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Painlevé transcendents in two-dimensional topological field theory

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Contents

Introduction

This paper is devoted to the theory of WDVV equations of associativity. This remarkable system of nonlinear differential equations was discovered by E.Witten [Wi1] and R.Dijkgraaf, E.Verlinde and H.Verlinde [DVV] in the beginning of '90s. It was first derived as equations for the so-called primary free energy of a family of two-dimensional topological field theories. Later it proved to be an efficient tool in solution of problems of the theory of Gromov - Witten invariants, reflection groups and singularities, integrable hierarchies.

Here we mainly consider the relationships of WDVV to the theory of Painlevé equations. This is a two-way connection. First, any solution to WDVV satisfying certain semisimplicity conditions, can be expressed via Painlevé-type transcendents. Conversely, theory of WDVV works as a source of remarkble particular solutions of the Painlevé equations.

The paper is an extended version of the lecture notes of a course given at 1996 Cargèse summer school "The Painlevé property: one century later". It is organized as follows.

In Lecture 1 we give a sketch of the ideas of two-dimesnional topological field theory, we formulate WDVV and give main examples of solutions coming from quantum cohomology and from singularity theory. In Lecture 2 we give a coordinate-free reformulation of WDVV introducing the notion of Frobenius manifold. We also construct the first main geometrical object, namely, the deformed affine connection on a Frobenius manifold. The monodromy at the origin of the deformed connection gives us the first set of important invariants of Frobenius manifolds. In Lecture 3 we define the class of semisimple Frobenius manifolds. In physics they correspond to two-dimensional topological field theories with all relevant perturbations. We construct the so-called canonical coordinates on such manifolds. In Lecture 4 we complete the classification of semisimple Frobenius manifolds in terms of monodromy data of certain universal linear differential operator with rational coefficients. We give a nontrivial example of computation of the monodromy data in quantum cohomology. In the last Lecture we develop a "mirror construction" representing the principal geometrical objects on a semisimple Frobenius manifold by residues and oscillatory integrals of a family of analytic functions on Riemann surfaces.

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Lecture 1.

Algebraic properties of correlators in 2D topological field theories. Moduli of a 2D TFT and WDVV equations of associativity

By definition, a quantum field theory (QFT) on a D-dimensional oriented manifold Σ (in our case $D = 2$) consists of:

1) Local fields $\phi_\alpha(x)$, $x \in \Sigma$. The metric $g_{ij}(x)$ on Σ could be one of the fields. It is called gravity.

2) Lagrangian

$$L = L(\phi, \partial_x \phi, \ldots).$$

The equations of motion of the classical field theory have the form

$$\frac{\delta S}{\delta \phi_\alpha(x)} = 0$$

where

$$S[\phi] = \int_\Sigma L(\phi, \partial_x \phi, \ldots)$$

is the classical action.

3) In the path-integral quantization we are interested in the partition function

$$Z_\Sigma = \int [d\phi] e^{-S[\phi]}$$

and, more generally, in the (non-normalized) correlation functions

$$< \phi_\alpha(x) \phi_\beta(y) \ldots >_\Sigma = \int [d\phi] \phi_\alpha(x) \phi_\beta(y) \ldots e^{-S[\phi]}.$$

The integration in both cases is over the space of local fields ϕ on Σ with an appropriate measure $[d\phi]$. In the full theory we are also to take an integration over the space of manifolds Σ.

4) The theory admits topological invariance if an arbitrary change of the metric on Σ preserves the action

$$\frac{\delta S}{\delta g_{ij}(x)} \equiv 0.$$

In $D = 2$ case such a theory will be called 2D topological field theory (TFT). For example, in the 2D case the total curvature functional

$$S = \frac{1}{2\pi} \int_\Sigma R \sqrt{g} d^2 x$$

is topologically invariant. Indeed, due to Gauss - Bonnet theorem it is equal to the Euler character of the surface Σ.

For a topological field theory the partition function gives a topological invariant of Σ. The correlation functions depend only on the topology of Σ and on the fields (but not on their positions). Particularly, in the 2D case we have

$$< \phi_\alpha(x)\phi_\beta(y)\dots >_\Sigma \equiv < \phi_\alpha\phi_\beta\dots >_g .$$

In the r.h.s. there are just numbers depending on the genus g of the surface Σ and on the labels $\alpha,\ \beta,\dots$ of the fields.

5) In the *matter sector* of the QFT we integrate over the space of all fields but the metric $(g_{ij}(x))$. For a TFT the correlators of the matter sector have a nice algebraic description to be presented in a moment. To describe coupling of the QFT to gravity one is to integrate over the space of metrics. In TFTs coupling to gravity can be reduced to integration over the space of conformal classes of the metrics on Σ, i.e., over the moduli space of Riemann surfaces of the genus $g = g(\Sigma)$. This is a much more complicated procedure by now fixed only for the genera $g = 0,\ 1$.

We describe now the algebraic properties of the matter sector correators in a 2D TFT. We will consider simple theories having a finite number of observables in the matter sector

$$\phi_1, \dots, \phi_n$$

(the so-called primary chiral fields). One can easily derive all algebraic properties of the correlators using the general Atiyah axioms of a topological field theory. We present here only the summary of the properties.

Definition 1.1. A *Frobenius algebra* is a pair $(A,\ < \ ,\ >)$ where A is a commutative associative algebra (over $\mathbf{C}$) with a unity and $< \ ,\ >$ stands for a symmetric non-degenerate *invariant* bilinear form on A. The invariance means validity of the following identity

$$< a\, b, c > = < a, b\, c > \tag{1.1}$$

for arbitrary 3 vectors $a,\ b,\ c \in A$.

Theorem 1.1 (see [Dij1, Dij1, Du7]). *The matter sector correlators of any 2D TFT with n observables can be encoded by a Frobenius algebra $(A, < \ ,\ >)$ with a marked basis $e_1, \dots, e_n$. The genus g correlators of the observables have the form*

$$< \phi_{\alpha_1}\phi_{\alpha_2}\dots\phi_{\alpha_k} >_g = < e_{\alpha_1}\cdot e_{\alpha_2}\cdot \dots \cdot \phi_{\alpha_k}, H^g >$$

where

$$H = \eta^{\alpha\beta} e_\alpha \cdot e_\beta \in A$$

$$\left(\eta^{\alpha\beta}\right) = \left(\eta_{\alpha\beta}\right)^{-1}, \quad \eta_{\alpha\beta} := < e_\alpha, e_\beta > .$$

Physicists call $(A, < \ ,\ >)$ the primary chiral algebra of the TFT. Observe that the structure of the Frobenius algebra is uniquely determined by the genus zero two- and three-point correlators

$$< e_\alpha, e_\beta > = < \phi_\alpha\phi_\beta >_0, \quad < e_\alpha\cdot e_\beta, e_\gamma > = < \phi_\alpha\phi_\beta\phi_\gamma >_0 .$$

Usually the observables are chosen in such a way that the vector e_1 coincides with the unity of the algebra A. Then

$$< e_\alpha, e_\beta >=< \phi_1 \phi_\alpha \phi_\beta >_0 .$$

We give now the two main "physical" examples of Frobenius algebras.

Example 1.1. Let X be a 2d-dimensional closed oriented manifold without odd-dimensional cohomologies. Take the full cohomology algebra

$$A = H^*(X)$$

with the bilinear form

$$< \omega_1, \omega_2 >= \int_X \omega_1 \wedge \omega_2, \quad \omega_1,\ \omega_2 \in H^*(X) \tag{1.2}$$

(we realize cohomologies by classes of closed differential forms). Symmetry and invariance of this bilinear form are obvious. Nondegeneracy follows from the Poincaré duality theorem. This Frobenius algebra describes the matter sector of the topological sigma model (X is the target space).

Actually, in this example we have a certain graded structure on $(A, < \ , \ >)$. Generalizing, we give

Definition 1.2. The Frobenius algebra is called *graded* if a linear operator $Q : A \to A$ and a number d are defined such that

$$Q(a\,b) = Q(a)b + aQ(b), \tag{1.3a}$$

$$< Q(a), b > + < a, Q(b) >= d < a, b > \tag{1.3b}$$

for any $a,\ b \in A$. The operator Q is called *grading operator* and the number d is called *charge* of the Frobenius algebra. We will consider only the case of diagonalizable grading operators. Then we may assign degrees to the eigenvectors e_α of Q

$$\deg(e_\alpha) = q_\alpha \ \text{ if } \ Q(e_\alpha) = q_\alpha e_\alpha. \tag{1.4a}$$

For the topological example the vectors of a homogeneous basis are chosen in such a way that

$$e_\alpha \in H^{2q_\alpha}(X), \ \ \deg(e_\alpha) = q_\alpha. \tag{1.4b}$$

The charge d is equal to the half of the dimension of X.

Particular example: $X = \mathbf{CP}^d$. The full cohomology space has dimension $n = d + 1$. The natural basis in $A = H^*\left(\mathbf{CP}^d\right)$ is

$$1,\ \omega,\ \omega^2, \dots, \omega^d$$

where ω is the standard Kähler form on the projective space. We normalize it by the condition

$$\int_{\mathbf{CP}^d} \omega^d = 1.$$

Then $(A, < \, , \, >)$ is isomorphic to the quotient of the polynomial algebra

$$A = \mathbf{C}[\omega]/\left(\omega^{d+1}\right)$$

with the bilinear form

$$< \omega^k, \omega^l >= \delta_{k+l,d}.$$

Remark 1.1. We will consider below also graded Frobenius algebras $(A, < \, , \, >)$ over graded commutative associative rings R. In this case we have two grading operators $Q_R : R \to R$ and $Q_A : A \to A$ satisfying the properties

$$Q_R(\alpha\beta) = Q_R(\alpha)\beta + \alpha Q_R(\beta), \ \alpha, \ \beta \in R \tag{1.5a}$$

$$Q_A(ab) = Q_A(a)b + aQ_A(b), \ a, \ b \in A \tag{1.5b}$$

$$Q_A(\alpha a) = Q_R(\alpha)a + \alpha Q_A(a), \ \alpha \in R, \ a \in A \tag{1.5c}$$

$$Q_R < a, b > +d \ < a, b > =< Q_A(a), b > + < a, Q_A(b) >, \ a, \ b \in A. \tag{1.5d}$$

The number d is called *the charge* of the graded Frobenius algebra.

Example 1.2. of Frobenius algebra. Let $f(x)$ be a polynomial of $x \in \mathbf{C}^N$ with an isolated singularity at $x = 0$. This means that

$$df(x)|_{x=0} = 0$$

(we may also assume that $f(0) = 0$),

$$df(x)|_{x \neq 0} \neq 0$$

for x sufficiently close to the origin. Take the quotient of the polynomial algebra

$$A = \mathbf{C}[x]/_{\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right)}. \tag{1.6}$$

This is called the Jacobi ring, or the local algebra of the singularity. This is a finite-dimensional algebra if the singularity has finite multiplicity n. (The number $n = \dim A$ is also called Milnor number of the singularity.) We define bilinear form on A taking the residue

$$< p, q >= \frac{1}{(2\pi i)^N} \int_{\cap_i |\frac{\partial f}{\partial x_i}| = \epsilon} \frac{p(x)q(x)d^N x}{\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_N}}. \tag{1.7}$$

Here ϵ is sufficiently small positive number. Again, symmetry and invariance of the bilinear form are trivial. Nodegeneracy is less trivial; see the proof in [AGV], Volume 1, Section 5. To obtain a graded Frobenius algebra one is to take a quasihomogeneous polynomial $f(x)$.

This Frobenius algebra describes the matter sector of a topological Landau - Ginsburg model. The function $f(x)$ is called superpotential of the theory.

Particular example: the simple singularity of the A_n type. Here $N=1$, $f(x)=x^{N+1}$. The local algebra

$$A=\mathbf{C}[x]/\left(x^{N+1}\right)=\text{span }\left(1,x,x^2,\ldots,x^{n-1}\right)$$

$$x^k\cdot x^l=\begin{cases}x^{k+l}, & k+l<n\\ 0, & k+l\geq n\end{cases}$$

$$<x^k,x^l>=\text{res}\,\frac{x^{k+l}}{(n+1)x^n}=\begin{cases}0, & k+l\neq n-1\\ \frac{1}{n+1}, & k+l=n-1.\end{cases}$$

The grading operator is determined by

$$Q(x)=\frac{1}{n+1}x,$$

the charge is

$$d=\frac{n-1}{n+1}.$$

We have already said that the procedure of coupling to gravity of a 2D TFT is more complicated (not settled in full generality). For the genus zero case it can be done still in an axiomatic way. It turns out that the axioms of coupling to gravity can be reduced to WDVV equations of associativity. Here WDVV stands for Witten - Dijkgraaf - E.Verlinde - H.Verlinde. In the paper [Wi1] the equations of associativity were derived in the setting of topological sigma models. In [DVV] they were derived in a more general class of TFTs obtained by the so-called twisting from $N=2$ supersymmetric QFTs. Basically, the idea was to consider correlators of a particular n-dimensional family of TFTs

$$S\mapsto S-\sum_{\alpha=1}^{n}\int_\Sigma \phi_\alpha^{(2)}$$

as functions of the coupling constants $t=\left(t^1,\ldots,t^n\right)$. Here $\phi_1^{(2)},\ldots,\phi_n^{(2)}$ are certain two-forms on Σ being in one-to-one correspondence with the observables $\phi_1,\ldots,\phi_n$. The deformation preserves the topological invariance (not the grading!). So one obtains a n-dimensional deformation $(A_t,<\ ,\ >_t)$ of n-dimensional Frobenius algebra $(A,<\ ,\ >)=(A_0,<\ ,\ >_0)$. A basis $e_1=1,e_2,\ldots,e_n$ corresponding to the chosen system of observables $\phi_1,\ldots,\phi_n$ is marked in all of the algebras A_t. The following properties of the family of Frobenius algebras $(A_t,<\ ,\ >_t)$ were proved by WDVV:

$$<e_\alpha,e_\beta>_t\equiv<e_\alpha,e_\beta> \qquad (WDVV1)$$

$$c_{\alpha\beta\gamma}(t):=<e_\alpha\cdot e_\beta,e_\gamma>_t=\frac{\partial^3F(t)}{\partial t^\alpha\partial t^\beta\partial t^\gamma}. \qquad (WDVV2)$$

Here $F(t) = \log Z_0(t)$ is the genus zero free energy of the family of TFTs (the so-called primary free energy).

The last is the quasihomogeneity condition: the structure constants $c_{\alpha\beta\gamma}(t)$ of the algebras A_t are weighted homogeneous functions of the degree $q_\alpha + q_\beta + q_\gamma - d$ where we assign degree $1 - q_\alpha$ to the variable t^α for each $\alpha = 1, \ldots, n$:

$$c_{\alpha\beta\gamma}\left(\lambda^{1-q_1}t^1, \ldots, \lambda^{1-q_n}t^n\right) = \lambda^{q_\alpha+q_\beta+q_\gamma-d}c_{\alpha\beta\gamma}\left(t^1, \ldots, t^n\right)$$

for an arbitrary $\lambda \neq 0$. Observe that $q_1 = 0$ if $e_1 = 1$. All the quasihomogeneity equations can be written as a one for the primary free energy

$$F\left(\lambda^{1-q_1}t^1, \ldots, \lambda^{1-q_n}t^n\right) = \lambda^{3-d}F\left(t^1, \ldots, t^n\right) + \text{quadratic} \qquad (WDVV3)$$

where "quadratic" stands for at most degree two polynomial in $t^1, \ldots, t^n$. (Later we will slightly modify the quasihomogeneity requirement for those t^α where $q_\alpha = 1$ - see the beginning of Lecture 2.)

The WDVV equations of associativity is the problem of classification of n-dimensional families of n-dimensional Frobenius algebras satisfying the above properties WDVV1 - WDVV3. One can consider this problem as the first approximation to the problem of classification of 2D TFTs, at least of those obtained by twisting from N=2 supersymmetric theories. We do not present here other stores of the whole building of a 2D TFT (coupling to gravity [Wi2, Du3, Du7], Zamolodchikov-type Hermitean metric on the space of parameters t [CV1, Du4]). Probably, the upper stores can be put not on an arbitrary solution of WDVV as on the basement. However, before proceeding to the upper stores we will study the structure of the eventual basement, i.e., of a solution of WDVV. These my lectures are devoted just to this problem of classification of solutions of WDVV equations of associativity.

We finish this section with a sketch of construction of the deformed 2D TFTs for the two above examples. Observe first that for a graded Frobenius algebra $(A_0, < \ , \ >, Q, d)$ one can construct a trivial cubic solution of WDVV

$$F_0 = \frac{1}{6} < 1, (t)^3 >, \quad t = t^\alpha e_\alpha \in A_0. \tag{1.8}$$

In all the physical examples the free energy $F(t)$ is constructed as an analytic perturbation of a cubic F_0.

Example 1.3. We will additionally assume the $2d$-dimensional target space X to be Kähler. The deformation of F_0 is defined as the generating function of Gromov - Witten invariants. Let us consider the moduli space of instantons

$$X_{[\beta],l} := \left\{\text{holomorphic } \beta : \left(S^2, p_1, \ldots, p_l\right) \to X, \text{ given homotopy class } [\beta] \in H_2\left(X; \mathbf{Z}\right)\right\}. \tag{1.9}$$

The holomorphic maps β of the Riemann sphere S^2 with marked points $p_1, \ldots, p_l$ are considered up to holomorphic change of parameter. Under certain assumptions about the manifold X (see [KM, RT, MS, Au]) it can be shown that $X_{[\beta],l}$ can be compactified to produce an orbifold of the complex dimension

$$\dim_{\mathbf{C}} X_{[\beta],l} = d + \int_{S^2} \beta^* \left(c_1(X)\right) + l - 3.$$

Here $c_1(X) \in H^2(X)$ is the first Chern class of X.

Observe that any of the marked points p_i defines the evaluation map that we denote by the same symbol

$$p_i : \ X_{[\beta],l} \to X, \quad (\beta, p_1, \dots, p_l) \mapsto \beta(p_i). \tag{1.10}$$

For an element

$$a_1 \otimes a_2 \otimes \dots \otimes a_k \in \left(H^*(X)\right)^{\otimes k}$$

define the number

$$< a_1 \otimes \dots \otimes a_k >_{[\beta],l} = \begin{cases} 0, \ k \neq l \\ \int_{X_{[\beta],l}} p_1^*(a_1) \wedge \dots \wedge p_l^*(a_l), \ k = l \end{cases} \tag{1.11}$$

We extend this symbol linearly onto the infinite direct sum

$$\mathbf{C} \oplus H^* \oplus H^* \otimes H^* \oplus \left(H^*\right)^{\otimes 3} \oplus \dots$$

with $H^* := H^*(X)$.

Define now the function $F(t)$,

$$t = (t', t'') \in H^*(X) \tag{1.12a}$$

$$t' \in H^2(X)/2\pi i H^2(X, \mathbf{Z}), \quad t'' \in H^{* \neq 2}(X), \tag{1.12b}$$

$$F(t) = F_0(t) + \sum_{[\beta] \neq 0, l} \left\langle e^{t''} \right\rangle_{[\beta],l} e^{\int_{S^2} \beta^*(t')}. \tag{1.13}$$

Here $F_0(t)$ is the cubic (1.8) for the Frobenius algebra $A_0 = H^*(X)$. The exponential

$$e^t := 1 + \frac{t}{1!} + \frac{1}{2!} t \otimes t + \dots$$

is considered as an element of the infinite direct sum.

The numbers $< a_1 \otimes \dots \otimes a_k >_{[\beta],l}$ can be nonzero only if the following dimension condition holds true

$$\deg a_1 + \dots \deg a_l = \dim X_{[\beta],l} = d + \int_{S^2} \beta^* \left(c_1(X)\right) + l - 3.$$

This can be written in the form

$$\sum_{i=1}^{l} (1 - \deg a_i) = 3 - d - \int_{S^2} \beta^* \left(c_1(X)\right). \tag{1.14}$$

We see from this dimension condition that for any $[\beta], l$ the coefficient

$$\left\langle e^{t''} \right\rangle_{[\beta],l}$$

is a polynomial in $t'' \in H^{*\neq 2}(X)$. The coefficients of these polynomials proved to be independent on the complex structure on X [Gr, MS, RT] but only on the homotopy class of the symplectic structure on X given by the imaginary part Ω of the Kähler metric. They are called *Gromov - Witten invariants* of (X, Ω). (Actually, one can start with more general situation to define GW invariants of a compact symplectic manifold (X, Ω). To this end one is to consider pseudoholomorphic maps $\beta : S^2 \to X$ w.r.t. an appropriate almost complex structure on X. See details in [Gr, MS, RT].)

The family of algebras A_t with the parameter

$$t \in H^*(X)/2\pi i H^2(X, \mathbf{Z})$$

is called *quantum cohomology* of X. Sometimes they considered quantum cohomology in the restricted sense where the parameter $t = t'$ of the deformation belongs to

$$t \in H^{1,1}(X)/2\pi i H^2(X, \mathbf{Z}).$$

This restricted quantum cohomology is closely related to Floer symplectic cohomology of (X, Ω) (see [Sad, Pi, MS]). In the *point of classical limit* $t' \to -\infty$ (i.e., $\int_{S^2} \beta^*(t') \to -\infty$ for any $[\beta] \neq 0$) $F(t) \to f_0(t)$, so the quantum cohomology goes to the classical ones.

Particular example. Quantum cohomology of the projective plane $\mathbf{CP^2}$. For $t = t^1 + t^2\omega + t^3\omega^2 \in H^*(\mathbf{CP^2})$ the cubic function $F_0(t)$ is

$$F_0(t) = \frac{1}{2}\left(t^1\right)^2 t^3 + \frac{1}{2} t^1 \left(t^2\right)^2.$$

Here $t' = t^2\omega$, $t'' = t^1 + t^3\omega^2$. The series $F(t)$ has the form [KM1]

$$F(t) = F_0(t) + \sum_{k=1}^{\infty} \frac{N_k}{(3k-1)!} \left(t^3\right)^{3k-1} e^{kt^2}. \tag{1.15}$$

Here

$$N_k = \#\big\{\text{rational curves of degree } k \text{ on } \mathbf{CP^2}$$

$$\text{passing through } 3k-1 \text{ generic points.}\big\}$$

E.g., $N_1 = 1$ (one line through 2 points), $N_2 = 1$ (one conic through 5 points). One can see that the quasihomogeneity condition WDVV3 must be modified: the function $F(t)$ has degree $1 = 3 - 2$ (up to quadratic terms) if t^1 has degree 1, t^3 has degree -1, t^2 has degree 0 but $\exp t^2$ has degree 3. This quasihomogeneity anomaly comes from the term

$$\int_{S^2} \beta^* \left(c_1 \left(\mathbf{CP^2}\right)\right)$$

in the dimension condition (1.14).

The series $F(t)$ has nonempty domain of convergence

$$\mathrm{Re}\,\left(t^2 + 3\log t^3\right) < R \tag{1.16}$$

for some positive R. Numerical estimation for R was obtained by [DI]

$$R \simeq 1.981.$$

Actually, the following asymptotic ansatz was proposed in [DI]

$$\frac{N_k}{(3k-1)!} \simeq a^k b k^{-\frac{7}{2}}, \quad k- > \infty$$

with $a \simeq 0.138$, $b \simeq 6.1$. The exact values of the constants a, b are not known.

The structure constants of the restricted quantum cohomology ring are obtained by triple differentiation of $F(t)$ and setting $t^1 = t^3 = 0$. The resulting ring has very simple structure: this is the quotient of the polynomial ring

$$QH^*\left(\mathbf{CP^2}\right) = \mathbf{C}[e_2]/\left(e_2^3 = q\right)$$

with

$$q = e^{t^2}.$$

Clearly, at the point of classical limit $q \to 0$ one obtains the classical cohomology ring of the projective plane.

The function $F(t)$ proves to solve the WDVV equatons of associativity [KM1]. It was observed by Kontsevich that, plugging the ansatz (1.15) with $N_1 = 1$ into the equations of associativity one can compute recursively all the coefficients N_k. We leave as an exercise for the reader to derive these recursion relations for N_k.

Remark 1.2. Denote

$$\phi(x) = \sum_{k=1}^{\infty} \frac{N_k}{(3k-1)!} e^{kx}$$

and

$$\psi(x) = \frac{\phi''' - 27}{8(27 + 2\phi' - 3\phi'')}$$

(the prime stands for the x-derivative). Then the coefficients $N_K^{(1)}$ of the expansion

$$\psi(x) = -\frac{1}{8} + \sum_{k=1}^{\infty} \frac{k\, N_k^{(1)}}{(3k)!} e^{kx}$$

are the elliptic Gromov - Witten invariants of $\mathbf{CP}^2$, i.e., they are the numbers of elliptic curves of the degree k passing through $3k$ generic points on $\mathbf{CP}^2$. This was proved in [DZ2].

Also in the general situation of quantum cohomology of a manifold X one can prove validity of WDVV for a vast class of manifolds X [KM, MS, RT]. The quasihomogeneity conditions have the form WDVV3 for the dependence of $F(t)$ on the coordinates of the component $t'' \in H^{*\neq 2}(X)$. For the other component $t' = \sum t'^{\alpha} e'_{\alpha} \in H^2(X)$ of $t = (t', t'')$ the coordinates t'^{α} are dimensionless. We assign then the degrees to the exponentials

$$\deg e^{t'^{\alpha}} = r_{\alpha} \tag{1.17}$$

if

$$c_1(X) = \sum r_{\alpha} e'_{\alpha}. \tag{1.18}$$

Clearly, for $X = \mathbf{CP^2}$ we obtain the above condition $\deg \exp t^2 = 3$. For Calabi - Yau (CY) varieties X also the exponentials $\exp t'^{\alpha}$ are dimensionless since $c_1(X) = 0$. Particularly, for CY 3-folds all the GW polynomials

$$\left\langle e^{t''} \right\rangle_{[\beta], l}$$

are just numbers, as it follows from the dimension condition (1.14). That means that, essentially, the full quantum cohomology of a CY 3-fold is reduced to the restricted one (we do not consider here the contributions from the odd-dimensional classes of th CY). According to mirror conjecture [COGP], the free energy of a CY 3-fold X can be expressed via certain generalized hypergeometric functions. These hypergeometric functions are periods of the holomorphic three-form on the so-called dual CY 3-fold X^*. The mirror conjecture has been proved in [Gi2 - Gi4] for CY complete intersections in projective spaces. A general geometrical setting justifying mirror conjecture was proposed in [Wi4].

In the opposite case of Fano varieties, where $c_1(X) > 0$, nothing is known about the analytic structure of the free energy (besides the trivial example of the projective line where the full quantum cohomology is reduced to the restricted one). The restricted quantum cohomology can be often computed (actually, they are computed for all Fano complete intersections in [Beau]). For many examples of Fano varieties it was shown that, like in the above example of $\mathbf{CP^2}$, one can reconstruct all the GW invariants from the restricted quantum cohomology just solving recursively the WDVV equations of associativity. The restricted quantum cohomology serve as the initial data to specify uniquely the solution of WDVV.

We suggest that the success of this reconstruction of GW invariants of Fano varieties, unlikely CY varieties, where WDVV gives essentially no information about the GW invariants, is based on the following conjectural property [TX] of quantum cohomology of Fano varieties: the deformed Frobenius algebra A_t is semisimple for generic value of the parameter t. In these lectures we describe the general solution of WDVV satisfying the semisimplicity condition. We will show that they can be expressed via certain Painlevé-type transcendents. We will also discuss the problem of selection of the particular solutions of WDVV corresponding to free energies of physically motivated models of 2D TFT.

Example 1.4. In the topological Landau - Ginsburg models with the superpotential $f(x)$ the deformed Frobenius algebra is given by the formulae similar to (1.6), (1.7) where one is to use the versal deformation [AGV, Ar2]

$$f_s(x) = f(x) + \sum_{i=1}^{n} s^i p_i(x) \tag{1.19}$$

of the singularity. Here $p_1(x) = 1$, $p_2(x)$, $\ldots$, $p_n(x)$ is a basis of the local algebra of the singularity. (Actually, one is to choose properly the volume form $d^N x$ in (1.7). The construction of the needed volume form is given in [Sai2].) The metric

$$\sum \eta_{ij}(s)\, ds^i ds^j$$

on the space of the parameters $s = \left(s^1, \ldots, s^n\right)$ has the form

$$\eta_{ij}(s) = \frac{1}{(2\pi i)^N} \int_{\cap_j |\frac{\partial f_s(x)}{\partial x_j}|=\epsilon} \frac{p_i(x) p_j(x) d^N(x)}{\frac{\partial f_s(x)}{\partial x_1} \cdots \frac{\partial f_s(x)}{\partial x_n}}. \tag{1.20}$$

Under certain assumptions [Sai2] one can prove that this metric has zero curvature. Thus one can introduce new coordinates $\left(t^1, \ldots, t^n\right)$ on the space of parameters such that

$$\eta_{ij} ds^i ds^j = \eta_{\alpha\beta} dt^\alpha dt^\beta$$

with a constant matrix $\eta_{\alpha\beta}$. In these coordinates

$$c_{\alpha\beta\gamma}(t) = \frac{1}{(2\pi i)^N} \int_{\cap_j |\frac{\partial f_s(x)}{\partial x_j}|=\epsilon} \frac{\frac{\partial f_s(x)}{\partial t^\alpha} \frac{\partial f_s(x)}{\partial t^\beta} \frac{\partial f_s(x)}{\partial t^\gamma} d^N(x)}{\frac{\partial f_s(x)}{\partial x_1} \cdots \frac{\partial f_s(x)}{\partial x_n}}. \tag{1.21}$$

The explicit formulae for the A-D-E simple singularities see in [BV].

Particular case. Simple singularity of A_3 type. Here $p_1 = 1$, $p_2 = x$, $p_3 = x^2$ is a basis in the local algebra. So

$$f_s = x^4 + s_1 + s_2 x + s_3 x^2$$

(I use all lower indices in concrete examples). The metric (1.7) in the coordinates s_1, s_2, s_3 has the matrix depending on s

$$< p_i, p_j >_s = -4 \operatorname*{res}_{x=\infty} \frac{p_i(x) p_j(x)}{4x^3 + 2 s_3 x + s_2}.$$

We obtain the following matrix of the metric

$$\eta_{ij}(s) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} s_3 \end{pmatrix}.$$

Introducing the new coordinates

$$s_1 = t_1 + \frac{1}{8}t_3^2$$
$$s_2 = t_2$$
$$s_3 = t_3$$

we obtain the constant matrix

$$\eta_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The new parametrization of the versal deformation has the form

$$P_t(x) \equiv f_s(x) = x^4 + t_1 + \frac{1}{8}t_3^2 + t_2 x + t_3 x^2.$$

The only nontrivial "three-point functions"

$$c_{\alpha\beta\gamma} = -\operatorname*{res}_{\infty} \frac{\partial_\alpha P_t \partial_\beta P_t \partial_\gamma P_t}{\partial_x P_t} dx$$

are

$$c_{113} = c_{122} = 1, \quad c_{223} = -\frac{1}{4}t_3, \; c_{233} = -\frac{1}{4}t_2, \; c_{333} = \frac{1}{16}t_3^2.$$

This gives a polynomial solution of WDVV

$$F(t_1, t_2, t_3) = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 - \frac{1}{16}t_2^2 t_3^2 + \frac{1}{960}t_3^5. \tag{1.22}$$

We can continue our experiments with WDVV and try to find *all* polynomial solutions $F(t_1, t_2, t_3)$. This simple exercise gives only 4 polynomial solutions [Du6, Du7]! Besides (1.22) they are

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + \frac{1}{6}t_2^3 t_3 + \frac{1}{6}t_2^2 t_3^3 + \frac{1}{210}t_3^7 \tag{1.23}$$
$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + \frac{1}{6}t_2^3 t_3^2 + \frac{1}{20}t_2^2 t_3^5 + \frac{1}{3960}t_3^{11} \tag{1.24}$$
$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + t_2^4.$$

The last polynomial does not satisfy the semisimplicity condition. It turns out that the first two can be described in terms of singularities of the type B_3 and H_3 respectively. In Lecture 5 I will explain the construction of the polynomials (1.22) - (1.24) in terms of invariants of the Coxeter groups of the type A_3, B_3, H_3 resp. and the generalization of this construction to higher dimensions. Observe that the Coxeter groups A_3, B_3, H_3 are just all the groups of symmetries of Platonic solids (of tetrahedron, octahedron, and icosahedron resp.). So, WDVV equations of associativity "know" not only enumeration of rational plane curve, but they also "know" the list of Platonic solids! See also Conjecture of Lecture 5 below regarding polynomial solutions of WDVV.

Lecture 2

Equations of associativity and Frobenius manifolds
Deformed flat connection and its monodromy at the origin

We give first the precise formulation of WDVV equations of asociativity. Next, we will reformulate them in a coordinate-free form.

We look for a function $F\left(t^1,\ldots,t^n\right)\equiv F(t)$, a constant symmetric nondegenerate matrix $\left(\eta^{\alpha\beta}\right)$, numbers $q_1,\ldots,q_n$, $r_1,\ldots,r_n$, d such that

$$\partial_\alpha\partial_\beta\partial_\lambda F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\delta F(t)=\partial_\delta\partial_\beta\partial_\lambda F(t)\eta^{\lambda\mu}\partial_\mu\partial_\gamma\partial_\alpha F(t) \tag{$WDVV1$}$$

for any $\alpha,\ \beta,\ \gamma,\ \delta\ =\ 1,\ldots,n$. (We denote

$$\partial_\alpha:=\frac{\partial}{\partial t^\alpha}$$

etc., summation over repeated indices is assumed.) Equivalently, the algebra

$$A_t=\text{span }(e_1,\ldots,e_n)$$

with the multiplication law

$$\begin{aligned}e_\alpha\cdot e_\beta&=c_{\alpha\beta}^\gamma(t)e_\gamma\\ c_{\alpha\beta}^\gamma(t):&=\eta^{\gamma\epsilon}\partial_\epsilon\partial_\alpha\partial_\beta F(t)\end{aligned}\tag{2.1}$$

is to be associative for any t. The algebra will automatically be commutative.

The symmetric nondegenerate bilinear form $<\ ,\ >$ on A_t defined by

$$<e_\alpha,e_\beta>:=\eta_{\alpha\beta}\tag{2.2}$$

where the matrix $(\eta_{\alpha\beta})$ is the inverse one to $\left(\eta^{\alpha\beta}\right)$, is invariant (in the sense of (1.1)) since the expression

$$<e_\alpha\cdot e_\beta,e_\gamma>=\partial_\alpha\partial_\beta\partial_\gamma F(t)\tag{2.3}$$

is symmetric w.r.t. any permutation of α, β, γ.

The variable t^1 will be marked and we require that

$$\partial_\alpha\partial_\beta\partial_1 F(t)\equiv\eta_{\alpha\beta}.\tag{$WDVV2$}$$

This means that the first basic vector e_1 will be the unity of all the algebras A_t. From (WDVV1,2) we conclude that $(A_t,<\ ,\ >)$ is a Frobenius algebra for any t.

The last one is the quasihomogeneity condition that we write down in the infinitesimal form using the Euler identity for the quasihomogeneous functions. Introducing the Euler vector field

$$E=\sum_{\alpha=1}^n\left[(1-q_\alpha)t^\alpha+r_\alpha\right]\partial_\alpha\tag{2.4}$$

we require the function $F(t)$ to satisfy

$$\mathcal{L}_E F(t) := \sum_{\alpha=1}^{n} [(1-q_\alpha)t^\alpha + r_\alpha]\,\partial_\alpha F(t) = (3-d)F(t) + \frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C \quad (WDVV3)$$

for some constants $A_{\alpha\beta}$, B_α, C. The numbers q_α, r_α, d must satisfy the following normalization conditions

$$q_1 = 0, \quad r_\alpha \neq 0 \text{ only if } q_\alpha = 1. \tag{2.5}$$

Loosely speaking, we assign the degree $1 - q_\alpha$ to the variable t^α. But if $q_\alpha = 1$ then the degree r_α is assigned to $\exp t^\alpha$. With respect to this assignment the function $F(t)$ has degree $3-d$ up to quadratic terms.

