EXTENDED AFFINE WEYL GROUPS AND FROBENIUS MANIFOLDS – II

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ABSTRACT. For the root system of type B_l and C_l, we generalize the result of [5] by showing the existence of a Frobenius manifold structure on the orbit space of the extended affine Weyl group that corresponds to any vertex of the Dynkin diagram instead of a particular choice of [5].

1. Introduction

For an irreducible reduced root system R defined in l-dimensional Euclidean space V with Euclidean inner product $(\ ,\ )$, we fix a basis of simple roots $\alpha_1, \ldots, \alpha_l$ and denote by $\alpha_j^\vee,\ j = 1, 2, \cdots, l$ the corresponding coroots. The Weyl group W is generated by the reflections

$$\mathbf{x} \mapsto \mathbf{x} - (\alpha_j^\vee, \mathbf{x})\alpha_j, \quad \forall\, \mathbf{x} \in V,\ j = 1, \ldots, l. \tag{1.1}$$

The semi-direct product of W by the lattice of coroots yields the affine Weyl group W_a that acts on V by the affine transformations

$$\mathbf{x} \mapsto w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee, \quad w \in W,\ m_j \in \mathbb{Z}. \tag{1.2}$$

We denote by $\omega_1, \ldots, \omega_l$ the fundamental weights that are defined by relations

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}, \quad i, j = 1, \ldots, l. \tag{1.3}$$

Fixing a simple root α_k, we define an *extended affine Weyl group* $\widetilde{W} = \widetilde{W}^{(k)}(R)$ as in [5]. It acts on the extended space

$$\widetilde{V} = V \oplus \mathbb{R}$$

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and is generated by the transformations

$$x = (\mathbf{x}, x_{l+1}) \mapsto (w(\mathbf{x}) + \sum_{j=1}^{l} m_j \alpha_j^\vee,\ x_{l+1}), \quad w \in W,\ m_j \in \mathbb{Z}, \tag{1.4}$$

and

$$x = (\mathbf{x}, x_{l+1}) \mapsto (\mathbf{x} + \omega_k,\ x_{l+1} - 1). \tag{1.5}$$

Let us introduce coordinates $x_1, \dots, x_l$ on the space V by

$$\mathbf{x} = x_1\, \alpha_1^\vee + \cdots + x_l\, \alpha_l^\vee. \tag{1.6}$$

Denote by f the determinant of the Cartan matrix of the root system R.

Definition 1.1 ([5])**.** $\mathcal{A} = \mathcal{A}^{(k)}(R)$ *is the ring of all* $\widetilde{W}$*-invariant Fourier polynomials of the form*

$$\sum_{m_1,\dots,m_{l+1}\in\mathbb{Z}} a_{m_1,\dots,m_{l+1}} e^{2\pi i(m_1 x_1 + \cdots + m_l x_l + \frac{1}{f}\, m_{l+1} x_{l+1})}$$

that are bounded in the limit

$$\mathbf{x} = \mathbf{x}^0 - i\ \omega_k \tau, \quad x_{l+1} = x_{l+1}^0 + i\ \tau, \quad \tau \to +\infty \tag{1.7}$$

for any $x^0 = (\mathbf{x}^0, x_{l+1}^0)$.

For the fixed simple root α_k, we introduce a set of numbers

$$d_j = (\omega_j, \omega_k), \quad j = 1, \dots, l \tag{1.8}$$

and define the following Fourier polynomials [5]

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}}\, y_j(\mathbf{x}), \quad j = 1, \dots, l, \tag{1.9}$$

$$\tilde{y}_{l+1}(x) = e^{2\pi i x_{l+1}}. \tag{1.10}$$

Here $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ are the W_a-invariant Fourier polynomials defined by

$$y_j(\mathbf{x}) = \frac{1}{n_j} \sum_{w\in W} e^{2\pi i(\omega_j, w(\mathbf{x}))}, \quad n_j = \#\{w \in W | e^{2\pi i(\omega_j, w(\mathbf{x}))} = e^{2\pi i(\omega_j, \mathbf{x})}\}. \tag{1.11}$$

It was shown in [5] that for some particular choices of the simple root α_k, a Chevalley type theorem holds true for the ring $\mathcal{A}$, i.e., it is generated by $\tilde{y}_1, \dots, \tilde{y}_{l+1}$, and thus the orbit space defined as $\mathcal{M} = \mathrm{Spec}\mathcal{A}$ of the extended affine Weyl group $\widetilde{W}$ is an affine algebraic variety of dimension $l+1$. Furthermore, in [5] it was proved that on such an orbit space there exists a *Frobenius manifold structure* whose potential is a polynomial of $t^1, \dots, t^l, e^{t^{l+1}}$. Here $t^1, \dots, t^{l+1}$ are the flat coordinates of the Frobenius manifold. For the root system of type A_l, there is in fact no restrictions on the choice of α_k. However, for the root systems of type $B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$ there is only one choice for each. Recall that the geometric structures on the orbit spaces $\mathcal{M}$ generalize those that live on the orbit spaces of the finite Coxeter groups discovered in [9, 8] and [4].

In [10] P. Solodowy showed that the Chevalley type theorem of [5] can also be derived from the results of K. Wirthmüller [11], and in fact it holds true for any choice of the base element α_k, or equivalently, for any fixed vertex of the Dynkin diagram. So we have

Theorem 1.2 ([10, 11])**.** *The ring $\mathcal{A}$ is isomorphic to the ring of polynomials of $\tilde{y}_1(x)$, $\cdots$, $\tilde{y}_{l+1}(x)$.*

A natural question, as it was pointed out in [5, 10], is whether the geometric structures revealed in [5] also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of α_k? The purpose of the present paper is to give an affirmative answer to this question for the root systems of type B_l, C_l. We will show that on the corresponding orbit spaces there also exist Frobenius manifold structures with potentials that are polynomials in $t^1, \ldots, t^{l-1}, t^l, \frac{1}{t^l}, e^{t^{l+1}}$. Here $t^1, \ldots, t^{l+1}$ are the flat coordinates of the resulting Frobenius manifold.

The paper is organized as follows: in Sec.2, we construct a flat pencil of metrics on each orbit space of the extended affine Weyl group of the root system of type C_l for any fixed vertex of the Dynkin diagram, then in Sec.3 we study properties of the flat coordinates of the flat metric (η^{ij}). In Sec.4, we prove the existence of a Frobenius manifold structure on each orbit space. In Sec.5 we give some examples. In Sec.6 we show that to the root system of type B_l we can apply a similar construction as the one for the root system of type C_l. The resulting Frobenius manifolds are isomorphic to those that are obtained from C_l. Some concluding remarks are given in the last section.

