nature of integrability of equations (0.1) and (0.9) is substantially different from that of the Yang-Baxter equation.

Remark. Within the ambiguity (0.4), the function

$$q(x, y) = \frac{\sigma(x - y - z_0)}{\sigma(x_0)\sigma(x - y)}$$

solving the functional equation (0.1) coincides with the Baker-Akhiezer function

$$\psi(x, y; z_0) = \exp(z_0\zeta(x) - z_0\zeta(y))q(x, y).$$

We recall that $\psi(x, y; z_0)$ as a function of x is a single-valued function on the elliptic curve

$$x \in \mathbb{C}/\{2m\omega + 2n\omega'\},$$

and it has an essential singularity in $x = 0$

$$\psi(x, y; z_0) \simeq \exp\frac{z_0}{x} \qquad \text{for } x \to 0.$$

A generalization of the Baker-Akhiezer function (essentially, a generalization of the exponential function on the Riemann sphere) plays a very important role in the machinery of the algebraic-geometric method of integration of nonlinear equations [11]. We have found that Baker-Akhiezer functions on Riemann surfaces of genus g are related to the integrable functional equation[1]

$$\frac{q(x, y)q(y, z)}{q(x, z)} = r(x, y) - r(z, y) + \sum_{k=1}^{g} s_k(y)p_k(x, z). \tag{0.10}$$

(See Appendix for the precise formulations.)

It would be interesting to derive all the standard facts of the theory of Baker-Akhiezer functions from the functional equation (0.10). We are also tempted to consider solutions of the integro-functional equation

$$\frac{q(x, y)q(y, z)}{q(x, z)} = r(x, y) - r(z, y) + \int_M s(y, t)p(x, z, t)\, dt \tag{0.11}$$

(integral over a space M with a measure dt) as Baker-Akhiezer functions of infinite genus. We will consider these problems in subsequent publications.

1. Commutation representation for the functional equation. Let X be a set of at least 5 elements. Let $p(x, y)$, $q(x, y)$, $r(x, y)$ be complex-valued functions on $X \times X\backslash$(diagonal) satisfying the functional equation

$$q(x, y)q(y, z) = q(x, z)[r(x, y) - r(z, y)] + p(x, z), \tag{1.1}$$

$$x, y, z \text{ distinct}.$$

[1] In [15] it was shown that the Baker-Akhiezer function on a Riemann surface of any genus g as a function of z_0 (i.e., of a point of the Jacobian of the Riemann surface) satisfies certain functional equations. These coincide with ours only for $g = 1$.

Let $x_1, \ldots, x_n$ be arbitrary distinct points of the set X, $n \geqslant 4$. Let us fix n numbers $\gamma_1, \ldots, \gamma_n$ satisfying

$$\gamma_1 + \cdots + \gamma_n = 0, \tag{1.2a}$$

$$|\gamma_1|^2 + \cdots + |\gamma_n|^2 \neq 0, \tag{1.2b}$$

and the following condition: there exist n pairwise distinct numbers $a_1, \ldots, a_n$ such that

$$a_1 + \cdots + a_n = 0 \qquad \text{and } \gamma_1 a_1 + \cdots + \gamma_n a_n = 0. \tag{1.2c}$$

We introduce $n \times n$ off-diagonal matrices

$$p_{ij} := p(x_i, x_j), \qquad q_{ij} := q(x_i, x_j)\gamma_j, \qquad r_{ij} := r(x_i, x_j)\gamma_j. \tag{1.5}$$

In these formulae $i \neq j$. We put zeros on the diagonals of the matrices.

LEMMA 1.1. *If $p(x, y)$, $q(x, y)$, $r(x, y)$ satisfy the functional equation (1.1), then the matrices (1.3) satisfy the following system of algebraic equations:*

$$q_{ij} q_{jk} = q_{ik}[r_{ij} - r_{kj}] + \gamma_j p_{ik}, \tag{1.4}$$

$$i, j, k \text{ distinct}.$$

Proof. Obvious.

We construct now a "Lax pair" (more precisely, a *commutation representation*) for the system of algebraic equations (1.4). This system will be represented as the condition of commutativity of a family of $n \times n$ matrices depending on an additional spectral parameter λ. We will call these matrices *λ-matrices*. The idea to consider equations of commutativity of a pair of λ-matrices as "integrable algebraic equations" was proposed by one of the authors (P. S.) in [13]. Here we develop this idea further by considering multidimensional commutative algebras of λ-matrices.