We will consider the class of equivalence of solutions modulo additions of quadratic polynomials in t.

Exercise 2.1. For any α, β prove that

$$(q_\alpha + q_\beta - d)\eta_{\alpha\beta} = 0. \tag{2.6}$$

Exercise 2.2. Prove that, by adding a quadratic polynomial to $F(t)$, the coefficients $A_{\alpha\beta}$, B_α, C in (WDVV3) can be normalized in such a way that

$$\begin{aligned}
A_{\alpha\beta} &\neq 0 \text{ only if } q_\alpha + q_\beta = d-1\\
A_{1\alpha} &= \sum_\alpha \eta_{\alpha\epsilon} r_\epsilon\\
B_\alpha &\neq 0 \text{ only if } q_\alpha = d-2\\
B_1 &= 0\\
C &\neq 0 \text{ only if } d = 3.
\end{aligned} \tag{2.7}$$

Normalized in such a way coefficients $A_{\alpha\beta}$, B_α, C must also be considered as the unknown parameters of the WDVV problem.

Trivial solutions are cubics corresponding to graded Frobenius algebras $(A_0, < \, , \, >)$

$$\text{cubic } = \frac{1}{6}c_{\alpha\beta\gamma}t^\alpha t^\beta t^\gamma = \frac{1}{6} < 1, (t)^3 > . \tag{2.8}$$

Solutions needed are analytic perturbations of cubics. That means that

$$F(t) = \text{cubic} + \sum_{k,l\geq 0} a_{k,l}\,(t'')^l\, e^{kt'} \tag{2.9}$$

where the vector argument t is subdivided in two parts

$$t = (t', t''), \ \deg t' = 0, \ \deg t'' \neq 0, \tag{2.10}$$

k, l are multiindices with all nonnegative coordinates. For

$$t'' \to 0, \quad t' \to -\infty \tag{2.11}$$

$F(t)$ goes to the cubic. In quantum cohomology this is called the point of classical limit.

There are two main approaches in the theory WDVV.

1. Algebraic approach. We study the formal series solutions (2.9) to WDVV analyzing, say, the recursion relations for the coefficients $a_{k,l}$. An example of the algebraic approach are Kontsevich's recursion relations for the numbers of plane curves, and also our discovery of Platonic solids when classifying polynomial solutions to WDVV. A general approach to construct solutions of WDVV in the class of formal series was recently proposed in [BS]. Some family of formal power series solutions (not satisfying the quasihomogeneity (WDVV3)) was very recently constructed in [Los].

2. Analytic approach. First to describe *all* solutions to WDVV and then to select the solutions of the needed class (2.9).

In my lectures I will speak on the analytic approach to WDVV. This can be applied to the solutions of the form (2.9) only if the series converges near the point of classical limit (2.11). The convergence can be easily verified in concrete examples of quantum cohomologies. However, the general proof of convergence is still missing.

Let me first be more specific about the explicit form of WDVV.

Exercise 2.3. Let Q be the grading operator in $\mathbf{C}^n = \operatorname{span}(e_1, \ldots, e_n)$ defined by

$$Q(e_\alpha) = q_\alpha e_\alpha, \ \alpha = 1, \ldots, n.$$

Show that WDVV remain invariant under the linear transformations of the variables t

$$t \mapsto Mt, \ t = \left(t^1, \ldots, t^n\right)^T$$
$$(e_1, \ldots, e_n) \mapsto (e_1, \ldots, e_n) M^{-1}$$

(the upper label ^{T} stands for the transposition) where the matrix M satisfies the two conditions

$$Me_1 = e_1$$
$$MQ = QM.$$

Prove that, if $d \neq 0$ then the matrix $\eta = (\eta_{\alpha\beta})$ by a transformation of the above form can be reduced to the antidiagonal form

$$\eta_{\alpha\beta} = \delta_{\alpha+\beta,\, n+1}. \tag{2.12}$$

Derive from (WDVV2) that, in these coordinates, the function $F(t)$ can be represented in the form

$$F(t) = \frac{1}{2} {t^1}^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1} + f\left(t^2, \ldots, t^n\right) \tag{2.13}$$

for some function f of $n-1$ variables. WDVV can be written as a system of differential equations for this function.

We will usually consider only the case $d \neq 0$, although this is not important for the mathematical theory of WDVV.

Example 2.1. $n = 2$. Here $f = f(t_2)$. The equations WDVV1 are empty. The quasihomogeneity condition WDVV3 gives that

$$
\begin{aligned}
&f = t_2{}^{\frac{3-d}{1-d}}, \ d \neq 1 \\
&f = e^{\frac{2t_2}{r}}, \ d = 1, \ E = t_1\partial_1 + r\partial_2, \\
&f = -\frac{1}{2} c \log t_2, \ d = 3, \ E = t_1\partial_1 - 2t_2\partial_2, \ \mathcal{L}_E F = F + c.
\end{aligned}
$$

Example 2.2. For $n = 3$ the function $f = f(x, y)$, $x = t_2$, $y = t_3$ must satisfy the following PDE

$$f_{xxy}^2 = f_{yyy} + f_{xxx} f_{xyy}.$$

For generic d the variables t_1, t_2, t_3 have the scaling dimensions 1, $1 - \frac{d}{2}$, $1 - d$ respectively, and the scaling dimension of the function f is $3 - d$. So

$$f(x, y) = \frac{x^4}{y} \phi\left(\log(y x^q)\right), \quad q = 2\frac{d-1}{2-d}.$$

Plugging this to the above PDE one obtains the following complicated 3d order ODE for the function ϕ

$$
\begin{gathered}
-6\,\phi + 48\,\phi^2 + 11\,\phi' + 88\,q\,\phi\,\phi' - (144 + 144\,q - 3\,q^2)\,\phi'^2 - 6\,\phi'' + 48(2 + 2\,q + q^2)\,\phi\,\phi'' \\
-4\,q(16 + 16\,q + q^2)\,\phi'\,\phi'' - (13\,q^2 + 13\,q^3 + q^4)\,\phi''^2 \\
+\phi''' + 8q(3 + 3q + q^2)\,\phi\,\phi''' + 2\,q^2(1 + q + q^2)\,\phi'\,\phi''' - q^3(1 + q)\,\phi''\,\phi''' = 0.
\end{gathered}
$$

The nongeneric are the integer values $-2 \leq d \leq 4$. In this case the ansatz for f is to be modified. For example, for $d = 2$, $r_2 = r$

$$f(x, y) = \frac{1}{y}\phi(x + r \log y)$$

where the function f satisfies the ODE

$$\phi'''[r^3 + 2\phi' - r\phi''] - (\phi'')^2 - 6r^2\phi'' + 11r\phi' - 6\phi = 0. \tag{2.14}$$

The case of quantum cohomology of $\mathbf{CP^2}$ corresponds to $r = 3$ (see Lecture 1 above). In this case the equation (2.14) has a unique solution $\phi = \phi(x)$ of the form

$$\phi = \sum_{k \geq 1} A_k e^{k\,x} \tag{2.15}$$

normalized by the condition $A_1 = \frac{1}{2}$. Plugging the series (2.15) into the equation (2.14) one obtains the recursion relations for the numbers

$$N_k = (3k-1)!A_k$$

of rational curves of degree k on $\mathbf{CP^2}$ passing through $3k-1$ generic points.

Exercise 2.4. Derive from the recursion relations that the series (2.15) converges if

$$\operatorname{Re} x < \log \frac{6}{5}.$$

Recall that the numerical estimate of [DI] guarantees convergence for

$$\operatorname{Re} x < 1.981.$$

We conclude that in the first nontrivial case $n = 3$ the general solution of WDVV depends on 3 arbitrary parameters. However, this parametrization does not say anything about the analytic properties of solutions. For the next case $n = 4$ the situation looks to be even worse: the function $f = f(t_2, t_3, t_4)$ is to be found from an overdetermined system of 6 PDEs. With n growing the overdeterminancy of the system WDVV1 grows rapidly.

In these lectures we will give complete classification of the solutions of WDVV satisfying the semisimplicity condition. Recall that this condition means that the algebra A_t is semisimple for generic t. The solution will be expressed via certain Painlevé-type transcendents (via particular transcendents of the Painlevé-VI type in the first nontrivial case $n = 3$).

In this lecture we will develop some preliminary geometric constructions of the theory of WDVV. First we will give a coordinate-free reformulation of WDVV equations of asociativity. The basic idea is to identify the algebra A_t with the tangent space T_tM to the space of the parameters $t \in M$,

$$A_t \ni e_\alpha \leftrightarrow \partial_\alpha \in T_tM, \ \alpha = 1, \ldots, n.$$

The space of parameters M acquires a new geometrical structure: the tangent spaces T_tM are Frobenius algebras w.r.t. the multiplication

$$\partial_\alpha \cdot \partial_\beta = c_{\alpha\beta}^\gamma(t)\partial_\gamma \tag{2.16}$$

and metric

$$< \partial_\alpha, \partial_\beta >= \eta_{\alpha\beta} \tag{2.17}$$

We arrive [Du5] at the following main

Definition 2.1. (Smooth, analytic) *Frobenius structure* on the manifold M is a structure of Frobenius algebra on the tangent spaces $T_tM = (A_t, < \ , \ >_t)$ depending (smoothly, analytically) on the point t. This structure must satisfy the following axioms.

FM1. The metric on M induced by the invariant bilinear form $< \, , \, >_t$ is flat. Denote ∇ the Levi-Civita connection for the metric $< \, , \, >_t$. The unity vector field e must be covariantly constant,

$$\nabla e = 0. \tag{2.18}$$

We use here the word 'metric' as a synonim of a symmetric nondegenerate bilinear form on TM, not necessarily a positive one. Flatness of the metric, i.e., vanishing of the Riemann curvature tensor, means that locally a *system of flat coordinates* $(t^1, \dots, t^n)$ exists such that the matrix $< \partial_\alpha, \partial_\beta >$ of the metric in these coordinates becomes constant.

FM2. Let c be the following symmetric trilinear form on TM

$$c(u, v, w) :=< u \cdot v, w > . \tag{2.19}$$

The four-linear form

$$(\nabla_z c)(u, v, w), \; u, v, w, z \in TM$$

must be also symmetric.

Before formulating the last axiom we observe that the space $Vect(M)$ of vector fields on M acquires a structure of a Frobenius algebra over the algebra $Func(M)$ of (smooth, analytic) functions on M.

FM3. A linear vector field $E \in Vect(M)$ must be fixed on M, i.e.,

$$\nabla\nabla E = 0. \tag{2.20}$$

The operators

$$\begin{aligned} Q_{Func(M)} &:= E \\ Q_{Vect(M)} &:= \mathrm{id} + \mathrm{ad}_E \end{aligned} \tag{2.21}$$

introduce in $Vect(M)$ a structure of graded Frobenius algebra of a given charge d over the graded ring $Func(M)$ (see above Remark 1.1 after Definition 1.2).

Lemma 2.1. *Locally a Frobenius manifold with diagonalizable ∇E is described by a solution of WDVV and vice versa.*

Proof. 1. Starting from a solution of WDVV define the multiplication (2.16) and the metric (2.17) on the tangent planes to the parameter space. In the original coordinates $(t^1 \dots, t^n)$ the metric is manifestly flat. In these coordinates the covariant derivatives coincide with the partial ones

$$\nabla_\alpha = \partial_\alpha.$$

Since $e = \partial_1$ we have $\nabla e \equiv 0$. The first axiom FM1 is proved. The tensor c in (2.19) has the components

$$c_{\alpha\beta\gamma}(t) \equiv c(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial_\alpha\partial_\beta\partial_\gamma F(t).$$

So

$$(\nabla_{\partial_\delta} c)\,(\partial_\alpha, \partial_\beta, \partial_\gamma) = \partial_\alpha\partial_\beta\partial_\gamma\partial_\delta F(t)$$

is totally symmetric. This proves FM2.

Let us now prove FM3. The equations

$$Q_{Vect(M)}(a \cdot b) = Q_{Vect(M)}(a) \cdot b + a \cdot Q_{Vect(M)}(b)$$
$$Q_{Func(M)} < a, b > + d < a, b > = < Q_{Vect(M)}(a), b > + < a, Q_{Vect(M)}(b) >$$

can be recasted in the form

$$\mathcal{L}_E(a \cdot b) - \mathcal{L}_E(a) \cdot b - a \cdot \mathcal{L}_E(b) = a \cdot b \tag{2.22}$$
$$\mathcal{L}_E < a, b > - < \mathcal{L}_E a, b > - < a, \mathcal{L}_E b > = (2 - d) < a, b > . \tag{2.23}$$

We will prove the last two equations.

The Euler vector field is clearly a linear one. The gradient ∇E is a diagonal constant matrix

$$\nabla E = \mathrm{diag}\ (1 - q_1, \ldots, 1 - q_n) . \tag{2.24}$$

Triple differentiating of the quasihomogeneity equation (WDVV3) w.r.t. t^α, t^β, t^γ gives

$$\sum_\epsilon \left[(1 - q_\epsilon)t^\epsilon + r_\epsilon\right] \partial_\epsilon \left(c_{\alpha\beta\gamma}(t)\right) = (q_\alpha + q_\beta + q_\gamma - d)c_{\alpha\beta\gamma}(t). \tag{2.25}$$

From this and from (2.6) easily follow the identities (2.22), (2.23) of the definition of the graded Frobenius algebra over graded ring of functions.

2. Choose locally flat coordinates $(t^1, \ldots, t^n)$ on a Frobenius manifold. We can choose them in a particular way such that ∂_1, ..., ∂_n are the eigenvectors of the linear operator $\nabla E : TM \to TM$

$$(\nabla E)\partial_\alpha = \lambda_\alpha \partial_\alpha$$

for some constant λ_α (in the flat coordinates the matrix of the covariantly constant tensor ∇E is constant). This will be the homogeneous basis for the grading operator $Q_{Vect(M)}$

$$Q_{Vect(M)}\partial_\alpha = (1 - \lambda_\alpha)\partial_\alpha.$$

So

$$E = \sum_{\alpha=1}^n (\lambda_\alpha t^\alpha + r_\alpha)\partial_\alpha$$

for some constants r_α. We can kill by a shift all these constants but those for which $\lambda_\alpha = 0$. This gives the form (2.4) of the Euler vector field with

$$q_\alpha := 1 - \lambda_\alpha.$$

From the obvious equation

$$Q_{Vect(M)} e = 0$$

we immediately obtain that $\lambda_1 = 1$, i.e., $q_1 = 0$.

From the symmetry w.r.t. α, β, γ, δ of partial derivatives

$$\partial_\delta c_{\alpha\beta\gamma}(t)$$

of the symmetric tensor

$$c_{\alpha\beta\gamma}(t) =< \partial_\alpha \cdot \partial_\beta, \partial_\gamma >$$

we conclude that locally a function $F(t)$ exists such that

$$c_{\alpha\beta\gamma}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t).$$

For the invariant metric we obtain

$$\eta_{\alpha\beta} =< \partial_\alpha, \partial_\beta >=< \partial_\alpha \cdot \partial_\beta, \partial_1 >= \partial_1 \partial_\alpha \partial_\beta F(t).$$

We have proved WDVV1 and WDVV2.

Spelling the last axiom FM3 out we obtain the following two formulae

$$\mathcal{L}_E \eta_{\alpha\beta} = \partial_\alpha E^\epsilon \eta_{\epsilon\beta} + \partial_\beta E^\epsilon \eta_{\alpha\epsilon} = (2-d)\eta_{\alpha\beta}$$
$$\mathcal{L}_E c^\gamma_{\alpha\beta} = E^\epsilon \partial_\epsilon c^\gamma_{\alpha\beta} - \partial_\epsilon E^\gamma c^\epsilon_{\alpha\beta} + \partial_\alpha E^\epsilon c^\gamma_{\epsilon\beta} + \partial_\beta E^\epsilon c^\gamma_{\alpha\epsilon} = c^\gamma_{\alpha\beta}$$

From this it follows

$$(q_\alpha + q_\beta - d)\eta_{\alpha\beta} = 0 \tag{2.26}$$

$$E^\epsilon \partial_\epsilon c^\gamma_{\alpha\beta} = (q_\alpha + q_\beta - q_\gamma) c^\gamma_{\alpha\beta}. \tag{2.27}$$

Using (2.6) we lower the index γ in the last equation to obtain

$$E^\epsilon \partial_\epsilon c_{\alpha\beta\gamma} = (q_\alpha + q_\beta + q_\gamma - d) c_{\alpha\beta\gamma}, \quad \alpha,\, \beta,\, \gamma = 1, \dots, n.$$

Triple integration gives

$$E^\epsilon \partial_\epsilon F = (3-d)F + \text{ quadratic.}$$

Lemma is proved.

Remark 2.1. The definition of Frobenius manifold can be easily translated into algebraic language as a graded Frobenius algebra structure on the module of derivations of a graded commutative associative algebra. An important extension of this definition for the case of $\mathbf{Z}_2$-graded algebras was done by Kontsevich and Manin [KM1]. Such *Frobenius supermanifolds* are necessary to deal with Gromov - Witten invariants of manifolds with nontrivial odd-dimensional cohomologies. In this paper we will not discuss this extension.

Exercise 2.5. Prove that the direct product $M' \times M''$ of two Frobenius manifolds of *the same* charge d carries a natural structure of a Frobenius manifold of the charge d, the unity vector field $e' \oplus e''$, and the Euler vector field $E' \oplus E''$.

We now address the problem of (local) classification of Frobenius manifolds coinciding with local clasification of solutions of WDVV. To be more specific we give

Definition 2.2. A (local) diffeomorphism

$$\phi : M \to \tilde{M}$$

of two Frobenius manifolds is called *(local) equivalence* if the differential

$$\phi_* : T_t M \to T_{\phi(t)} \tilde M$$

is an isomorphism of algebras for any $t \in M$ and

$$\phi^* <\ ,\ >_{\tilde M} = c^2 <\ ,\ >_M$$

where c is a nonzero constant not depending on the point of M.

The corresponding free energies F and $\tilde F$ are related by

$$\tilde F\left(\phi(t)\right) = c^2 F(t) + \text{quadratic}.$$

Definition 2.3. A Frobenius manifold is called *reducible* if it is equivalent to the direct product of two Frobenius manifolds (see Exercise 2.5 above).

The first main tool in dealing with Frobenius manifolds is a deformation of the Levi-Civita connection ∇. We put

$$\tilde\nabla_u v := \nabla_u v + z\, u \cdot v. \tag{2.28a}$$

Here u, v are two vector fields on M, z is the parameter of the deformation. We extend this up to a meromorphic connection on the direct product $M \times \mathbf{C}$ by the formulae

$$\begin{aligned} \tilde\nabla_u \frac{d}{dz} &= 0 \\ \tilde\nabla_{\frac{d}{dz}} \frac{d}{dz} &= 0 \\ \tilde\nabla_{\frac{d}{dz}} v &= \partial_z v + E \cdot v - \frac{1}{z}\mu\, v \end{aligned} \tag{2.28b}$$

where

$$\mu := \frac{2-d}{2} - \nabla E = \operatorname{diag}\left(\mu_1, \dots, \mu_n\right) \tag{2.29a}$$

$$\mu_\alpha := q_\alpha - \frac{d}{2}. \tag{2.29b}$$

Here u, v are tangent vector fields on $M \times \mathbf{C}$ having zero component along $\mathbf{C}$. Observe that $\tilde\nabla$ is a symmetric connection.

Proposition 2.1. *For a Frobenius manifold M the curvature of the conection $\tilde\nabla$ equals zero. Conversely, if on the tangent spaces to M a structure of a Frobenius algebra is defined satisfying FM1, and the curvature of the connection $\tilde\nabla$ vanishes and the Euler vector field E satisfies*

$$\mathcal{L}_E <\ ,\ > = (2-d) <\ ,\ > \tag{2.30}$$

with a constant d, then M is a Frobenius manifold.

Proof. For a covector

$$\xi = \xi_\alpha dt^\alpha + 0\, dz$$

one has

$$\tilde{\nabla}_\alpha \xi_\beta = \partial_\alpha \xi_\beta - z\, c_{\alpha\beta}^\gamma \xi_\gamma$$

$$\tilde{\nabla}_{\frac{d}{dz}} \xi_\beta = \partial_z \xi_\beta - E^\gamma c_{\gamma\beta}^\alpha \xi_\alpha + \frac{1}{z} M_\beta^\epsilon \xi_\epsilon$$

where we denote M_β^ϵ the matrix entries of the linear operator $\mu = \frac{1}{2}(2-d) - \nabla E$. Any solution of the system $\tilde{\nabla}\xi = 0$ is a (local) horizontal section of $T^* (M \times \mathbf{C})$ for the connection $\tilde{\nabla}$. A basis of horizontal sections is given by dz and by n linearly independent solutions of the system

$$\partial_\alpha \xi_\beta = z\, c_{\alpha\beta}^\gamma \xi_\gamma \tag{2.31}$$

$$\partial_z \xi_\beta = E^\gamma c_{\gamma\beta}^\alpha \xi_\alpha - \frac{1}{z} M_\beta^\epsilon \xi_\epsilon. \tag{2.32}$$

Such a basis exists *iff* the compatibility conditions

$$\partial_\alpha \partial_\gamma = \partial_\gamma \partial_\alpha,\ \ \partial_\alpha \partial_z = \partial_z \partial_\alpha$$

hold true. Differentiating (2.31) w.r.t. t^γ and subtracting the same expression with α and γ permuted we obtain the first compatibility condition in the form

$$z\ \left(\partial_\gamma c_{\alpha\beta}^\epsilon - \partial_\alpha c_{\gamma\beta}^\epsilon\right) \xi_\epsilon + z^2 \left(c_{\alpha\beta}^\lambda c_{\lambda\gamma}^\epsilon - c_{\gamma\beta}^\lambda c_{\lambda\alpha}^\epsilon\right) \xi_\epsilon = 0.$$

This must vanish for arbitrary ξ. We obtain

$$c_{\alpha\beta}^\lambda c_{\lambda\gamma}^\epsilon = c_{\gamma\beta}^\lambda c_{\lambda\alpha}^\epsilon$$

(associativity) and

$$\partial_\gamma c_{\alpha\beta}^\epsilon = \partial_\alpha c_{\gamma\beta}^\epsilon$$

(local existence of $F(t)$). Similarly, from compatibility of (2.31) and (2.32) we obtain, first, that

$$\partial_\alpha M_\beta^\epsilon = 0.$$

So ∇E is a constant matrix. Assume, for simplicity, ∇E to be diagonal, $\nabla E = \mathrm{diag}\,(1 - q_1, \ldots, 1 - q_n)$. Then we further obtain

$$E^\epsilon \partial_\epsilon c_{\alpha\beta}^\gamma = (q_\alpha + q_\beta - q_\gamma) c_{\alpha\beta}^\gamma.$$

As we already know, this together with (2.30) (i.e., with (2.6)) is equivalent to FM3. Proposition is proved.

Remark 2.2. Due to Proposition, one can alternatively define Frobenius manifolds as those carrying a metric and a linear pencil of affine connections (2.28a) satisfying the above conditions (such a definition was explicitly used in [Du4]). It is interesting that manifolds with a metric and a linear pencil of affine connections deforming the Levi-Civita connection are known also in mathematical statistics - see the book [Ama]. This structure appeares in the parametric statistics that studies families of probabilistic measures depending on a finite number of parameters. The metric was introduced by Rao about 1945 using the classical Fischer matrix of the family. The deformed Levi-Civita connection was discovered by N.N.Chentsov in 1972. It has the form (2.28a). However, the curvature of the deformed connection does not vanish identically but it vanishes for two values of the parameter z.

Exercise 2.6. Prove that the solutions of the linear system (2.31), (2.32) are all closed differential forms

$$\xi_\alpha dt^\alpha = d\,\tilde t$$

(the differential along M only).

Choosing a basis of n linearly independent solutions $\xi_\alpha^{(1)}, \dots, \xi_\alpha^{(n)}$ of the system we obtain n functions $\tilde t_1(t,z), \dots, \tilde t_n(t,z)$. Together with z they give a system of flat coordinates for the connection $\tilde\nabla$ on a domain in $M \times \mathbf{C}$. This means that, in these coordinates, the covariant derivatives coincide with partial ones.

How to choose a basis of the deformed flat coordinates $\tilde t_1(t,z), \dots, \tilde t_n(t,z)$? Let us forget first about the last component (2.28b) of the connection $\tilde\nabla$. The first part (2.28a) can be considered as a deformation of the affine structure on M with z being the parameter of the deformation. We can look for the deformed flat coordinates in the form of the series

$$\tilde t_\alpha = \sum_{p=0}^\infty h_{\alpha,p}(t) z^p =: h_\alpha(t;z),\ \ \alpha = 1, \dots, n. \tag{2.33}$$

Lemma 2.2. *The coefficients $h_{\alpha,p}(t)$ can be determined recursively from the relations*

$$\begin{aligned} h_{\alpha,0} &= t_\alpha \equiv \eta_{\alpha\epsilon} t^\epsilon \\ \partial_\beta\partial_\gamma h_{\alpha,p+1} &= c_{\beta\gamma}^\epsilon \partial_\epsilon h_{\alpha,p},\ \ p = 0,\, 1,\, 2, \dots \end{aligned} \tag{2.34}$$

uniquely up to a transformation of the form

$$\tilde t_\alpha \mapsto \sum_{\beta=1}^n \tilde t_\beta G_\alpha^\beta(z)$$

where the coefficients G_1, G_2, … of the matrix-valued series

$$G(z) = \left(G_\alpha^\beta(z)\right) = 1 + z\,G_1 + z^2 G_2 + \dots$$

do not depend on t.

Proof. We are only to show that the right-hand sides of (2.34) are second derivatives along t^β and t^γ. This can be proved inductively using the identity

$$\partial_\alpha\left(c_{\beta\gamma}^\epsilon \partial_\epsilon h_{\lambda,p}\right) - \partial_\beta\left(c_{\alpha\gamma}^\epsilon \partial_\epsilon h_{\lambda,p}\right) = \left(c_{\beta\gamma}^\epsilon c_{\epsilon\alpha}^\rho - c_{\alpha\gamma}^\epsilon c_{\epsilon\beta}^\rho\right)\partial_\rho h_{\lambda,p-1} = 0$$

due to associativity.

The gradients $\nabla h_{\alpha,p}(t)$ and their inner products $< \nabla h_{\alpha,p}, \nabla h_{\beta,q} >$ play a very important role in the theory and applications of Frobenius manifolds. Before discussing how to normalize them uniquely I will give here two important identities for these coefficients.

Exercise 2.7. Prove that

$$t_\alpha =< \nabla h_{\alpha,0}, \nabla h_{1,1} > \tag{2.35}$$

$$\begin{aligned} F(t) = \frac{1}{2} \{ &< \nabla h_{\alpha,1}, \nabla h_{1,1} > \eta^{\alpha\beta} < \nabla h_{\beta,0}, \nabla h_{1,1} > \\ &- < \nabla h_{1,1}, \nabla h_{1,2} > - < \nabla h_{1,3}, \nabla h_{1,0} > \} . \end{aligned} \tag{2.36}$$

Exercise 2.8. Prove the identity

$$\nabla \langle \nabla h_\alpha(t;z), \nabla h_\beta(t;w) \rangle = (z+w) \nabla h_\alpha(t;z) \cdot \nabla h_\beta(t;w).$$

Observe that $\langle \nabla h_\alpha(t;z), \nabla h_\beta(t;-z) \rangle$ does not depend on t.

To choose canonically the system of deformed flat coordinates $\tilde t_1(t;z), \ldots, \tilde t_n(t;z)$ we will now use the last equation (2.32) of the horizontality of the gradients $\xi_\alpha = \partial_\alpha \tilde t(t;z)$

$$\partial_z \xi_\alpha = \mathcal{U}_\alpha^\beta \xi_\beta - \frac{1}{z} \mu_\alpha \xi_\alpha. \tag{2.37}$$

Here

$$\mathcal{U}_\alpha^\beta(t) := E^\epsilon c_{\epsilon\alpha}^\beta \tag{2.38}$$

is the matrix of multiplication by the Euler vector field. The choice of the basis can be done by carefully looking at the behaviour of the solutions at $z = 0$ where the connection $\tilde\nabla$ has logarithmic singularity. The analysis of this behaviour will provide us with some numerical invariants of the Frobenius manifold.

Let us introduce the numbers

$$\mu_\alpha = q_\alpha - \frac{d}{2}, \ \alpha = 1, \ldots, n.$$

Recall that they are the entries of the diagonal matrix

$$\mu = \frac{2-d}{2} 1 - \nabla E = \mathrm{diag}\,(\mu_1, \ldots, \mu_n). \tag{2.39}$$

The operator μ is antisymmetric w.r.t. the metric $< \, , \, >$

$$< \mu\, a, b > + < a, \mu\, b >= 0. \tag{2.40}$$

We say that the operator μ is *resonant* if some of the differences $\mu_\alpha - \mu_\beta$ is a nonzero integer. Otherwise it is called *nonresonant.* We will also use expressions resonant/nonresonant Frobenius manifold if the corresponding operator μ is resonant/nonresonant. For example, any Frobenius manifold related to quantum cohomology is resonant (all the numbers q_α

are integers). The Frobenius manifold on the space of versal deformations of A_3 singularity is nonresonant one

$$\mu = \operatorname{diag}\left(-\frac{1}{4}, 0, \frac{1}{4}\right).$$

We consider first the nonresonant case. In this case the system of the deformed flat coordinates can be uniquely chosen in such a way that

$$\tilde{t}_\alpha(t;z) = [t_\alpha + O(z)]\ z^{\mu_\alpha},\ \ \alpha = 1, \ldots, n$$

(here, as usual, $t_\alpha = \eta_{\alpha\epsilon} t^\epsilon$). The coordinates are multivalued analytic functions of z defined for sufficiently small $z \neq 0$. Going along a closed loop around $z = 0$ one obtains the monodromy transformation

$$\left(\tilde{t}_1, \ldots, \tilde{t}_n\right) \mapsto \left(\tilde{t}_1, \ldots, \tilde{t}_n\right)\ M_0,$$

$$M_0 = \operatorname{diag}\left(e^{2\pi i \mu_1}, \ldots, e^{2\pi i \mu_n}\right).$$

For the resonant case such a choice is not possible. The monodromy matrix M_0 cannot be diagonalized.

We will first rewrite the equation (2.32) in the matrix form. Doing the linear change

$$\xi = (\xi_1, \ldots, \xi_n) \mapsto \eta^{-1} \xi^T$$

rewrite (2.31), (2.32) as follows

$$\partial_\alpha \xi = z\, C_\alpha \xi \tag{2.41a}$$

$$\partial_z \xi = \left(\mathcal{U} + \frac{1}{z}\mu\right)\xi \tag{2.41b}$$

where

$$(C_\alpha)_\beta^\gamma = c_{\alpha\beta}^\gamma.$$

The matrix $\mathcal{U}$ is η-symmetric

$$\mathcal{U}^T \eta = \eta\, \mathcal{U} \tag{2.42}$$

and μ is η-antisymmetric

$$\mu\, \eta + \eta\, \mu = 0. \tag{2.43}$$

The solutions of the system (2.41) are gradients of the deformed flat coordinates $\xi = \nabla\, \tilde{t}$.

Lemma 2.3. *The bilinear form*

$$\langle \xi_1, \xi_2 \rangle := \xi_1^T(-z)\, \eta\, \xi_2(z) \tag{2.44}$$

on the space of solutions of (2.41b) does not depend on z.

Proof is obvious.

Let us study the classes of equivalence of the system (2.41b) under gauge transforms

$$\xi \mapsto G(z)\,\xi \tag{2.45a}$$

of the form

$$G(z) = 1 + z\,G_1 + z^2 G_2 + \dots \tag{2.45b}$$

$$G^T(-z)\eta G(z) \equiv \eta. \tag{2.45c}$$

Lemma 2.4. *After an arbitrary gauge transform (2.45) the vector-function*

$$\tilde{\xi} = G(z)\xi \tag{2.46}$$

satisfies the system

$$\partial_z \tilde{\xi} = \left(\frac{1}{z}\mu + \tilde{U}_1 + z\,\tilde{U}_2 + \dots\right)\tilde{\xi}$$

where the matrices $\tilde{U}_{2k+1}$ are η-symmetric and the matrices $\tilde{U}_{2k}$ are η-antisymmetric.

Proof. After the gauge transform the vector function $\tilde{\xi}$ satisfies

$$\partial_z \tilde{\xi} = A(z)\tilde{\xi}$$

with

$$\begin{aligned} A(z) &= G(z)\left(\frac{1}{z}\mu + \mathcal{U}\right)G^{-1}(z) + G'(z)G^{-1}(z) \\ &=: \frac{1}{z}\mu + \tilde{U}_1 + z\,\tilde{U}_2 + \dots \end{aligned}$$

where the matrix coefficients $\tilde{U}_k$ are defined by this equation. Using (2.42), (2.43) and (2.45c) one obtains

$$A^T(-z) = \eta\,A(z)\,\eta^{-1}.$$

This gives

$$\tilde{U}_k^T = (-1)^{k+1}\eta\,\tilde{U}_k\,\eta^{-1},\ k = 1, 2, \dots.$$

Lemma is proved.

In the nonresonant case one can choose the gauge transform in such a way that all $\tilde{U}_1 = \tilde{U}_2 = \dots = 0$. The only gauge invariant of the system (2.41b) near the logarithmic singularity $z = 0$ is the diagonal matrix μ.

Let us consider slightly more general system

$$\partial_z \xi = \left(\frac{1}{z}\mu + U_1 + z\,U_2 + z^2\,U_3 + \dots\right)\xi \tag{2.47}$$

with the coefficients satisfying

$$U_k^T = (-1)^{k+1}\eta\,U_k\,\eta^{-1},\ k = 1, 2, \dots. \tag{2.48}$$

Lemma 2.5. *By a gauge transformation of the form (2.45) the system (2.47) can be reduced to the canonical form*

$$\partial_z \tilde{\xi} = \left(\frac{1}{z}\mu + R_1 + z\, R_2 + z^2\, R_3 + \dots \right) \tilde{\xi} \tag{2.49}$$

where the matrices $R_1,\ R_2,\ \dots$ *satisfy*

$$R_k^T = (-1)^{k+1} \eta\, R_k\, \eta^{-1} \tag{2.50}$$

$$(R_k)^\alpha_\beta \neq 0 \text{ only if } \mu_\alpha - \mu_\beta = k,\ \ k = 1,\, 2, \dots. \tag{2.51}$$

Observe that there is only a finite number of nonzero matrices R_k.

Proof. Gauge equivalence of the system (2.47) to a system (2.49) with the matrices R_k satisfying (2.51) is a wellknown fact (see, e.g. [Ga]). Namely, from the recursion relations

$$R_n = U_n + n\, G_n + [G_n, \mu] + \sum_{k=1}^{n-1} (G_{n-k} U_k - R_k G_{n-k}) \tag{2.52}$$

we determine uniquely the matrix entries

$$(R_n)^\alpha_\beta \ \text{ for } \mu_\alpha - \mu_\beta = n$$

and

$$(G_n)^\alpha_\beta \ \text{ for } \mu_\alpha - \mu_\beta \neq n$$

and we put

$$(G_n)^\alpha_\beta = 0 \text{ for } \mu_\alpha - \mu_\beta = n.$$

Using induction it can be easily seen that the matrices R_n satisfy the η-symmetry/antisymmetry conditions (2.50) and the matrices G_n satisfy the orthogonality conditions (2.53)

$$G_n^T = (-1)^{n+1} \eta\, G_n\, \eta^{-1} + \sum_{k=1}^{n-1} (-1)^{n+k+1} G_k^T \eta\, G_{n-k} \eta^{-1}. \tag{2.53}$$

Lemma is proved.

We will call (2.49) *the normal form* of the system (2.47). The ambiguity in the choice of the normal form will be described below.