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2. Flat pencils of metrics on the orbit spaces of $\widetilde{W}^{(k)}(C_l)$

Let $\mathcal{M}$ be the orbit space defined as Spec$\mathcal{A}$ of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$ for any fixed $1 \le k \le l$. We choose the standard base $\alpha_1, \dots, \alpha_l$ of simple roots for the root system C_l as given in [2]. As in [5] we define an indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V} = V \oplus \mathbb{R}$ such that $\widetilde{V}$ is the orthogonal direct sum of V and $\mathbb{R}$, V endowed with the W-invariant Euclidean metric

$$(dx_m, dx_n)^{\sim} = \frac{m}{4\pi^2}, \quad 1 \le m \le n \le l \tag{2.1}$$

and $\mathbb{R}$ endowed with the metric

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4\pi^2 d_k} = -\frac{1}{4k\,\pi^2}. \tag{2.2}$$

Here the numbers d_j are defined in (1.8) and take the values

$$d_1 = 1, \dots, d_{k-1} = k-1,\ d_j = k, \quad k \le j \le l. \tag{2.3}$$

The W_a-invariant Fourier polynomials $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ that are defined in (1.11) have the expressions

$$y_j(\mathbf{x}) = \sigma_j(\xi_1, \cdots, \xi_l), \tag{2.4}$$

where

$$\xi_j = e^{2\,i\pi\,(x_j - x_{j-1})} + e^{-2\,i\pi\,(x_j - x_{j-1})},\ x_0 = 0,\ j = 1, \cdots, l \tag{2.5}$$

and $\sigma_j(\xi_1, \dots, \xi_l)$ is the j-th elementary symmetric polynomial of $\xi_1, \cdots, \xi_l$. For the reason that will be clear later, we will use in what follows the following set of generators for the the ring of W_a-invariant Fourier polynomials which we still denote by $y_1, \dots, y_l$:

$$y_j(\mathbf{x}) = \sigma_j(\zeta_1, \cdots, \zeta_l) \tag{2.6}$$

with

$$\zeta_j = \xi_j + 2, \quad j = 1, \dots, l. \tag{2.7}$$

Consequently we have a set of generators for the ring $\mathcal{A} = \mathcal{A}^{(k)}(C_l)$

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}}\, y_j(\mathbf{x}), \quad j = 1, \dots, l, \tag{2.8}$$

$$\tilde{y}_{l+1}(x) = e^{2\pi i x_{l+1}}, \tag{2.9}$$

defined in the same way as in (1.9), (1.10). They form a global coordinates on $\mathcal{M}$. As in [5], we introduce the following local coordinates on $\mathcal{M}$:

$$y^1 = \tilde{y}_1, \dots, y^l = \tilde{y}_l,\ y^{l+1} = \log \tilde{y}_{l+1} = 2\pi i\, x_{l+1}. \tag{2.10}$$

They live on a covering $\widetilde{\mathcal{M}}$ of $\mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\}$. The projection

$$P : \widetilde{V} \to \widetilde{\mathcal{M}} \tag{2.11}$$

induces a symmetric bilinear form on $T^*\widetilde{\mathcal{M}}$

$$(dy^i, dy^j)^{\sim} \equiv g^{ij}(y) := \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}(dx^a, dx^b)^{\sim}. \tag{2.12}$$

Denote

$$\Sigma = \{y | \det(g^{ij}(y)) = 0\}. \tag{2.13}$$

It turns out that Σ is an analogue of the discriminant. Namely, as it was shown in [5], Σ is the P-image of the union of hyperplanes

$$\{(\mathbf{x}, x_{l+1}) | (\beta, \mathbf{x}) = m \in \mathbb{Z},\ x_{l+1} = \text{arbitrary}\}, \quad \beta \in \Phi^+, \tag{2.3}$$

where Φ^+ is the set of all positive roots.

Lemma 2.1. *The functions $g^{ij}(y)$ are weighted homogeneous polynomials in $y^1, \cdots, y^l, e^{y^{l+1}}$ of the degree*

$$\deg g^{ij} = \deg y^i + \deg y^j, \tag{2.14}$$

where $\deg y^j = d_j$ and $\deg y^{l+1} = d_{l+1} = 0$.

Proof The assertion follows immediately from the Theorem 1.2 which says that the functions $\tilde{y}_1, \dots, \tilde{y}_{l+1}$ form a set of generators of the ring $\mathcal{A}$. $\square$

From this lemma we see that Σ is an algebraic subvariety in $\mathcal{M}$ and the matrix (g^{ij}) is invertible in $\mathcal{M} \setminus \Sigma$. The inverse matrix $(g^{ij})^{-1}$ defines a flat metric on $\mathcal{M} \setminus \Sigma$.

Let us introduce the following new coordinates on $\mathcal{M}$

$$\theta^j = \begin{cases} e^{k\, y^{l+1}}, & j = 0, \\ y^j e^{(k-j) y^{l+1}}, & j = 1, \cdots, k-1, \\ y^j, & j = k, \cdots, l \end{cases} . \tag{2.15}$$

Denote

$$\mu_j = 2\pi i(x_j - x_{j-1}), \quad \mu_{l+1} = y^{l+1}, \quad j = 1, \cdots, l. \tag{2.16}$$

In the coordinates $\mu_1, \dots, \mu_{l+1}$ the indefinite metric on $\widetilde{V}$ has the form

$$\left((d\mu_i, d\mu_j)^{\sim}\right) = \mathrm{diag}(-1, \dots, -1, \frac{1}{k}). \tag{2.17}$$

Define

$$P(u) := \sum_{j=0}^{l} u^{l-j}\theta^j = e^{k\mu_{l+1}} \prod_{j=1}^{l} (u + \zeta_j). \tag{2.18}$$

We can easily verify that the function $P(u)$ satisfies

$$\frac{\partial P(u)}{\partial \mu_a} = \frac{1}{u+\zeta_a} P(u)(e^{\mu_a} - e^{-\mu_a}), \quad 1 \le a \le l; \tag{2.19}$$

$$\frac{\partial P(u)}{\partial \mu_{l+1}} = kP(u), \quad P'(u) := \frac{\partial P(u)}{\partial u} = P(u) \sum_{a=1}^{l} \frac{1}{u+\zeta_a}. \tag{2.20}$$

By using these identities, we have

Lemma 2.2. *The following formulae hold true for the generating functions of the metric (g^{ij}) and the contravariant components of its Levi-Civita connection in the coordinates $\theta^0, \dots, \theta^l$*

$$\begin{aligned}
&\sum_{i,j=0}^{l} (d\theta^i, d\theta^j)^{\sim} u^{l-i} v^{l-j} = (dP(u), dP(v))^{\sim} \\
&\quad = (k-l)P(u)P(v) + \frac{u^2+4u}{u-v} P'(u)P(v) - \frac{v^2+4v}{u-v} P(u)P'(v),
\end{aligned}$$

$$\begin{aligned}
&\sum_{i,j,m=0}^{l} \Gamma^{ij}_m(\theta) d\theta^m u^{l-i} v^{l-j} = \sum_{a,b,m=1}^{l+1} \frac{\partial P(u)}{\partial \mu_a} \frac{\partial^2 P(v)}{\partial \mu_b \partial \mu_m} d\mu_m (d\mu_a, d\mu_b) \\
&\quad = (k-l)P(u)dP(v) + \frac{u^2+4u}{u-v} P'(u)dP(v) - \frac{v^2+4v}{u-v} P(u)dP'(v) \\
&\qquad + \frac{2u+uv+2v}{(u-v)^2} P(v)dP(u) - \frac{2u+uv+2v}{(u-v)^2} P(u)dP(v).
\end{aligned}$$

Here $\Gamma^{ij}_m(\theta) = -\sum_{s=1}^{l+1} g^{is}(\theta)\Gamma^j_{sm}(\theta)$.