Let us fix n numbers $\gamma = (\gamma_1, \ldots, \gamma_n)$ satisfying (1.2). Let $\mathscr{L}^\gamma$ be a commutative algebra of $n \times n$ λ-matrices L generated by the subspace $\mathscr{L}_1^\gamma$ of the form

$$\mathscr{L}_1^\gamma = \{L = \lambda A + U \mid A = \operatorname{diag}(a_1, \ldots, a_n), U = (u_{ij})\} \tag{1.5a}$$

where $a_1, \ldots, a_n$ are *arbitrary* complex numbers satisfying (1.2c). We assume the algebra $\mathscr{L}^\gamma$ to be *irreducible*; i.e., there exists no nontrivial subspace $V \subset \mathbb{C}^n$ which is invariant with respect to $\mathscr{L}^\gamma$. This is guaranteed by the inequalities

$$\sum_{s \neq i} (|u_{is}|^2 + |u_{si}|^2) \neq 0 \tag{1.5b}$$

for any $i = 1, \ldots, n$ and for some matrix $U = (u_{ij})$ such that $L = \lambda A + U \in \mathscr{L}_1^\gamma$.

LEMMA 1.2. *There exist off-diagonal matrices* $Q = (q_{ij})$ *and* $R = (r_{ij})$ *such that all the elements of* $\mathscr{L}_1^\gamma$ *can be represented in the form*

$$L \equiv L_A := \lambda A + [A, Q] + D_A + cI \tag{1.6}$$

where D_A *is a diagonal matrix with entries given by the vector* $[R, A]\,(1, 1, \ldots, 1)^T$, *i.e.,*

$$D_A = \operatorname{diag}([R, A](1, 1, \ldots, 1)^T), \tag{1.7}$$

c is an arbitrary constant, and I is the unity matrix.

Proof. Let $\mathscr{A}$ be the space of matrices $A = \operatorname{diag}(a_1, \ldots, a_n)$ satisfying (1.2c). The map

$$\mathscr{L}_1^\gamma/\{cI\} \to \mathscr{A}, \qquad \lambda A + U \to A \tag{1.8}$$

is an isomorphism of linear spaces. Indeed, let $A \in \mathscr{A}$ be a matrix with pairwise distinct diagonal elements. Let $L = \lambda A + U$ be an element of $\mathscr{L}_1^\gamma$. We define an off-diagonal matrix Q by the equality

$$U = [A, Q] + D, \qquad D = \operatorname{diag}(d_1, \ldots, d_n). \tag{1.9}$$

For any $L' = \lambda A' + U' \in \mathscr{L}$, the commutativity of L and L' implies $[A, U'] = [A', U]$, or using (1.9)

$$U' = [A', Q] + D', \qquad D' = \operatorname{diag}(d_1', \ldots, d_n').$$

Let $L_i = \lambda A + [A, Q] + D_i$, $i = 1, 2$ be two elements of $\mathscr{L}_1^\gamma$. Here D_1, D_2 are some diagonal matrices. Then $D_1 - D_2 \in \mathscr{L}^\gamma$. So $D_1 - D_2 = cI$ because of the irreducibility condition (1.5b). So the linear map (1.8) is an isomorphism. The inverse map is just given by (1.6). (The formula (1.7) gives the general form of a diagonal matrix $D = D_A$ linearly depending on the traceless matrix A.) Lemma 1.2 is proved.

The matrices Q, R can be considered as parameters of commutative algebras of λ-matrices of the above form. The matrix Q is determined uniquely by the commutative algebra, while the ambiguity in determining R is of the form

$$r_{ij} \sim r_{ij} + p_i \gamma_j \tag{1.10}$$

for arbitrary p_i.

In the coordinate form, the formulae of Lemma 1.2 mean that for any matrix A satisfying (1.2) there exists a unique matrix $L_A \in \mathscr{L}$ of the form

$$(L_A)_{ij} = \left[\lambda a_i + \sum_s r_{is}(a_s - a_i)\right]\delta_{ij} + (a_i - a_j)q_{ij}, \qquad i, j = 1, \ldots, n. \tag{1.11}$$