Lemma 2.6. *The matrix solution of the system (2.49) is*

$$\xi = z^\mu z^R \tag{2.54}$$

where

$$R := R_1 + R_2 + \dots. \tag{2.55}$$

Proof. From (2.51) we obtain the identity

$$z^\mu R_k z^{-\mu} = z^k R_k,\ \ k = 1,\, 2, \dots. \tag{2.56}$$

So, differentiating the matrix-valued function (2.54) one obtains

$$\begin{aligned}\partial_z \xi &= \frac{\mu}{z} z^\mu z^R + \frac{1}{z} z^\mu R z^R \\ &= \left(\frac{\mu}{z} + R_1 + R_2 z^2 + \dots\right) z^\mu z^R.\end{aligned}$$

Lemma is proved.

Exercise 2.9. Prove that the monodromy around $z = 0$ of the solution

$$\Xi_0 := z^\mu z^R$$

has the form

$$\begin{aligned}\Xi_0\left(z\, e^{2\pi i}\right) &= \Xi_0(z) M_0 \\ M_0 &= \exp 2\pi i(\mu + R).\end{aligned} \tag{2.57}$$

We will now represent the parameters of the normal form (2.49) in a geometric way. Let $\mathcal{V}$ be a linear space equipped with a symmetric nondegenerate bilinear form $<\, ,\, >$ and an antisymmetric operator

$$\mu : \mathcal{V} \to \mathcal{V},\ \ \ < \mu\, a, b > + < a, \mu\, b >= 0.$$

Let us assume, for simplicity, the operator μ to be diagonalizable. Let

$$\text{spec} := \{\mu_1, \dots, \mu_n\}$$

be the spectrum of μ. Denote $e_1,\ \dots,\ e_n$ the corresponding eigenvectors. We define a filtration on V

$$\begin{aligned}0 = F_0 \subset F_1 \subset F_2 \subset \dots \subset \mathcal{V} \\ F_k := \text{span}\,\{e_\alpha | \mu_\alpha + k \notin \text{ spec}\}.\end{aligned} \tag{2.58}$$

Obviously, for a non-resonant μ the filtration consists of two terms $0 = F_0 \subset F_1 = \mathcal{V}$.

The asociated graded space

$$\begin{aligned}\mathcal{V}_* &= \oplus_{k\geq 1} \mathcal{V}_k \\ \mathcal{V}_k &= F_k / F_{k-1}\end{aligned}$$

is isomorphic to $\mathcal{V}$ due to the natural isomorphism

$$\mathcal{V}_k \simeq \text{Ker}\,(\mu + k - 1) \cap F_k \subset \mathcal{V}.$$

A linear operator

$$R : \mathcal{V} \to \mathcal{V}$$

is called μ-*nilpotent* if it commutes with $\exp 2\pi i\mu$

$$R\, e^{2\pi i\mu} = e^{2\pi i\mu} R \tag{2.59}$$

and

$$R(F_k) \subset F_{k-1},\ k = 1,\, 2, \dots . \tag{2.60}$$

The associated operator

$$R_* : \mathcal{V}_* \to \mathcal{V}_*$$

has a natural grading

$$R_* = \oplus_{k\geq 1} R_k \tag{2.61}$$

where the operator R_k shifts the grading by $-k$

$$R_k\left(\mathcal{V}_m\right) \subset \mathcal{V}_{m-k} \text{ for any } m > k. \tag{2.62}$$

Writing all the operators by matrices in the basis of eigenvectors of μ one obtains

$$(R_k)^\alpha_\beta = \begin{cases} R^\alpha_\beta \text{ if } \mu_\alpha - \mu_\beta = k \\ 0 \ \text{ otherwise.} \end{cases} \tag{2.63}$$

We say that the μ-nilpotent operator is μ-*skew-symmetric* if

$$\{R\,x, y\} + \{x, R\,y\} = 0 \text{ for any } x,\ y \in \mathcal{V} \tag{2.64}$$

where

$$\{x, y\} := \left\langle e^{\pi i\mu} x, y\right\rangle . \tag{2.65}$$

The corresponding graded components R_k satisfy the following conditions

$$\langle R_k x, y\rangle = (-1)^{k+1} \langle x, R_k y\rangle \ \text{ for any } x,\ y \in \mathcal{V}. \tag{2.66}$$

We conclude that the normal form of the system (2.47) is a quadruple

$$(\mathcal{V}, <\ ,\ >, \mu, R) \tag{2.67}$$

where

$$\mathcal{V} = \operatorname{span}\left(e_1, \dots, e_n\right)$$

is n-dimensional space with a bilinear symmetric form

$$< e_\alpha, e_\beta >= \eta_{\alpha\beta}$$

and an antisymmetric operator

$$\mu = \operatorname{diag}\left(\mu_1, \dots, \mu_n\right)$$

and a μ-nilpotent μ-skew-symmetric operator $R : \mathcal{V} \to \mathcal{V}$.

Let us describe now the ambiguity in the choice of the normal form data. We say that

$$G : \mathcal{V} \to \mathcal{V} \tag{2.68}$$

is a *μ-parabolic orthogonal operator* if

$$G = 1 + \Delta \tag{2.69}$$

where Δ is a μ-nilpotent operator and G satisfies the following orthogonality condition w.r.t. the bilinear form (2.65)

$$\{G\,x, G\,y\} = \{x, y\}. \tag{2.70}$$

Representing Δ as a sum of the graded components

$$\Delta \simeq \Delta_* = \oplus_{k\geq 1}\Delta_k \tag{2.71}$$

one rewrites the orthogonality condition in the form

$$\left(1 - \Delta_1^T + \Delta_2^T - \dots\right)\,\eta\,\left(1 + \Delta_1 + \Delta_2 + \dots\right) = \eta. \tag{2.72}$$

Exercise 2.10. Prove that the monodromy operator (2.57) is orthogonal w.r.t. the bilinear form (2.65).

Clearly all μ-parabolic orthogonal operators G form a group denoted $\mathcal{G}\,(\mu, < \ , \ >)$. The space of all μ-nilpotent μ-skew-symmetric operators R coincides with the Lie algebra of the nilpotent group. The group $\mathcal{G}\,(\mu, < \ , \ >)$ acts on this space by conjugations

$$R \mapsto G^{-1} R\,G. \tag{2.73}$$

In the grading components one has

$$\begin{aligned} &R_1 \mapsto R_1 \\ &R_2 \mapsto R_2 + [R_1, \Delta_1] \\ &R_3 \mapsto R_3 + [R_2, \Delta_1] - \Delta_1 R_1 \Delta_1 + [R_1, \Delta_2] + \Delta_1^2 R_1 \end{aligned} \tag{2.74}$$

etc. Two μ-nilpotent μ-skew-symmetric operators related by a conjugation (2.73) will be called *equivalent*.

Lemma 2.7. *The set of all normal forms at $z = 0$ of a given system (2.47) is in one-to-one correspondence with the orbit of one normal form w.r.t. the action (2.73) of the group $\mathcal{G}\,(\mu, < \ , \ >)$.*

Proof. Let us consider two normal forms of the same system (2.47)

$$\begin{aligned} \partial_z \xi &= \left(\frac{\mu}{z} + R_1 + R_2 z + \dots\right)\xi \\ \partial_z \tilde{\xi} &= \left(\frac{\mu}{z} + \tilde{R}_1 + \tilde{R}_2 z + \dots\right)\tilde{\xi}. \end{aligned}$$

They must be related by a gauge transformation

$$\tilde{\xi} = G(z)\xi$$

where

$$G(z) = 1 + \Delta_1 z + \Delta_2 z^2 + \dots$$

satisfies (2.45c). Explicitly we obtain a system of relations identical to (2.53)

$$\tilde{R}_n = R_n + n\,\Delta_n + [\Delta_n, \mu] + \sum_{k=1}^{n-1} \left(\Delta_{n-k} R_k - \tilde{R}_k \Delta_{n-k}\right),\ n = 1,\, 2, \dots$$

From this system we recursively prove that

$$(\Delta_n)^\alpha_\beta = 0 \text{ unless } \mu_\alpha - \mu_\beta = n.$$

So

$$G := G(1) = 1 + \Delta_1 + \Delta_2 + \dots$$

is a μ-parabolic orthogonal operator $G : \mathcal{V} \to \mathcal{V}$. From (2.45c) we derive the orthogonality condition (2.70) in the form (2.72).

Let us derive the relation (2.73) for the operators

$$R = R_1 + R_2 + \dots, \quad \tilde{R} = \tilde{R}_1 + \tilde{R}_2 + \dots.$$

Since $\xi = z^\mu z^R$ is a solution to (2.49) (see Lemma 4),

$$\tilde{\xi} = G(z) z^\mu z^R$$

must be a solution to the system with tilde. So we must have

$$G(z) z^\mu z^R = z^\mu z^{\tilde{R}} C$$

for an invertible matrix C. Let us rewrite the last equation in the form

$$z^{-\mu} G(z) z^\mu z^R = z^{\tilde{R}} C.$$

Using the identities

$$z^{-\mu} \Delta_k z^\mu = z^{-k} \Delta_k$$

we finally obtain

$$(1 + \Delta_1 + \Delta_2 + \dots)\, z^R = G\, z^R = z^{\tilde{R}}.$$

Expanding

$$G\, z^R = G \left(1 + R \log z + \frac{R^2}{2!} \log^2 z + \dots\right)$$
$$= z^{\tilde{R}} C = \left(1 + \tilde{R} \log z + \frac{\tilde{R}^2}{2!} \log^2 z + \dots\right) C$$

and equating the coefficients in front of various powers of $\log z$ we obtain

$$C = G$$
$$G\,R = \tilde{R}\,G.$$

Lemma is proved.

Definition 2.4. A quadruple

$$(\mathcal{V}, <\ ,\ >, \mu, [R]) \tag{2.75}$$

where $\mathcal{V}$ is n-dimensional linear space with a bilinear symmetric form $<\ ,\ >$, an antisymmetric diagonalizable operator μ and a class of equivalence $[R]$ of normal forms (2.49) of the system (2.47) will be called *monodromy data at* $z = 0$ of this system.

Lemma 2.8. *Two systems of the form*

$$\partial_z \xi^{(i)} = \left(\frac{\mu}{z} + \sum_{k\geq 1} U_k^{(i)} z^{k-1}\right) \xi^{(i)}, \quad i = 1,\, 2 \tag{2.76}$$

satisfying

$$U_k^{(i)\,T} = (-1)^{k+1} \eta\, U_k^{(i)} \eta^{-1}$$

are equivalent w.r.t. a gauge transform of the form (2.45) iff *they have the same monodromy data (2.75).*

Proof. Let the gauge transformations

$$\tilde{\xi}^{(i)} = G^{(i)}(z) \xi^{(i)}, \quad i = 1,\, 2$$

reduce the systems (2.76) to the normal forms

$$\partial_z \tilde{\xi}^{(i)} = \left(\frac{\mu}{z} + \sum_{k\geq 1} R_k^{(i)} z^{k-1}\right) \tilde{\xi}^{(i)}, \quad i = 1,\, 2.$$

If

$$R^{(1)} = G\,R^{(2)} G^{-1}$$

with

$$G = 1 + \Delta_1 + \ldots \in \mathcal{G}\,(\mu, <\ ,\ >)$$

then the gauge transformation

$$\tilde{\xi}^{(2)} = (1 + z\,\Delta_1 + z^2 \Delta_2 + \ldots)\, \tilde{\xi}^{(1)}$$

establishes a gauge equivalence of the systems (2.76) for $i = 1$ and $i = 2$. Thus the systems (2.76) are gauge equivalent with

$$\xi^{(2)} = {G^{(2)}}^{-1}(z) \left(1 + z\,\Delta_1 + z^2 \Delta_2 + \ldots\right) G^{(1)}(z)\, \xi^{(1)}.$$

Conversely, from Lemma 2.7 it follows that gauge equivalent systems have the same monodromy data. Lemma is proved.

We will now return to Frobenius manifolds. The last component (2.41b) of the system determining horizontal sections of the connection $\tilde\nabla$ is a linear system of ODEs with rational coefficients of the form (2.47). The coefficients $\mathcal{U}^\alpha_\beta(t) = E^\epsilon(t)c^\alpha_{\epsilon\beta}(t)$ depend on the point t of the Frobenius manifold as on the parameter. The solutions ξ take values in the space $\mathcal{V} = T_tM$. We may identify the tangent planes at different points t using the Levi-Civita connection ∇ on M. Actually, the space $\mathcal{V}$ is equipped with an additional structure, namely, a marked vector $e \in \mathcal{V}$. This is an eigenvector of the linear operator μ with the eigenvalue $-d/2$.

We now show

Isomonodromicity Theorem (first part). *The monodromy data at $z = 0$ of the system (2.41b) do not depend on $t \in M$.*

Proof. The matrix μ and the bilinear form $< \, , \, >$ are t-independent by construction. We will now construct a t-independent representative R of the normal form (2.49) of the equation (2.41b)

$$\partial_z\xi = \left(\mathcal{U} + \frac{\mu}{z}\right)\xi$$

$$\mathcal{U}^\alpha_\beta(t) = E^\epsilon(t)c^\alpha_{\epsilon\beta}(t).$$

Let us choose a basis $h_1(t;z)$, ..., $h_n(t;z)$ of solutions of the system

$$\partial_\alpha\partial_\beta h_\gamma(t;z) = z\, c^\epsilon_{\alpha\beta}(t)\partial_\epsilon h_\gamma(t;z)$$

of the form (2.33), (2.34). Multiplying, if necessary, the series

$$h_\alpha(t;z) = t_\alpha + \sum_{p\geq 1} h_{\alpha,p}(t)z^p$$

by a matrix-valued series

$$M(z) \equiv \left(M^\alpha_\beta(z)\right) = 1 + \sum z^k M_k$$

$$h_\alpha(t;z) \mapsto \sum_\epsilon h_\epsilon(t;z)M^\epsilon_\alpha(z)$$

with t-independent coefficients M_1, M_2, ... we obtain the identity

$$\langle\nabla\, h_\alpha(t;-z), \nabla\, h_\beta(t;z)\rangle = \eta_{\alpha\beta}$$

(see Exercise 2.8). Let us construct a $n \times n$-matrix series $G(t;z) = \left(G^\alpha_\beta(t;z)\right)$

$$G^\alpha_\beta(t;z) = \eta^{\alpha\epsilon}\partial_\epsilon h_\beta(t;z)$$
$$G(t;z) = 1 + \sum_{k\geq 1} G_k(t)z^k.$$

The identity (2.45c) reads

$$G^T(t;-z)\,\eta\,G(t;z)\equiv\eta.$$

We do now a gauge transform in the system (2.41)

$$\xi = G(t;z)\,\tilde\xi.$$

Since G is a matrix solution of the equations

$$\tilde\nabla_\alpha G = 0, \quad \alpha = 1,\ldots,n,$$

we obtain

$$\partial_\alpha\tilde\xi = 0, \quad \alpha = 1,\ldots,n.$$

The system (2.41b) after the gauge transform will read in a form (2.47) with the matrices U_1, U_2, ... satisfying the symmetry/antisymmetry conditions (2.48)

$$\partial_z\tilde\xi = \left(\frac{\mu}{z} + U_1 + z\,U_2 + \ldots\right)\tilde\xi.$$

The full system of the last two equations remains to be compatible after the gauge transform. This implies t-independency of the coefficients U_1, U_2, ... Hence the normal form of the system (2.41b) does not depend on t. Theorem is proved.

Definition 2.5. The monodromy data (2.75) of the system (41b) are called *monodromy data of the Frobenius manifold at $z=0$.*

Explicitly,

$$R_1{}^\alpha_\beta = \mathcal{U}^\alpha_\beta \text{ for } \mu_\alpha-\mu_\beta = 1 \tag{2.77a}$$

$$R_2{}^\alpha_\beta = \sum_{\mu_\alpha-\mu_\gamma\neq 1}\frac{\mathcal{U}^\alpha_\gamma\mathcal{U}^\gamma_\beta}{\mu_\alpha-\mu_\gamma-1} \text{ for } \mu_\alpha-\mu_\beta = 2 \tag{2.77b}$$

etc.

Exercise 2.11. Prove, using the formula (2.36), that the normalized coefficients $A_{\alpha\beta}$, B_α, C in the quasihomogeneity equation (WDVV3) for the free energy $F(t)$ have the form

$$A_{\alpha\beta} = R_1{}^\epsilon_\alpha\eta_{\epsilon\beta}, \tag{2.78}$$

particularly,

$$r_\alpha = R_1{}^\alpha_1, \tag{2.79}$$

$$B_\alpha = R_2{}^\epsilon_\alpha\eta_{\epsilon 1} \tag{2.80}$$

$$C = -\frac{1}{2}R_3{}^\epsilon_1\eta_{\epsilon 1}. \tag{2.81}$$

Remark 2.3. We defined our monodromy data as *formal invariants*, i.e., all the gauge transforms (2.45) were defined by formal power series $G(z) = 1 + G_1 z + \ldots$. It is wellknown, however, that at a regular singularity of the system (2.41b) formal invariants coincide with analytic ones [CL]. In other words, all the normalizing transformations are convergent series for sufficiently small $|z|$.

We obtain

Theorem 2.1. *For any Frobenius manifold with the monodromy data* $(\mathcal{V}, < \ >, \mu, [R])$ *there exists a fundamental matrix*

$$\begin{aligned} \Xi^0(t;z) &= H(t;z) z^\mu z^R \\ H(t;z) &= 1 + H_1(t) z + H_2(t) z^2 = \dots \end{aligned} \tag{2.82}$$

for the system (2.41) defining horizontal sections of $\tilde{\nabla}$. *The power series converges for sufficiently small* $|z|$. *Here* R *is a representative of the class* $[R]$.

A change of the representative

$$R \mapsto \tilde{R} = G^{-1} R G$$

where

$$G = 1 + \Delta_1 + \Delta_2 + \dots$$

is a μ*-parabolic orthogonal operator, transforms*

$$\Xi^0 \mapsto \tilde{\Xi}^0 = \Xi^0 G.$$

The power series transforms as follows

$$\tilde{H}(t;z) \left(1 + z\,\Delta_1 + z^2 \Delta_2 + \dots\right) = H(t;z).$$

A choice of the fundamental matrix $\Xi^0(t;z)$ *determines a system of deformed flat coordinates* $\tilde{t}_1(t;z)$, $\dots$, $\tilde{t}_n(t;z)$ *such that the gradients* $\nabla \tilde{t}_1$, $\dots$, $\nabla \tilde{t}_n$ *are the columns of the matrix* Ξ^0

$$\left(\tilde{t}_1(t;z), \dots, \tilde{t}_n(t;z)\right) = \left(h_1(t;z), \dots, h_n(t;z)\right) z^\mu z^R. \tag{2.83}$$

The functions $\tilde{t}_\alpha(t;z)$ *are determined uniquely up to* t*-independent shifts and* μ*-parabolic transformations.*

We will call the functions $\tilde{t}_\alpha(t;z)$ *the normalized deformed flat coordinates* on the Frobenius manifold. Recall that the columns of the matrix $H(t;z)$ are gradients of the functions $h_1(t;z)$, $\dots$, $h_n(t;z)$. The Taylor expansions of these functions for small $|z|$ have the form

$$h_\alpha(t;z) = t_\alpha + \sum_{p=1}^{\infty} h_{\alpha,p}(t) z^p$$

with the coefficients $h_{\alpha,p}(t)$ satisfying the system of recursion relations (2.34). But now the coefficients are determined uniquely within the ambiguity given by the action of the group of μ-parabolic orthogonal transformations and up to a t-independent shift.

Exercise 2.12. For a μ-nilpotent operator

$$R = R_1 + R_2 + \dots$$

define the operators $R_{k,l}$ putting

$$\begin{aligned} R_{0,0} &= 1 \\ R_{k,0} &= 0, \quad k > 0 \\ R_{k,l} &= \sum_{i_1+\dots i_l = k} R_{i_1} \dots R_{i_l} \end{aligned} \tag{2.84}$$

Prove that the normalized deformed flat coordinates have the following expansion near $z = 0$:

$$\tilde{t}_\alpha(t; z) = \sum_{k,\, l \geq 0} \sum_{p=0}^{k} \sum_{\epsilon} h_{\epsilon,p}(t) \left(R_{k-p,l}\right)_\alpha^\epsilon z^{k+\mu_\alpha} \frac{\log^l z}{l!}. \tag{2.85}$$

Exercise 2.13. Derive the following quasihomogeneity conditions for the gradients of the functions $h_{\alpha,k}(t)$

$$\mathcal{L}_E \nabla h_{\alpha,k} = \left(k + \frac{1}{2}(d-2) + \mu_\alpha\right) \nabla h_{\alpha,k} + \sum_{\epsilon,\, p} \nabla h_{\epsilon,k-p} \left(R_p\right)_\alpha^\epsilon. \tag{2.86}$$

These quasihomogeneity conditions together with the relations (2.34) can serve as the recursive definition of the functions $h_{\alpha,k}$ starting from $h_{\alpha,0} = t_\alpha$.

We conclude this Lecture with descrption of the monodromy data of Frobenius manifold with good analytic properties (partuclarly, of quantum cohomologies).

Proposition 2.2. *For the Frobenius manifold of the form (2.9) all the matrices R_2, R_3, ... vanish and*

$$\left(R_1\right)_\beta^\alpha = \sum_\epsilon r_\epsilon c_{\epsilon\beta}^\alpha. \tag{2.87}$$

Here the numbers r_ϵ enter into the Euler vector field (2.4), $c_{\alpha\beta}^\gamma$ are the structure constants of the cubic part of $F(t)$ (2.9).

Proof. Due to Isomonodromicity Theorem it is sufficient to compute the monodromy data of the operator (2.41b) at the point $t = t_0$ of "classical limit". We have

$$\mathcal{U}_\beta^\alpha(t_0) = \sum_\epsilon r_\epsilon c_{\epsilon\beta}^\alpha.$$

The algebra A_{t_0} is graded by the degree

$$\deg e_\alpha = q_\alpha.$$

On the other hand, the vector

$$\sum_\epsilon r_\epsilon e_\epsilon$$

has degree one. Thus the operator (2.38) of multiplication in A_{t_0} by this vector increases degrees by one.Hence

$$\mathcal{U}^\alpha_\beta \neq 0 \text{ only if } q_\alpha - q_\beta = 1,$$

and the system

$$\partial_z \xi = \left(\mathcal{U}(t_0) + \frac{\mu}{z}\right)\xi$$

is already in the normal form (2.49) with $R = R_1 = \mathcal{U}(t_0)$. Proposition is proved.

Corollary. *In the quantum cohomology of a manifold X the monodromy data at $z = 0$ is the operator $R = R_1$ of multiplication by the first Chern class $c_1(X)$ acting in the classical cohomologies $H^*(X)$.*

Example 2.3. Let us explain who is the deformed connection (2.28) in the case of quantum cohomology of a “sufficiently good” (for example, smooth projective) $2d$-dimensional manifold X (the assumptions and notations are as in Lecture 1 above). Let $\phi \in H^*(X; \mathbf{C})$ be an arbitrary element. We will construct function $\tilde t_\phi(t; z)$, $t = (t', t'')$ as in (2.10), such that for any ϕ it satisfies

$$\tilde\nabla d\tilde t_\phi = 0. \tag{2.88}$$

Taking $\phi = \phi_1, \ldots, \phi = \phi_n$ for a basis in $H^*(X, \mathbf{C})$ we will obtain a system of flat coordinates of the deformed connection.

Denote Q the grading operator (1.4). We introduce a line bundle $\mathcal{L}$ on the moduli space $X_{[\beta],l}$. The fibre of this bundle in the point $(\beta, p_1, \ldots, p_l) \in X_{[\beta],l}$ is the cotangent line to the Riemann sphere at the first marked point p_1. Let

$$\sigma_1 := c_1(\mathcal{L}) \in H^2\left(X_{[\beta],l}\right).$$

Put

$$\tilde t_\phi(t; z) = z^{-\frac{d}{2}} \sum_{[\beta],l} \left\langle \frac{z^Q z^{c_1(X)}\phi}{1 - z\sigma_1} \otimes 1 \otimes e^{t''}\right\rangle_{[\beta],l} e^{\int_{S^2}\beta^*(t')}. \tag{2.89}$$

Here we define the symbols

$$\left\langle \frac{a_1}{1 - z\sigma_1} \otimes a_2 \otimes \ldots \otimes a_k \right\rangle_{[\beta],l}$$

as the formal series in z using the expansion

$$\frac{1}{1 - z\sigma_1} = 1 + z\sigma_1 + z^2\sigma_1^2 + \ldots,$$

and

$$\langle a_1\sigma_1^m \otimes a_2 \otimes \ldots \otimes a_k\rangle_{[\beta],l} = \begin{cases} 0, \ k \neq l \\ \int_{X_{[\beta],l}} \sigma_1^m \wedge p_1^*(a_1) \wedge p_2^*(a_2) \wedge \ldots \wedge p_l^*(a_l), \ k = l. \end{cases} \tag{2.90}$$

These are particular *gravitational descendants* arising in the description of coupling of the topological sigma-model (=quantum cohomology) to topological gravity [Wi2, Dij1, Dij2, Du3].

Theorem 2.2. *The function $\tilde t_\phi(t;z)$ for any $\phi \in H^*(X)$ satisfies the equation (2.88).*

Proof. Let us choose some basis $\phi_1, \ldots, \phi_n$ in $H^*(X)$. The formula (2.89) for the functions $\left(\tilde t_{\phi_1}(t;z), \ldots, \tilde t_{\phi_n}(t;z)\right)$ can be rewritten in the form

$$\left(\tilde t_{\phi_1}(t;z), \ldots, \tilde t_{\phi_n}(t;z)\right) = \left(h_1(t;z), \ldots, h_n(t;z)\right) z^\mu z^R$$

where the formal series $h_\alpha(t;z)$ have the form

$$h_\alpha(t;z) = t_\alpha + \sum_{p=1}^\infty h_{\alpha,p}(t) z^p, \ \alpha = 1, \ldots, n$$

$$h_{\alpha,p}(t) = \sum_{[\beta],l} \left\langle \sigma_1^p \phi_\alpha \otimes 1 \otimes e^{t''} \right\rangle_{[\beta],l} e^{\int_{S^2} \beta^*(t')}, \tag{2.91}$$

$$\mu = \mathrm{diag}\,(\mu_1, \ldots, \mu_n), \quad \mu_\alpha = q_\alpha - \frac{d}{2}, \ \phi_\alpha \in H^{2q_\alpha}(X),$$

R is the matrix of the operator of multiplication by the first Chern class $c_1(X)$ (cf. (2.83) above). To demonstrate (2.88) it is sufficient to prove that the coefficients $H_{\alpha,p}(t)$ satisfy the recursion relations

$$\partial_\lambda \partial_\mu h_{\alpha,p}(t) = c_{\lambda\mu}^\nu \partial_\nu h_{\alpha,p-1}(t), \ p \geq 1, \tag{2.92}$$

(see (2.34)) and

$$\mathcal{L}_E \nabla h_{\alpha,p} = \left(p + \frac{d-2}{2} + \mu_\alpha\right) h_{\alpha,p} + \nabla h_{\epsilon,p-1} (R)_\alpha^\epsilon. \tag{2.93}$$

The last equation is the particular case of (2.86) for the case of quantum cohomology where $R = R_1$ is the matrix of multiplication by the first Chern class and $R_2 = R_3 = \ldots = 0$ (see above Corollary from Proposition 2.2). The first relation (2.92) follows from the genus 0 topological recursion relations of Dijkgraaf and Witten [DW] (see the derivation in [Du3, Du7]). The second follows from the recursion relations of Hori [Ho]. Theorem is proved.

In general we do not know anything about analytic properties of the series (2.89) for big $|z|$. But if the quantum cohomology algebra is semisimple (conjecturally, this is the case for Fano varieties X, see below Lecture 3) then the asymptotic behaviour of the series (2.89) for big $|z|$ is under control. The Stokes parameters of this asymptotic behaviour will give us in Lecture 4 additional parameters of the Frobenius manifold to determine it uniquely.

Example 2.4. We will now compute the flat coordinates of the deformed connection $\tilde\nabla$ for the Frobenius manifolds arising in the singularity theory. We consider here only the case of simple singularities $f(x)$ (see the definition in [AGV, Ar2]). (The formulation of K.Saito's theory of primitive forms in the setting of Frobenius manifolds for more general singularities can be found in [Sab], [Man3], [Ta].) Simple singularities are labelled by the simply-laced Dynkin diagrams A_n, D_n, E_6, E_7, E_8. Denote $f_t(x)$ the corresponding versal deformation. The variable x is one-dimensional for A_n or $x = (x_1, x_2)$ for other simple singularities. The parameters $t = (t^1, \dots, t_n)$ for A_n, D_n, E_n. Explicitly

$$
\begin{aligned}
A_n :&\ f_t(x) = x^{n+1} + a_n x^{n-1} + \ldots + a_1\\
D_n :&\ f_t(x) = x_1^{n-1} + x_1 x_2^2 + a_{n-1} x_1^{n-2} + \ldots + a_1 + b\, x_2\\
E_6 :&\ f_t(x) = x_1^4 + x_2^3 + a_6 x_1^2 x_2 + a_5 x_1 x_2 + a_4 x_1^2 + a_3 x_2 + a_2 x_1 + a_1\\
E_7 :&\ f_t(x) = x_1^3 x_2 + x_2^3 + a_1 + a_2 x_2 + a_3 x_1 x_2 + a_4 x_1 x_2^2 + a_5 x_1 + a_6 x_1^2 + a_7 x_2^2\\
E_8 :&\ f_t(x) = x_1^5 + x_2^3 + a_8 x_1^3 x_2 + a_7 x_1^2 x_2 + a_6 x_1^3 + a_5 x_1 x_2 + a_4 x_1^2 + a_3 x_2 + a_2 x_1 + a_1.
\end{aligned}
$$

The coefficients a_i are certain polynomials of the flat coordinates $t^1, \dots, t^n$. The dependence on the flat coordinates satisfies the following two remarkable identities [EYY2]

$$\phi_\alpha(x;t)\phi_\beta(x;t) = c_{\alpha\beta}^\gamma(t)\phi_\gamma(x;t) + K_{\alpha\beta}^a(x;t)\frac{\partial f_t(x)}{\partial x^a} \tag{2.94}$$

$$\partial_\alpha \phi_\beta(x;t) = \frac{\partial K_{\alpha\beta}^a(x;t)}{\partial x^a}. \tag{2.95}$$

Here

$$\phi_\alpha(x;t) = \frac{\partial f_t(x)}{\partial t^\alpha},\ \ \alpha = 1, \dots, n,$$

$K_{\alpha\beta}^a(x;t)$ are certain polynomials, the index a takes only one value for A_n or two values for D_n, E_n. The coefficients $c_{\alpha\beta}^\gamma(t)$ coincide with the structure constants of the corresponding Frobenius manifold.

Theorem 2.3. *The oscillatory integrals*

$$\tilde t_C(t;z) = z^{\frac{N-2}{2}} \int_C e^{z\, f_t(x)} d^N x \tag{2.96}$$

($N = 1$ for A_n and $N = 2$ for D_n, E_n) are flat coordinates of the deformed connection $\tilde\nabla$. Here C is any N-dimensional cycle in $\mathbf{C}^N$ that goes to infinity along the directions where $\mathrm{Re}\, z\, f_t(x) \to -\infty$.

Proof. We are to prove that the functions

$$\xi_\alpha = \partial_\alpha \tilde t(t;z) = z^{\frac{N}{2}} \int_C \phi_\alpha(x;t) e^{z\, f_t(x)} d^N x$$

satisfy the system (2.88). Using (2.94), (2.95) we obtain

$$\begin{aligned}
\partial_\alpha \xi_\beta &= z^{\frac{N}{2}} \int_C \frac{\partial K^a_{\alpha\beta}(x;t)}{\partial x^a} e^{z\, f_t(x)} d^N x \\
&+ z^{\frac{N+2}{2}} \int_C \left[c^\gamma_{\alpha\beta}(t)\phi_\gamma(x;t) + K^a_{\alpha\beta}(x;t) \frac{\partial f_t(x)}{\partial x^a} \right] e^{z\, f_t(x)} d^N x \\
&= z\, c^\gamma_{\alpha\beta}(t)\xi_\gamma + z^{\frac{N}{2}} \int_C \frac{\partial}{\partial x^a} \left[K^a_{\alpha\beta}(x;t) e^{z\, f_t(x)} \right] d^N x \\
&= z\, c^\gamma_{\alpha\beta}(t)\xi_\gamma
\end{aligned}$$

(we used Stokes formula

$$\int_C \frac{\partial}{\partial x^a} v^a d^N x = 0$$

for any vector field v^a vanishing on the boundary of C at infinity).

To demonstrate the second equation (2.41b) it is sufficient to prove that

$$z\partial_z \tilde{t}_C = \mathcal{L}_E \tilde{t}_C + \frac{d-2}{2} \tilde{t}_C.$$

Here the Euler vector field and d have the form

$$E = \sum_{\alpha=1}^n \frac{d_\alpha}{h} t^\alpha \partial_\alpha, \;\; d_\alpha = m_\alpha + 1$$

$$d = 1 - \frac{2}{h}$$

where h is the Coxeter number and m_α are the exponents of the corresponding Weyl group $W(A_n)$, $W(D_n)$, $W(E_n)$ (see Lecture 5 below). One can assign certain degrees r_1, r_2 to the variables x_1, x_2 (r_1 only for A_n) in such a way that the whole deformation $f_t(x)$ be a quasihomogeneous function of $t^1, \ldots, t^n$, x_1, x_2 of the degree 1. Explicitly,

$$\begin{aligned}
A_n &: \; r_1 = \frac{1}{n+1} \\
D_n &: \; r_1 = \frac{1}{n-1}, \; r_2 = \frac{n-2}{2n-2} \\
E_6 &: \; r_1 = \frac{1}{4}, \; r_2 = \frac{1}{3} \\
E_7 &: \; r_1 = \frac{2}{9}, \; r_2 = \frac{1}{3} \\
E_8 &: \; r_1 = \frac{1}{5}, \; r_2 = \frac{1}{3}
\end{aligned}$$

Note that the coefficients a_α of the versal deformations are quasihomogeneous polynomials of $t^1, \ldots, t^n$ of the degrees $d_\alpha/h = \deg t^\alpha$. The quasihomogeneity can be recasted in the form of the following Euler identity

$$\sum_a r_a x^a \frac{\partial f_t(x)}{\partial x^a} + \sum_\alpha \frac{d_\alpha}{h} t^\alpha \frac{\partial f_t(x)}{\partial t^\alpha} = f_t(x).$$

Using this identity we obtain

$$\begin{aligned}
\partial_z \tilde{t}_C &= \frac{N-2}{2z}\tilde{t}_C + z^{\frac{N-2}{2}} \int_C f_t(x) e^{z\, f_t(x)} d^N x \\
&= \frac{N-2}{2z}\tilde{t}_C + z^{\frac{N-2}{2}} \int_C \left[\sum_a r_a x^a \frac{\partial f_t(x)}{\partial x^a} + \sum_\alpha \frac{d_\alpha}{h} t^\alpha \frac{\partial f_t(x)}{\partial t^\alpha} \right] e^{z\, f_t(x)} d^N x \\
&= \frac{N-2}{2z}\tilde{t}_C + \frac{1}{z}\mathcal{L}_E \tilde{t}_C + z^{\frac{N-4}{2}} \int_C \sum_a \frac{\partial}{\partial x^a} \left(r_a x^a e^{z\, f_t(x)} \right) d^N x - \frac{r_1 + r_2}{z}\tilde{t}_C \\
&= \frac{N-2}{2z}\tilde{t}_C - \frac{1}{z}\left(r_1 + r_2 - \frac{N-2}{2} \right) \tilde{t}_C.
\end{aligned}$$

It remains to check that in all these cases

$$r_1 + r_2 - \frac{N-2}{2} = \frac{2-d}{2} = \frac{h+2}{2h}.$$

Theorem is proved.