Proof By using (2.19), (2.20) we have

$$
\begin{aligned}
&(dP(u), dP(v))^{\sim} = \frac{1}{k}\frac{\partial P(u)}{\partial \mu_{l+1}}\frac{\partial P(v)}{\partial \mu_{l+1}} - \sum_{a=1}^{l}\frac{\partial P(u)}{\partial \mu_a}\frac{\partial P(v)}{\partial \mu_a}\\
&= kP(u)P(v) - \sum_{a=1}^{l} P(u)P(v)\frac{\zeta_a^2 - 4\zeta_a}{(u+\zeta_a)(v+\zeta_a)}\\
&= \sum_{s=1}^{l} P(u)P(v)\frac{v}{u-v}(1-\frac{v}{v+\zeta_a}) - \sum_{a=1}^{l} P(u)P(v)\frac{u}{u-v}(1-\frac{u}{u+\zeta_a})\\
&\quad + \sum_{a=1}^{l} P(u)P(v)\frac{1}{u+\zeta_a}\frac{4u}{u-v} - \sum_{a=1}^{l} P(u)P(v)\frac{1}{v+\zeta_a}\frac{4v}{u-v} + kP(u)P(v)\\
&= (k-l)P(u)P(v) + \frac{u^2+4u}{u-v}P'(u)P(v) - \frac{v^2+4v}{u-v}P(u)P'(v).
\end{aligned}
$$

So we proved the first formula, the second formula can be proved in the same way. The lemma is proved. □

The above lemma shows that in the coordinates $\theta^0, \dots, \theta^l$ the functions $g^{ij}(\theta)$ are quadratic polynomials, and the contravariant components Γ^{ij}_m are homogeneous linear functions[1]. It reveals the following important properties of the flat metric:

Corollary 2.3. *In the coordinates $y^1, \dots, y^{l+1}$ the functions $\Gamma^{ij}_m(y)$ are weighted homogeneous polynomials of degree*

$$\deg \Gamma^{ij}_m(y) = d_i + d_j - d_m. \tag{2.21}$$

Corollary 2.4. *In the coordinates $y^1, \dots, y^{l+1}$ the polynomials $g^{ij}(y)$ and $\Gamma^{ij}_m(y)$ are at most linear in y^k.*

Now let us define a symmetric bilinear form on $T^*\mathcal{M}$ by

$$< dy^i, dy^j >:= \eta^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^k}. \tag{2.22}$$

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[1]These metrics give rise to a quadratic Poisson structure on the space of "loops" $\{S^1 \to M\}$ (see [3] for the details):

$$\{\theta^i(s), \theta^j(s')\} = g^{ij}(\theta(s))\delta'(s-s') + \Gamma^{ij}_m(\theta(s))\theta^m_s\delta(s-s').$$

We plan to study such important class of quadratic metrics and Poisson structures in a separate publication.

Lemma 2.5. *The matrix $(\eta^{ij}(y))$ has the form*

$$\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & k & P_1 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \cdots & k & R_1 & P_2 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \cdots & R_1 & R_2 & P_3 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
k & R_1 & R_2 & \cdots & & R_{k-2} & P_{k-1} & 0 & 0 & \cdots & 0 & 0\\
P_1 & P_2 & P_3 & \cdots & P_{k-2} & P_{k-1} & P_k & 0 & 0 & \cdots & 0 & 1\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & Q_1 & Q_2 & \cdots & Q_{l-k} & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & Q_2 & Q_3 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & Q_{l-k} & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}. \tag{2.23}$$

Here

$$\begin{aligned}
&R_j = 4(k-j+1)y^{j-1}e^{y^{l+1}} + (k-j)y^j,\\
&P_j = 4(k-j+1)y^{j-1}e^{y^{l+1}},\\
&Q_m = 4my^{k+m} + (1-\delta_{m,l-k})(m+1)y^{k+m+1},\\
&1 \le j \le k, \quad 1 \le m \le l-k
\end{aligned}$$

and we assume $y^0 = 1$.

It follows from the above lemma that

$$\det(\eta^{ij}) = (-1)^l k^{k-1} 4^{l-k}(l-k)^{l-k}(y^l)^{l-k},$$

so $(\eta^{ij})^{-1}$ defines a metric on $\mathcal{M} \setminus \{y \in \mathcal{M} | y^l = 0\}$.

Theorem 2.6. *The space $\mathcal{M}$ carries a flat pencil of metrics (bilinear forms on T^*M)*

$$g^{ij}(y) \ \ \text{and} \ \ \eta^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^k},$$

i.e., any linear combination $g^{ij} + \lambda\,\eta^{ij}$ defines a flat metric on certain open subset of $\mathcal{M}$ and the contravariant components of the Levi-Civita connection of (η^{ij}) is given by

$$\gamma^{ij}_m(y) = \frac{\partial \Gamma^{ij}_m(y)}{\partial y^k}. \tag{2.24}$$

The metric $(g^{ij}(y))$ does not degenerate on $\mathcal{M} \setminus \Sigma$ and the metric $(\eta^{ij}(y))$ does not degenerate on $\mathcal{M} \setminus \{y \in \mathcal{M} | y^l = 0\}$.

Proof The result follows, applying Lemma D.1 of [4], from the fact that in the coordinates $y^1, \ldots, y^{l+1}$ the flat metric g^{ij} and the contravariant components of its Levi-Civita connection depend at most linearly on y^k . The theorem is proved. $\square$

Remark 2.7. *Our particular choice of the basis of the* $\widetilde{W}^{(k)}(C_l)$*-invariant Fourier polynomials (2.6)–(2.9) ensures that the components of the flat metric* $(g^{ij}(y))$ *are at most linear in* y^k*. This linearity is the most crucial step in the construction of the above flat pencil of metrics. If we choose the basis of the* $\widetilde{W}^{(k)}(C_l)$*-invariant Fourier polynomials by using* (2.4), (2.8) *and* (2.9), *then we lose this linearity property of the functions* $(g^{ij}(y))$ *and the construction of the flat metric* $(\eta^{ij}(y))$ *becomes obscure.*

3. Flat coordinates of the metric (η^{ij})

In this section, we will show that the flat coordinates of the metric (η^{ij}) are algebraic functions of $y^1, \dots, y^{l+1}, e^{y^{l+1}}$. To this end, we first perform changes of coordinates to simplify the matrix (η^{ij}).

Lemma 3.1. *There exists a system of new coordinates* $z^1, \dots, z^{l+1}$ *of the form*

$$z^j = y^j + p_j(y^1, \dots, y^{j-1}, e^{y^l+1}),\ 1 \le j \le k,\ z^{l+1} = y^{l+1}, \quad (3.1)$$

$$z^j = y^j + \sum_{m=j+1}^{l} c^j_m y^m, \quad k+1 \le j \le l, \quad (3.2)$$

where p_j *are homogeneous polynomials of degree* d_j *and* c^j_m *are some constants such that in the new coordinates the metric* (η^{ij}) *still has the form (2.23) with the entries replaced by*

$$R_j = 0, \quad P_j = 0, \quad Q_m = 4mz^{k+m}, \quad 1 \le j \le k, \quad 1 \le m \le l-k. \quad (3.3)$$

Proof Let us first note that the $(k+1) \times (k+1)$ matrix $(\tilde{\eta}^{ij})$ which has elements

$$\tilde{\eta}^{ij} = \eta^{ij}(y),\ \tilde{\eta}^{k+1,m} = \tilde{\eta}^{m,k+1} = \delta_{j,k}, \quad 1 \le i, j \le k,\ 1 \le m \le k+1$$

coincides, under renaming of the coordinate $y^{l+1} \mapsto y^{k+1}$, with the matrix $(\eta^{ij}(y))_{(k+1)\times(k+1)}$ that is constructed as in the last section with respect to the extended affine Weyl group $\widetilde{W}^{(k)}(C_k)$. Thus by using the results of [5] we can find homogeneous polynomials $p_j, 1 \le j \le k$ such that under the change of coordinates (3.1) and $z^j = y^j$, $k+1 \le j \le l$

the matrix $(\eta^{ij}(z)$ has the form (2.23) with entries

$$R_j = 0, \quad P_j = 0, \quad Q_m = 4mz^{k+m} + (1-\delta_{m,l-k})(m+1)z^{k+m+1}, \\ 1 \le j \le k, \quad 1 \le m \le l-k.$$

To finish the proof of the lemma, we need to perform a second change of coordinates. To this end, denote by Ψ a $n \times n$ matrix with entries as linear functions of $a^1, \dots, a^n$

$$\psi^{ij}(a) = 4(i+j-1)a^{i+j-1} + (i+j)a^{i+j}, \quad i, j \ge 1.$$

Here $a^m = 0$ for $m \ge n+1$. We are to find a linear transformation of the triangular form

$$a^j = \sum_{m=j}^{n} B^j_m b^m, \quad B^j_j = 1, \ j \ge 1$$

such that

$$\sum_{r,s=1}^{n} 4(r+s-1)b^{r+s-1}\frac{\partial a^i}{\partial b^r}\frac{\partial a^j}{\partial b^s} \\ = 4(i+j-1)\sum_{m=i+j-1}^{n} B^{i+j-1}_m b^m + (i+j)\sum_{m=i+j}^{n} B^{i+j}_m b^m.$$

Equivalently, the constants B^i_j must satisfy the relations

$$4(i+j-1)B^{i+j-1}_m + (i+j)B^{i+j}_m = 4m\sum_{\alpha+\beta=m+1} B^i_\alpha B^j_\beta, \\ i+j \le m \le n. \tag{3.4}$$

Introduce the generating functions

$$f^i(t) = \sum_{\alpha \ge 0} B^i_{i+\alpha} t^\alpha, \quad i = 1, 2, \dots. \tag{3.5}$$

Then the relations in (3.4) can be encoded into the following equations of $f^i(t)$:

$$4(i+j-1)t^{i+j-2}f^{i+j-1} + (i+j)t^{i+j-1}f^{i+j} = 4\frac{d}{dt}\left(t^{i+j-1}f^if^j\right). \tag{3.6}$$

This system of equations has the following solution which was obtain by Si-Qi Liu

$$f^i(t) = \cosh\left(\frac{\sqrt{t}}{2}\right)\left(\frac{2\sinh\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t}}\right)^{2i-1}. \tag{3.7}$$

From the above result we derive the existence of constants $c^j_m,\ k+1\le j\le l,\ j+1\le m\le l$ such that under the change of coordinates

$$z^i\mapsto z^i,\ i=1,\dots,k,l+1,\quad z^j\mapsto z^j+\sum_{m=j+1}^{l}c^j_m z^m,\quad k+1\le j\le l,$$

the matrix $(\eta^{ij}(z)$ has the form (2.23) and with entries given by (3.3). The lemma is proved. $\square$

Lemma 3.2. *Under the change of coordinates*

$$w^i=z^i,\quad i=1,\dots,k,\ l+1, \tag{3.8}$$

$$w^{k+1}=z^{k+1}(z^l)^{-\frac{1}{2(l-k)}},\ w^j=z^j(z^l)^{-\frac{j-k}{l-k}},\ w^l=(z^l)^{\frac{1}{2(l-k)}}, \tag{3.9}$$

$$j=k+2\cdots,l-1,$$

the components of the metric $(\eta^{ij}(z))$ are transformed to the form

$$\operatorname{diag}(A_{(k-1)\times(k-1)},B_{(l-k+2)\times(l-k+2)}),$$

where the matrix A has entries $A^{ij}=\delta_{i,k-i}k$ and the upper triangular matrix B has the form

$$B=\begin{pmatrix} 0&0&0&0&0&\cdots&0&0&1\\ 0&0&0&0&0&\cdots&0&2&\\ 0&0&S_{k+3}&S_{k+4}&\cdots&S_{l-1}&S_l&&\\ 0&0&S_{k+4}&S_{k+5}&\cdots&S_l&&&\\ \vdots&\vdots&\vdots&&&&&&\\ 0&0&S_l&&&&&&\\ 0&2&&&&&&&\\ 1&&&&&&&& \end{pmatrix}. \tag{3.10}$$

with

$$S_{k+j}=4j(w^l)^{-2}w^{k+j},\ S_l=4(l-k)(w^l)^{-2},\quad 3\le j\le l-k-1. \tag{3.11}$$