Let us show that, for some cycles $C_1, \ldots, C_n$ the oscillatory integrals $\tilde{t}_{C_1}(t; z), \ldots, \tilde{t}_{C_n}(t; z)$ give independent flat coordinates of the deformed connection $\tilde{\nabla}$. First we will rewrite, following [AGV], the integral (2.96) as a Laplace-type transform of an appropriate function $p_C(\lambda; t)$:

$$\tilde{t}_C(t; z) = z^{\frac{N-2}{2}} \int_0^\infty e^{z\lambda} p_C(\lambda; t)\, d\lambda \tag{2.97}$$

where the integration is to be taken along any ray in the half-plane $\mathrm{Re}\, z\lambda < 0$. Put

$$p_C(\lambda; t) := \oint_{C(\lambda)} \frac{d^N x}{d f_t(x)}. \tag{2.98}$$

Here the Gelfand - Leray form $d^N x / d f_t(x)$ is defined by the equation

$$d^N x = d f_t(x) \wedge \frac{d^N x}{d f_t(x)},$$

the $(N-1)$-cycle $C(\lambda)$ is the intersection of C with the level surface

$$V_\lambda(t) = \{ x \mid f_t(x) = \lambda \}. \tag{2.99}$$

Fixing a noncritical for $f_t(x)$ value λ_0 we obtain the *period map*

$$t \mapsto \left[\frac{d^N x}{d f_t(x)} \right] \in H^{N-1}\left(V_{\lambda_0}(t) \right) \tag{2.100}$$

where the square brackets denote the cohomology class of the form. The map is defined for those t when λ_0 is not a critical value of $f_t(x)$. In the coordinates the map reads

$$t \mapsto \left(p_{\sigma_1}(\lambda_0; t), \ldots, p_{\sigma_n}(\lambda_0; t) \right) \tag{2.101}$$

for a basis

$$\sigma_1, \ldots, \sigma_n \in H_{N-1}\left(V_{\lambda_0}(t); \mathbf{Z}\right).$$

The period map is known to be a local diffeomorphism [Loo]. Now, choosing a basis of N-cycles $C_1, \ldots, C_n$ such that the $(N-1)$-cycles $C_1(\lambda), \ldots, C_n(\lambda)$ are linearly independent we obtain independent flat coordinates $\tilde{t}_{C_1}(t; z), \ldots, \tilde{t}_{C_n}(t; z)$.

Finally, we note that nodegeneracy of the period map was not essential to prove independency of the oscillatory integrals. One could use, instead, the analysis of the asymptotic behaviour of the integrals when the λ goes to one of the critical values of $f_t(x)$ and C is the corresponding vanishing cycle. We hope, however, that this digression into the singularity theory would help to understand the constructions of Lecture 5.

Remark 2.4. The two main classes of examples of Frobenius manifolds look so different. There are, however, some unexpected relationships between these two classes of two-dimensional topological field theories. That is, main playing characters of a two-dimensional topological field theory constructed from quantum cohomology turn out to coincide with those coming from singularity theory. This phenomenon was first discovered in quantum cohomologies of Calabi - Yau varieties [COGP]. It was called *mirror conjecture* (now partially proved [Gi2 - Gi4]). In the last lecture we will present our version of mirror construction for semisimple Frobenius manifolds. Particularly, we will express the deformed flat coordinates of $\tilde{\nabla}$ on any semisimple Frobenius manifold satisfying certain nondegeneracy condition by oscillatory integrals, and we also obtain an analogue of the residue formulae (1.20), (1.21). Some general approaches to mirror conjecture were recently proposed in [Wi4, Gi3, LLY, BK].

Lecture 3

Semisimplicity and canonical coordinates.

In this Lecture we introduce the class of semisimple Frobenius manifolds and obtain the main geometrical tool of dealing with them: the canonical coordinates [Du3].

We recall that a commutative associative algebra A is called *semisimple* if it contains no nilpotents, i.e., nonzero vectors $a \in A$ such that

$$a^m = 0$$

for some positive integer m.

Lemma 3.1. *Any semisimple finite-dimensional Frobenius algebra over* $\mathbf{C}$ *is isomorphic to orthogonal direct sum*

$$\mathbf{C} \oplus \ldots \oplus \mathbf{C} \tag{3.1}$$

of one-dimensional algebras.

Proof. Let $\lambda_1, \ldots, \lambda_k \in A^*$ be the roots of the commutative algebra A, i.e., such linear functions $\lambda_i : A \to \mathbf{C}$ that for any $a \in A$ the eigenvalues of the operator of multiplication by a are $\lambda_1(a), \ldots, \lambda_k(a)$. None of these linear functions equals identical zero since the algebra has a unity. Let

$$A_j := \cap_a \mathrm{Ker}\ (a \cdot -\lambda_j(a))^n\,,\ j = 1, \ldots, k$$

be the corresponding root subspaces. It is easy to see that the algebra is decomposed into the orthogonal direct sum of the root subspaces

$$A = \oplus_{j=1}^k A_j$$

$$A_j \cdot A_i = 0 \ \text{ for } i \neq j \tag{3.2}$$

and, thus,

$$\lambda_i\,(A_j) = 0, \quad \text{for } i \neq j.$$

For some i let $0 \neq v_i \in A_i$ be an eigenvector, i.e., such a vector that

$$v\, v_i = \lambda_i(v)\, v_i$$

for any $v \in A$. If $\lambda_i(v_i) = 0$ then

$$v_i^2 = 0.$$

This is not possible due to absence of nilpotents in the algebra A. Hence $\lambda_i(v_i) \neq 0$. Put

$$\pi_i = \frac{v_i}{\lambda_i(v_i)}.$$

We obtain that

$$\pi_i^2 = \pi_i.$$

Let us prove that each A_i is one-dimensional. If $w_i \neq 0$ is another eigenvector in A_i then

$$w_i\,\pi_i = \lambda_i(w_i)\pi_i = \lambda_i(\pi_i)w_i = w_i.$$

So w_i is proportional to π_i. Similarly, one can see that in A_i there are no vectors adjoint to π_i. Indeed, if π_i' is an adjoint vector of the height one, i.e.,

$$(\pi_i - \lambda_i(\pi_i))\ \pi_i' = \pi_i$$

then

$$\begin{aligned}\pi_i\,\pi_i' &= \pi_i + \pi_i'\\ &= \lambda_i(\pi_i')\,\pi_i.\end{aligned}$$

This contradicts to linear independence of π_i and π_i'. So, any of A_i for $i = 1, \ldots, k$ is a one-dimensional subalgebra in A generated by the vector π_i such that

$$\pi_i^2 = \pi_i. \tag{3.3}$$

From (3.2) it follows that

$$\pi_i\pi_j = 0 \quad \text{for } i \neq j. \tag{3.4}$$

So $\pi_1, \ldots, \pi_k$ are the basic idempotents of the semisimple algebra $A_1 \oplus \ldots \oplus A_k = A$. Particularly, $k = n$. Lemma is proved.

Corollary 3.1. *For any n-dimensional semisimple Frobenius algebra the basic idempotents $\pi_1\ \ldots,\ \pi_n$ of the algebra are determined uniquely up to reordering.*

Definition 3.1. A Frobenius manifold M is called *semisimple* if the algebras T_tM are semisimple for generic $t \in M$.

Semisimplicity of an algebra is an open property. So, if at some point $t = t_0 \in M$ the algebra T_tM is semisimple then it remains semisimple in some neighborghood of t_0.

Exercise 3.1. Prove that the function

$$F(t_1, t_2, t_3, t_4) = \frac{1}{2}t_1^2 t_4 + t_1 t_2 t_3 + f(t_2)$$

and the Euler vector field

$$E = t_1\partial_1 - t_3\partial_3 - 2t_4\partial_4$$

give a solution of WDVV (with $d = 3$) for an arbitrary function $f(t_2)$.

So, nonsemisimple Frobenius manifolds may depend on functional parameters. This fact is wellknown to experts in mirror symmetry: WDVV equations of asociativity provide no information about GW invariants of Calabi - Yau three-folds. For a Calabi - Yau three-fold the Frobenius structure is identicaly nilpotent.

In the opposite case semisimple Frobenius manifolds depend on finite number of parameters. Part of these parameters were described in Lecture 2: they are monodromy data at $z = 0$ of the connection $\tilde\nabla$. In Lecture 4 for semi-simple Frobenius manifolds we will define also monodromy data at $z = \infty$. We will show that the full list of the monodromy data is a complete local invariant of a semisimple Frobenius manifold. We will also describe the global structure of these manifolds (i.e., the analytic continuation of the local structure) in terms of the monodromy data.

Conjecturally, semisimplicity holds true for quantum cohomology of Fano varieties (see below the example for $\mathbf{CP}^2$). This conjecture is partially supported by the results of [TX].

Theorem 3.1. *Let $u_1(t), \ldots, u_n(t)$ be the eigenvalues of the operator of multiplication by the Euler vector field*

$$\det\left(\mathcal{U}(t) - \lambda \cdot 1\right) = (-1)^n \prod_{i=1}^n \left(\lambda - u_i(t)\right), \tag{3.5}$$

$$\mathcal{U}^\alpha_\beta(t) = E^\epsilon(t) c^\alpha_{\epsilon\beta}(t). \tag{3.6}$$

Near a semisimple point $t_0 \in M$ they can serve as local coordinates. In these coordinates

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i} \tag{3.7}$$

$$e = \sum_{i=1}^n \frac{\partial}{\partial u_i} \tag{3.8}$$

$$E = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i} \tag{3.9}$$

$$< \, , \, > = \sum_{i=1}^n \eta_{ii}(u) du_i^2 \quad \text{where } \eta_{ii}(u) = \frac{\partial t_1}{\partial u_i}, \quad t_1 := \eta_{1\epsilon} t^\epsilon. \tag{3.10}$$

Main Lemma. *Let M be a complex-analytic manifold with a structure of Frobenius algebras on the tangent planes $T_t M$ depending analytically on t and satisfying FM1 and FM2 (the quasihomogeneity FM3 not included). Then local coordinates $u_1, \ldots, u_n$ exist near a semisimple point $t_0 \in M$ such that*

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij} \frac{\partial}{\partial u_i}. \tag{3.11}$$

Proof. Near a semisimple point t_0 one can choose a frame of basic idempotents π_1, $\ldots, \pi_n$

$$\pi_i \cdot \pi_j = \delta_{ij} \pi_i$$

depending analytically on the point. It is sufficient to show that the Lie brackets $[\pi_i, \pi_j]$ of these vector fields vanish. Let us use the deformed flat connection

$$\tilde{\nabla}_u v = \nabla_u v + z\, u \cdot v$$

(no $\tilde\nabla_{d/dz}$ component since we do not assume quasihomogeneity). Let us introduce the coefficients Γ_{ij}^k and f_{ij}^k from the expansions

$$\nabla_{\pi_j}\pi_i = \sum_k \Gamma_{ij}^k \pi_k$$
$$[\pi_i, \pi_j] = \sum_k f_{ij}^k \pi_k.$$

Computing linear in z terms of the curvature

$$\tilde\nabla_{\pi_i}\tilde\nabla_{\pi_j} - \tilde\nabla_{\pi_j}\tilde\nabla_{\pi_i} - \tilde\nabla_{[\pi_i,\pi_j]} = 0$$

we obtain the equation

$$\Gamma_{kj}^l\delta_i^l + \Gamma_{ki}^l\delta_{kj} - \Gamma_{ki}^l\delta_j^l - \Gamma_{kj}^l\delta_{ki} = f_{ij}^l\delta_k^l$$

valid for arbitrary four indices i, j, k, l (no summation w.r.t. repeated indices in these formulas!). For $l = k$ we obtain

$$f_{ij}^k = 0.$$

Lemma is proved.

Proof of Theorem. Let us prove that

$$\mathcal{L}_E\left(\frac{\partial}{\partial u_i}\right) = -\frac{\partial}{\partial u_i} \tag{3.12}$$

where $u_1, \ldots, u_n$ are the local coordinates constructed in Main Lemma. We use the equation (2.23) of the axiom FM3. This reads

$$\mathcal{L}_E(a\cdot b) - \mathcal{L}_E(a)\cdot b - a\cdot\mathcal{L}_E b = a\cdot b \tag{3.13}$$

for any vector fields a and b. Applying this to the basic idempotents $\pi_i = \partial/\partial u_i$ for $a = \pi_i$, $b = \pi_j$, $i \neq j$ we obtain

$$\mathcal{L}_E(\pi_i)\cdot\pi_j + \pi_i\cdot\mathcal{L}_E(\pi_j) = 0.$$

Hence

$$\mathcal{L}_E(\pi_i)\cdot\pi_j = 0 \text{ for } i \neq j.$$

So $\mathcal{L}_E(\pi_i) = \lambda_i\,\pi_i$ with some factor λ_i. Applying now (3.13) to the case $a = b = \pi_i$ we obtain

$$\lambda_i\pi_i - 2\lambda_i\pi_i = \pi_i.$$

So $\lambda_i = -1$. This proves (3.12). Writing the vector-field E in the coordinates $(u_1, \ldots, u_n)$

$$E = \sum_{i=1}^n E^i(u)\frac{\partial}{\partial u_i}$$

we obtain from (3.12)

$$\frac{\partial E^i}{\partial u_j} = 0, \quad i \neq j, \quad \frac{\partial E^i}{\partial u_i} = 1.$$

Doing, if necessary, a shift of the coordinates $(u_1, \dots, u_n)$ we arrive at the formula (3.9). The eigenvectors of the operator of multiplication by this vector-field are $\partial/\partial u_1$, ..., $\partial/\partial u_n$. The corresponding eigenvalues are u_1, ..., u_n. To complete the proof of Theorem we observe that the basic idempotents of a Frobenius algebra are pairwise orthogonal

$$< \pi_i, \pi_j >=< e, \pi_i \cdot \pi_j >= 0 \text{ for } i \neq j.$$

Consider the 1-form dt_1. By definition for any vector v

$$\partial_v t_1 = dt_1(v) =< e, v >$$

where e is the unity of the algebra. Hence

$$< \pi_i, \pi_i >=< e, \pi_i \cdot \pi_i >= \left\langle e, \frac{\partial}{\partial u_i} \right\rangle = \frac{\partial t_1}{\partial u_i}.$$

Theorem is proved.

Corollary 3.2. *All the points $t \in M$ where the eigenvalues of $(E(t)\cdot)$ are pairwise distinct are semisimple.*

Definition 3.2. The coordinates $(u_1, \dots, u_n)$ constructed in Theorem 1 are called *canonical coordinates* of the Frobenius manifold.

The canonical coordinates near any point are defined uniquely up to permutations. We will use Latin indices for canonical coordinates and we put

$$\partial_i := \frac{\partial}{\partial u_i}.$$

We will also show explicitly all the sums w.r.t. Latin indices not distinguishing between upper and lower indices. Recall that Greek indices are used for flat coordinates and

$$\partial_\alpha = \frac{\partial}{\partial t^\alpha}.$$

The rules of tensor algebra (raising and lowering indices using $\eta^{\alpha\beta}$ and $\eta_{\alpha\beta}$, the Einstein summation rule etc.) will be applied only to Greek indices.

We make now an algebraic digression about semisimple Frobenius algebras over $\mathbf{C}$. Let $(A, < \ , \ >)$ be such an algebra with a basis $e_1 = e$, e_2, ..., e_n and the multiplication table

$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta} e_\gamma.$$

Let $\pi_1, \ldots, \pi_n$ be the idempotents of A. Introduce the basis of *normalized* idempotents

$$f_i = \frac{\pi_i}{\sqrt{<\pi_i, \pi_i>}}, \quad i = 1, \ldots, n$$

choosing arbitrary signs of the square roots. Let us introduce the matrix $\Psi = (\psi_{i\alpha})$ putting

$$e_\alpha = \sum_{i=1}^n \psi_{i\alpha} f_i, \quad \alpha = 1, \ldots, n.$$

Exercise 3.2. Prove the following formulae

$$\Psi^T \Psi = \eta \tag{3.14}$$
$$\psi_{i1} = \sqrt{<\pi_i, \pi_i>} \tag{3.15}$$
$$f_i = \sum_{\alpha,\,\beta=1}^n \psi_{i1}\psi_{i\beta}\eta^{\beta\alpha} e_\alpha \tag{3.16}$$
$$c_{\alpha\beta\gamma} = \sum_{i=1}^n \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}. \tag{3.17}$$

On a semisimple Frobenius manifold the matrix Ψ depends on the point. The above formula give

$$<\,,\,> = \sum_{i=1}^n \psi_{i1}^2(u) du_i^2 \tag{3.18}$$
$$\partial_\alpha = \sum_{i=1}^n \frac{\psi_{i\alpha}(u)}{\psi_{i1}(u)} \partial_i \tag{3.19}$$
$$\partial_i = \sum_{\alpha,\,\epsilon} \eta^{\alpha\epsilon} \psi_{i\epsilon}(u)\psi_{i1}(u)\partial_\alpha \tag{3.20}$$

or, equivalently,

$$dt^\alpha = \sum_{i=1}^n \psi_i^\alpha(u)\psi_{i1}(u) du_i, \quad \text{where } \psi_i^\alpha := \eta^{\alpha\epsilon}\psi_{i\epsilon}. \tag{3.21}$$

We will now rewrite the connection ∇ in the frame of normalized idempotents

$$f_i = \frac{\partial_i}{\sqrt{<\partial_i, \partial_i>}}. \tag{3.22}$$

We recall that the horizontal sections ξ satisfy the compatible system

$$\partial_\alpha \xi = z\, C_\alpha \xi, \quad (C_\alpha)_\gamma^\beta := c_{\alpha\gamma}^\beta \tag{3.23}$$
$$\partial_z \xi = \left(\mathcal{U} + \frac{\mu}{z}\right) \xi, \quad \mathcal{U}_\gamma^\beta := E^\epsilon c_{\epsilon\gamma}^\beta, \ \mu = \operatorname{diag}(\mu_1, \ldots, \mu_n). \tag{3.24}$$

The operator $\mathcal{U}$ of multiplication by the Euler vector field becomes diagonal in the basis $f_1, \ldots, f_n$

$$\Psi\,\mathcal{U}\,\Psi^{-1} =: U = \operatorname{diag}(u_1, \ldots, u_n). \tag{3.25}$$

We introduce also the matrix

$$V := \Psi\,\mu\,\Psi^{-1} \tag{3.26}$$

of the operator μ in the same basis. From antisymmetry

$$< \mu\, a, b > + < a, \mu\, b >= 0$$

it follows antisymmetry of the matrix V

$$V^T + V = 0. \tag{3.27}$$

Lemma 3.2. *After the gauge transformation*

$$y = \Psi\xi \tag{3.28}$$

the system (2.41) reads

$$\partial_i y = (z\,E_i + V_i)\; y, \quad i = 1, \ldots, n \tag{3.29}$$

$$\partial_z y = \left(U + \frac{V}{z}\right) y. \tag{3.30}$$

Here E_i are the matrix unities

$$\left(E_i\right)_{ab} = \delta_{ia}\delta_{ib}, \tag{3.31}$$

V_i are skew-symmetric matrices uniquely determined by the equations

$$[U, V_i] = [E_i, V]. \tag{3.32}$$

The matrices V and Ψ satisfy the differential equations

$$\partial_i \Psi = V_i \Psi \tag{3.33}$$

$$\partial_i V = [V_i, V]. \tag{3.34}$$

Observe that the matrices V_i are defined in those points where the canonical coordinates are pairwise distinct. Symbolically, (3.32) can be recasted in the form

$$V_i = ad_{E_i} ad_U{}^{-1}\,(V)\,. \tag{3.35}$$

Proof. Using (3.25) one obtains

$$\partial_i \xi = z\,\Pi_i \xi$$

where Π_i is the operator of multiplication by π_i. By definition of Ψ

$$\Psi\,\Pi_i \Psi^{-1} = E_i.$$

So

$$\partial_i y = z\, E_i y + \tilde{V}_i y$$

where

$$\tilde{V}_i := \partial_i \Psi \cdot \Psi^{-1}. \tag{3.36}$$

Using the orthogonality (3.14) we obtain antisymmetry of $\tilde{V}_i$. From compatibility

$$\partial_i \partial_j y = \partial_j \partial_i y$$

it follows

$$\left[E_i, \tilde{V}_j\right] = \left[E_j, \tilde{V}_i\right]$$

for any i, j. This implies existence of a symmetric matrix Γ such that

$$\tilde{V}_i = [E_i, \Gamma]\,, \quad i = 1, \ldots, n.$$

The off-diagonal entries of Γ are determined uniquely. In the points of M where $u_i \neq u_j$ for any $i \neq j$ we thus obtain a uniquely determined skew-symmetric matrix $\tilde{V}$ such that

$$\left[U, \tilde{V}_i\right] = \left[E_i, \tilde{V}\right], \quad i = 1, \ldots, n.$$

Doing the gauge transformation (3.28) in the system (2.41b) we obtain

$$\partial_z y = \left(U + \frac{V}{z}\right) y.$$

The compatibility $\partial_i \partial_z = \partial_z \partial_i$ implies

$$V = \tilde{V},$$

$$\partial_i V = [V_i, V], \quad i = 1, \ldots, n.$$

The definition (3.36) of the matrix $\tilde{V}_i = V_i$ reads

$$\partial_i \Psi = V_i \Psi.$$

Lemma is proved.

Exercise 3.3. Let us consider $V = (V_{ij}(u))$ as a function of $u = (u_1, \ldots . u_n)$ with the values in the Lie algebra $so(n)$. Prove that the equations (3.34) can be considered as time-dependent Hamiltonian systems

$$\frac{\partial V}{\partial u_i} = \{V, H_i(V; u)\}\,, \quad i = 1, \ldots, n \tag{3.37}$$

with the quadratic Hamiltonians

$$H_i(V;u) = \frac{1}{2}\sum_{j\neq i}\frac{V_{ij}^2}{u_i - u_j}, \quad i = 1, \dots, n \tag{3.38}$$

w.r.t. the standard linear Poisson bracket on $so(n)$

$$\{V_{ij}, V_{kl}\} = V_{il}\delta_{jk} - V_{jl}\delta_{ik} + V_{jk}\delta_{il} - V_{ik}\delta_{jl}. \tag{3.39}$$

The canonical coordinates $u_1, \dots, u_n$ play the role of the time variables of these Hamiltonian systems.

Exercise 3.4. Prove that $\{H_i, H_j\} = 0$ for any i, j. From this and from commutativity of the flows (3.37) derive that the form

$$\sum_{i=1}^n H_i(V;u)du_i$$

is closed for any solution $V(u)$ of the system (3.34). That means that (locally) there exists a function $\tau(u)$ such that

$$\frac{\partial \log\tau(u)}{\partial u_i} = H_i(V(u);u), \; i = 1, \dots, n. \tag{3.40}$$

This is called *tau-function* of the solution of the system $V(u)$. In the next Lecture we will show that the system (3.33), (3.34) can be solved by reducing to certain linear Riemann - Hilbert boundary value problem. The tau-function will coincide with the Fredholm determinant of the corresponding system of integral equations (see [JM, Mi]).

Importance of the tau-function in topological field theory is clear from the following

Theorem 3.2 [DZ2]. *Let X be a smooth projective manifold such that the quantum cohomology of X is semisimple. Then the generating function $F^{(1)}(t)$ of elliptic Gromov - Witten invariants of X is given by the formula*

$$F^{(1)}(t) = \log\frac{\tau(u)}{J^{1/24}}|_{u=u(t)}$$

where $\tau(u)$ is the above tau-function and

$$J = \det\left(\frac{\partial t^\alpha}{\partial u_i}\right) = \psi_{11}\dots\psi_{n1}.$$

Particularly, from this theorem it follows validity of Conjectures 0.1 and 0.2 of recent paper of Givental [Gi5].

We prove now the converse to Lemma 2 statement.

Let $V(u)$, $\Psi(u)$ be a solution of the system (3.33), (3.34) with a diagonalizable matrix $V(u)$. We observe first that the product

$$\Psi^{-1}(u)V(u)\,\Psi(u)$$

does not depend on u. We can therefore find a constant matrix C in such a way that

$$\Psi^{-1}(u)V(u)\,\Psi(u) = C\,\mu\,C^{-1}$$

where

$$\mu = \mathrm{diag}\,(\mu_1, \dots, \mu_n)$$

is a constant diagonal matrix. Doing a change

$$\Psi(u) \mapsto \Psi(u)C$$

we obtain another solution of the linear system (3.33) such that

$$\Psi^{-1}(u)V(u)\,\Psi(u) = \mathrm{diag}\,(\mu_1, \dots, \mu_n). \tag{3.41}$$

After these preliminaries we formulate

Lemma 3.3. *Let $V(u) = (V_{ij}(u))$, $\Psi(u) = (\psi_{i\alpha}(u))$ be a solution of the system (3.33), (3.34) satisfying (3.41). Then the formulae (3.18), (3.21), (3.17) define a Frobenius structure on the domain*

$$u_i \neq u_j \text{ for } i \neq j, \quad \psi_{11}(u) \dots \psi_{n1}(u) \neq 0. \tag{3.42}$$

Proof. From the antisymmetry of the matrices V_i it follows that

$$\partial_i \left(\Psi^T \Psi\right) = 0, \ i = 1 \dots, n.$$

Put

$$\eta = (\eta_{\alpha\beta}) = \Psi^T \Psi, \quad \left(\eta^{\alpha\beta}\right) = \eta^{-1}. \tag{3.43}$$

Next step is to prove that the 1-forms

$$\sum_{i=1}^n \psi_i^\alpha \psi_{i1} du_i, \quad \text{where } \psi_i^\alpha = \eta^{\alpha\epsilon}\psi_{i\epsilon}$$

are closed. From (3.33) we obtain

$$\partial_j \psi_{i\alpha} = \frac{V_{ij}}{u_j - u_i}\psi_{j\alpha} \ \text{ for any } i \neq j, \text{ any } \alpha.$$

From this the identity

$$\partial_j \left(\psi_i^\alpha \psi_{i1}\right) = \partial_i \left(\psi_j^\alpha \psi_{j1}\right)$$

follows. This proves local existence of the functions t^α such that

$$dt^\alpha = \sum_{i=1}^n \psi_i^\alpha \psi_{i1} du_i.$$

The differentials $dt^1, \ldots, dt^n$ are independent on the domain (3.42). So $t^1, \ldots, t^n$ serve as local coordinates on the domain. From the orthogonality (3.14) we obtain that

$$\partial_\alpha = \sum_{i=1}^n \frac{\psi_{i\alpha}}{\psi_{i1}} \partial_i.$$

The last step is to prove the symmetry

$$\partial_\delta \left(\sum_{i=1}^n \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}} \right) = \partial_\gamma \left(\sum_{i=1}^n \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\delta}}{\psi_{i1}} \right).$$

To prove this we are to use another consequence of (3.33)

$$\partial_i \psi_{i\alpha} = -\sum_{k\neq i} \partial_k \psi_{i\alpha}$$

valid for any i, any α. We leave this computation as an exercise for the reader. Lemma is proved.

Corollary 3.3. *Classes of local equivalence of semisimple Frobenius manifolds such that 1 is an eigenvalue of ∇E of the multiplicity k depend on*

$$k - 1 + \frac{n(n-1)}{2}$$

parameters.

Proof. Take the initial data

$$V(u^0) = \left(V_{ij}(u^0)\right) \tag{3.44}$$

of the antisymmetric matrix V in a point $u^0 = (u_1^0, \ldots, u_n^0)$ with

$$u_i^0 \neq u_j^0 \text{ for } i \neq j.$$

Solving the system (3.34) of commuting ODEs we obtain locally uniquely the matrix-valued function $V(u)$ and, therefore, the matrices $V_i(u)$. The solution $\Psi(u)$ of the linear system (3.33) such that

$$\Psi^{-1}(u)V(u)\,\Psi(u) = \mathrm{diag}\,(\mu_1, \ldots, \mu_n) = \mu$$

is determined uniquely up to multiplication by a matrix

$$\Psi(u) \mapsto \Psi(u)C$$
$$C^{-1}\mu\, C = \mu.$$

The matrices C preserving the direction of the eigenvector e of μ with the eigenvalue $\mu_1 = -d/2$ produce equivalences of the Frobenius manifolds. The assumption about the multiplicity of the eigenvalue 1 of ∇E means that the eigenvalue $\mu_1 = -d/2$ of μ has also the multiplicity k. The vectors $C\, e$ considered up to rescalings must be eigenvectors of μ with the same eigenvalue μ_1. The directions of these vectors give $k-1$ parameters additional to the initial data (3.44). Corollary is proved.

Lecture 4

Stokes matrices and classification of semisimple Frobenius manifolds.

In the previous Lecture we parametrized semisimple Frobenius manifolds M by initial data of the system (3.33), (3.34) of differential equations in a point $t \in M$ such that $u_i(t) \neq u_j(t)$ for $i \neq j$. Typically, however, one has no "natural" point in the Frobenius manifold to specify the initial data (3.44). E.g., for Frobenius manifolds with good analytic properties the "natural" point would be $t_0 = (t'' = 0, t' = -\infty)$. But in this point the Frobenius algebra $T_{t_0}M$ typically is nilpotent (one can keep in mind the example of quantum cohomology where t_0 is the point of classical limit). So in this point $u_1(t_0) = u_2(t_0) = \ldots = u_n(t_0)$. This is a singular point for the system (3.33), (3.34).

Instead, we will use [Du3, Du7] the monodromy data of the system (3.30) as the parameters. Recall that the system is gauge equivalent to the equations (2.41) determinining the horizontal sections of the connection $\tilde{\nabla}$. Part of the monodromy data has already been defined in Lecture 2. Namely, this part is the monodromy at $z = 0$ of the system (3.30) gauge equivalent to (2.41b). Recall that for a system (3.30) with the monodromy data at $z = 0$ of the form $(\mathcal{V}, <\ ,\ >, \mu, [R])$ a fundamental matrix solution $Y_0(z)$ exists such that

$$Y_0 = \Phi(z) z^\mu z^R \tag{4.1}$$

where

$$\Phi(z) = \Psi + z\,\Phi_1 + z^2 \Phi_2 + \ldots \tag{4.2}$$

is an invertible matrix holomorphic for small $|z|$ satisfying

$$\Phi^T(-z)\Phi(z) = \eta. \tag{4.3}$$

Let us describe the ambiguity in the choice of the normalized solution (4.1).

Let $\mathcal{C}_0(\mu, R)$ be the group of all invertible matrices C such that

$$z^\mu z^R C\, z^{-R} z^{-\mu} = C_0 + z\, C_1 + \ldots \tag{4.4}$$

is a matrix-valued polynomial in z.

Lemma 4.1. *Two solutions $Y(z)$, $\tilde{Y}(z)$ of the system (3.30) have the same form (4.1)* iff *they are related by a right multiplication by a matrix $C \in \mathcal{C}_0(\mu, R)$.*

Proof. If

$$Y = \Phi(z) z^\mu z^R, \quad \tilde{Y} = \tilde{\Phi}(z) z^\mu z^R$$

satisfy (3.30) then

$$\tilde{Y}(z) = Y(z) C$$

for a constant matrix C. We have

$$\Phi^{-1}(z)\tilde{\Phi}(z) = z^\mu z^R C\, z^{-R} z^{-\mu}.$$

Hence the r.h.s. must be a polynomial. The converse statement is obvious. Lemma is proved.

Exercise 4.1. Show that the matrices in $\mathcal{C}_0(\mu, R)$ commute with $\exp 2\pi i\mu$ and that they must have the form

$$C = C_0 + C_1 + C_2 + \dots$$

with

$$(C_k)^\alpha_\beta \neq 0 \ \text{ only if } \mu_\alpha - \mu_\beta = k, \quad k = 0,\, 1, \dots.$$

Particularly, the matrix C_0 commutes with μ.

Remark 4.1. In the case of a Frobenius manifold we have an additional structure of the monodromy data of (3.30) at $z = 0$. Namely, an eigenvector e of the matrix V with the eigenvalue $\mu_1 = -d/2$ must be marked. It corresponds to the unity of M. We must therefore to impose an additional constraint on the matrix C in (4.4): the component C_0 must preserve the marked vector. Observe that the marked vector corresponds to the first column ψ_{i1} of the matrix Ψ.

The second part is the monodromy data at $z = \infty$ that we are going to define now.

We first describe the monodromy data at $z = \infty$ of the system

$$\frac{dy}{dz} = \left(U + \frac{1}{z}V\right) y \tag{4.5}$$

with arbitrary $n \times n$ matrices of the form

$$U = \mathrm{diag}\,(u_1, \dots, u_n), \quad u_i \neq u_j \tag{4.6}$$

$$V^T = -V \tag{4.7}$$

being a diagonalizable matrix

$$\Psi^{-1} V\, \Psi = \mu = \mathrm{diag}\,(\mu_1, \dots, \mu_n). \tag{4.8}$$

The point $z = \infty$ is an irregular singularity of the system (3.30). So in the problem of normal form of the system (3.30) we are to distinguish formal gauge equivalences

$$y \mapsto G\left(\frac{1}{z}\right) y$$

$$G\left(\frac{1}{z}\right) = 1 + \frac{G_1}{z} + \frac{G_2}{z^2} + \dots$$

and analytic ones, where the series converges for sufficiently large $|z|$.

Definition 4.1. Two systems

$$\frac{dy^{(i)}}{dz} = \left(U + \frac{1}{z}V^{(i)}\right) y^{(i)}, \quad i = 1,\, 2 \tag{4.9}$$

are called *analytically equivalent at* $z = \infty$ if there exists a gauge transform

$$y^{(2)} = G\,(z)\; y^{(1)} \tag{4.10}$$

with the matrix-valued function $G(z)$ analytic at $z = \infty$ satisfying $G(\infty) = 1$ and the orthogonality condition

$$G^T(-z)\, G(z) = 1.$$

The *monodromy at $z = \infty$ of the system (3.30)* is the class of analytic equivalence of this system.

Below we will explain how one can parametrize the monodromy at infinity by Stokes matrices of the system (3.30). But first we will show that the system is, to some extent, uniquely determined by the monodromy at $z = 0$ and $z = \infty$.

Lemma 4.2. *Let (4.9) be two systems analytically equivalent at $z = \infty$. Then the matrix G establishing the gauge equivalence is a rational function of z of the form*

$$G = 1 + \frac{G_1}{z} + \frac{G_2}{z^2} + \ldots + \frac{G_m}{z^m}. \tag{4.11}$$

Proof. Let the given gauge transform (4.10) be analytic for $|z| > M$ for some constant M. Choose a point z_0 with $|z_0| > M$ and the fundamental matrix solutions $Y^{(i)}(z)$ of the systems (4.9) with the initial data

$$Y^{(i)}(z_0) = 1,\ i = 1, 2.$$

For any z with $|z| > M$ we must have

$$G(z)Y^{(1)}(z) = Y^{(2)}(z)C$$

for some constant nondegenerate matrix C. The solutions $Y^{(1,2)}(z)$ can be continued analytically along any path in $\mathbf{C} \setminus 0$. The formula

$$G(z) = Y^{(2)}(z)C\, {Y^{(1)}(z)}^{-1}$$

gives analytic continuation of $G(z)$ (recall that $\det Y^{(i)}(z) = \exp(z - z_0) \sum_i u_i \neq 0$). We obtain a single-valued analytic function in $\bar{\mathbf{C}} \setminus 0$ such that $G(\infty) = 1$. Near the point of regular singularity $z = 0$ the entries of the matrices $Y^{(1,2)}(z)$ grow not faster than some power of $|z|$. Hence also $G(z)$ has at most power growth at $z = 0$. So it must be a rational function having a pole only at $z = 0$. Lemma is proved.

Exercise 4.2. Prove that the determinant of the matrix (4.11) is identically equal to 1.

Remark 4.2. Gauge transformations with rational $G(z)$ are called *Schlesinger transformations* [JM]. For the case of Frobenius manifolds they induce certain symmetries of WDVV, i.e., changes of variables

$$t \mapsto \hat{t}$$

$$F \mapsto \hat{F}$$

mapping solutions to solutions. We give here the explicit form [Du7] of such symmetries for the case $m \leq 1$ in (4.11).