Proof By a straightforward calculation. $\square$

Proposition 3.3. *In the coordinates $w^1,\dots,w^{l+1}$ the Christoffel symbols of the metric (η^{ij}) have the following properties*

(1) $\gamma^m_{ij}=0$ *for* $m=1,\dots,k,l,l+1,\ i,j=1,\dots,l+1;$
(2) $\gamma^{k+1}_{ij}=-\frac{\partial\eta_{ij}(w)}{\partial w^l}$ *are weighted homogenous polynomials in* $w^{k+3},\dots,w^l;$
(3) γ^m_{ij} *are weighted homogenous polynomials in* $w^{k+3},\dots,w^l$ *for* $k+2\le m\le l-1$ *and* $i,j=1,\dots,l-1,l+1;$
(4) $\gamma^m_{lj}=\frac{1}{w^l}\delta^m_j$, *for* $k+2\le m\le l-1,\ 1\le j\le l+1.$

Proof The first three properties of γ_{ij}^m follow easily from the simple form of the matrix $(\eta^{ij}(w))$. To prove the last property, we only need to note that

$$\gamma_{lj}^m = \frac{1}{2}\sum_{s=1}^{l+1}\eta^{ms}\frac{\partial\eta_{sj}}{\partial w^l} = \frac{1}{2}\sum_{s=1}^{l+1}\frac{2}{w^l}\eta^{ms}\eta_{sj} = \frac{1}{w^l}\delta_j^m. \tag{3.12}$$

The proposition is proved. ☐

Theorem 3.4. *We can choose the flat coordinates of the metric* $(\eta^{ij}(w)$ *in the form*

$$t^1 = w^1, \dots, t^k = w^k,\ t^l = w^l,\ t^{l+1} = w^{l+1}, \tag{3.13}$$

$$t^{k+1} = w^{k+1} + w^l\, h_{k+1}(w^{k+2}, \dots, w^{l-1}), \tag{3.14}$$

$$t^j = w^l(w^j + h_j(w^{j+1}, \dots, w^{l-1})). \tag{3.15}$$

Here h_j *are weighted homogeneous polynomials of degree* $\frac{k\,(l-j)}{l-k}$ *for* $j = k+1, \dots, l-2$ *and* $h_{l-1} = 0$.

Proof To find the flat coordinates $t = t(w)$, we need to solve the following system of PDEs

$$\frac{\partial^2 t}{\partial w^i \partial w^j} - \sum_{m=1}^{l+1}\gamma_{ij}^m\frac{\partial t}{\partial w^m} = 0, \quad i, j = 1, \dots, l+1. \tag{3.16}$$

From the above proposition we easily see that $t^1, \dots, t^k, t^{l+1}$ are $k+1$ solutions of the above system. We still need to find $l-k$ independent solutions $t^{k+1}, \dots, t^l$. Introduce the $(l-k)\times(l-k)$ matrix

$$\Phi = (\phi_j^i), \quad \phi_j^i = \frac{\partial t^{k+i}}{\partial w^{k+j}}. \tag{3.17}$$

Then the system (3.16) is reduced to

$$\partial_m \Phi = \Phi A_m, \quad \partial_m = \frac{\partial}{\partial w^m}, \quad m = k+1, \dots, l. \tag{3.18}$$

They are regular at $\mathbf{w} = (w^{k+1}, \dots, w^l) = 0$ except the system with $m = l$. In this case the coefficient matrix has the simple form

$$A_l = \mathrm{diag}(0, \frac{1}{w^l}, \dots, \frac{1}{w^l}, 0). \tag{3.19}$$

Now assume that Φ has the form

$$\Phi = \Psi\,\mathrm{diag}(1, w^l, \dots, w^l, 1). \tag{3.20}$$

Then the systems in (3.18) are converted to

$$\partial_m \Psi = \Psi B_m, \quad \partial_l \Psi = 0, \quad m = k+1, \dots, l-1. \tag{3.21}$$

The entries of the coefficient matrices B_m are now weighted homogeneous polynomials of $w^{k+1}, \ldots, w^l$. Thus we can find a unique analytic at $\mathbf{w} = 0$ solution Ψ of the above systems such that

$$\Psi|_{\mathbf{w}=0} = \mathrm{diag}(1, \ldots, 1). \tag{3.22}$$

From the weighted homogeneity of the coefficient matrices B_m it follows that the elements of Ψ are also weighted homogeneous. Since $\deg w^j > 0$ for $j = k+1, \ldots, l$ we see that they are in fact polynomials of $w^{k+1}, \ldots, w^l$. Thus the result of the theorem follows. The theorem is proved. □

Due to the above construction, we can associate the following natural degrees to the flat coordinates

$$\tilde{d}_j = \deg t^j := \frac{j}{k}, \quad 1 \le j \le k, \tag{3.23}$$

$$\tilde{d}_m = \deg t^m := \frac{2l - 2m + 1}{2(l-k)}, \quad k+1 \le m \le l, \tag{3.24}$$

$$\tilde{d}_{l+1} = \deg t^{l+1} := 0, \tag{3.25}$$

and we readily have the following corollary

Corollary 3.5. *In the flat coordinates $t^1, \ldots, t^{l+1}$, the entries of the matrix (η^{ij}) has the form*

$$\eta^{ij} = \begin{cases} k, & j = k - i, \quad 1 \le i \le k-1, \\ 4(l-k), & j = k + l + 1 - i, \quad k+2 \le i \le l-1, \\ 1, & i = l+1, j = k \quad \text{or } i = k,\ j = l+1, \\ 2, & i = l, j = k+1 \quad \text{or } i = k+1,\ j = l. \end{cases} \tag{3.26}$$

The entries of the matrix $(g^{ij}(t))$ and the Christoffel symbols $\Gamma^{ij}_m(t)$ are weighted homogeneous polynomials of $t^1, \ldots, t^l, \frac{1}{t^l}, e^{t^{l+1}}$ of degrees $\tilde{d}_i + \tilde{d}_j$ and $\tilde{d}_i + \tilde{d}_j - \tilde{d}_m$ respectively. In particular,

$$\begin{aligned} g^{m,l+1} &= \tilde{d}_m t^m, \quad 1 \le m \le l, \quad g^{l+1,l+1} = \frac{1}{k}, \\ \Gamma^{l+1,i}_j &= \tilde{d}_j\, \delta_{i,j}, \quad 1 \le i, j \le l+1. \end{aligned} \tag{3.27}$$

Remark 3.6. *For the orbit spaces of finite reflection groups flat coordinates were constructed by Saito, Yano and Sekiguchi in* [9] *(see also* [8]*).*

The numbers $\tilde{d}_1, \tilde{d}_{l+1}$ satisfy a duality relation that is similar to that of [5]. To describe this duality relation, let us delete the k-th vertex of the Dynkin diagram $\mathcal{R}$ and obtain two components $\mathcal{R} \setminus \alpha_k = \mathcal{R}_1 \cup \mathcal{R}_2$. On each component we have an involution

$$i \mapsto i^*, \quad i = 1, \ldots, k-1 \text{ and } i = k+1, \ldots, l \tag{3.28}$$

given by the reflection with respect to the center of the component. We also define

$$k^* = l+1, \quad (l+1)^* = k, \tag{3.29}$$

then we have

$$\tilde{d}_i + \tilde{d}_{i^*} = 1, \quad i = 1, \ldots, l+1, \tag{3.30}$$

and from the above corollary we see that η^{ij} is a nonzero constant iff $j = i^*$.