Type 1. $G = \text{const}$, $G\mu = \mu G$, G permutes the two eigenvectors of μ with the numbers 1 and κ. Then

$$\begin{aligned}
\hat{t}_\alpha &= \partial_\alpha \partial_\kappa F(t) \\
\frac{\partial^2 \hat{F}}{\partial \hat{t}^\alpha \partial \hat{t}^\beta} &= \frac{\partial^2 F}{\partial t^\alpha \partial t^\beta} \\
\hat{\eta}_{\alpha\beta} &= \eta_{\alpha\beta}.
\end{aligned} \tag{4.12}$$

Type 2.

$$G = 1 + \frac{A}{z}$$

where

$$A_{ij} = \frac{\psi_{i1}\psi_{j1}}{t_1}.$$

Then

$$\begin{aligned}
\hat{t}^1 &= \frac{1}{2} \frac{t_\sigma t^\sigma}{t_1} \\
\hat{t}^\alpha &= \frac{t^\alpha}{t_1}, \ \alpha \neq 1, \, n \\
\hat{t}^n &= -\frac{1}{t_1} \\
\hat{F} &= t_1^{-2} \left[F - \frac{1}{2} t^1 t_\sigma t^\sigma \right] \\
\hat{\eta}_{\alpha\beta} &= \eta_{\alpha\beta}.
\end{aligned} \tag{4.13}$$

Also one may take superposition of (4.13) with any transformation of the form (4.12)

We classify now the systems of the form (3.30) having the same monodromies at $z = 0$ and $z = \infty$. We will show that, generically, these systems must coincide. There remain, however, some subtleties in the nongeneric situation. The ambiguity of the reconstruction of the system (3.30) starting from the monodromy data at $z = 0$ and $z = \infty$ will be completely described in terms of the monodromy at $z = 0$.

Let us choose a representative R in the class of equivalence $[R]$ of the monodromy data at $z = 0$ of the system (3.30). Let us consider the centralizer of the monodromy matrix

$$M_0 = \exp 2\pi i(\mu + R) \tag{4.14}$$

in the group of invertible matrices, i.e., the matrices C commuting with M_0

$$C^{-1} M_0 C = M_0. \tag{4.15}$$

For any such a matrix C the product

$$z^\mu z^R C z^{-R} z^{-\mu} = \sum_k A_k z^k \tag{4.16}$$

is a matrix-valued Laurent polynomial in z. Particularly, for the matrix $C \in \mathcal{C}_0(\mu, R)$ the r.h.s. of (4.16) contains only nonnegative powers of z. Denote $\mathcal{C}(\mu, R)$ the quotient group of the centralizer (4.15) over the subgroup $\mathcal{C}_0(\mu, R)$.

Example 4.1. For a nonresonant μ the group $\mathcal{C}(\mu, R)$ consists of one element.

Example 4.2. The group $\mathcal{C}(\mu, R)$ with a resonant μ and $R = 0$ is not trivial. It is isomorphic to the subgroup of "upper triangular" parabolic matrices in the centralizer of $\exp 2\pi i\mu$

$$C = \ldots + C_{-2} + C_{-1} + 1 \tag{4.17}$$

where

$$(C_k)^\alpha_\beta \neq 0 \ \text{ only if } \mu_\alpha - \mu_\beta = k, \ k = -1, \, -2, \ldots. \tag{4.18}$$

Let the two systems of the form (4.9) have the same monodromy data at $z = 0$ and $z = \infty$. We will asociate with such a pair a matrix $C \in \mathcal{C}(\mu, R)$ where μ, R are the monodromy data of the systems (4.9) at $z = 0$. Let $Y^{(1)}(z)$ be the matrix solution of the system

$$\partial_z Y^{(1)} = \left(U + \frac{1}{z} V^{(1)}\right) Y^{(1)} \tag{4.19}$$

of the form

$$Y^{(1)}(z) = \Phi(z) z^\mu z^R.$$

Let $G_0(z) = 1 + O(z)$ and $G_\infty(z) = 1 + O\left(1/z\right)$ be the gauge transformations of the system (4.19) to another system of the same form

$$\partial_z Y^{(2)} = \left(U + \frac{1}{z} V^{(2)}\right) Y^{(2)}. \tag{4.20}$$

The matrix-valued functions $G_0(z)$ and $G_\infty(z)$ are assumed to be analytic near $z = 0$ and $z = \infty$ resp. Near $z = 0$ we obtain a solution

$$Y_0^{(2)}(z) = G_0(z) Y^{(1)}(z)$$

of (4.20). Continuing $Y^{(1)}(z)$ analytically along a ray ρ in the neighborhood of infinity we produce another solution of the system (4.20)

$$Y_\infty^{(2)}(z) = G_\infty(z) Y^{(1)}(z).$$

Continuing $Y_\infty^{(2)}(z)$ back along the same ray ρ, we obtain two matrix solutions of (4.20) defined in a neighborhood of $z = 0$. They must be related by a multiplication by an invertible matrix C_{12}

$$Y_\infty^{(2)}(z) = Y_0^{(2)}(z) C_{12}.$$

We rewrite the last equation in the form

$$G_0^{-1}(z) G_\infty(z) = \Phi(z) z^\mu z^R C_{12} z^{-R} z^{-\mu} \Phi^{-1}(z). \tag{4.21}$$

The r.h.s. must be a meromorphic function near $z = 0$. That means, particularly, that the matrix C_{12} commutes with the monodromy matrix M_0. We arrive at

Theorem 4.1. *The set of all systems*

$$\partial_z \tilde{Y} = \left(U + \frac{1}{z} \tilde{V} \right) \tilde{Y}$$

of the form (3.30) having the monodromy data at $z = 0$ and $z = \infty$ coinciding with those of the given system

$$\partial_z Y = \left(U + \frac{1}{z} V \right) Y$$

is in one-to-one correspondence with the elements of the group $\mathcal{C}(\mu, R)$.

Proof. The above construction associates with the pair of this systems an element $C = C_{12}$ of the centralizer of M_0. It remains to show that the two systems coincide *iff*

$$z^\mu z^R C\, z^{-R} z^{-\mu}$$

is a polynomial in z. Indeed, if this is the case then the r.h.s. of (4.21) is analytic at $z = 0$. Hence $G_\infty(z)$ is analytic at $z = 0$. Using the normalization $G_\infty(\infty) = 1$ we conclude that $G_\infty(z) \equiv 1$. The converse statement is obvious. Theorem is proved.

We proceed now to a "quantative" description of the monodromy at infinity of systems of the form (3.30). We first show that all the systems (3.30) with given pairwise distinct values of $u_1, \ldots, u_n$ are gauge equivalent at $z = \infty$ w.r.t. *formal* gauge transformations. It is sufficient to construct a gauge transformation

$$\tilde{Y} = G(z)\, Y \tag{4.22}$$

of the system (3.30) to the system with constant coefficients

$$\partial_z \tilde{Y} = U\, \tilde{Y}. \tag{4.23}$$

Lemma 4.3. *For any system (3.30) there exists a unique formal series*

$$G(z) = 1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \ldots \tag{4.24}$$

satisfying

$$G^T(-z) G(z) = 1 \tag{4.25}$$

such that (4.22) transforms (3.33) to the system (4.23) with constant coefficients.

Proof. For the coefficients of the formal series (4.24) one obtains the recursion relations

$$\begin{aligned} [U, A_1] &= V \\ [U, A_{k+1}] &= A_k V - k\, A_k, \quad k = 1, 2, \ldots. \end{aligned}$$

Representing

$$A_k = B_k + D_k$$

with an off-diagonal matrix B_k and a diagonal one D_k we obtain

$$\begin{aligned} B_1 &= {ad_U}^{-1}(V) \\ D_k &= \frac{1}{k}\mathrm{diag}\,(B_k V) \\ B_{k+1} &= {ad_U}^{-1}\,(A_k V - k\,A_k) \end{aligned}$$

where 'diag' stands for the diagonal part of the matrix. This proves existence and uniqueness of the series $G(z)$.

Let us choose a fundamental matrix $Y(z)$ for the system (3.30) such that

$$Y^T(-z)Y(z) \equiv 1.$$

Then

$$G(z)Y(z)$$

is a formal solution of the system (4.23). Hence for an appropriate constant invertible matrix C

$$G(z)Y(z) = e^{z\,U}C.$$

Computing the product

$$\left(G^{-1}(z)\right)^T G^{-1}(-z) = e^{z\,U}\left(C\,C^T\right)^{-1} e^{-z\,U}$$

we conclude that $C\,C^T = 1$ since the l.h.s. is a formal series in inverse powers of z of the form $1 + O(1/z)$. This proves the orthogonality relation (4.25). Lemma is proved.

The series $G(z)$ typically diverges. However, in certain sectors of the complex z-plane near $z = \infty$ it serves as the asymptotic development of an actual solution of the original system.

We recall that a series

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

is an asymptotic expansion of the function $f(z)$ for $|z| \to \infty$ in the sector

$$\alpha < \arg z < \beta$$

if for any n

$$z^n\left[f(z) - \sum_{k=0}^{n}\frac{a_k}{z^k}\right] \to 0$$

as $|z| \to \infty$ uniformly in the sector

$$\alpha + \varepsilon < \arg z < \beta - \varepsilon$$

for any sufficiently small positive ε. This fact will be denoted briefly

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots, \quad |z| \to \infty, \quad \alpha < \arg z < \beta.$$

Let us denote

$$Y_{\mathrm{formal}}(z) = \left(1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots\right) e^{z\,U} \tag{4.26}$$

where the coefficients of the formal series are defined in Lemma 4.3. We say that a matrix solution $Y(z)$ of the system (3.30) has asymptotic development

$$Y(z) \sim Y_{\mathrm{formal}}(z), \quad |z| \to \infty, \quad \alpha < \arg z < \beta$$

if in the same sector

$$Y(z)\, e^{-z\,U} \sim Y_{\mathrm{formal}}(z)\, e^{-z\,U} = 1 + \frac{A_1}{z} + \frac{A_2}{z^2} + \dots.$$

Definition 4.2. A line ℓ through the origin in the complex z-plane is called *admissible* for the system (3.30) if

$$\mathrm{Re}\, z\,(u_i - u_j)|_{z \in \ell \setminus 0} \neq 0 \ \text{ for any } i \neq j. \tag{4.27}$$

Let us fix an admissible line ℓ and an orientation on it. According to the orientation the line splits into the negative and positive parts ℓ_- and ℓ_+ resp. Let the parts have the equations

$$\begin{aligned} \ell_+ &= \{z\,|\arg z = \phi\} \\ \ell_- &= \{z\,|\arg z = \phi - \pi\}. \end{aligned} \tag{4.28}$$

We construct two sectors

$$\begin{aligned} \Pi_{\mathrm{right}}: &\quad \phi - \pi - \varepsilon < \arg z < \phi + \varepsilon \\ \Pi_{\mathrm{left}}: &\quad \phi - \varepsilon < \arg z < \phi + \pi + \varepsilon \end{aligned} \tag{4.29}$$

for sufficiently small positive ε.

Theorem 4.2. *There exist unique solutions $Y_{\mathrm{right/left}}(z)$ of (3.30) analytic in the sectors $\Pi_{\mathrm{right/left}}$ resp. having the asymptotic development*

$$Y_{\mathrm{right/left}}(z) \sim Y_{\mathrm{formal}}(z) \tag{4.30}$$

as $|z| \to \infty$ in these sectors.

Proof see in [BJL1].

We are now ready to define Stokes matrices of the system (3.30). In the narrow sector

$$\Pi_+: \quad \phi - \varepsilon < \arg z < \phi + \varepsilon \tag{4.31}$$

we have two solutions. They must be related by multiplication by a matrix

$$Y_{\text{left}}(z) = Y_{\text{right}}(z)S, \quad z \in \Pi_+. \tag{4.32}$$

Similarly, in the opposite narrow sector Π_-

$$Y_{\text{left}}(z) = Y_{\text{right}}(z)S_-, \quad z \in \Pi_-. \tag{4.33}$$

Definition 4.3. The matrices S, S_- are called *Stokes matrices of the system (3.30).*

Lemma 4.4. *Two systems with equal Stokes matrices w.r.t. the same admissible oriented line ℓ are analytically equivalent near $z = \infty$.*

Proof. Let $Y^{(1)}_{\text{left/right}}(z)$, $Y^{(2)}_{\text{left/right}}(z)$ be the solutions of the corresponding systems with the needed asymptotic developments in the sectors $\Pi_{\text{left/right}}$ resp. Let us consider the following piecewise analytic matrix-valued function $G(z)$ defined for sufficiently large $|z|$ such that

$$G(z) = \begin{cases} Y^{(2)}_{\text{right}}(z){Y^{(1)}_{\text{right}}}^{-1}(z), & z \in \Pi_{\text{right}} \\ Y^{(2)}_{\text{left}}(z){Y^{(1)}_{\text{left}}}^{-1}(z), & z \in \Pi_{\text{left}}. \end{cases}$$

In the sectors Π_+, Π_- we have

$$\begin{aligned} Y^{(1,2)}_{\text{left}}(z) &= Y^{(1,2)}_{\text{right}}(z)S, \quad z \in \Pi_+ \\ Y^{(1,2)}_{\text{left}}(z) &= Y^{(1,2)}_{\text{right}}(z)S_-, \ z \in \Pi_-. \end{aligned}$$

So $G(z)$ is a single-valued analytic function for $|z| > M$ for some big constant M. In the sectors $\Pi_{\text{right/left}}$

$$G(z) \sim 1 + O\left(\frac{1}{z}\right).$$

Hence $z = \infty$ is a removable singularity for this function, and

$$G(\infty) = 1.$$

This function $G(z)$ establishes the needed gauge transformation between the systems. Lemma is proved.

We will now describe the algebraic properties of the Stokes matrices. We first describe explicitly all non-admissible lines. Each of them consists of two *Stokes rays*

$$R_{ij} := \{z \,|z = -ir(\bar{u}_i - \bar{u}_j), \ r \geq 0\}, \quad i \neq j \tag{4.34}$$

(some of them may coincide). We explain: for $z \in R_{ij}$

$$\left|e^{z\,u_i}\right| = \left|e^{z\,u_j}\right|;$$

on the right of R_{ij}

$$\left|e^{z\,u_i}\right| < \left|e^{z\,u_j}\right|$$

and on the left of R_{ij}

$$\left|e^{z\,u_i}\right| > \left|e^{z\,u_j}\right|.$$

The ray R_{ji} is the opposite one to R_{ij}. An admissible line ℓ must contain no Stokes rays. The sectors $\Pi_{\rm right/left}$ can be extended up to the first nearest Stokes ray (see [BJL1]).

Theorem 4.3. *The Stokes matrices $S = (s_{ij})$, S_- of the system (3.30) satisfy the following properties*

$$S_- = S^T. \tag{4.35}$$

$$s_{ii} = 1,\ \ i = 1, \dots, n \tag{4.36a}$$

$$s_{ij} \neq 0 \ \text{ only if } \ R_{ij} \subset \Pi_{\rm left}. \tag{4.36b}$$

Proof. We know that, for any two matrix solutions $Y_1(z)$, $Y_2(z)$ of the system (3.30), the product

$$Y_1^T(-z) Y_2(z)$$

does not depend on z. Let us choose for $z \in \Pi_{\rm right}$ $Y_2(z) = Y_{\rm right}(z)$, $Y_1(z) = Y_{\rm left}(z)$. Using the asymptotic developments

$$Y_{\rm right}(z) \sim G(z)\, e^{z\,U}$$
$$Y_{\rm left}(-z) \sim G(-z)\, e^{-z\,U}$$

valid for $z \in \Pi_+$, with $G(z)$ being defined in Lemma 4.3, and the orthogonality condition (4.25) we obtain

$$Y_{\rm left}^T(-z) Y_{\rm right}(z) \equiv 1, \quad z \in \Pi_+.$$

Let us continue analytically this formula in the counter-clockwise direction through the ray ℓ_+. We obtain after the analytic continuation

$$Y_{\rm right}(z) \mapsto Y_{\rm left}(z) S^{-1}$$
$$Y_{\rm left}(-z) \mapsto Y_{\rm right}(-z) S_-.$$

So

$$S_-^T Y_{\rm right}^T(-z) Y_{\rm left}(z) S^{-1} \equiv 1, \quad z \in \Pi_-.$$

As above we show that

$$Y_{\rm right}^T(-z) Y_{\rm left}(z) \equiv 1, \quad z \in \Pi_-.$$

Hence

$$S_-^T = S.$$

Let us now prove (4.36). Comparing the asymptotic developments of the both sides of (4.32) for $z \in \Pi_+$ we conclude that

$$e^{z\,U} S\, e^{-z\,U} \sim 1, \quad |z| \to \infty, \; z \in \Pi_+.$$

This means that

$$e^{z(u_i - u_j)} s_{ij} \sim \delta_{ij}, \quad |z| \to \infty, \; z \in \Pi_+.$$

For the diagonal terms this implies $s_{ii} = 1$. For the off-diagonal terms we have

$$\left| e^{z(u_i - u_j)} \right| \to \infty \;\text{ for }\; |z| \to \infty, \; z \in \Pi_+$$

if $R_{ij} \subset \Pi_{\rm right}$. So, for those pairs $i \neq j$ for which $R_{ij} \subset \Pi_{\rm right}$ we must have $s_{ij} = 0$. The opposite ray $R_{ji} \subset \Pi_{\rm left}$. And

$$\left| e^{z(u_j - u_i)} \right| \to 0 \;\text{ for }\; |z| \to \infty, \; z \in \Pi_+.$$

So s_{ji} need not to be zero. Lemma is proved.

We see that the Stokes matrix S contains $n(n-1)/2$ independent parameters.

To complete the list of the monodromy data we define the central connection matrix

$$Y_0(z) = Y_{\rm right}(z) C, \quad z \in \Pi_+ \tag{4.37}$$

(observe: the branchcut in the definition of $Y_0(z)$ is to be chosen along ℓ_-).

The monodromy (μ, R) at $z = 0$, the monodromy S at $z = \infty$, and the central connection matrix C are not independent. First of all, we have the following *cyclic relation*

$$C^{-1} S^T S^{-1} C = M_0 = \exp 2\pi i (\mu + R). \tag{4.38}$$

This expresses a simple topological fact: on the punctured plane $\mathbf{C}\backslash 0$ a loop around infinity is homotopic to a loop around the origin. Another property comes from the orthogonality relations

$$\begin{aligned} S &= C\, e^{-\pi i R} e^{-\pi i \mu} \eta^{-1} C^T \\ S^T &= C\, e^{\pi i R} e^{\pi i \mu} \eta^{-1} C^T. \end{aligned} \tag{4.39}$$

We leave the proof of these identities as an exercise for the reader.

The matrix C is defined up to transformations of the form

$$C \mapsto B\, C, \quad B\, S\, B^T = B \tag{4.40a}$$

preserving the relations (4.38), (4.39), and

$$C \mapsto C\, C_0, \quad C_0 \in \mathcal{C}_0(\mu, R) \tag{4.40b}$$

corresponding to a change of the solution $Y_0(z)$.

Exercise 4.3. Prove that classes of equivalence (4.40) of central connection matrices of systems (3.30) with a given monodromy (μ, R) at the origin and a given monodromy S at infinity are in one-to-one correspondence with the group $\mathcal{C}(\mu, R)$.

The properties (4.38) and (4.39) typically specify the central connection matrix C of the system with given μ, R, S essentially uniquely with an ambiguity (4.40) that does not affect the Frobenius structure. This reflects the claim of Theorem 4.1 (here "typically" means triviality of the group $\mathcal{C}(\mu, R)$). Anyhow, the following uniqueness theorem holds.

Lemma 4.5. *If two systems*

$$\partial_z Y^{(1,2)} = \left(U + \frac{1}{z} V^{(1,2)}\right) Y^{(1,2)}$$

have the same matrices μ, R, S (w.r.t. the same admissible oriented line ℓ), C then $V^{(2)} = V^{(1)}$.

The proof is similar to that of Lemma 4.4. We leave it as an exercise.

Let us return to semisimple Frobenius manifolds. Starting from a point $t_0 \in M$ such that the eigenvalues $u_1(t_0), \ldots, u_n(t_0)$ of the operator $\mathcal{U}(t_0) = (E(t_0)\cdot)$ are pairwise distinct, ordering these eigenvalues, and choosing signs of the square roots of $< \partial_i, \partial_i >$, and fixing an oriented line ℓ on the complex z-plane admissible for the points $u_1(t_0), \ldots, u_n(t_0)$ we define the Stokes matrix $S = S(t_0)$ and the central connection matrix $C = C(t_0)$. We will now prove that these matrices do not change under small variations of t_0. Observe that the property of admissibility of the line ℓ is stable under small perturbations of t_0.

Isomonodromicity Theorem (second part). *The Stokes matrix S and the central connection matrix C do not depend on the point of a semisimple Frobenius manifold.*

Proof. Due to Lemma 4.3 the coefficients $A_1, A_2, \ldots$ of the solution $Y_{\text{formal}}(z; u)$ are analytic functions on u. From the uniqueness of $Y_{\text{formal}}(z; u)$ it easily follows that

$$\partial_i Y_{\text{formal}}(z; u) = (z\, E_i + V_i)\, Y_{\text{formal}}(z; u), \ \ i = 1, \ldots, n.$$

The same statements are true for the solutions $Y_{\text{right/left}}(z; u)$ and, as we already know from Lecture 2, for the solution $Y_0(z; u)$. Using the definitions

$$S = Y_{\text{right}}^{-1}(z; u) Y_{\text{left}}(z; u), \quad z \in \Pi_+$$

$$C = Y_{\text{right}}^{-1}(z; u) Y_0(z; u), \quad z \in \Pi_{\text{right}}$$

we obtain

$$\partial_i S = 0, \quad \partial_i C = 0.$$

Theorem is proved.

Together with the results of Lecture 2 we conclude that the monodromy data μ, R, S, C do not depend on the point of the Frobenius manifold.

We will now show how to reconstruct the semisimple Frobenius manifiold starting from the monodromy data.

To reconstruct the operator (3.30) and the solutions $Y_{\rm right/left}$, Y_0 for given $u_1, \ldots,$ u_n, (μ, R, S, C) one is to solve certain Riemann - Hilbert boundary value problem . Let D be the disk

$$|z| < \rho$$

for some $\rho > 0$, $P_{\rm right}$ and $P_{\rm left}$ the two components of $\mathbf{C} \setminus \ell$ intersected with the external parts of the disk. We are to construct a piecewise-analytic function

$$\Phi(z) = \begin{cases} \Phi_{\rm right}(z), & z \in P_{\rm right} \\ \Phi_{\rm left}(z), & z \in P_{\rm left} \\ \Phi_0(z), & z \in D \end{cases}$$

continues in the closures of $P_{\rm right}$, $P_{\rm left}$, D resp. such that:

1). on the positive (i.e., that belonging to ℓ_+) part of the common boundary of $P_{\rm right}$ and $P_{\rm left}$ the boundary values of the functions are related by

$$\Phi_{\rm left}(z) = \Phi_{\rm right}(z) e^{z\,U} S\, e^{-z\,U}. \tag{4.41}$$

2). on the negative part of the common boundary of $P_{\rm right}$ and $P_{\rm left}$ the boundary values of the functions are related by

$$\Phi_{\rm left}(z) = \Phi_{\rm right}(z) e^{z\,U} S^T e^{-z\,U}. \tag{4.42}$$

3). on the common boundary of D and $P_{\rm right}$ the boundary values of the functions are related by

$$\Phi_0(z) = \Phi_{\rm right}(z) e^{z\,U} C z^{-R} z^{-\mu}. \tag{4.43}$$

4). on the common boundary of D and $P_{\rm left}$ the boundary values of the functions are related by

$$\Phi_0(z) = \Phi_{\rm left}(z) e^{z\,U} S^{-1} C z^{-R} z^{-\mu}. \tag{4.44}$$

5). for $|z| \to \infty$ within $P_{\rm right/left}$

$$\Phi_{\rm right/left}(z) \to 1. \tag{4.45}$$

Theorem 4.4. *If the Riemann - Hilbert boundary value problem 1 - 5 has a unique solution at a point $u^0 = \left(u_1^0, \ldots, u_n^0\right)$, $u_i^0 \neq u_j^0$ for $i \neq j$, then the unique solution $\Phi = \Phi(z; u_1, \ldots, u_n)$ exists for u sufficiently close to u^0 and it is an analytic function of u. It can be continued analytically to a meromorphic function on the universal covering of the space*

$$\mathbf{C}^n \setminus \mathrm{diag} := \{(u_1, \ldots, u_n) \,|\, u_i \neq u_j \text{ for } i \neq j\}. \tag{4.46}$$

Proof follows from the general theory of Riemann - Hilbert boundary value problems (see in [Mi, Ma]).

Having a solution $\Phi = (\Phi_{\text{right}}(z;u), \Phi_{\text{left}}(z;u), \Phi_0(z;u))$ of the Riemann - Hilbert boundary value problem we can reconstruct the solutions

$$\begin{gathered} Y_{\text{right/left}}(z;u) = \Phi_{\text{right/left}}(z;u) e^{z\,U} \\ Y_0(z;u) = \Phi_0(z;u) z^{\mu} z^R. \end{gathered} \tag{4.47}$$

Let us introduce notations for the coefficients of the expansion of the matrix $\Phi_0(z;u) = (\Phi_{0\,i\alpha}(z;u))$ near $z=0$

$$\Phi_{0\,i\alpha}(z;u) = \sum_{p=0}^{\infty} \phi_{i\alpha,\,p}(u) z^p. \tag{4.48}$$

Observe

$$\phi_{i\alpha,\,0}(u) = \psi_{i\alpha}(u). \tag{4.49}$$

Isomonodromicity Theorem (third part). *Let the Riemann - Hilbert boundary value problem (4.41) - (4.45) for given μ, R, S, C satisfying (4.36), (4.38), (4.39) have a unique solution $\Phi = \Phi(z;u^0)$ at a point $u^0 = (u_1^0, \ldots, u_n^0)$, $u_i^0 \neq u_j^0$ for $i \neq j$ such that*

$$\prod_{i=1}^{n} \phi_{i1,\,0}(u^0) \neq 0. \tag{4.50}$$

Then the formulae

$$\eta_{\alpha\beta} = \sum_{i=1}^{n} \phi_{i\alpha,\,0}(u) \phi_{i\beta,\,0}(u) \tag{4.51}$$

$$e = \sum_{i=1}^{n} \partial_i \tag{4.52}$$

$$E = \sum_{i=1}^{n} u_i \partial_i \tag{4.53}$$

$$t_\alpha = \sum_{i=1}^{n} \phi_{i1,\,1}(u) \phi_{i\alpha,\,0}(u) \tag{4.54}$$

$$c_{\alpha\beta\gamma} = \sum_{i=1}^{n} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}} \tag{4.55}$$

$$F = \frac{1}{2} \sum_{i=1}^{n} \left[\eta^{\alpha\beta} \phi_{i\alpha,\,1} \phi_{i\beta,\,0} \phi_{i1,\,1}^2 - \phi_{i1,\,2}\phi_{i1,\,1} - \phi_{i1,\,0}\phi_{i1,\,3} \right] \tag{4.56}$$

define a semisimple Frobenius structure on a small neighborhood of u^0.

Proof. Let us define the matrix-valued functions $Y_{\text{right/left}}(z;u)$, $Y_0(z;u)$ by the formulae (4.47) and prove that they satisfy the linear system (3.29), (3.30) with

$$V(u) = [U, A_1(u)] \tag{4.57}$$

$$V_i(u) = [E_i, A_1(u)] \tag{4.58}$$

where the matrix $A_1(u)$ is defined from the asymptotic development

$$A_1(u) := \lim_{|z|\to\infty,\, z\in\Pi_+} z\,(\Phi_{\rm right}(z;u) - 1)\,. \tag{4.59}$$

Let us consider the piecewise-analytic function

$$Y(z;u) = \begin{cases} Y_{\rm right}(z;u), & z \in \Pi_{\rm right} \\ Y_{\rm left}(z;u), & z \in \Pi_{\rm left} \\ Y_0(z;u), & z \in D. \end{cases}$$

We prove first that the matrix $Y(z;u)$ is invertible for any z, u. Indeed, $\det Y(z;u)\, e^{-z\sum u_i}$ is a piecewise-analytic function of z having no jumps on the intersections of the domains $\Pi_{\rm right}$, $\Pi_{\rm left}$, D and going to 1 when $|z|\to\infty$. Thus

$$\det Y(z;u) \equiv e^{z(u_1+\ldots+u_n)}.$$

We introduce now piecewise-analytic functions

$$G_i(z;u) := \partial_i Y(z;u)\cdot Y^{-1}(z;u).$$

From construction of S, C it follows that $G_i(z;u)$ has no jumps on the intersections of the domains $\Pi_{\rm right}$, $\Pi_{\rm left}$, D. So it is an analytic matrix-valued function on $\mathbf{C}\setminus 0$. At $|z|\to\infty$ it has the asymptotic development

$$\begin{aligned} G_i(z;u) &= \partial_i\left[\left(1+\frac{A_1}{z}+\ldots\right)e^{z\,U}\right]e^{-z\,U}\left(1-\frac{A_1}{z}+\ldots\right) \\ &\sim z\,E_i + V_i + O\left(\frac{1}{z}\right). \end{aligned}$$

At $z=0$ the function $G_i(z;u)$ goes to a finite limit

$$\begin{aligned} G_i(z;u) &= \partial_i\left[(\Psi(u)+O(z))\,z^{\mu}z^{R}\right]z^{-R}z^{-\mu}\left[\Psi^{-1}(u)+O(z)\right] \\ &= \partial_i\Psi(u)\cdot\Psi^{-1}(u)+O(z) \end{aligned}$$

due to constancy of μ, R. Hence

$$G_i(z;u) = z\,E_i + V_i$$

and

$$\partial_i Y = (z\,E_i + V_i)\,Y,\quad i=1,\ldots,n.$$

Particularly,

$$\partial_i\Psi = V_i\Psi.$$

Similarly, considering the piecewise-analytic function

$$G_z := \partial_z Y(z;u)\cdot Y^{-1}(z;u)$$

we obtain that

$$G_z = U + \frac{V}{z}$$

where the matrix $V = V(u)$ is defined in (4.57).

To prove the orthogonality conditions

$$\begin{aligned} \Phi^T_{\text{right/left}}(-z;u)\Phi_{\text{right/left}}(z;u) &\equiv 1 \\ \Phi^T_0(-z;u)\Phi_0(z;u) &\equiv \eta \end{aligned}$$

we will consider the piecewise-analytic matrix-valued function

$$G(z) := \begin{cases} Y_{\text{right}}(z;u)Y^T_{\text{left}}(-z;u), & z \in \Pi_{\text{right}} \\ Y_{\text{left}}(z;u)Y^T_{\text{right}}(-z;u), & z \in \Pi_{\text{left}}. \end{cases}$$

For $z \in \Pi_+ \cap \Pi_{\text{right}}$

$$\begin{aligned} G(z) &= Y_{\text{right}}(z;u)Y^T_{\text{left}}(-z;u) \\ &= Y_{\text{right}}(z;u)SY^T_{\text{right}}(-z;u). \end{aligned}$$

For $z \in \Pi_+ \cap \Pi_{\text{left}}$

$$\begin{aligned} G(z) &= Y_{\text{left}}(z;u)Y^T_{\text{right}}(-z;u) \\ &= Y_{\text{right}}(z;u)SY^T_{\text{right}}(-z;u). \end{aligned}$$

So, $G(z)$ has no jumps on ℓ_+. Similarly, it has no jumps on ℓ_-. Using (4.39) one obtains that for $z \in \Pi_{\text{right}}$ near $z = 0$

$$\begin{aligned} G(z) &= Y_0(z;u)e^{\pi i R}e^{\pi i \mu}\eta^{-1}Y^T_0(-z;u) = \Phi_0(z;u)\eta^{-1}\Phi^T_0(-z;u) \\ &= 1 + O(z). \end{aligned}$$

A similar computation gives the same behaviour of $G(z)$ at $z \to 0$, $z \in \Pi_{\text{left}}$. So $G(z) \equiv 1$. This proves the orthogonality conditions.

The equations (4.54), (4.56) is the spelling of (2.35), (2.36). Note that the functions $t_1(u)$, ..., $t_n(u)$ are independent coordinates in the points u where the product

$$\prod_{i=1}^n \psi_{i\alpha}(u) \neq 0.$$

Theorem is proved.

Exercise 4.4. Show that the product (4.50) does not vanish identically unless the matrix

$$e^{z\,U} S\, e^{-z\,U}$$

is independent on one of the variables $(u_1, \ldots, u_n)$.

The Isomonodromicity Theorem gives a structure of semisimple Frobenius manifold on small domains in the space of isomonodromy deformations of the operator

$$L = \frac{d}{dz} - \left(U + \frac{V}{z}\right)$$

with rational coefficients. The parameters of these Frobenius manifold are the monodromy data

$$(\mu, e, R, S, C) \tag{4.60}$$

of the operator satisfying the above properties (4.6), (4.7). Here e is a marked eigenvector of the matrix V with the eigenvalue μ_1 (μ_1 being the marked diagonal entry of the matrix μ). The choice of e corresponds to the choice of the first column of the matrix Ψ in the formulae (4.50) - (4.56). (We need not to fix the bilinear form $< \, , \, >$. It is given by (4.51).) It also demonstrates that, locally, any semisimple Frobenius manifold can be realized in such a way.

Exercise 4.5. We say that the Stokes matrix S is *reducible* if it has the form $S = S' \oplus S''$ w.r.t. some decomposition of the set of indices $\{1, \ldots, n\} = I' \cup I''$ into a union of two non-empty non-intersecting subsets. Prove that a reducible matrix S can make a part of the monodromy data only if $\exp 2\pi i \mu_1$ is the eigenvalue of both the matrices ${S'}^T {S'}^{-1}$ and ${S''}^T {S''}^{-1}$. Prove that the Stokes matrix of a reducible Frobenius manifold is reducible (see Exercise 2.5).

We will now describe the structure of analytic continuation of semisimple Frobenius manifolds. According to Theorem 4.4 and due to the formulae (4.54), (4.56) the functions t_α and F can be continued analytically to meromorphic functions on the universal covering of $\mathbf{C}^n \setminus \mathrm{diag}$. Since the canonical coordinates are defined up to reordering, the structure of analytic continuation of the Frobenius manifold with given monodromy data (4.60) is described by an action of the fundamental group

$$\pi_1 \left((\mathbf{C}^n \setminus \mathrm{diag}\,) / S_n, (u_1^0, \ldots, u_n^0) \right) = \mathcal{B}_n$$

(the braid group) on the monodromy data computed at a given point u^0. The global structure of the Frobenius manifold is described by the stationary subgroup ${\mathcal{B}_n}^0 \subset \mathcal{B}_n$ of the given monodromy data (4.60).

To compute the action of the braid group $\mathcal{B}_n$ on the monodromy data, and also to describe the dependence of the monodromy data on the admissible oriented line ℓ, we will briefly present here the theory of Stokes factors (see [BJL1]).

Let us label all the Stokes rays (4.34) of the system (3.30) in the counter-clockwise order starting from the first one in Π_{right}. We obtain the rays

$$\begin{array}{c} R^{(1)}, \ldots, R^{(m)} \text{ in } \Pi_{\mathrm{right}} \\ R^{(m+1)}, \ldots, R^{(2m)} \text{ in } \Pi_{\mathrm{left}}. \end{array} \tag{4.61}$$

We will use the cyclic labelling $R^{(k \pm 2m)} = R^{(k)}$. Observe that the narrow sectors Π_+ and Π_- contain no Stokes rays. For generic $(u_1, \ldots, u_n)$ one has

$$m = \frac{n(n-1)}{2}$$

but some coincidences of the Stokes rays may happen in the nongeneric situation when there are three u_i, u_j, u_k on a line or two pairs u_i, u_j and u_k, u_l on two parallel lines. Let us consider the sector of z-plane from $R^{(k)}e^{\frac{-i\varepsilon}{2}}$ to $R^{(m+k)}e^{-i\varepsilon}$. According to Theorem 4.2 there exists a unique solution $Y^{(k)}(z)$ of (3.30) such that

$$Y^{(k)}(z) \sim Y_{\mathrm{formal}}(z), \quad |z| \to \infty \tag{4.62}$$

within the above sector. This solution can be extended preserving the asymptotics into the open sector

$$\Pi_k: \text{ from } R^{(k-1)} \text{ to } R^{(m+k)}. \tag{4.63}$$

On the intersection of two subsequent sectors one has a constant matrix K_j defined by

$$Y^{(j+1)}(z) = Y^{(j)}(z)K_j, \quad z \in \Pi_j \cap \Pi_{j+1}. \tag{4.64}$$

Lemma 4.6.

$$Y_{\mathrm{right}} = Y^{(1)}, \quad Y_{\mathrm{left}} = Y^{(m+1)} \tag{4.65}$$

$$S = K_1 \dots K_m. \tag{4.66}$$

Proof is obvious.