4. The Frobenius manifold structure on the orbit space of $\widetilde{W}^{(k)}(C_l)$

Now we are ready to describe the Frobenius manifold structure on the orbit space of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$. Let us first recall the definition of Frobenius manifold, see [4] for details.

Definition 4.1. *A Frobenius algebra is a pair $(A, <\ ,\ >)$ where A is a commutative associative algebra with a unity e over a field $\mathcal{K}$ (in our case $\mathcal{K} = \mathbb{C}$) and $<\ ,\ >$ is a $\mathcal{K}$-bilinear symmetric nondegenerate invariant form on A, i.e.,*

$$< x \cdot y, z > = < x, y \cdot z >, \quad \forall\ x, y, z \in A.$$

Definition 4.2. *A Frobenius structure of charge d on an n-dimensional manifold M is a structure of Frobenius algebra on the tangent spaces $T_tM = (A_t, <\ ,\ >_t)$ depending (smoothly, analytically etc.) on the point t. This structure satisfies the following axioms:*

- FM1. *The metric $<\ ,\ >_t$ on M is flat, and the unity vector field e is covariantly constant, i.e., $\nabla e = 0$. Here we denote ∇ the Levi-Civita connection for this flat metric.*
- FM2. *Let c be the 3-tensor $c(x, y, z) := < x \cdot y, z >$, $x, y, z \in T_tM$. Then the 4-tensor $(\nabla_w c)(x, y, z)$ is symmetric in $x, y, z, w \in T_tM$.*
- FM3. *The existence on M of a vector field E, called the Euler vector field, which satisfies the conditions $\nabla\nabla E = 0$ and*

$$[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y,$$

$$E < x, y > - < [E, x], y > - < x, [E, y] > = (2-d) < x, y >$$

for any vector fields x, y on M.

A manifold M equipped with a Frobenius structure on it is called a Frobenius manifold.

Let us choose locally flat coordinates $t^1, \cdots t^n$ for the invariant flat metric, then locally there exists a function $F(t^1, \cdots, t^n)$, called the *potential* of the Frobenius manifold, such that

$$< u \cdot v, w >= u^i v^j w^s \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^s} \tag{4.1}$$

for any three vector fields $u = u^i \frac{\partial}{\partial t^i}$, $v = v^j \frac{\partial}{\partial t^j}$, $w = w^s \frac{\partial}{\partial t^s}$. Here and in what follows summations over repeated indices are assumed. By definition, we can also choose the coordinates t^1 such that $e = \frac{\partial}{\partial t^1}$. Then in the flat coordinates the components of the flat metric can be expressed in the form

$$\frac{\partial^3 F}{\partial t^1 \partial t^i \partial t^j} = \eta_{ij} =< \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} >, \quad i, j = 1, \ldots, n. \tag{4.2}$$

The associativity of the Frobenius algebras is equivalent to the following overdetermined system of equations for the function F

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^k \partial t^m} = \frac{\partial^3 F}{\partial t^k \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^i \partial t^m} \tag{4.3}$$

for arbitrary indices i, j, k, m from 1 to n.

In the flat coordinates the Euler vector field E has the form

$$E = \sum_{i=1}^{n} (d^i_j t^j + r_i) \frac{\partial}{\partial t^i} \tag{4.4}$$

for some constants d^i_j, r_i, $i = 1, \ldots, n$ which satisfy

$$d^i_1 = \delta^i_1, \quad r_1 = 0.$$

From the axiom FM3 it follows that the potential F satisfies the quasi-homogeneity condition

$$\mathcal{L}_E F = (3 - d) F + \frac{1}{2} A_{ij}\, t^i t^j + B_i\, t^i + \text{constant}. \tag{4.5}$$

The system (4.2)–(4.5) is called the $WDVV$ *equations of associativity* which is equivalent to the above definition of Frobenius manifold in the chosen system of local coordinates.

In our examples the constant matrix d^j_i is always diagonal, $d^j_i = \hat{d}_i \delta^j_i$.

Let us also recall an important geometrical structure on a Frobenius manifold M, the *intersection form* of M. This is a symmetric bilinear form $(\ ,\ )^*$ on T^*M defined by the formula

$$(w_1, w_2)^* = i_E(w_1 \cdot w_2), \tag{4.6}$$

here the product of two 1-forms w_1, w_2 at a point $t \in M$ is defined by using the algebra structure on T_tM and the isomorphism

$$T_tM \to T_t^*M \tag{4.7}$$

established by the invariant flat metric $< \, , \, >$. In the flat coordinates $t^1, \cdots, t^n$ of the invariant metric, the intersection form can be represented by

$$(dt^i, dt^j)^* = \mathcal{L}_E F^{ij} = (d + 1 - \hat{d}_i - \hat{d}_j)F^{ij} + A^{ij}, \tag{4.8}$$

where

$$A^{ij} = \eta^{ii'}\eta^{jj'}A_{i'j'}, \quad F^{ij} = \eta^{ii'}\eta^{jj'}\frac{\partial^2 F}{\partial t^{i'}\partial t^{j'}} \tag{4.9}$$

and $F(t)$ is the potential of the Frobenius manifold. Denote by $\Sigma \subset M$ the *discriminant* of M on which the intersection form degenerates, then an important property of the intersection form is that on $M \setminus \Sigma$ its inverse defines a new flat metric.

Theorem 4.3. *There exists a unique Frobenius structure of charge* $d = 1$ *on the orbit space* $\mathcal{M} \setminus \{t^l = 0\}$ *of* $\widetilde{W}^{(k)}(C_l)$ *polynomial in* $t^1, t^2, \cdots, t^l, \frac{1}{t^l}, e^{t^{l+1}}$ *such that*

(1) *The unity vector field* e *coincides with* $\frac{\partial}{\partial y^k} = \frac{\partial}{\partial t^k}$*;*
(2) *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^{l} \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{\partial}{\partial t^{l+1}} \tag{4.10}$$

where $\tilde{d}_1, \ldots, \tilde{d}_l$ *are defined in (3.23)–(3.25).*
(3) *The invariant flat metric and the intersection form of the Frobenius structure coincide respectively with the metric* (η^{ij}) *and* $(g^{ij}(t))$ *on* $\mathcal{M} \setminus \{t^l = 0\}$.