Definition 4.4. The matrices K_j are called *Stokes factors* of the matrix S.

Exercise 4.6. Prove that

$$K_{m+j}K_j^T = 1. \tag{4.67}$$

How to find the Stokes factors knowing the Stokes matrix S and the configuration of pairwise distinct complex numbers $u_1, \dots, u_n$? The clue is in the following property of Stokes factors (see [BJL1]).

Lemma 4.7. *All the diagonal entries of K_j equal 1. Of the off-diagonal entries $(K_j)_{ab}$ all equal zero but those for which the Stokes ray R_{ba} coincides with $R^{(j)}$.*

Proof. On $z \in \Pi_j \cap \Pi_{j+1}$ one must have

$$e^{z\,U}K_j e^{-z\,U} \to 1 \text{ as } |z| \to \infty.$$

Hence $(K_j)_{aa} = 1$ (as in the proof of Theorem 4.3). On the intersection the absolute values

$$\left|e^{z(u_a-u_b)}\right|$$

can go to either $+\infty$ or 0 for any pair $a \neq b$ but those for which R_{ab} or R_{ba} coincides with $R^{(j)}$. Indeed, the whole intersection $\Pi_j \cap \Pi_{j+1}$ lies on the right from the oriented line

$$R^{(m+j)} \cup \left(-R^{(j)}\right).$$

If

$$R_{ab} = R^{(m+j)}, \quad R_{ba} = R^{(j)}$$

then on the right from the oriented line one has

$$\left|e^{z(u_a-u_b)}\right| \to 0 \quad \text{as } |z| \to \infty.$$

Lemma is proved.

Theorem 4.5. *Any Stokes matrix S with the above properties can be uniquely factorized into the product $S = K_1 \dots K_m$ of Stokes factors of the above form.*

Proof see in [BJL1].

From the factorization (4.66) it follows that the Stokes matrix does not change if one deforms the admissible line ℓ not intersecting any of the Stokes rays. We describe now what happens if the oriented admissible line $\ell = \ell_+ \cup (-\ell_-)$ passes through the Stokes ray R moving counter-clockwise. Instead, one may consider a deformation of one of the Stokes rays R passing through ℓ_+ moving clockwise.

Lemma 4.8. *After the above deformation the new solutions $Y'_{\text{right/left}}$, the new Stokes matrix S', and the new connection matrix C' have the form*

$$Y_{\text{right}} = Y'_{\text{right}} K_R^T \tag{4.68a}$$

$$Y'_{\text{left}} = Y_{\text{left}} K_R \tag{4.68b}$$

$$S' = K_R^T S\, K_R \tag{4.68c}$$

$$C' = K_R^T C \tag{4.68d}$$

(the last formula holds true modulo the ambiguity (4.40). Here K_R is the Stokes factor corresponding to the Stokes ray R.

Proof follows from Lemma 4.6 and from Exercise 4.6.

We are now ready to compute the action of the braid group $\mathcal{B}_n$ on the monodromy data describing the analytic continuation of the Frobenius manifold. First, the action of $\mathcal{B}_n$ on the monodromy at $z = 0$ is trivial. We now compute the action on the Stokes matrix S. Let us assume that the canonical coordinates $(u_1, \dots, u_n)$ are ordered in such a way that S is an upper triangular matrix. We choose the standard generators $\sigma_1, \dots, \sigma_{n-1}$ of the braid group $\mathcal{B}_n$. The generator σ_i is given by a deformation of $(u_1, \dots, u_n)$ such that:

1). u_k remains fixed for $k \neq i,\ i+1$.

2). u_i and u_{i+1} are permuted moving counterclockwise.

Let us deform $(u_1, \dots, u_n)$ in the coefficients of the operator

$$L = \frac{d}{dz} - \left(U + \frac{V(u)}{z}\right).$$

Due to isomonodromicity the matrices S and C remain unchanged until some of the Stokes rays passes through ℓ. After this we are to reorder the canonical coordinates to preserve upper triangularity of the Stokes matrix and, then, to compute the new matrices S' and C' using Lemma 4.8. We are to recall here that the operator L for a given ordering of the canonical coordinates $(u_1, \dots, u_n)$ is determined up to a transformation

$$L \mapsto J\, L\, J \tag{4.69}$$

where J is an arbitrary diagonal matrix of the form

$$J = \mathrm{diag}\,(\pm 1, \dots, \pm 1). \tag{4.70}$$

Thus the matrices

$$S \text{ and } J\,S\,J, \quad C \text{ and } J\,C \tag{4.71}$$

must be identified. So what we need is actually an action of $\mathcal{B}_n$ on the classes of equivalence of the matrices S and C w.r.t. the identifications (4.71).

The result is given by

Theorem 4.6. *The analytic continuation of a semisimple Frobenius manifold is described by the following action*

$$\begin{aligned} S &\mapsto \beta(S) \\ C &\mapsto \beta(C) \end{aligned} \tag{4.72}$$

of the braid group $\mathcal{B}_n \ni \beta$ on the Stokes matrix $S = (s_{ij})$ and the central connection matrix C. For the standard generator $\beta = \sigma_i$ the action has the form

$$\begin{aligned} \sigma_i(S) &= K^{(i)}(S)\, S\, K^{(i)}(S) \\ \sigma_i(C) &= K^{(i)}(S)\, C \end{aligned} \tag{4.73}$$

where

$$\begin{aligned} \left(K^{(i)}(S)\right)_{kk} &= 1, \;\; k = 1, \dots, n, \quad k \neq i,\, i+1 \\ \left(K^{(i)}(S)\right)_{i+1,\, i+1} &= -s_{i,\, i+1} \\ \left(K^{(i)}(S)\right)_{i,\, i+1} &= \left(K^{(i)}(S)\right)_{i+1,\, i} = 1 \end{aligned} \tag{4.74}$$

all other entries of the matrix $K^{(i)}(S)$ are equal to zero.

Proof. Let us assume that, during the deformation σ_i, the coordinates u_i and u_{i+1} remain sufficiently close to each other. Then all the Stokes rays but $R_{i,\,i+1}$ and $R_{i+1,\,i}$ will be only slightly deformed and they will return to their original positions (with renumbering $i \leftrightarrow i+1$) after the end of the deformation. But the rays $R_{i,\,i+1}$ and $R_{i+1,\,i}$ interchange their positions rotating clockwise. Particularly, it is the ray $R = R_{i+1,\,i}$ who passes through the positive half-line ℓ_+ rotating clockwise. At the very last moment before the collision the configuration of the Stokes rays is such that $R^{(1)} = R_{i+1,\,i}$ and $R^{(m+1)} = R_{i,\,i+1}$, and we may assume that $R^{(1)}$ and $R^{(m+1)}$ contain no other Stokes rays. From Theorem 4.5 we obtain a factorization of S into the product of upper triangular Stokes factors

$$S = K_1 K_2 \dots K_m$$

where the only nonzero off-diagonal entry of the matrix K_1 sits in the $(i, i+1)$ box, and all the factors $K_2, \dots, K_m$ have zero on the $(i, i+1)$ place. From this we obtain that

$$(K_1)_{i,\,i+1} = s_{i,\,i+1}.$$

We are now to apply the formulae (4.68) to compute the new matrices S', C' with

$$K_R = K_{m+1} = \left(K_1^T\right)^{-1}.$$

After this applying the permutation $i \leftrightarrow i+1$ we arrive at the formulae (4.74). Theorem is proved.

Example 4.3. For $n=3$ the generators σ_1, σ_2 of $\mathcal{B}_3$ act as follows in the space of Stokes matrices

$$S = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_1(x,y,z) = (-x, z, y - x\,z), \quad \sigma_2(x,y,z) = (y, x - y\,z, -z). \tag{4.75}$$

Exercise 4.7. Prove that the braid

$$\zeta = (\sigma_1 \dots \sigma_{n-1})^n \tag{4.76}$$

acts trivially on Stokes matrices.

The braid ζ generates the center of $\mathcal{B}_n$ (see [Bi]). So the quotient $\mathcal{B}_n$/center acts on the space of Stokes matrices. For $n=3$ the quotient is isomorphic to the modular group $PSL_2(\mathbf{Z})$ [*ibid*].

Let $\mathcal{B}_n(S,C) \subset \mathcal{B}_n$ be the stationary subgroup of the class of equivalence (4.71) of the pair S, C. We realize it as a subgroup in the fundamental group

$$\pi_1\left([\mathbf{C}^n \setminus \mathrm{diag}\,]/S_n, (u_1^0, \dots, u_n^0)\right)$$

and construct the corresponding covering

$$M(S,C) \to [\mathbf{C}^n \setminus \mathrm{diag}\,]/S_n$$

i.e., such a covering that the group of deck transformations of the fiber is isomorphic to $\mathcal{B}_n(S,C)$. From Theorem 4.6 it follows

Theorem 4.7. *1). For a given monodromy data (μ, e, R, S, C) the Frobenius structure extends from a small neighborhood of u^0 to a dense open subset in the manifold $M(S,C)$. This Frobenius structure on $M(S,C)$ we denote $Fr(\mu, e, R, S, C)$.*

2). Let (μ, e, R, S, C) be the monodromy data of a semisimple Frobenius manifold M computed at the point $u^0 = (u_1^0, \dots, u_n^0)$ w.r.t. an admissible oriented line ℓ. Let M^0 be the open part of the Frobenius manifold M consisting of all points $t \in M$ such that all the eigenvalues $u_1(t)$, ..., $u_n(t)$ of the operator of multiplication by the Euler vector field are pairwise distinct. Then the map

$$M^0 \to Fr(\mu, e, R, S, C)$$

is well-defined and it is an equivalence of Frobenius manifolds.

Example 4.4. Let us compute the monodromy data of quantum cohomology of $\mathbf{CP}^2$, i.e., of the solution (1.15) of WDVV equations of asociativity . The monodromy at $z=0$ is completely determined by the classical cohomology $H^*\left(\mathbf{CP}^2\right)$ together with the first Chern class $c_1\left(\mathbf{CP}^2\right)$ (see Lecture 2). We obtain

$$\mu = \operatorname{diag}(-1,0,1), \quad R = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}.$$

Let us compute the Stokes matrix in the semisimple point

$$t_1 = t_3 = 0, \quad \text{arbitrary } t_2 \text{ with } \operatorname{Re} t_2 < R. \tag{4.77}$$

Here R is the radius of convergence (1.16). Let us denote $q = \exp t_2$. The system (2.41) for horizontal sections $(\xi_1, \xi_2, \xi_3) = \left(\partial_1 \tilde{t}, \partial_2 \tilde{t}, \partial_3 \tilde{t}\right)$ of the connection $\tilde{\nabla}$ can be reduced to two third-order equations

$$\begin{aligned} \partial_2^3 \phi &= z^3 q\, \phi \\ (z\partial_z)^3 \phi &= 27 z^3 q\, \phi \end{aligned} \tag{4.78}$$

for the function

$$\phi = \phi(t_2 z) = \frac{\xi_1}{z},$$

$$(\xi_1, \xi_2, \xi_3) = \left(z\,\phi, \frac{1}{3} z\partial_z \phi, \frac{1}{9} \partial_z (z\partial_z \phi) \right).$$

The system (4.78) is equivalent to one equation

$$(z\partial_z)^3 \Phi = 27 z^3 \Phi \tag{4.79}$$

using the quasihomogeneity

$$\phi(t_2, z) = \Phi\left(z\, q^{1/3}\right). \tag{4.80}$$

The problem is reduced to computation of the Stokes matrix of the generalized hypergeometric equation (see [DM]). We are to carefully select the basis of formal solutions of (4.79) at $z \to \infty$ corresponding to the basis of columns of $Y_{\text{formal}}(z)$ of the solution (4.26) of the gauge-equivalent system (3.30).

The operator $\mathcal{U}$ of multiplication by the Euler vector field in the basis $e_1 = \partial_1$, $e_2 = \partial_2$, $e_3 = \partial_3$ has the matrix

$$\mathcal{U}(t) = \begin{pmatrix} 0 & 0 & 3q \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \quad t = (0, t_2, 0), \;\; q = e^{t_2}. \tag{4.81}$$

The canonical coordinates (i.e., the eigenvalues of $\mathcal{U}$) in the point (4.77) take the values

$$u_1 = 3q^{1/3}, \;\; u_2 = 3\bar{\epsilon}^2 q^{1/3}, \;\; u_3 = 3\epsilon^2 q^{1/3} \tag{4.82}$$

where

$$\epsilon = \exp\frac{\pi i}{3}.$$

The corresponding idempotents of the quantum cohomology algebra are

$$\begin{aligned}\pi_1 &= \frac{1}{3}\left(e_1 + q^{-1/3}e_2 + q^{-2/3}e_3\right)\\ \pi_2 &= \frac{1}{3}\left(e_1 + \epsilon^2 q^{-1/3}e_2 + \bar\epsilon^2 q^{-2/3}e_3\right).\\ \pi_3 &= \frac{1}{3}\left(e_1 + \bar\epsilon^2 q^{-1/3}e_2 + \epsilon^2 q^{-2/3}e_3\right)\end{aligned}$$

The invariant metric

$$<\pi_1,\pi_1>=\frac{1}{3}q^{-2/3},\quad <\pi_2,\pi_2>=\frac{1}{3}\bar\epsilon^2 q^{-2/3},\quad <\pi_3,\pi_3>=\frac{1}{3}\epsilon^2 q^{-2/3}.$$

Evaluating the square root we obtain the normalized idempotents

$$\begin{aligned}f_1 &= \frac{1}{\sqrt{3}}\left(q^{1/3}e_1 + e_2 + q^{-1/3}e_3\right)\\ f_2 &= \frac{1}{\bar\epsilon\sqrt{3}}\left(q^{1/3}e_1 + \epsilon^2 e_2 + \bar\epsilon^2 q^{-1/3}e_3\right).\\ f_3 &= \frac{1}{\epsilon\sqrt{3}}\left(q^{1/3}e_1 + \bar\epsilon^2 e_2 + \epsilon^2 q^{-1/3}e_3\right)\end{aligned}$$

This gives the matrix $\Psi = (\psi_{i\alpha})$

$$\Psi = \frac{1}{\sqrt{3}}\begin{pmatrix} q^{-1/3} & 1 & q^{1/3}\\ \bar\epsilon q^{-1/3} & -1 & \epsilon q^{1/3}\\ \epsilon q^{-1/3} & -1 & \bar\epsilon q^{1/3}\end{pmatrix}. \tag{4.83}$$

We can easily compute the matrix V in the point of interest (cf. [MM]). But what we need is to determine the asymptotic structure of the solutions of (4.79) at $z \to \infty$. We must choose the basis $\tilde t_1^\infty$, $\tilde t_2^\infty$, $\tilde t_3^\infty$ of the coordinates $\tilde t$ such that the matrix

$$Y_{ij} := \frac{\partial_i \tilde t_j^\infty}{\psi_{i1}}$$

has the development (4.26), i.e.,

$$Y_{ij} \sim \left(\delta_{ij} + O\left(\frac{1}{z}\right)\right) e^{z\,u_j},\quad i,\, j = 1,\, 2,\, 3. \tag{4.84}$$

This gives the three solutions ϕ_1, ϕ_2, ϕ_3 of the system (4.78) such that

$$\phi_j = \frac{1}{z}\frac{\partial}{\partial t^1}\tilde t_j^\infty = \frac{1}{z}\sum_{i=1}^3 \partial_i \tilde t_j^\infty = \frac{1}{z}\sum_{i=1}^3 \psi_{i1} Y_{ij}.$$

For the corresponding basic solutions of (4.79) we obtain the needed developments

$$\begin{aligned}
\Phi_1 &\sim \frac{1}{\sqrt{3}} \frac{e^{3z}}{z} \left(1 + O\left(\frac{1}{z}\right)\right) \\
\Phi_2 &\sim \frac{\bar{\epsilon}}{\sqrt{3}} \frac{e^{3\bar{\epsilon}^2 z}}{z} \left(1 + O\left(\frac{1}{z}\right)\right) \\
\Phi_3 &\sim \frac{\epsilon}{\sqrt{3}} \frac{e^{3\epsilon^2 z}}{z} \left(1 + O\left(\frac{1}{z}\right)\right).
\end{aligned} \tag{4.85}$$

We are now to compute the Stokes matrix of the equation (4.79) with respect to the bases of solutions having the asymptotic developments (4.85) in the right/left half-planes $\Pi_{\text{right/left}}$ with some admissible oriented line ℓ.

The Stokes rays of equation (4.79) have the form

$$\begin{aligned}
R_{12} &= \{-\rho\,\epsilon \,|\, \rho \geq 0\} \\
R_{13} &= \{\rho\,\bar{\epsilon} \,|\, \rho \geq 0\} \\
R_{23} &= \{\rho \;\;|\, \rho \geq 0\}
\end{aligned} \tag{4.86}$$

the rays R_{21}, R_{31}, R_{32} are the opposite to the above. We choose the admissible line

$$\ell = \{r\, e^{i\alpha} \,|\, -\infty < r < \infty\} \tag{4.87}$$

for a fixed small $\alpha > 0$ oriented according to the positive direction of r. We will use now a suitable Meijer function [Lu] to compute the Stokes matrix.

Lemma 4.9. *The function*

$$g(z) = \frac{1}{(2\pi)^2 i} \int_{-c-i\infty}^{-c+i\infty} \Gamma^3(-s) e^{\pi i s} z^{3s} ds \tag{4.88}$$

defined for $z \neq 0$,

$$-\frac{5\pi}{6} < \arg z < \frac{\pi}{6} \tag{4.89}$$

where c is any positive number, satisfies (4.79). The analytic continuation of this function has the asymptotic development

$$g(z) \sim \frac{1}{\sqrt{3}} \bar{\epsilon} \frac{e^{3\bar{\epsilon}^2 z}}{z} = \Phi_2(z), \quad |z| \to \infty \tag{4.90}$$

in the sector

$$-\frac{5\pi}{3} < \arg z < \pi. \tag{4.91}$$

It satisfies the identity

$$g\left(z\, e^{2\pi i}\right) - 3\, g\left(z\, e^{\frac{4\pi i}{3}}\right) + 3\, g\left(z\, e^{\frac{2\pi i}{3}}\right) - g\left(z\right) = 0. \tag{4.92}$$

Proof (cf [Lu]). Using the Stirling formula

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{z}\right)$$

and

$$\lim_{|y|\to\infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}y} |y|^{\frac{1}{2}-x} = \sqrt{2\pi},\ x,\ y \text{ real}$$

we prove uniform convergence of the integral in the domain (4.89) and independence it on c. Differentiation gives

$$(z\partial_z)^3 g = \frac{27}{(2\pi)^2 i}\int_{-c-i\infty}^{-c+i\infty} s^3\Gamma^3(-s)e^{\pi i s}z^{3s}ds.$$

Using the property of gamma-function

$$s\,\Gamma(-s) = -\Gamma(1-s)$$

and doing a shift $s \to s+1$ we obtain for the r.h.s. the integral

$$\frac{27}{(2\pi)^2 i}\int_{-c-i\infty}^{-c+i\infty} \Gamma^3(-s)e^{\pi i s}z^{3(s+1)}ds = 27z^3 g.$$

To derive the asymptotic development we use Laplace method. Representing the integrand in the form $\exp phase$ and using Stirling formula one obtains the following asymptotic development for

$$\begin{aligned} phase &= 3\log\Gamma(-s) + \pi i s + 3s\log z \\ &\sim -3\left(s+\frac{1}{2}\right)\log s + 3s\log z + (3-2\pi i)s \end{aligned}$$

valid for

$$-\frac{3\pi}{2} < \arg s < -\frac{\pi}{2}. \tag{4.93}$$

For big $|z|$ the phase has critical point at

$$s \sim z\,e^{-\frac{2\pi i}{3}} - \frac{1}{2}.$$

This critical point is in the domain (4.89) if

$$-\frac{5\pi}{6} < \arg z < \frac{\pi}{6}.$$

In the critical point the value of the phase is

$$phase_0 \sim -\frac{3}{2}\log z + 3ze^{-\frac{2\pi i}{3}} + \frac{3}{2}\log 2\pi + \pi i$$

and the second s-derivative at this point

$$phase_0'' \sim -\frac{3e^{\frac{2\pi i}{3}}}{z}.$$

Applying Laplace formula for the integral

$$g(z) \sim \frac{1}{(2\pi)^2 i} \frac{1}{\sqrt{2\pi}} \frac{e^{phase_0}}{\sqrt{phase_0''}}$$

we obtain the asymptotics (4.90). The asymptotics remains valid in a wider sector

$$-\frac{5\pi}{3} < \arg z < \pi.$$

Indeed, during this analytic continuation, i.e., counterclockwise until R_{32} and clockwise until R_{21} the exponential $e^{z\, u_2}$ remains dominant.

To derive the identity (4.92) we observe that the equation is invariant w.r.t. the rotation

$$z \mapsto z\, e^{\frac{2\pi i}{3}}.$$

This generates a linear operator, A, in the 3-dimensional space of solutions of (4.79). Let us prove that all the eigenvalues of A are equal to 1. Indeed, near $z = 0$ all the solutions have the form

$$\Phi(z) = \sum_{m=0}^{\infty} \frac{z^{3m}}{(m!)^3} \left[a_m + b_m \log z + c_m \log^2 z\right] \tag{4.94}$$

where a_0, b_0, c_0 are arbitrary parameters and the coefficients a_m, b_m, c_m for $m > 0$ are uniquely determined from the recursion relations

$$\begin{aligned} c_m &= c_{m-1} \\ b_m + \frac{2}{m} c_m &= b_{m-1} \\ a_m + \frac{1}{m} b_m + \frac{2}{3m^2} c_m &= a_{m-1}. \end{aligned}$$

The operator

$$(A\phi)(z) = \Phi\left(z\, e^{\frac{2\pi i}{3}}\right)$$

in the basis of solutions of the form (4.94) with only one nonzero of a_0, b_0, c_0 is given by a triangular matrix with all 1 on the diagonals. Writing Cayley - Hamilton theorem

$$(A-1)^3 = 0$$

we obtain

$$A^3 g - 3A^2 g + 3A\ g - g = 0.$$

This gives the identity (4.92). Lemma is proved.

Let us construct the three solutions $\Phi^{\text{right}}(z) = \left(\Phi_1^{\text{right}}(z), \Phi_2^{\text{right}}(z), \Phi_3^{\text{right}}(z)\right)$ having the asymptotic behaviour of the form (4.85)

$$\Phi_j^{\text{right}}(z) \sim \Phi_j(z), \quad |z| \to \infty, \ -\pi < \arg z < \frac{\pi}{3}, \ j = 1, 2, 3.$$

We can take

$$\Phi^{\text{right}}(z) = \left(-g\left(e^{\frac{2\pi i}{3}} z\right), g(z), g\left(e^{-\frac{2\pi i}{3}} z\right)\right). \tag{4.95}$$

Similarly, the components of the vector-function $\Phi^{\text{left}}(z)$ must have the asymptotics

$$\Phi_j^{\text{left}}(z) \sim \Phi_j(z), \quad |z| \to \infty, \ 0 < \arg z < \frac{4\pi}{3}, \ j = 1, 2, 3.$$

We take

$$\Phi^{\text{left}}(z) = \left(-g\left(e^{-\frac{4\pi i}{3}} z\right), g\left(e^{-2\pi i} z\right) - 3g\left(e^{-\frac{4\pi i}{3}} z\right), g\left(e^{-\frac{2\pi i}{3}} z\right)\right). \tag{4.96}$$

The only novelty to be proved is the formula for Φ_2^{left}. Indeed, from Lemma 4.9 it follows that

$$\Phi_2^{\text{left}}(z) = g\left(e^{-2\pi i} z\right) - 3g\left(e^{-\frac{4\pi i}{3}} z\right) \sim \Phi_2(z), \ |z| \to \infty, \ \frac{\pi}{3} < \arg z < \frac{4\pi}{3}.$$

Using the identity (4.92) we may rewrite this function as

$$\Phi_2^{\text{left}}(z) = g(z) - 3g\left(e^{-\frac{2\pi i}{3}} z\right) \sim \Phi_2(z), \ |z| \to \infty, \ 0 < \arg z < \frac{\pi}{3}.$$

Applying again the identity (4.92) we obtain that in the sector

$$0 < \arg z < \frac{\pi}{3}$$

$$\left(\Phi_1^{\text{left}}(z), \Phi_2^{\text{left}}(z), \Phi_3^{\text{left}}(z)\right) = \left(\Phi_1^{\text{right}}(z), \Phi_2^{\text{right}}(z), \Phi_3^{\text{right}}(z)\right) S$$

with

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -3 & -3 & 1 \end{pmatrix}. \tag{4.97}$$

This is the Stokes matrix of the quantum cohomology of $\mathbf{CP}^2$. Changing the sign of the normalized idempotent f_3 we can reduce S to the form

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix}. \tag{4.97$'$}$$

The matrix (4.97) was obtained from physical considerations in [CV2]. The main argument was that, in Landau - Ginzburg models of 2D topological field theory the entries of the Stokes matrix must be integers. Then, since the eigenvalues of $S^T S^{-1}$ must all be 1, one arrives at the following Diophantine equation for the entries

$$x^2 + y^2 + z^2 - xyz = 0$$

where

$$S = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

All the integer solutions to the equation have the form

$$x = 3x_1, \ y = 3y_1, \ z = 3z_1$$

where x_1, y_1, z_1 are integer solutions to Markoff equations

$$x_1^2 + y_1^2 + z_1^2 - 3x_1 y_1 z_1 = 0.$$

The solutions of Markoff equation are known to be all equivalent to $(1, 1, 1)$ modulo the action (4.75) of the braid group. This solution of Markoff equation just corresponds to the Stokes matrix (4.97')

In the next Lecture we will construct polynomial Frobenius manifolds starting from an arbitrary finite Coxeter group. Particularly, for the Coxeter groups with simply-laced Dynkin diagrams these coincides with the Frobenius manifolds of the singularity theory. It can be shown, using this construction, that, in this examples, S is the variation operator of the singularity computed in the so-called marked basis of vanishing cycles [AGV].

Here we define a remarkable operation of *tensor product* of Frobenius manifolds. We are motivated by the results of Kaufmann, Kontsevich, and Manin [KM, Ka] describing quantum cohomology of the direct product of two varieties.

Let M', M'' be two Frobenius manifolds of the dimensions n' and n'' resp. We say that a Frobenius manifold M of the dimenion $n' n''$ is the tensor product $M = M' \otimes M''$ if it has the following structure.

1). The tangent planes TM with the bilinear form $< \, , \, >$ and the unity vector field e are represented as

$$(TM, < \, , \, >, e) = (TM' \otimes TM'', < \, , \, >' \otimes < \, , \, >'', e' \otimes e'')$$

(as usually, we identify the tangent planes in different points using the Levi-Civita flat connection). Thus, the flat coordinates on M have double labels

$$t = \left(t^{\alpha' \alpha''} \right), \quad 1 \leq \alpha' \leq n', \ 1 \leq \alpha'' \leq n''.$$

The unity vector field is

$$e = \frac{\partial}{\partial t^{1' 1''}}.$$

The matrix of $<\,,\,>$ has the form

$$\eta_{\alpha'\alpha''\,\beta'\beta''} = \eta_{\alpha'\beta'}\eta_{\alpha''\beta''}.$$

2). In the points

$$t \in M, \quad t^{\alpha'\alpha''} = 0 \text{ for } \alpha' > 1,\ \alpha'' > 1 \tag{4.98}$$

the algebra T_tM is the tensor product

$$T_tM = T_{t'}M' \otimes T_{t''}M'',$$

$$t' = \left(t^{2'1''}, \ldots, t^{n'1''}\right)$$

$$t'' = \left(t^{1'2''}, \ldots, t^{1'n''}\right)$$

i.e.,

$$c^{\gamma'\gamma''}_{\alpha'\alpha''\,\beta'\beta''}(t) = c^{\gamma'}_{\alpha'\beta'}(t')c^{\gamma''}_{\alpha''\beta''}(t'').$$

In these formulae $\eta_{\alpha'\beta'}$, $c^{\gamma'}_{\alpha'\beta'}$ and $\eta_{\alpha''\beta''}$, $c^{\gamma''}_{\alpha''\beta''}$ are the invariant bilinear form and the structure constants of the Frobenius manifolds M' and M'' resp.

3). The charge

$$d_M = d_{M'} + d_{M''}$$

and the Euler vector field on M has the form

$$E = \sum_{\alpha',\,\alpha''} t^{\alpha'\alpha''}\left(1 - q_{\alpha'} - q_{\alpha''}\right)\frac{\partial}{\partial t^{\alpha'\alpha''}} + \sum r_{\alpha'}\frac{\partial}{\partial t^{\alpha'1''}} + \sum r_{\alpha''}\frac{\partial}{\partial t^{1'\alpha''}}. \tag{4.99}$$

Here

$$E' = \sum_{\alpha'=1}^{n'}\left[(1 - q_{\alpha'})t^{\alpha'} + r_{\alpha'}\right]\partial_{\alpha'}$$

$$E'' = \sum_{\alpha''=1}^{n''}\left[(1 - q_{\alpha''})t^{\alpha''} + r_{\alpha''}\right]\partial_{\alpha''}$$

are the Euler vector fields on M' and M'' resp.

For any two semisimple Frobenius manifolds M', M'' we will now describe their tensor product $M = M' \otimes M''$ in terms of the monodromy data of the factors.

Lemma 4.10. *1). If $M = M' \otimes M''$ with semisimple M' and M'' then M is semisimple.*

2). Let $t'_0 \in M'$, $t''_0 \in M''$ be two points such that a) $t_0^{1'} = t_0^{1''}$, *and* b) *the values of the canonical coordinates $u_{i'} = u_{i'}(t'_0)$, $i' = 1, \ldots, n'$, $u_{i''} = u_{i''}(t''_0)$, $i'' = 1, \ldots, n''$ satisfy the properties*

$$u_{i'} \neq u_{j'},\ i' \neq j'$$

$$u_{i''} \neq u_{j''},\ i'' \neq j''$$

$$u_{i'} + u_{i''} \neq u_{j'} + u_{j''},\ (i',\, i'') \neq (j',\, j'').$$

Let ℓ be a line on the z-plane such that for any $z \in \ell \setminus 0$

$$\begin{gathered}\mathrm{Re}\,[z\,(u_{i'} - u_{j'})] \neq 0,\ i' \neq j'\\ \mathrm{Re}\,[z\,(u_{i''} - u_{j''})] \neq 0,\ i'' \neq j''\\ \mathrm{Re}\,[z\,(u_{i'} - u_{j'})] + \mathrm{Re}\,[z\,(u_{i''} - u_{j''})] \neq 0,\ (i',\, i'') \neq (j',\, j'').\end{gathered}$$

Then the Stokes matrix S of M in the point t_0 with the coordinates

$$\begin{aligned} t^{\alpha' 1''} &= t_0^{\alpha'},\ \alpha' = 1, \dots, n'\\ t^{1'\alpha''} &= t_0^{\alpha''},\ \alpha'' = 1, \dots, n''\\ t^{\alpha'\alpha''} &= 0,\ \alpha' > 1,\ \alpha'' > 1\end{aligned} \tag{4.100}$$

is the tensor product of the Stokes matrices S' of M' in the point t_0' and S'' of M'' in the point t_0''

$$S = S' \otimes S''.$$

Proof. If $t_0' \in M'$, $t_0'' \in M''$ are semisimple points of the Frobenius manifolds then the point (4.100) will be a semisimple point of M. The idempotents of the algebra

$$T_{t_0'}M' \otimes T_{t_0''}M''$$

are tensor products $\pi_{i'} \otimes \pi_{i''}$, $i' = 1, \dots, n'$, $i'' = 1, \dots, n''$. The operator of multiplication by the Euler vector field (4.99) in the point (4.100) has the form

$$\mathcal{U} = 1' \otimes \mathcal{U}'' + \mathcal{U}' \otimes 1'' - t^1\, 1' \otimes 1''$$

where $t^1 = t_0^{1'} = t_0^{1''}$. The eigenvalues of this operator are

$$u_{i'} + u_{i''} - t^1, \quad 1 \leq i' \leq n',\ 1 \leq i'' \leq n''.$$

These are the values of the canonical coordinates on M in the points of the $(n' + n'' - 1)$-dimensional locus (4.98).

Let $Y'_{\text{right/left}}(z; t_0')$, $Y''_{\text{right/left}}(z; t_0'')$ be the solutions of the system (3.30) for M' and M'' resp. with the asymptotic behaviour (4.26) in the right/left half-planes w.r.t. the admissible line ℓ. Then the solutions of the system (3.30) for M with the needed asymptotic development (4.26) are

$$\begin{gathered} Y_{\text{right}}(z; t_0) = e^{-z\,t^1}\, Y'_{\text{right}}(z; t_0') \otimes Y''_{\text{right}}(z; t_0''),\ z \in \Pi_{\text{right}}\\ Y_{\text{left}}(z; t_0) = e^{-z\,t^1}\, Y'_{\text{left}}(z; t_0') \otimes Y''_{\text{left}}(z; t_0''),\ z \in \Pi_{\text{left}}\end{gathered}$$

This proves Lemma.

Theorem-Definition 4.8. *Let*

$$M = Fr\,(\mu' \otimes 1 + 1 \otimes \mu'', e' \otimes e'', R' \otimes 1 + 1 \otimes R'', S' \otimes S'', C' \otimes C'')$$

$$M' = Fr\,(\mu', e', S', C')$$

$$M'' = Fr\,(\mu'', e'', S'', C'')\,.$$

Then

$$M = M' \otimes M''.$$

Proof. Let $u' = (u_{1'}\ldots, u_{n'}) \in M'$, $u'' = (u_{1''}\ldots, u_{n''}) \in M''$ be two regular points of these Frobenius manifolds, i.e., such points that the Riemann - Hilbert boundary value problem of the form (4.41) - (4.45) for each of the manifolds has unique solution $\left(Y_0', Y_{\text{right}}', Y_{\text{left}}'\right)$ and $\left(Y_0'', Y_{\text{right}}'', Y_{\text{left}}''\right)$ resp. Doing, if necessary, a diagonal shift

$$u_{i'} \mapsto u_{i'} + c,\ i' = 1, \ldots, n'$$

we may also assume that

$$t^{1'}(u') = t^{1''}(u'') =: t^1.$$

Then the functions

$$Y_0 = e^{-z\,t^1} Y_0' \otimes Y_0''$$

$$Y_{\text{right}} = e^{-z\,t^1} Y_{\text{right}}' \otimes Y_{\text{right}}''$$

$$Y_{\text{left}} = e^{-z\,t^1} Y_{\text{left}}' \otimes Y_{\text{left}}''$$

will give the solution of the Riemann - Hilbert boundary value problem for the manifold M. It follows that the matrix Ψ in the points (4.100) is also a tensor product

$$\Psi = \left(\psi_{i'\alpha'}(u')\psi_{i''\alpha''}(u'')\right).$$

Using the formulae of Isomonodromicity Theorem we conclude that $M = M' \otimes M''$. Theorem is proved.

Example 4.5. Let M be the Frobenius manifold corresponding to quantum cohomology of $\mathbf{CP}^1$, i.e.,

$$F = \frac{1}{2} t_1^2 t_2 + e^{t_2}$$

$$E = t_1 \partial_1 + 2\partial_2.$$

The monodromy data are

$$\mu = \text{diag}\,(-1/2, 1/2)\,,\ R = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

(the computation of S is similar to the above computation of the Stokes matrix of quantum cohomology of $\mathbf{CP}^2$, but it is simpler). The tensor square of this Frobenius manifold computed according to Theorem describes the quantum cohomology of $\mathbf{CP}^1 \times \mathbf{CP}^1$.