Proof By following the lines of the proof of Lemma 2.6 that is given in [5] we can show the existence of a unique weighted homogeneous polynomial

$$G := G(t^1, \ldots, t^{k-1}, t^{k+1}, \ldots, t^l, \frac{1}{t^l}, e^{t^{l+1}})$$

of degree 2 such that the function

$$F = \frac{1}{2}(t^k)^2 t^{l+1} + \frac{1}{2}t^k \sum_{i,j \neq k} \eta_{ij}\, t^i t^j + G \tag{4.11}$$

satisfies the equations

$$g^{ij} = \mathcal{L}_E F^{ij}, \quad \Gamma^{ij}_m = \tilde{d}_j\, c^{ij}_m, \quad i, j, m = 1, \dots, l+1, \tag{4.12}$$

where $c^{ij}_m = \frac{\partial F^{ij}}{\partial t^m}$. Obviously, the function F satisfies the equations

$$\frac{\partial^3 F}{\partial t^k \partial t^i \partial t^j} = \eta_{ij}, \quad i, j = 1, \dots, l+1 \tag{4.13}$$

and the quasi-homogeneity condition

$$\mathcal{L}_E F = 2F. \tag{4.14}$$

From the properties of a flat pencil of metrics [4] it follows that F also satisfies the associativity equations

$$c^{ij}_m\, c^{mp}_q = c^{ip}_m\, c^{mj}_q \tag{4.15}$$

for any set of fixed indices i, j, p, q. Now the theorem follows from above properties of the function F and the simple identity $\mathcal{L}_E e = -e$. The theorem is proved. □

5. Some examples

In this section we give some examples to illustrate the above construction of the Frobenius manifold structures. For the sake of simplicity of notations, instead of $t^1, \dots, t^{l+1}$ we will redenote the flat coordinates of the metric η^{ij} by $t_1, \dots, t_{l+1}$, and we will also denote $\partial_i = \frac{\partial}{\partial t_i}$ in the the following examples.

Example 5.1. $[C_3, k = 1]$ *Let R be the root system of type C_3, take $k = 1$, then $d_1 = d_2 = d_3 = 1$, and*

$$\begin{aligned} y^1 &= e^{2\,i\pi\,x_4}\left(\zeta_1 + \zeta_2 + \zeta_3\right); \\ y^2 &= e^{2\,i\pi\,x_4}\left(\zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_2\zeta_3\right); \\ y^3 &= e^{2\,i\pi\,x_4}\zeta_1\zeta_2\zeta_3; \\ y^4 &= 2\,i\pi\,x_4, \end{aligned}$$

where $\zeta_j = e^{2\,i\pi\,(x_j - x_{j-1})} + e^{-2\,i\pi\,(x_j - x_{j-1})} + 2$ and $x_0 = 0$, $j = 1, 2, 3$. The metric $(\ ,\ )^{\sim}$ has the form

$$((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The flat coordinates are

$$t_1 = y^1 - 2\,e^{y_4},\ t_2 = (y^2 - \frac{1}{6}\,y^3)(y^3)^{-\frac{1}{4}},\ t_3 = (y^3)^{\frac{1}{4}},\ t_4 = y^4$$

and the intersection form is given by

$$
\begin{aligned}
g^{11} &= 2\,t_2 t_3\, e^{t_4} + \frac{1}{3}\, {t_3}^4 e^{t_4} + 4\, e^{2t_4};\\
g^{12} &= \frac{7}{3}\, {t_3}^3 e^{t_4} + \frac{7}{2}\, t_2 e^{t_4};\;\; g^{13} = \frac{5}{2}\, t_3 e^{t_4};\;\; g^{14} = t_1;\\
g^{22} &= 12\, {t_3}^2 e^{t_4} - \frac{1}{4}\, {t_2}^2 + \frac{1}{12}\, {t_3}^3 t_2 - \frac{1}{108}\, {t_3}^6 + \frac{1}{4}\, \frac{{t_2}^3}{{t_3}^3};\\
g^{23} &= 2\, t_1 + 4\, e^{t_4} - \frac{1}{3}\, t_2 t_3 + \frac{1}{72}\, {t_3}^4 - \frac{1}{4}\, \frac{{t_2}^2}{{t_3}^2};\\
g^{24} &= \frac{3}{4}\, t_2;\;\; g^{33} = \frac{1}{4}\, \frac{t_2}{t_3} - \frac{1}{12}\, {t_3}^2;\;\; g^{34} = \frac{1}{4}\, t_3;\;\; g^{44} = 1.
\end{aligned}
$$

The potential has the expression

$$
\begin{aligned}
F = {}& \frac{1}{2}\, {t_1}^2 t_4 + \frac{1}{2}\, t_1 t_2 t_3 - \frac{1}{48}\, {t_2}^2 {t_3}^2 + \frac{1}{1440}\, t_2 {t_3}^5 - \frac{1}{36288}\, {t_3}^8\\
& + t_2 t_3 e^{t_4} + \frac{1}{6}\, {t_3}^4 e^{t_4} + \frac{1}{2}\, e^{2t_4} + \frac{1}{48}\, \frac{{t_2}^3}{t_3}
\end{aligned}
$$

and the Euler vector field is given by

$$
E = t_1 \partial_1 + \frac{3}{4} t_2 \partial_2 + \frac{1}{4} t_3 \partial_3 + \partial_4.
$$

Remark 5.2. *If we take $k = 2$ or 3 for the C_3 root system, we obtain a Frobenius manifold structure that is isomorphic to the one given in Example 2.7 [$B_3, k = 2$] or Example 2.8 [$C_3, k = 3$] of* [5].

Example 5.3. [$C_4, k = 1$] *Let R be the root system of type C_4. Take $k = 1$, then $d_1 = d_2 = d_3 = d_4 = 1$, and*

$$
\begin{aligned}
y^1 &= e^{2\, i\pi\, x_5} \left(\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4\right);\\
y^2 &= e^{2\, i\pi\, x_5} \sum_{1 \le a < b \le 4} \zeta_a \zeta_b;\\
y^3 &= e^{2\, i\pi\, x_5} \sum_{1 \le a < b < c \le 4} \zeta_a \zeta_b \zeta_c;\\
y^4 &= e^{2\, i\pi\, x_5} \zeta_1 \zeta_2 \zeta_3 \zeta_4;\\
y^5 &= 2\, i\pi\, x_5,
\end{aligned}
$$

where $\zeta_j = e^{2\,i\pi\,(x_j - x_{j-1})} + e^{-2\,i\pi\,(x_j-x_{j-1})} + 2$ *and* $x_0 = 0,\ j = 1, 2, 3, 4.$ *The metric* $(\ ,\ )^{\sim}$ *has the form*

$$((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 2 & 3 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