Example 4.6. Let M_h be the polynomial two-dimnsional Frobenius manifolds of the form

$$F = \frac{1}{2}t_1^2 t_2 + t_2^{h+1},\ h \in \mathbf{Z},\ h \geq 3.$$

Tensor product of the form $M_{h'} \otimes M_{h''}$ is a polynomial 4-dimensional Frobenius manifold only in the following three cases: $M_3 \otimes M_3$, $M_3 \otimes M_4$, $M_3 \otimes M_5$.

In the next Lecture we will establish a relation between polynomial Frobenius manifolds and finite Coxeter groups. We will see that the manifolds M_h correspond to the groups $I_2(h)$ of symmetries of regular h-gon on the plane. Particularly, for $h = 3$ we obtain $I_2(3) =$ the Weyl group of the type A_2, for $h = 4$ $I_2(4) =$ the Weyl group of the type B_2. Their tensor products also correspond to certain finite Coxeter groups. Namely,

$$M_{A_2} \otimes M_{A_2} = M_{D_4} \tag{4.101}$$

$$M_{A_2} \otimes M_{B_2} = M_{F_4} \tag{4.102}$$

$$M_{A_2} \otimes M_{I_2(5)} = M_{H_4} \tag{4.103}$$

the notations for finite Coxeter groups as in [Bou]; see also the next Lecture). Besides these there are only two more cases where tensor products of two polynomial Frobenius manifolds is again a polynomial Frobenius manifold. They correspond to the following Coxeter groups

$$M_{A_2} \otimes M_{A_3} = M_{E_6} \tag{4.104}$$

$$M_{A_2} \otimes M_{A_4} = M_{E_8}. \tag{4.105}$$

More generally, in the singularity theory our operation of tensor products of the Frobenius structures on the parameter space of versal deformation of an isolated quasihomogeneous singularity corresponds to the operation of the direct sum of singularities. Denoting $M_{f(x)}$ the Frobenius structure on the parameter space of versal deformation of the singularity of a function $f(x)$ we obtain

$$M_{f(x)+g(y)} = M_{f(x)} \otimes M_{g(y)}.$$

Indeed, according to Deligne (see in [AGV]) the variation operator of the direct sum of the singularities is the tensor product of the variation operators of the summands. From this point of view the identifications (4.101), (4.104), (4.105) become obvious. The equalities (4.102) and (4.103) seem not to admit simple explanation within the framework of the singularity theory. However, they are in the agreement with the embeddings of Frobenius manifolds obtained by folding of Dynkin diagrams explained in the next Lecture (I am thankful to J.-B.Zuber for bringing my attention to this point).

Lecture 5

Monodromy group and mirror construction for semisimple Frobenius manifolds

We will introduce a new metric [Du5, Du7] on an open subset of a Frobenius manifold M. The inverse of this metric will be a symmetric bilinear form on the cotangent bundle T^*M defined everywhere.

Definition 5.1. The *intersection form* of the Frobenius manifold M is the bilinear form on T^*M defined by the formula

$$(\omega_1, \omega_2) := i_{E(t)}(\omega_1 \cdot \omega_2), \quad \omega_1,\, \omega_2 \in T_t^*M. \tag{5.1}$$

In the r.h.s. the product of one-forms $T_t^*M \times T_t^*M \to T_t^*M$ is defined using the algebra structure on T_tM and the isomorphism

$$<\ ,\ >: T_tM \to T_t^*M.$$

In the flat coordinates the components of the intersection form are given by the formula

$$\begin{aligned} g^{\alpha\beta}(t) := \left(dt^\alpha, dt^\beta\right) &= E^\epsilon(t) c_\epsilon^{\alpha\beta}(t) \\ &= (d + 1 - q_\alpha - q_\beta) F^{\alpha\beta}(t) + A^{\alpha\beta}. \end{aligned} \tag{5.2}$$

Here

$$\begin{aligned} c_\epsilon^{\alpha\beta}(t) &= \eta^{\alpha\gamma} c_{\gamma\epsilon}^\beta(t) \\ F^{\alpha\beta}(t) &= \eta^{\alpha\lambda}\eta^{\beta\mu} \frac{\partial^2 F(t)}{\partial t^\lambda \partial t^\mu} \\ A^{\alpha\beta} &= \eta^{\alpha\lambda}\eta^{\beta\mu} A_{\lambda\mu} \end{aligned}$$

where the constant matrix $A_{\lambda\mu}$ was defined in (WDVV3).

From (5.2) one obtains

$$g^{\alpha\beta}(t) = t^1\, \eta^{\alpha\beta} + \tilde{g}^{\alpha\beta}\left(t^2, \dots, t^n\right)$$

with

$$\tilde{g}^{\alpha\beta}\left(t^2, \dots, t^n\right) = \sum_{\epsilon=2}^n E^\epsilon(t) c_\epsilon^{\alpha\beta}(t).$$

So the bilinear form does not degenerate identically.

Definition 5.2. The locus $\Sigma \subset M$

$$\Sigma = \left\{ t \in M \mid \det\left(g^{\alpha\beta}(t)\right) = 0 \right\} \tag{5.3}$$

is called *discriminant* of the Frobenius manifold M.

Exercise 5.1. Prove that the discriminant is specified by the equation

$$\det \mathcal{U}(t) = 0$$

where $\mathcal{U}(t)$ is the operator of multiplication by the Euler vector field.

The inverse

$$(g_{\alpha\beta}) = \left(g^{\alpha\beta}\right)^{-1} \tag{5.4}$$

defines a metric on the open subset $M \setminus \Sigma$.

Lemma 5.1. *1). The Christoffel coefficients of the Levi-Civita connection for the metric (5.4) in the flat coordinates t^α are uniquely determined from the equation*

$$\Gamma^{\alpha\beta}_{\gamma} := -g^{\alpha\epsilon}\Gamma^{\beta}_{\epsilon\gamma} = \left(\frac{d+1}{2} - q_\beta\right) c^{\alpha\beta}_{\gamma}. \tag{5.5}$$

2). The metric (5.4) on $M \setminus \Sigma$ is flat.

Proof can be found in [Du7].

For brevity we will call the bilinear form $g^{\alpha\beta}(t)$ on $T^*_t M$ *contravariant metric* and the expressions $\Gamma^{\alpha\beta}_{\gamma} := -g^{\alpha\epsilon}\Gamma^{\beta}_{\epsilon\gamma}$ *contravariant Christoffel coefficients* of the Levi-Civita connection for the metric.

We make a digression about linear pencils of contravariant metrics.

Let $\left(g_1^{ij}(x), {\Gamma_1}^{ij}_k(x)\right)$ and $\left(g_2^{ij}(x), {\Gamma_2}^{ij}_k(x)\right)$ be two contravariant metrics invertible on an open subset of a manifold M together with the corresponding contravariant Christoffel coefficients.

Definition 5.3. We say that the two contravariant metrics form a *linear quasihomogeneous pencil* of the charge d if

1). For any $\lambda \in \mathbf{C}$ the metric

$$g_1^{ij}(x) - \lambda\, g_2^{ij}(x)$$

does not degenerate on an open subset in M.

2). The functions

$${\Gamma_1}^{ij}_k(x) - \lambda\, {\Gamma_2}^{ij}_k(x)$$

are the contravariant Christoffel coefficients of the metric (5.4).

3). There exists a function $\varphi(x)$ on M such that the vector fields

$$E^i(x) := g_1^{ij}(x)\frac{\partial\varphi}{\partial x^j}, \quad e^i(x) := g_2^{ij}(x)\frac{\partial\varphi}{\partial x^j} \tag{5.6}$$

satisfy the following properties

$$[e, E] = e \tag{5.7}$$

$$\mathcal{L}_E g_1^{ij}(x) = (d-1)\, g_1^{ij}(x), \ \mathcal{L}_E g_2^{ij}(x) = (d-2)\, g_2^{ij}(x), \ \mathcal{L}_e g_1^{ij}(x) = g_2^{ij}(x), \ \mathcal{L}_e g_2^{ij}(x) = 0. \tag{5.8}$$

Theorem 5.1. *The intersection form of a Frobenius manifold together with the flat metric $<\ ,\ >$ form a flat pencil of the charge d.*

Proof can be derived from Lemma 5.1 (see [Du7]). The function $\varphi(t)$ equals $\varphi = t_1 = \eta_{1\epsilon}t^\epsilon$.

It can be shown [Du8] that, vice versa, a manifold with a flat pencil of contravariant metrics satisfying certain assumptions about eigenvalues of the linear operator ∇E carries a natural Frobenius structure such that, in the flat coordinates for g_2^{ij}, the metric g_1^{ij} has the form (5.2) (cf. [Du7], [DZ1]).

Definition 5.4. A function $x = x(t)$ is called *flat coordinate* of a metric if the differential dx is covariantly constant w.r.t. the Levi-Civita connection for the metric.

The flat coordinates of the intersection form on a Frobenius manifold are determined from the system of linear differential equations

$$g^{\alpha\epsilon}\partial_\beta\xi_\epsilon + \sum_\epsilon \left(\frac{1}{2} - \mu_\epsilon\right) c_\beta^{\alpha\epsilon}\xi_\epsilon = 0 \tag{5.9}$$

where $\xi_\beta = \partial_\beta x$.

Definition 5.5. The equations (5.9) are called *Gauss - Manin system of the Frobenius manifold.*

Exercise 5.2. Prove that the flat coordinates of the linear pencil $g^{\alpha\beta}(t) - \lambda\,\eta^{\alpha\beta}$ have the form

$$x(t^1 - \lambda, t^2, \dots, t^n) \tag{5.10}$$

where $x(t^1, t^2, \dots, t^n)$ are flat coordinates of the intersection form. Prove that the gradients $\xi^\alpha = \eta^{\alpha\beta}\partial_\beta x(t^1 - \lambda, t^2, \dots, t^n)$ satisfy the system of equations

$$(\mathcal{U} - \lambda)\,\partial_\beta\xi + C_\beta\left(\frac{1}{2} + \mu\right)\xi = 0 \tag{5.11}$$

$$(\mathcal{U} - \lambda)\,\partial_\lambda\xi = \left(\frac{1}{2} + \mu\right)\xi. \tag{5.12}$$

This is an extension of the Gauss - Manin system (5.9) onto $M \times \mathbf{C}_\lambda$. The second equation (5.12) has rational coefficients in λ. As above, compatibility of the full system will imply isomonodromicity of the Fuchsian system (5.12).

Digression. One can see by a straightforward computation that also the system

$$(\mathcal{U} - \lambda)\,\partial_\beta\phi + C_\beta\mu\,\phi = 0 \tag{5.13a}$$

$$(\mathcal{U} - \lambda)\,\partial_\lambda\phi = \mu\,\phi \tag{5.13b}$$

is compatible. We will use it to reduce WDVV for $n = 3$, $d \neq 0$ to a particular case of Painlevé-VI equation (semisimplicity is assumed). For $n = 3$ the matrix μ degenerates

$$\mu = \mathrm{diag}\,(\mu_1,\, 0,\, -\mu_1).$$

So, the equations (5.13b) for the vector-function $\phi = (\phi_1, \phi_2, \phi_3)^T$ splits into a 2×2 subsystem for $\chi = (\phi_1, \phi_3)^T$ and a quadrature for ϕ_2

$$\frac{d\chi}{d\lambda} = -\mu_1 A(\lambda)\,\chi \tag{5.14}$$

$$A(\lambda) = \frac{A_1}{\lambda - u_1} + \frac{A_2}{\lambda - u_2} + \frac{A_3}{\lambda - u_3}.$$

Here u_1, u_2, u_3 are the eigenvalues of $U(t)$ (i.e., the canonical coordinates), the 2×2-matrices have the form

$$A(\lambda) = \mu_1 \begin{pmatrix} v_1^1(\lambda;t) & -v_3^1(\lambda;t) \\ v_1^3(\lambda;t) & -v_3^3(\lambda;t) \end{pmatrix}$$

where the matrix $\left(v_\beta^\alpha(\lambda;t)\right) := (\mathcal{U}(t) - \lambda)^{-1}$,

$$A_i = \begin{pmatrix} \psi_{i1}\psi_{i3} & -\psi_{i3}^2 \\ \psi_{i1}^2 & -\psi_{i1}\psi_{i3} \end{pmatrix}, \quad i = 1,\, 2,\, 3. \tag{5.16}$$

Clearly, the matrices satisfy the conditions

$$\det A_i = \operatorname{tr} A_i = 0, \;\; i = 1,\, 2,\, 3 \tag{5.17a}$$

$$A_1 + A_2 + A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.17b}$$

Following [JM] we introduce coordinates p, q, k on the space of matrices A_1, A_2, A_3 satisfying (5.17). The coordinate q is the root of the linear equation

$$[A(q)]_{12} = 0; \tag{5.18a}$$

the coordinate p is the value

$$p = [A(q)]_{11}\,. \tag{5.18b}$$

Explicitly,

$$\begin{aligned} q &= \left(g^{11}g^{22} - {g^{12}}^2\right)/g^{11} \\ p &= \mu_1 \frac{g^{11}g^{22}}{{g^{12}}^3 + g^{11}g^{12}g^{13} - g^{11}g^{12}g^{22} - {g^{11}}^2 g^{23}} \end{aligned} \tag{5.19}$$

The entries of the matrices A_i can be expressed via the coordinates p, q, k as follows

$$\begin{aligned} \psi_{i1}\psi_{i3} &= -\frac{q - u_i}{2\mu_1^2 P'(u_i)} \left[P(q)p^2 + 2\mu_1 \frac{P(q)}{q - u_i} p + \mu_1^2 (q + 2u_i - \sum u_j) \right] \\ \psi_{i3}^2 &= -k \frac{q - u_i}{P'(u_i)} \\ \psi_{i1}^2 &= -k^{-1} \frac{q - u_i}{4\mu_1^4 P'(u_i)} \left[P(q)p^2 + 2\mu_1 \frac{P(q)}{q - u_i} p + \mu_1^2 (q + 2u_i - \sum u_j) \right]^2 \end{aligned} \tag{5.20}$$

where the polynomial $P(\lambda)$ has the form

$$P(\lambda) := (\lambda - u_1)(\lambda - u_2)(\lambda - u_3). \tag{5.21}$$

Compatibility of the system (5.13) implies

$$\begin{aligned}\partial_i q &= \frac{P(q)}{P'(u_i)}\left[2p + \frac{1}{q-u_i}\right]\\ \partial_i p &= -\frac{P'(q)p^2 + \left(2q + u_i - \sum u_j\right)p + \mu_1(1-\mu_1)}{P'(u_i)}\end{aligned} \tag{5.22}$$

and it gives a quadrature for the function $\log k$

$$\partial_i \log k = (2\mu_1 - 1)\frac{q-u_i}{P'(u_i)}. \tag{5.23}$$

Eliminating p from the system we obtain a second order differential equation for the function $q = q(u_1, u_2, u_3)$

$$\begin{aligned}\partial_i{}^2 q &= \frac{1}{2}\frac{P'(q)}{P(q)}\left(\partial_i q\right)^2 - \left[\frac{1}{2}\frac{P''(u_i)}{P'(u_i)} + \frac{1}{q-u_i}\right]\partial_i q\\ &+\frac{1}{2}\frac{P(q)}{\left(P'(u_i)\right)^2}\left[(2\mu_1-1)^2 + \frac{P'(u_i)}{(q-u_i)^2}\right], \quad i = 1,\, 2,\, 3.\end{aligned}$$

The system (5.22) is invariant w.r.t. transformations of the form

$$\begin{aligned}u_i &\mapsto a u_i + b\\ q &\mapsto a q + b.\end{aligned}$$

Introducing the invariant variables

$$\begin{aligned}x &= \frac{u_3 - u_1}{u_2 - u_1}\\ y &= \frac{q}{u_2 - u_1} - \frac{u_1}{u_2 - u_1}\end{aligned}$$

we obtain for the function $y = y(x)$ the following particular Painlevé-VI equation

$$\begin{aligned}y'' &= \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right](y')^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]y'\\ &+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu_1-1)^2 + \frac{x(x-1)}{(y-x)^2}\right].\end{aligned} \tag{$PVI(\mu)$}$$

Conversely, for a solution $y(x)$ of the equation $PVI(\mu)$ we construct functions $q = q(u_1, u_2, u_3)$ and $p = p(u_1, u_2, u_3)$ putting

$$\begin{aligned}q &= (u_2 - u_1)y\left(\frac{u_3 - u_1}{u_2 - u_1}\right) + u_1\\ p &= \frac{1}{2}\frac{P'(u_3)}{P(q)}y'\left(\frac{u_3 - u_1}{u_2 - u_1}\right) - \frac{1}{2}\frac{1}{q - u_3}.\end{aligned}$$

Then we compute the quadrature (5.23) determining the function k (this provides us with one more arbitrary integration constant). After this we are able to compute the matrix $(\psi_{i\alpha}(u))$ from the equations (5.20) and

$$(\psi_{12}, \psi_{22}, \psi_{32}) = \pm i \left(\psi_{21}\psi_{33} - \psi_{23}\psi_{31}, \psi_{13}\psi_{31} - \psi_{11}\psi_{33}, \psi_{11}\psi_{23} - \psi_{13}\psi_{21}\right).$$

The last step is in reconstructing the flat coordinates $t = t(u)$ and the tensor $c_{\alpha\beta\gamma}$ using the formulae (3.21) and (3.17).

Example 5.1. Applying the above procedure to the three polynomial solutions (1.22) - (1.24) of WDVV we obtain the following three algebraic solutions of $PVI(\mu)$ with $\mu = -1/4, -1/3, -2/5$ resp. [Du7,DM] represented in a parametric form

$$\begin{aligned} y &= \frac{(s-1)^2(1+3s)(9s^2-5)^2}{(1+s)(25-207s^2+1539s^4+243s^6)}, \\ x &= \frac{(s-1)^3(1+3s)}{(s+1)^3(1-3s)} \end{aligned} \tag{5.24}$$

$$\begin{aligned} y &= \frac{(2-s)^2(1+s)}{(2+s)(5s^4-10s^2+9)} \\ x &= \frac{(2-s)^2(1+s)}{(2+s)^2(1-s)} \end{aligned} \tag{5.25}$$

$$\begin{aligned} y &= \frac{(s-1)^2(1+3s)^2(-1+4s+s^2)(7-108s^2+314s^4-588s^6+119s^8)^2}{(1+s)^3(-1+3s)P(s^2)} \\ x &= \frac{(-1+s)^5\,(1+3\,s)^3\,\left(-1+4\,s+s^2\right)}{(1+s)^5\,(-1+3\,s)^3\,\left(-1-4\,s+s^2\right)} \end{aligned} \tag{5.26a}$$

where

$$\begin{aligned} P(z) =& 49 - 2133z + 34308z^2 - 259044z^3 + 16422878z^4 - 7616646z^5 + 13758708z^6 \\ &+ 5963724z^7 - 719271z^8 + 42483z^9. \end{aligned} \tag{5.26b}$$

Some other particular solutions of Painlevé-VI in a relation with Frobenius manifolds were constructed in [Se].

Let us return to the intersection form of a Frobenius manifold. Due to Lemma 5.1 in a neighborhood of a point $t_0 \in M \setminus \Sigma$ one can choose n independent flat coordinates $x^1(t)$, ..., $x^n(t)$ of the intersection form. In these coordinates the matrix

$$g^{ab} = \left(dx^a, dx^b\right) = \frac{\partial x^a}{\partial t^\alpha}\frac{\partial x^b}{\partial t^\beta} g^{\alpha\beta}(t) \tag{5.27}$$

becomes constant, and the Christoffel coefficients vanish. The flat coordinates are determined uniquely up to shifts and orthogonal transformations $\in O\left(n, g^{ab}\right)$.

Exercise 5.3. Show that fo $d \neq 1$ the flat coordinates $x(t)$ can be chosen in such a way that

$$\mathcal{L}_E x(t) = \frac{1-d}{2} x(t).$$

So, for $d \neq 1$, the flat coordinates of the intersection form satisfying the quasihomogeneity condition of Exercise 5.3 are determined uniquely up to a transformation from $O\left(n, g^{ab}\right)$.

The solutions of the Gauss - Manin system can be continued analyticaly along any path in $M \setminus \Sigma$. We obtain a multivalued *period map*

$$t \mapsto \left(x^1(t), \dots, x^n(t)\right) \tag{5.28}$$

defined on $M \setminus \Sigma$ (cf. the end of Lecture 2 above). The multivaluedness of the period map is described by a representation of the fundamental group of the complement to the discriminant

$$\pi_1\left(M \setminus \Sigma; t_0\right) \to O\left(n, g^{ab}\right) \tag{5.29}$$

(for $d = 1$ instead of the orthogonal group we obtain a representation into the group of affine isometries of the metric g^{ab}).

Definition 5.6. The image $W(M)$ of the representation (5.29) is called *monodromy group* of the Frobenius manifold.

Our main aim now is to compute the monodromy group of a semisimple Frobenius manifold in terms of the Stokes matrix of the manifold.

In the semisimple case doing the gauge transform

$$\phi = \Psi\, \xi \tag{5.30}$$

we rewrite the extended Gauss - Manin system (5.11), (5.12) in the form

$$(U - \lambda) \frac{d\phi}{d\lambda} = \left(\frac{1}{2} + V\right) \phi \tag{5.31a}$$

or, equivalently,

$$\frac{d\phi}{d\lambda} = \sum_{i=1}^{n} \frac{B_i}{\lambda - u_i} \phi \tag{5.31b}$$

with

$$B_i = -E_i \left(\frac{1}{2} + V\right) \tag{5.31c}$$

where E_i is the matrix unity (3.31),

$$\partial_i \phi = -\frac{B_i}{\lambda - u_i} \phi + V_i \phi, \;\; i = 1, \dots, n. \tag{5.32}$$

We obtain a Fuchsian system (5.31b) with the matrix residues $B_1, \ldots, B_n$ of a particular form (5.31c). Compatibility of (5.31a) with (5.31b) provides isomonodromicity of the dependence of the coefficients of the system on the position of the poles $u_1, \ldots, u_n$. We will now relate the structure of the monodromy of the Fuchsian system (5.31b) to the Stokes matrix of the Frobenius manifold.

Lemma 5.2. *Let $\phi^{(1)}$, $\phi^{(2)}$ be two solutions of the system (5.31). Then the bilinear form*

$$\left(\phi^{(1)}, \phi^{(2)}\right) := {\phi^{(1)}}^T \left(U - \lambda\right) \phi^{(2)} \tag{5.33}$$

does not depend neither on λ nor on $u_1, \ldots, u_n$.

Proof can be obtained by straightforward differentiation.

Remark 5.1. We remember that the solutions $\phi = (\phi_1 \ldots, \phi_n)^T$ of the system (5.31) correspond to flat coordinates $x(t)$ of the intersection form

$$\phi_i = \sum \psi_{i\alpha} \eta^{\alpha\beta} \partial_\beta x \left(t^1 - \lambda, t^2, \ldots, t^n\right). \tag{5.34}$$

If $\phi^{(1)}$ corresponds to $x_1(t)$, $\phi^{(2)}$ to $x_2(t)$ then the bilinear form (5.33) equals

$$\left(\phi^{(1)}, \phi^{(2)}\right) = (dx_1, dx_2) - \lambda < dx_1, dx_2 > . \tag{5.35}$$

We will now construct, essentially following [BJL], a particular system of solutions of the Fuchsian system (5.31b). Let us choose an argument φ in such a way that

$$\arg\left(u_i - u_j\right) \neq \frac{\pi}{2} + \varphi \,(\mathrm{mod}\, 2\pi) \text{ for any } i \neq j. \tag{5.36}$$

We make n distinct parallel branchcuts $L_1, \ldots, L_n$ on the complex λ-plane of the form

$$L_j = \{\lambda = u_j + i\rho e{-}i\varphi, \quad \rho \geq 0\}, \; j = 1, \ldots, n. \tag{5.37}$$

Each branchcut has positive and negative sides

$$L_j^+ = \left\{\lambda \,|\, \arg\left(u_j - \lambda\right) = -\frac{\pi}{2} - \varphi + 0\right\}, \quad L_j^- = \left\{\lambda \,|\, \arg\left(u_j - \lambda\right) = -\frac{\pi}{2} - \varphi + 2\pi - 0\right\}.$$

On the complement

$$\mathbf{C} \setminus \cup_j L_j \tag{5.38}$$

the single-valued functions $\sqrt{u_1 - \lambda}, \ldots, \sqrt{u_n - \lambda}$ are well-defined. We specify them uniquely requiring that

$$\text{on } L_j^+ \quad \arg\sqrt{u_j - \lambda} = -\frac{\pi}{4} - \frac{\phi}{2} + 0. \tag{5.39}$$

Let us choose small loops $\gamma_1, \ldots, \gamma_n$ going around the points $u_1, \ldots, u_n$ in the counterclockwise direction. Let $R_1^*, \ldots, R_n^*$ be the monodromy transformations in the space of solutions of (5.31b) corresponding to the loops $\gamma_1, \ldots, \gamma_n$.

Lemma 5.3. *1). There exist unique solutions $\phi^{(1)}(\lambda)$, ..., $\phi^{(n)}(\lambda)$ of (5.31b) analytic in (5.38) such that*

$$R_j^* \phi^{(j)} = -\phi^{(j)}, \quad j = 1, \ldots, n \tag{5.40}$$

$$\phi_a^{(j)}(\lambda) = \frac{\delta_a^j}{\sqrt{u_j - \lambda}} + O(\sqrt{u_j - \lambda}), \quad \lambda \to u_j. \tag{5.41}$$

2). Introduce a symmetric matrix $G = (G_{ij})$

$$G_{ij} = \left(\phi^{(i)}, \phi^{(j)}\right). \tag{5.42}$$

The monodromy transformations R_j^* are the reflections

$$R_j^* \phi^{(i)} = \phi^{(i)} - 2\, G_{ij} \phi^{(j)}, \quad i, j = 1, \ldots, n \tag{5.43}$$

in the hyperplanes orthogonal to $\phi^{(j)}$ w.r.t. the bilinear form (5.33).

Proof. The matrix residue B_j in (5.31b) has one eigenvalue $-1/2$ and $n-1$ eigenvalues 0. So one can construct a fundamental system of solutions $\phi^{(j)}(\lambda)$, $r_2(\lambda)$, ..., $r_n(\lambda)$ such that $R_j^* \phi^{(j)} = -\phi^{(j)}$, $R_j^* r_k = r_k$, $k = 2, \ldots, n$. That means that the last $n-1$ solutions are analytic at $\lambda = u_j$. The solution $\phi^{(j)}(\lambda)$ is determined uniquely up to a nonzero factor. From this it easily follows the first part of Lemma.

To prove the second part let us represent $\phi^{(i)}(\lambda)$ as a linear combination of $\phi^{(j)}(\lambda)$ and of the solutions analytic at $\lambda = u_j$

$$\phi^{(i)}(\lambda) = C_{ij} \phi^{(j)}(\lambda) + r_{ij}(\lambda).$$

Here C_{ij} is some constant, the solution $r_{ij}(\lambda)$ is analytic at $\lambda = u_j$. Computing the bilinear form (5.42) and using Lemma 5.2 we obtain

$$G_{ij} = \lim_{\lambda \to u_j} \sum_{a=1}^n (u_a - \lambda) \phi_a^{(i)}(\lambda) \phi_a^{(j)}(\lambda) = C_{ij}.$$

We obtain

$$C_{ij} = G_{ij} \text{ for } i \neq j.$$

Similar computation gives

$$G_{ii} = 1.$$

We obtain a representation

$$\phi^{(i)}(\lambda) = G_{ij} \phi^{(j)}(\lambda) + r_{ij}(\lambda). \tag{5.44}$$

So

$$\begin{aligned} R_j^* \phi^{(i)} &= -G_{ij} \phi^{(j)}(\lambda) + r_{ij}(\lambda) \\ &= \phi^{(i)}(\lambda) - 2 G_{ij} \phi^{(j)}(\lambda) \\ &= \phi^{(i)}(\lambda) - 2 \frac{\left(\phi^{(i)}, \phi^{(j)}\right)}{\left(\phi^{(j)}, \phi^{(j)}\right)} \phi^{(j)}(\lambda). \end{aligned}$$

Lemma is proved.

We will now establish, using the technique of [BJL], a simple relation between the matrix G (5.42) for the system (5.31b) and the Stokes matrix of the operator (3.30).

Let us assume that the angle φ is chosen in such a way that the order of the rays L_1, $\dots$, L_n on the complex λ-plane corresponds to the order of the complex numbers $u_1, \dots,$ u_n in the following sense: looking along the ray L_j from the endpoint $\lambda = u_j$ we must see L_{j-1} as the nearest ray on the left and L_{j+1} as the nearest one on the right, $2 \leq j \leq n-1$.

Lemma 5.4. *The oriented line* $\ell = \ell_+ \cup (-\ell_-)$

$$\ell_+ = \{z \,|\, \arg z = \varphi\}$$

is admissible for the operator (3.30). The corresponding Stokes matrix S is upper triangular. It satisfies the relation

$$S + S^T = 2\, G \tag{5.45}$$

where G is the matrix (5.42).

Proof. Admissibility is obvious from (5.36). Let us construct the fundamental matrices $Y_{\text{right}}(z)$, $Y_{\text{left}}(z)$ having the needed asymptotic development (5.26) in the half-planes Π_{right}, Π_{left}. We will construct them taking an appropriate inverse Laplace transform of the solutions $\phi^{(j)}(\lambda)$ defined in Lemma 5.3. Put

$$Y_{aj}(z) = -\frac{\sqrt{z}}{2\sqrt{\pi}} \int_{C_j} \phi_a^{(j)}(\lambda)\, e^{\lambda\, z} d\lambda, \quad a,\, j = 1, \dots, n. \tag{5.46}$$

Here C_j is an infinite contour coming from infinity along the positive side of the branchcut L_j, then encircling the point $\lambda = u_j$ and, after, returning to infinity along the negative side of the branchcut L_j. Since $\lambda = \infty$ is a regular singularity of the system (5.31b), the solutions $\phi^{(j)}(\lambda)$ grow at $\lambda \to \infty$ not faster than a certain power of $|\lambda|$. We conclude that the integral converges absolutely for $z \in \Pi_{\text{left}}$. Using (5.31b) and integrating by parts we prove that the matrix $Y(z) = \left(Y_a^j(z)\right)$ satisfies the equation (3.30). To obtain the asymptotic development of this solution as $|z| \to \infty$ we can, due to Watson Lemma [WW], integrate the terms of the convergent expansion (5.41) of the solution $\phi^{(j)}(\lambda)$ near $\lambda = u_j$. Doing so we easily see that the solution $Y_{\text{left}}(z) := Y(z)$ has the needed asymptotic development

$$Y_{aj}(z) \sim \left(\delta_{aj} + O\left(\frac{1}{z}\right)\right) e^{z u_j}$$

as $|z| \to \infty$, $z \in \Pi_{\text{left}}$.

Let us now construct the fundamental matrix $Y_{\text{right}}(z)$. We are to choose the system of the opposite branchcuts

$$L_j' = \left\{\lambda = u_j - i\rho e^{-i\varphi},\ \rho \geq 0\right\}, \quad j = 1, \dots, n \tag{5.47}$$

to construct the corresponding solutions $\phi^{(j)'}(\lambda)$ and to define

$$Y_{\text{right}\,aj}(z) = -\frac{\sqrt{z}}{2\sqrt{\pi}} \int_{C_j'} \phi_a^{(j)'}(\lambda)\, e^{\lambda\, z} d\lambda, \quad a,\, j = 1, \dots, n. \tag{5.48}$$

Here the contour C'_j goes around the branchcut L'_j. As above we prove that the solution $Y_{\rm right}(z) := \left(Y_{\rm right}{}^j_a(z)\right)$ of (3.30) has the needed asymptotic development in $\Pi_{\rm right}$ as $|z| \to \infty$. It remains to establish a relation between the integrals (5.46) and (5.48). To continue analytically $Y_{\rm left}(z)$ through ℓ_+ in the clockwise direction into $\Pi_{\rm right}$ we are to rotate the branchcuts L_j in the counterclockwise direction until they take the places of L'_j, $j = 1, \ldots, n$. For $j = 1$ such a deformation does not meet obstructions. So

$$\phi^{(1)'} = \phi^{(1)}.$$

To deform L_2 to L'_2 we are to pass through the branchcut L_1. This is equivalent to the action of monodromy transformation R_1^*. So

$$\phi^{(2)'} = R_1^* \phi^{(2)}.$$

Continuing this process we obtain that

$$\phi^{(k)'} = R_1^* R_2^* \ldots R_{k-1}^* \phi^{(k)}, \; k = 2, \ldots, n.$$

Using the computation in the proof of Coxeter identity (see [Bou]) we obtain

$$\phi^{(k)} = 2G_{k1}\phi^{(1)'} + 2G_{k2}\phi^{(2)'} + \ldots + 2G_{k\,k-1}\phi^{(k-1)'} + \phi^{(k)'}, \; k = 1, \ldots, n.$$

Lemma is proved.

Corollary 5.1. *If*

$$\det(S + S^T) \neq 0 \tag{5.49}$$

then the functions $\phi^{(1)}(\lambda)$, ..., $\phi^{(n)}(\lambda)$ form a basis of the space of solutions of (5.31b).

Exercise 5.4. Prove that the Stokes matrix of quantum cohomology of a manifold X of an even complex dimension (assuming semisimplicity of the quantum cohomology) satisfies the nondegeneracy condition (5.49).

In the rest of this Lecture I will assume that $d \neq 1$ and that the Stokes matrix satisfies the nondegeneracy condition (5.49).

All the above constructions of the basis $\phi^{(1)}(\lambda)$, ..., $\phi^{(n)}(\lambda)$ were done for a given fixed point $(u_1, \ldots, u_n)$ of the Frobenius manifold. Since the solutions $\phi^{(j)}(\lambda)$ are determined uniquely, they become locally well-defined analytic functions of $(u_1, \ldots, u_n)$.

Lemma 5.5. *They satisfy also the equations (5.32).*

Proof. Let us consider the vector-function

$$\tilde\phi^{(j)} := \partial_i \phi^{(j)} + \frac{B_i}{\lambda - u_i}\phi^{(j)} - V_i \phi^{(j)}$$

for some i between 1 and n. Because of compatibility of (5.31) and (5.32) the vector-function $\tilde\phi^{(j)}$ satisfies (5.31). It is easy to see that this solution is regular near the points $\lambda = u_1$, ..., $\lambda = u_n$. Hence $\tilde\phi^{(j)} = 0$. Lemma is proved.

Let $\tilde{M} = \tilde{Fr}(e, \mu, R, S, C; u^0)$ be the Frobenius structure on the universal covering of $\mathbf{C}^n \setminus \mathrm{diag}$ defined by the given monodromy data (e, μ, R, S, C) with $\mu_1 \neq -1/2$, $\det(S + S^T) \neq 0$. The discriminant $\tilde{\sigma}$ of this Frobenius manifold consists of the lifts of the coordinate hyperplanes $u_i = 0$, $i = 1, \ldots, n$.

Let $\mathcal{E}$ be n-dimensional linear space equipped with a symmetric nondegenerate bilinear form $(\ ,\ )$ on the dual space $\mathcal{E}^*$ having in some basis $e^1, \ldots, e^n$ the Gram matrix

$$(e^i, e^j) = \left(S + S^T\right)_{ij}. \tag{5.50}$$

For any $1 \leq i \leq n$ denote

$$R_i : \mathcal{E} \to \mathcal{E} \tag{5.51}$$

the transformation dual to the reflection $R_i^* : \mathcal{E}^* \to \mathcal{E}^*$ in the hyperplanes orthogonal to e^i:

$$R_i^*(x) = x - (x, e^i)\, e^i. \tag{5.52}$$

Theorem 5.2. *The image of the monodromy representation in the group $O(\mathcal{E}, (\ ,\ ))$ of orthogonal transformations of the space $\mathcal{E}$*

$$\pi_1(\tilde{M}; u^0) \to O(\mathcal{E}, (\ ,\ )) \tag{5.53}$$

is the group generated by the reflections $R_1, \ldots, R_n$.

Proof. According to Lemma 5.3, locally we have a basis $\phi^{(1)}(\lambda; u), \ldots, \phi^{(n)}(\lambda; u)$ of solutions of (5.31), (5.32). The formula

$$x_j(\lambda; u) = \frac{2\sqrt{2}}{1-d} \sum_a (u_a - \lambda)\psi_{a1}(u)\phi_a^{(j)}(\lambda; u) \tag{5.54}$$

gives flat coordinates of the linear pencil $(\ ,\ ) - \lambda < , >$. Due to Lemma 5.4 we have

$$(dx_i, dx_j) - \lambda < dx_i, dx_j >= \left(S + S^T\right)_{ij}. \tag{5.55}$$

We obtain a locally well-defined isometry (the period map of the Frobenius manifold, cf. (2.101))

$$\tilde{M} \setminus \tilde{\Sigma} \to \mathcal{E}, \tag{5.56a}$$

$$u \mapsto (x_1(u), \ldots, x_n(u)) \tag{5.56b}$$

where

$$x_j(u) := x_j(\lambda; u)|_{\lambda=0}.$$

The monodromy around the branch $u_i = 0$ of $\tilde{\Sigma}$ in the given chart of the universal covering of $\mathbf{C}^n \setminus \mathrm{diag}$ is equivalent to the monodromy of the vector-function

$$(x_1(\lambda; u), \ldots, x_n(\lambda; u))$$

corresponding to a small loop around $\lambda = u_i$ in the λ-plane. We obtain the transformation

$$x_k(u) \mapsto x_k(u) - \left(S + S^T\right)_{ki} x_i(u), \quad k = 1, \dots, n \tag{5.57}$$

where we identify the coordinates in $\mathcal{E}$ with the dual basis in $\mathcal{E}^*$. That means that, locally, the monodromy group is generated by the reflections (5.57).

What happens with the analytic continuation into another chart of $\tilde{M}$? We arrive in another chart when some of the Stokes rays (4.34) passes through the admissible line ℓ. Simultaneously, two of the branchcuts $L_1, \dots, L_n$ pass one through another one. The Stokes matrix changes according to the rule (4.68). It is sufficient to understand what happens with the flat coordinates $x_1(\lambda; u), \dots, x_n(\lambda; u)$ with an elementary transformation (4.74) of the braid group.

Lemma 5.6. *The elementary braid σ_i permuting the points $\lambda = u_i$ and $\lambda = u_{i+1}$ in the complex λ-plane gives the following transformation of the solutions $\phi^{(1)}(\lambda), \dots, \phi^{(n)}(\lambda)$*

$$\begin{aligned}
\sigma_i\left(\phi^{(k)}\right) &= \phi^{(k)}, \quad k \neq i,\, i+1 \\
\sigma_i\left(\phi^{(i)}\right) &= \phi^{(i+1)} \\
\sigma_i\left(\phi^{(i+1)}\right) &= R^*_{i+1}\phi^{(i)} = \phi^{(i)} - S_{i,i+1}\phi^{(i+1)}
\end{aligned} \tag{5.58}$$

We leave the proof as an exercise to the reader.

We obtain, after the analytic continuation along the braid σ_i, that the new monodromy transformations in $\mathcal{E}^*$ are reflections in the hyperplanes orthogonal to the vectors

$$\left(e^1, \dots, e^{i-1}, e^{i+1}, R^*_{i+1}(e^i), e^{i+2}, \dots, e^n\right).$$

But reflection w.r.t. the hyperplane orthogonal to $R^*_{i+1}(e^i)$ is equal to $R^*_{i+1}R^*_i R^*_{i+1}$. This transformation belongs to the group generated by $R^*_1, \dots, R^*_n$. Theorem is proved.

To complete our description of an arbitrary semisimple Frobenius manifold in terms of an appropriate "singularity theory" we are to construct an analogue of versal deformation. This can be done (at least, under the nondegeneracy assumption (5.49)) in the following way ([Du7], Appendix I). We will construct a family of functions $\lambda(p; u)$, $u = (u_1, \dots, u_n)$ of complex variable p defined in an open domain $\mathcal{D}$ of a Riemann surface $\mathcal{R}$ realized as branched covering over complex plane with finite number of sheets. The Riemann surface may depend on u. However, the projection of the domain $\mathcal{D}$ on the complex plane will be fixed. These functions depend on complex pairwise distinct parameters $u_1, \dots, u_n$ belonging to a sufficiently small domain $\Omega \subset \mathbf{C}^n$.

The first main property is that $\lambda(p; u)$ as function of $p \in \mathcal{D}$ has critical values just $u_1, \dots, u_n$. The corresponding critical points must not be degenerate. The second condition we require from the function $\lambda(p; u)$ is that, for any two points $p_i^{(1,2)} \in \mathcal{D}$ with the same critical value u_i, we must have

$$\lambda''(p_i^{(1)}; u) = \lambda''(p_i^{(2)}; u).$$

Here the prime denotes the p-derivative.

Definition 5.7. The function $\lambda(p;u)$ on $\mathcal{D}\times\Omega$ satisfying the above two properties is called *superpotential* of some domain M_Ω in the Frobenius manifold M if:

1). The canonical coordinates $(u_1,\dots,u_n)$ map M_Ω to $\Omega\subset\mathbf{C}^n$.

2). For any critical points $p_1, \dots, p_n\in\mathcal{D}$ of $\lambda(p;u)$ with the critical values $u_1, \dots, u_n$ resp. the following expressions for the flat metric $<\ ,\ >$ on T_tM, the intersection form $(\ ,\ )$ (outside the discriminant Σ), and the multiplication of tangent vectors hold true

$$\langle\partial',\partial''\rangle_t=-\sum_{i=1}^n\operatorname*{res}_{p=p_i}\frac{\partial'\left(\lambda(p;u(t))dp\right)\partial''\left(\lambda(p;u(t))dp\right)}{d\lambda(p;u(t))}\tag{5.59}$$

$$(\partial',\partial'')_t=-\sum_{i=1}^n\operatorname*{res}_{p=p_i}\frac{\partial'\left(\log\lambda(p;u(t))dp\right)\partial''\left(\log\lambda(p;u(t))dp\right)}{d\log\lambda(p;u(t))}\tag{5.60}$$

$$\langle\partial'\cdot\partial'',\partial'''\rangle_t=-\sum_{i=1}^n\operatorname*{res}_{p=p_i}\frac{\partial'\left(\lambda(p;u(t))dp\right)\partial''\left(\lambda(p;u(t))dp\right)\partial'''\left(\lambda(p;u(t))dp\right)}{dp\,d\lambda(p;u(t))}\tag{5.61}$$

In these formulae ∂', ∂'', ∂''' are any three vector fields on M,

$$d\lambda(p;u):=\frac{\partial\lambda(p;u)}{\partial p}dp,\quad d\log\lambda(p;u):=\frac{\partial\log\lambda(p;u)}{\partial p}dp.$$

3). For some 1-cycles $Z_1, \dots, Z_n$ in $\mathcal{D}$ the integrals

$$\tilde t_j(u;z)=\frac{1}{\sqrt{z}}\int_{Z_j}e^{z\,\lambda(p;u)}dp,\quad j=1,\dots,n\tag{5.62}$$

converge and give a system of independent flat coordinates of the connection $\tilde\nabla$.

Example 5.2. For the polynomial Frobenius manifold corresponding to A_n singularity the versal deformation

$$\lambda=p^{n+1}+a_np^{n-1}+\dots,+a_1$$

gives the needed superpotential. The variables $u_1, \dots, u_n$ are the critical values of this function. Locally one can express the coefficients of the polynomial as single-valued functions of u_j. For other simple singularities the versal deformation is a family of polynomials of two variables. However, one can reduce double integrals for the residues (1.20), (1.21) and for the oscilatory integrals (2.96) to one-dimensional residues and single integrals of the above form. For the D_n case this was done in [DVV]. (the superpotential becomes a rational function). For the case of E_6 singularity the superpotentil is algebraic. It was found in [LW].

Example 5.3. For the case when the Riemann surfaces $\mathcal{R}$ can be compactified at infinity in such a way that there is exactly one branch point on the Riemann surface over any of the critical values u_j then the Frobenius manifold can be identified with a

Hurwitz space of branched coverings. The Frobenius structure on the Hurwitz spaces was constructed in [Du1, Du2] (see also [Du7]). In [Kr] the method of [Du1, Du2] was extended to produce also certain algebro-geometric solutions satisfying WDVV1 and WDVV2 but not WDVV3.

We will now construct a superpotential for semisimple Frobenius manifolds satisfying nodegeneracy assumption (5.49).

Let $u^0 = (u_1^0, \ldots, u_n^0)$, $u_i^0 \neq u_j^0$ for $i \neq j$, be any point of M (written in the canonical coordinates). We choose the branchcuts $L_1^0, \ldots, L_n^0$ as above. For $M \ni u$ sufficiently close to u^0 we will choose the branchcuts $L_1, \ldots, L_n$ coinciding with $L_1^0, \ldots, L_n^0$ outside some small neigborhoods of the points $u_1^0, \ldots, u_n^0$ resp. This allows us to construct solutions $\phi^{(1)}(\lambda; u), \ldots, \phi^{(n)}(\lambda; u)$ as in Lemma 5.3. Denote

$$G^{ij} = \left(\phi^{(i)}, \phi^{(j)}\right).$$

Let (G_{ij}) be the inverse matrix. Let us consider the solution (cf. [BJL2])

$$\phi(\lambda; u) = \sum_{i,\, j=1}^{n} G_{ij} \phi^{(j)}(\lambda; u). \tag{5.63}$$

Lemma 5.7. *The solution $\phi(\lambda; u) = (\phi_a(\lambda; u))$ for $\lambda \to u_j$ has the behaviour*

$$\phi_a(\lambda; u) = \frac{\delta_{aj}}{\sqrt{u_j - \lambda}} + O(1), \ a, \, j = 1, \ldots, n. \tag{5.64}$$

Proof follows from (5.44).

Denote $p = p(\lambda; u)$ the corresponding flat coordinate of the intersection form

$$p(\lambda; u) = \frac{\sqrt{2}}{1-d} \sum_{a=1}^{n} (u_a - \lambda) \psi_{a1}(u) \phi_a(\lambda; u) \tag{5.65}$$

i.e.,

$$\frac{\partial p(\lambda; u)}{\partial u_a} = \frac{1}{\sqrt{2}} \psi_{a1}(u) \phi_1(\lambda; u). \tag{5.66}$$

It is analytic in the domain

$$\mathbf{C} \setminus \cup_j L_j. \tag{5.67}$$

For $\lambda \to u_j$ it behaves as follows

$$p(\lambda; u) = p_j + \sqrt{2} \psi_{j1} \sqrt{u_j - \lambda} + O(u_j - \lambda) \tag{5.68}$$

where $p_j = p(u_j; u)$. For $\lambda \to \infty$ the function $p(\lambda; u)$ has a regular singularity. Hence $dp(\lambda; u)/d\lambda$ has at most finite number of zeroes $r_1, \ldots, r_N$ in (5.67). Without loss of generality we may assume that all these zeroes are simple and that they do not belong to the branchcuts L_j.

Let $\mathcal{D}_0$ be the image of the domain

$$\lambda \in \mathbf{C} \setminus \cup_j L_j^0$$

w.r.t. the map $p(\lambda; u^0)$. Particularly, the two sides of the branchcut L_j^0 open to produce a smooth boundary curve of $\mathcal{D}_0$ passing through p_j^0. Denote

$$\zeta_j^0 = p(r_j; u^0),\ j = 1, \dots, N.$$

Let us consider the inverse $\lambda = \lambda(p; u^0)$ to the function $p(\lambda; u^0)$. It lives on a certain branched covering $\hat{\mathcal{D}}_0$ of the domain $\mathcal{D}_0$ obtained by cutting $\mathcal{D}_0$ along some paths going from ζ_1^0, ..., ζ_N^0 to infinity and by subsequent glueing of a finite number of copies of $\mathcal{D}_0$ with the cuts. Near a point of the boundary of $\hat{\mathcal{D}}_0$ passing through p_j^0 we have

$$\lambda = u_j^0 - \frac{1}{2\psi_{j1}^2(u^0)}(p - p_j^0)^2 + O(p - p_j^0)^3.$$

Thus we can analytically continue $\lambda(p; u^0)$ through the boundary of $\mathcal{D}_0$ near a domain $\mathcal{D}$ containing p_j^0 as its internal point. We can repeat this construction for all u sufficiently close to u^0 (actually, it is sufficient to require the points u_j not to intersect the branchcuts L_i^0). In this way we will produce a family of Riemann surfaces with the branchpoints ζ_1, ..., ζ_N. We may also assume that the image of (5.67) w.r.t. the map $p(\lambda; u)$ for any u close to u^0 belongs to the projection of the domain $\mathcal{D}$ onto the complex p-plane. This completes the construction of the family of functions $\lambda(p; u)$.

The cycles Z_j we need to compute the integrals (5.62) have the form

$$Z_j = p(C_j; u) \tag{5.69}$$

(more precisely, an arbitrary lift of this cycle on the Riemann surface) where C_j was defined in (5.46).

Theorem 5.3. *The function $\lambda(p; u)$ is a superpotential of the Frobenius manifold for a sufficiently small neighborhood of of the point u^0.*

Proof. By the construction the function $\lambda(p; u)$ has critical values u_1, ..., u_n. For any critical point $p_j \in \mathcal{D}$ on the Riemann surface with the critical value u_j we obtained

$$\lambda = u_j - \frac{1}{2\psi_{j1}^2(u)}(p - p_j)^2 + O(p - p_j)^3.$$

So the second derivatives of $\lambda(p; u)$ do not depend on the choice of the critical point.

Let us prove the formulae (5.59) - (5.61). We take $\partial' = \partial_a$, $\partial'' = \partial_b$, $\partial''' = \partial_c$ the vector fields along the canonical coordinates. Then

$$\partial_a \left(\lambda(p; u) dp\right) = \left[\delta_{aj} + O(p - p_j)\right] dp,\ p \to p_j,$$

$$d\lambda(p;u) = -\left[\frac{p-p_j}{\psi_{j1}^2} + O(p-p_j)^2\right] dp,\ p \to p_j.$$

So

$$\operatorname*{res}_{p=p_j} \frac{\partial_a\left(\lambda(p;u)dp\right)\partial_b\left(\lambda(p;u)dp\right)}{d\,\lambda(p;u)} = -\psi_{j1}^2\delta_{aj}\delta_{bj}.$$

Thus the formula (5.59) gives

$$< \partial_a, \partial_b >= \delta_{ab}\,\psi_{a1}^2.$$

This coincides with (3.15).

Similarly, (5.60) for $u_a \neq 0$ gives

$$(\partial_a, \partial_b) = \delta_{ab}\frac{\psi_{a1}^2}{u_a}.$$

This coincides with the definition of the intersection form written in the canonical coordinates. Finally, the last formula (5.61) gives

$$< \partial_a \cdot \partial_b, \partial_c >= \delta_{ab}\delta_{ac}\psi_{a1}^2.$$

This is equivalent to (3.15) together with the definition of the canonical coordinates $\partial_a \cdot \partial_b = \delta_{ab}\partial_a$.

To prove (5.62) we use that the integrals

$$\tilde{t}_j = -\sqrt{z}\int_{C_j} p(\lambda;u)e^{z\lambda}\,d\lambda,\ j =, \ldots, n \tag{5.70}$$

give flat coordinates of the deformed connection. Let us first prove their independence. Indeed, the Jacobi matrix

$$Y_{aj}(z;u) = \frac{1}{\psi_{a1}}\frac{\partial\tilde{t}_j}{\partial u_a} = -\sqrt{z}\int_{C_j}\phi_a(\lambda;u)e^{z\lambda}d\lambda$$

coincides with the fundamental matrix $Y^{\rm left}(z;u)$ (up to a factor $2\sqrt{\pi}$) due to Lemma 5.7 and Lemma 5.4. The final step of the derivtion is integration by parts and change of the integration variable $\lambda \to p$:

$$\begin{aligned}\tilde{t}_j &= -\frac{1}{\sqrt{z}}\int_{C_j} p(\lambda;u)\,de^{\lambda z}\\ &= \frac{1}{\sqrt{z}}\int_{C_j} e^{\lambda z}\frac{dp(\lambda;u)}{d\lambda}\,d\lambda\\ &= \frac{1}{\sqrt{z}}\int_{Z_j} e^{z\,\lambda(p;u)}dp.\end{aligned}$$

Theorem is proved.

Example 5.4. Let the reflections $R_1^*, \ldots, R_n^*$ generate a finite group W acting in the Euclidean space $\mathcal{E}$ of dimension n. Recall [Bou] that finite groups generated by reflections (5.43) are called *Coxeter groups*. Let us assume the group W to be an irreducible one. We will construct, following [Du6], a Frobenius manifold M_W with the monodromy group W.

The underlying manifold of M_W will be the orbit space

$$M_W = \mathcal{E}/W. \tag{5.71}$$

The coordinte ring of M_W is, by definition, the ring of W-invariant polynomials

$$\mathbf{C}[x_1, \ldots, x_n]^W$$

where $x_1, \ldots, x_n$ are Euclidean coordinates on $\mathcal{E}$. Due to Chevalley theorem [Bour] M_W has a natural structure of a graded affine algebraic variety:

$$\mathbf{C}[x_1, \ldots, x_n]^W \simeq \mathbf{C}[y^1, \ldots, y^n] \tag{5.72}$$

where

$$y^i = y^i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$$

are certain homogeneous W-invariant polynomials of the degrees

$$d_i := \deg y^i(x) = m_i + 1, \tag{5.73}$$

$m_1, \ldots, m_n$ are the exponents of the Coxeter group. The basic invariant polynomials determine a coordinate system on the orbit space M_W. The are determined uniquely up to invertible transformations of the form

$$y^i(x) \mapsto y^{i'}\left(y^1(x), \ldots, y^n(x)\right), \quad i = 1, \ldots, n$$

with quasihomogeneous polynomials $y^{i'}\left(y^1, \ldots, y^n\right)$ of the same degree d_i.

We will construct a polynomial Frobenius structure on M_W. That means that the structure functions $c_{\alpha\beta}^{\gamma}$ will be elements of the ring (5.72). Important ingredients of this contsruction will be the Arnold's construction of convolution of invariants [Ar1, Gi1] and also the flat coordinates on the orbit space M_W discovered by K.Saito *et al.* in [Sai1, SYS].

Let $y^1(x)$ be the invariant polynomial of the maximal degree $h = d_1$. The number h is called *the Coxeter number* of the group W. We define the unity vector field

$$e := \frac{\partial}{\partial y^1} \tag{5.74}$$

and the Euler vector field

$$E := \frac{1}{h} \sum_a x_a \frac{\partial}{\partial x_a}. \tag{5.75}$$

The unity vector field is defined up to a constant factor.

The construction of the metric $< \, , \, >$ and of the multiplication law of tangent vectors is more complicated. Let $(\, , \,)$ denote the W-invariant Euclidean metric on the space $\mathcal{E}$. We will use the orthonormal coordinates $x_1, \ldots, x_n$ w.r.t. this metric.

Let us define a bilinear symmetric form on T^*M_W. In the coordinates $y^1, \ldots, y^n$ it has the matrix

$$(dy^i, dy^j) = \sum_{a=1}^{n} \frac{\partial y^i}{\partial x_a} \frac{\partial y^j}{\partial x_a} = g^{ij}(y) \tag{5.76}$$

for some polynomials $g^{ij}(y)$, $y = (y^1, \ldots, y^n)$ (these exist due to Chevalley theorem). The matrix $\left(g^{ij}(y)\right)$ is invertible on $M_W \setminus \Sigma$ where the discriminant Σ consists of all singular orbits.

Theorem 5.4. *There exists a unique, up to an equivalence, polynomial Frobenius structure on the space of orbits of a finite Coxeter group with the unity vector field (5.74), the Euler vector field (5.75), and the intersection form (5.76).*

Sketch of the proof. We put

$$< \, , \, > := \mathcal{L}_e(\, , \,) \tag{5.77}$$

(cf. (5.8)). This gives *Saito metric* on the orbit space M_W. According to [Sai1, SYS] this metric is flat, and there exists a distinguished system of basic homogeneous W-invariant polynomials $t^1(x), \ldots, t^n(x)$ such that

$$\eta^{\alpha\beta} := < dt^\alpha, dt^\beta >$$

is a constant nondegenerate matrix. Our main observation is that the metrics $(\, , \,)$ and $< \, , \, >$ form a flat pencil (see above the definition). This allows us to reconstruct the Frobenius structure inverting the formula (5.2) (in the present case $A^{\alpha\beta} = 0$ in (5.2)). Namely, we define a W-invariant homogeneous polynomial F of the degree $2h+2$ from the equations

$$\eta^{\alpha\lambda}\eta^{\beta\mu} \frac{\partial^2 F}{\partial t^\lambda \partial t^\mu} = \frac{h\,(dt^\alpha, dt^\beta)}{\deg t^\alpha + \deg t^\beta - 2}, \quad \alpha, \beta = 1, \ldots, n \tag{5.78}$$

(cf. (5.2)). Such a polynomial exists and it satisfies WDVV. Theorem is proved.

Observe that for the Frobenius structure on M_W

$$d = 1 - \frac{2}{h}, \quad q_\alpha = 1 - \frac{\deg t^\alpha}{h}, \ \alpha = 1, \ldots, n. \tag{5.79}$$

Exercise 5.5. Prove that the monodromy group of the Frobenius manifold M_W is isomorphic to W. (Hint: prove that the flat coordinates of the intersection form coincide with the Euclidean coordinates in the space W.)

Exercise 5.6. Prove that the Frobenius manifolds M_W satisfy the semisimplicity condition.

Particularly, for $n = 2$ the polynomial Frobenius manifold corresponding to the group $I_2(k)$ of symmetries of regular k-gon has the form

$$F = \frac{1}{2} t_1^2 t_2 + t^{k+1}.$$

For $n = 3$ there are three irreducible finite Coxeter groups $W(A_3)$, $W(B_3)$ and $W(H_3)$. They are the groups of symmetries of regular tetrahedron, octahedron and icosahedron resp. The corresponding polynomial Frobenius manifolds have the form (1.22), (1.23), and (1.24) resp. We give here aalso the list of all our polynomial Frobenius manifolds in the dimension 4. Group $W(A_4)$.

$$F = \frac{1}{2} t_1^2\, t_4 + t_1\, t_2\, t_3 + \frac{1}{2} t_2^3 + \frac{1}{3} t_3^4 + 6 t_2 t_3^2 t_4 + 9 t_2^2 t_4^2 + 24 t_3^2 t_4^3 + \frac{216}{5} t_4^6.$$

Group $W(B_4)$.

$$F = \frac{1}{2} t_1^2\, t_4 + t_1\, t_2\, t_3 + {t_2}^3 + \frac{t_2\, {t_3}^3}{3} + 3\, {t_2}^2\, t_3\, t_4 + \frac{{t_3}^4\, t_4}{4}$$

$$+ 3\, t_2\, {t_3}^2\, {t_4}^2 + 6\, {t_2}^2\, {t_4}^3 + {t_3}^3\, {t_4}^3 + \frac{18\, {t_3}^2\, {t_4}^5}{5} + \frac{18\, {t_4}^9}{7}.$$

Group $W(D_4)$.

$$F = \frac{1}{2} t_1^2\, t_4 + t_1\, t_2\, t_3 + t_2^3 t_4 + t_3^3 t_4 + 6 t_2 t_3 t_4^3 + \frac{54}{35} t_4^7.$$

Group $W(F_4)$.

$$F = \frac{1}{2} t_1^2\, t_4 + t_1\, t_2\, t_3 + \frac{t_2^3\, t_4}{18} + \frac{3\, t_3^4\, t_4}{4} + \frac{t_2\, t_3^2\, t_4^3}{2}$$

$$+ \frac{t_2^2\, t_4^5}{60} + \frac{t_3^2\, t_4^7}{28} + \frac{t_4^{13}}{2^4 \cdot 3^2 \cdot 11 \cdot 13}.$$

Group $W(H_4)$

$$F = t_1\, t_2\, t_3 + \frac{{t_1}^2\, t_4}{2} + \frac{2\, {t_2}^3\, t_4}{3} + \frac{{t_3}^5\, t_4}{240} + \frac{t_2\, {t_3}^3\, {t_4}^3}{18} + \frac{{t_2}^2\, t_3\, {t_4}^5}{15} + \frac{{t_3}^4\, {t_4}^7}{2^3 \cdot 3^3 \cdot 5} +$$

$$\frac{t_2\, {t_3}^2\, {t_4}^9}{2 \cdot 3^4 \cdot 5} + \frac{8\, {t_2}^2\, {t_4}^{11}}{3^4 \cdot 5^2 \cdot 11} + \frac{{t_3}^3\, {t_4}^{13}}{2^2 \cdot 3^6 \cdot 5^2} + \frac{2\, {t_3}^2\, {t_4}^{19}}{3^8 \cdot 5^3 \cdot 19} + \frac{32\, {t_4}^{31}}{3^{13} \cdot 5^6 \cdot 29 \cdot 31}.$$

As it was shown in [Bl] these are all semisimple polynomial solutions of WDVV for $n = 4$ satisfying the conditions

$$0 < q_\alpha \leq d < 1,\ \alpha = 2,\, 3,\, 4.$$

Other examples of polynomial solutions of WDVV associated with finite Coxeter groups can be found in [Zu].

Remark 5.2. There are certain inclusions between the polynomial Frobenius manifolds of the form

$$M_W := \mathbf{C}^n/W$$

(the orbit spaces) for a finite Coxeter group W acting in n-dimensional Euclidean space. These inclusions correspond to the operation of folding of Dynkin graphs [AGV]. As it is shown in [Ya], if the Dynkin graph of a Coxeter group W' is obtained by folding of the Dynkin graph of another Coxeter group W then the corresponding orbit space $M_{W'}$ is a (graded) linear subspace in M_W w.r.t. the Saito linear structure. From our construction we immediately conclude that the inclusion

$$M_{W'} \subset M_W$$

is also an embedding of Frobenius manifolds. We obtain the following list of embeddings (they were obtained in [Zu] by a straightforward computation)

$$\begin{aligned} M_{B_n} &\subset M_{A_{2n-1}} \\ M_{I_2(k)} &\subset M_{A_{k-1}} \\ M_{F_4} &\subset M_{E_6} \\ M_{H_3} &\subset M_{D_6} \\ M_{H_4} &\subset M_{E_8}. \end{aligned}$$

(The group W_{G_2} coincides with $W_{I_2(6)}$ and, therefore, $M_{G_2} \subset M_{A_5}$.) The inclusions mean that, for example,

$$F_{E_8}(t_1, 0, t_3, 0, 0, t_6, 0, t_8) = F_{H_4}(t_1, t_3, t_6, t_8).$$

Conjecture. *Any irreducible semisimple polynomial Frobenius manifold is equivalent to M_W for some finite irreducible Coxeter group W.*

Remark 5.3. According to our construction, the Frobenius structure depends not only on the monodromy group W but also on class of equivalence of the ordered system of generating reflections $R_1^*, \ldots, R_N^*$. The equivalence is established by simultaneous conjugations of the generators by (,)-orthogonal transformations and by the following action of the braid group

$$\begin{aligned} \sigma_i(R_k^*) &= R_k^*, \ k \neq i, \, i+1 \\ \sigma_i(R_i^*) &= R_{i+1}^* \\ \sigma_i(R_{i+1}^*) &= R_{i+1}^* R_i^* R_{i+1}^*. \end{aligned} \tag{5.80}$$

Here, as above, σ_i are the standard generators of the braid group $\mathcal{B}_n$. Any such class of equivalence is determined by the orbit of the Stokes matrix $S = (S_{ij})$

$$S_{ii} = 1, \ S_{ij} = (e_i^*, e_j^*) \text{ for } i < j \tag{5.81}$$

w.r.t. the $\mathcal{B}_n$-action (5.80). Here e_i^* is the basis of normals to the mirrors of the reflections normalized by the condition

$$(e_i^*, e_i^*) = 2$$

for any i. For example, for an algebraic Frobenius manifold the orbit of the given Stokes matrix S must be finite. For the first nontrivial case $n = 3$ the classification of finite orbits of the action of $\mathcal{B}_3$ on the space of 3×3 Stokes matrices satisfying the nondegeneracy condition (5.49) was obtained in [DM]. Namely, there are only five finite orbits. Three of them correspond to standard system of generating reflections in the groups $W(A_3)$, $W(B_3)$, $W(H_3)$ of symmetries of regular tetrahedron, octahedron and icosahedron respectively. Recall the construction of a standard system of generating reflections in the group of symmetries of a regular polyhedron. Let O be the center of the polyhedron, M the center of its face, A a vertex of the face, H the center of an edge of the face having A as an endpoint. Then the reflections w.r.t. the planes OMA, OMH and OAH generate the group of symmetries of the polyhedron [Cox]. Reordering of these generators give the same equivalence class.

One can repeat this construction with cube (the reciprocal of octahedron) or with dodecahedron (the reciprocal of icosahedron) just to obtain the same system of generators in $W(B_3)$ and in $W(H_3)$ resp. Now we are able to describe the remaining two finite orbits of the action of $\mathcal{B}_3$. The corresponding mirrors of the reflections are obtained by applying the above construction to the great icosahedron and great dodecahedron. The description of these regular Kepler - Poinsot star-polyhedra one can find in the Coxeter book [Cox]. As above, their reciprocals give the same equivalence class. All these regular star-polyhedra have icosahedral symmetry. Thus, we obtain three classes of triples of generating reflections in the group $W(H_3)$.

The above classification was applied in [DM] to the problem of classification of algebraic solutions of $PVI(\mu)$. One can show that the standard systems of generators in $W(A_3)$, $W(B_3)$, $W(H_3)$ correspond to the polynomial solutions (1.22) - (1.24) of WDVV (thus, to the algebraic solutions (5.24) - (5.26) of $PVI(\mu)$). The last two finite orbits give algebraic (non-polynomial) Frobenius manifolds.

The problem of classification for any n of finite orbits of the action of the braid group $\mathcal{B}_n$ in the space of $n \times n$ Stokes matrices remains open. The solution of this problem could be useful to prove the above Conjecture. For $n = 4$ one can prove that all finite orbits of irreducible Stokes matrices satisfying the nondegeneracy condition (5.49) correspond to a system of generating reflection of a finite Coxeter group acting in $\mathbf{R}^4$. In the groups $W(A_4)$ and $W(B_4)$ there is only one equivalence class of systems of generating reflections, namely, the standard one. In the groups $W(D_4)$ and $W(F_4)$ there are two classes. Finally, in the group $W(H_4)$ there are 10 classes of systems of generating reflections. One of them corresponds to the standard system of generators in the group of symmetries of the regular 600-cell, 6 others to 4-dimensional regular star-polyhedra considered modulo reciprocity (see the definitions in [Cox]) but the remaining 3 classes do not have clear geometrical meaning. Of the full list of 16 finite orbits we give here Stokes matrices of representatives in the finite $\mathcal{B}_4$-orbits corresponding to only nonstandard systems of generators

$$D_4: \ S = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F_4: \ S = \begin{pmatrix} 1 & -1 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H_4: \ S = \begin{pmatrix} 1 & -1 & 0 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 & -1 & -\frac{1+\sqrt{5}}{2} \\ 0 & 0 & 1 & \frac{-1+\sqrt{5}}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & -1 & 0 & \frac{1+\sqrt{5}}{2} \\ 0 & 1 & -\frac{1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & -1 & 0 & \frac{1-\sqrt{5}}{2} \\ 0 & 1 & -\frac{1-\sqrt{5}}{2} & -\frac{1-\sqrt{5}}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The construction of Theorem 5.4 was generalized in [DZ1] to produce quasipolynomial solutions of WDVV, i.e., solutions with $d = 1$ of the form

$$F(t^1, \dots, t^n) = \text{cubic } + f(t^2, \dots, t^{n-1}, \exp t^n)$$

with a polynomial f. The monodromy group of these Frobenius manifolds are certain extensions of affine Weyl groups. Particularly, the monodromy group of quantum cohomology of $\mathbf{CP}^1$ is given by this construction (see [Du7]).

Example 5.5. Let us compute the monodromy group of the quantum cohomology of $\mathbf{CP}^2$. The Stokes matrix S (4.97) of this Frobenius manifold satisfies the nondegeneracy condition (5.49). The basic reflections in the monodromy group in the basis of the flat coordinates (x, y, z) corresponding to the basis $\Phi^{\text{left}} = \left(\Phi_1^{\text{left}}, \Phi_2^{\text{left}}, \Phi_3^{\text{left}}\right)$ of the solutions (4.96) have the matrices

$$R_1^* = \begin{pmatrix} -1 & -3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ R_2^* = \begin{pmatrix} 1 & 0 & 0 \\ -3 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \ R_3^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & -1 \end{pmatrix}.$$

The full monodromy group of the Frobenius manifold we obtain adding the monodromy transformation corresponding to the only nontrivial loop

$$t_2 \mapsto t_2 + 2\pi i.$$

This corresponds to the rotation
$$z \mapsto z\, e^{\frac{2\pi i}{3}}.$$
Using the explicit formulae (4.96) and the identity (4.92) we immediately obtain the needed transformation
$$\Phi^{\rm left}\left(z\, e^{\frac{2\pi i}{3}}\right) = \Phi^{\rm left}(z)\, T$$
or, equivalently,
$$(x, y, z) \mapsto (x, y, z)\, T \tag{5.82}$$
with
$$T = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{pmatrix}. \tag{5.83}$$
For the matrix
$$T_0 := T\, R_1^* = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{5.84}$$
we have the identities
$$T_0^3 = -1, \quad R_2^* = T_0^{-1} R_1^* T_0, \quad R_3^* = T_0^{-1} R_2^* T_0. \tag{5.85}$$
So, we introduce the new system of generators A, B, C in the full monodromy group putting
$$A = R_1^*, \quad B = T_0^4 = -T_0, \quad C = T_0^3 = -1. \tag{5.86}$$
All the transformations of the group preserve the integer lattice in $\mathbf{R}^3$. They also preserve the indefinite quadratic form with the Gram matrix $S + S^T$
$$q(x, y, z) = 2\left(x^2 + y^2 + z^2 + 3xy - 3xz - 3yz\right). \tag{5.87}$$
The group acts discretely on the complexification of the cone $q(x, y, z) > 0$.

Introducing the coordinates r, τ, $\bar\tau$
$$\begin{aligned} x &= \frac{ir}{2}\frac{2\tau\bar\tau - 3(\tau + \bar\tau) + 2}{\tau - \bar\tau} \\ y &= \frac{ir}{2}\frac{2\tau\bar\tau + \tau + \bar\tau - 2}{\tau - \bar\tau} \\ z &= \frac{ir}{2}\frac{2\tau\bar\tau - (\tau + \bar\tau) - 2}{\tau - \bar\tau} \end{aligned} \tag{5.88}$$
we obtain the action of the generating transformations
$$\begin{aligned} &A : \left(r \mapsto r,\ \tau \mapsto -\frac{1}{\tau},\ \bar\tau \mapsto -\frac{1}{\bar\tau}\right) \\ &B : \left(r \mapsto r,\ \tau \mapsto \frac{1}{1-\tau},\ \bar\tau \mapsto \frac{1}{1-\bar\tau}\right) \\ &C : (r \mapsto -r,\ \tau \mapsto \tau,\ \bar\tau \mapsto \bar\tau). \end{aligned} \tag{5.89}$$

We proved

Theorem 5.5. *The monodromy group of quantum cohomology of* $\mathbf{CP}^2$ *is isomorphic to* $PSL_2(\mathbf{Z}) \times \{\pm\}$.

It would be interesting to develop an appropriate theory of invariants for this action of the modular group. This could help to obtain analytic formulae for quantum cohomology of $\mathbf{CP}^2$.

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