To write down the flat coordinates, we first introduce the variables

$$w_1 = y^1 - 2e^{y^5},\ w_2 = (y^2 - \frac{1}{6}\,y^3 + \frac{1}{30}\,y^4)(y^4)^{-\frac{1}{6}},$$
$$w_3 = (y^3 - \frac{1}{4}\,y^4)(y^4)^{-\frac{2}{3}},\quad w_4 = (y^4)^{\frac{1}{6}},\ w_5 = y^5.$$

Then we have

$$t_1 = w_1,\ t_2 = w_2 - \frac{1}{12} w_3^2\, w_4,\ t_3 = w_3 w_4,\ t_4 = w_4,\ t_5 = w_5.$$

We omit the presentation of the long expression of the intersection form and only write down the potential F *here*

$$\begin{aligned} F = {} & \frac{1}{2}\,{t_1}^2 t_5 + \frac{1}{2}\,t_1 t_2 t_4 - \frac{1}{6912}\,{t_3}^4 + \frac{1}{17280}\,{t_3}^3{t_4}^3 \\ & -\frac{1}{288}\,t_2 t_4 {t_3}^2 - \frac{1}{34560}\,{t_4}^6{t_3}^2 + \frac{1}{24}\,t_1{t_3}^2 + \frac{1}{1440}\,t_3{t_4}^4 t_2 \\ & -\frac{1}{48}\,{t_2}^2{t_4}^2 - \frac{1}{60480}\,{t_4}^7 t_2 + \frac{1}{345600}\,{t_4}^9 t_3 - \frac{1}{7603200}\,{t_4}^{12} \\ & +\frac{1}{12}\,e^{t_5}{t_3}^2 + \frac{1}{6}\,e^{t_5}t_3{t_4}^3 + \frac{1}{120}\,e^{t_5}{t_4}^6 + t_2 t_4 e^{t_5} + \frac{1}{2}\,e^{2t_5} \\ & +\frac{1}{24}\,\frac{t_3{t_2}^2}{t_4} - \frac{1}{216}\,\frac{t_2{t_3}^3}{{t_4}^2} + \frac{1}{4320}\,\frac{{t_3}^5}{{t_4}^3}. \end{aligned}$$

The Euler vector field is given by

$$E = t_1\partial_1 + \frac{5}{6}t_2\partial_2 + \frac{1}{2}t_3\partial_3 + \frac{1}{6}t_4\partial_4 + \partial_5.$$

Example 5.4. $[C_4, k=2]$ *Let R be the root system of type C_4. Take $k=2$, then $d_1=1, d_2=d_3=d_4=2$, and*

$$\begin{aligned}
y^1 &= e^{2\,i\pi\,x_5}\,(\zeta_1+\zeta_2+\zeta_3+\zeta_4)\,;\\
y^2 &= e^{4\,i\pi\,x_5}\sum_{1\le a<b\le 4}\zeta_a\zeta_b;\\
y^3 &= e^{4\,i\pi\,x_5}\sum_{1\le a<b<c\le 4}\zeta_a\zeta_b\zeta_c;\\
y^4 &= e^{4\,i\pi\,x_5}\zeta_1\zeta_2\zeta_3\zeta_4;\\
y^5 &= 2\,i\pi\,x_5,
\end{aligned}$$

where ζ_j are defined as in the above example. The metric $(\ ,\ )^{\sim}$ has the form

$$((dx_i, dx_j)^{\sim}) = \frac{1}{4\pi^2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0\\ 1 & 2 & 2 & 2 & 0\\ 1 & 2 & 3 & 3 & 0\\ 1 & 2 & 3 & 4 & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{2}\end{pmatrix}.$$

and the flat coordinates are given by

$$\begin{aligned}
&t_1 = y^1 - 4e^{y^5},\ t_2 = y^2 - 2y^1e^{y^5} + 6e^{2y^5},\ t_3 = (y^3 - \frac{1}{6}\,y^4)(y^4)^{-\frac{1}{4}},\\
&t_4 = (y^4)^{\frac{1}{4}},\ t_5 = y^5.
\end{aligned}$$

The intersection form has the following components

$$\begin{aligned}
&g^{11} = 2t_2 - \frac{1}{2}t_1^2 + 4e^{2t_5};\\
&g^{12} = 6t_1e^{2t_5} + \frac{1}{2}t_4^4e^{t_5} + \frac{1}{2}t_3t_4e^{t_5};\\
&g^{13} = 5\,t_3e^{t_5} + \frac{10}{3}\,t_4^3e^{t_5};\ g^{14} = 3\,t_4e^{t_5};\ g^{15} = \frac{1}{2}t_1;\\
&g^{22} = 2\,e^{t_5}t_1t_3t_4 + 8\,e^{4t_5} + 8\,t_3t_4e^{2t_5} + \frac{16}{3}\,e^{2t_5}{t_4}^4 + 4\,e^{2t_5}{t_1}^2 + \frac{1}{3}\,e^{t_5}{t_4}^4t_1;\\
&g^{23} = \frac{56}{3}\,e^{2t_5}{t_4}^3 + 7\,e^{2t_5}t_3 + \frac{7}{3}\,t_1{t_4}^3e^{t_5} + \frac{7}{2}\,e^{t_5}t_1t_3;\\
&g^{24} = 5\,e^{2t_5}t_4 + \frac{5}{2}\,t_4e^{t_5}t_1;\ g^{25} = t_2;
\end{aligned}$$

$$g^{33} = 12\,{t_4}^2 e^{t_5} t_1 + 48\,{t_4}^2 e^{2t_5} - \frac{1}{4}\,{t_3}^2 + \frac{1}{12}\,t_3 {t_4}^3 - \frac{1}{108}\,{t_4}^6 + \frac{1}{4}\,\frac{{t_3}^3}{{t_4}^3};$$

$$g^{34} = 2\,t_2 + 4\,e^{t_5} t_1 + 4\,e^{2t_5} - \frac{1}{3}\,t_4 t_3 + \frac{1}{72}\,{t_4}^4 - \frac{1}{4}\,\frac{{t_3}^2}{{t_4}^2};$$

$$g^{35} = \frac{3}{4}\,t_3;\ g^{44} = \frac{1}{4}\,\frac{t_3}{t_4} - \frac{1}{12}\,{t_4}^2;\ g^{45} = \frac{1}{4} t_4;\ g^{55} = \frac{1}{2}.$$

The Euler vector field is given by

$$E = \frac{1}{2} t_1 \partial_1 + t_2 \partial_2 + \frac{3}{4} t_3 \partial_3 + \frac{1}{4} t_4 \partial_4 + \frac{1}{2} \partial_5.$$

Finally, we have the potential

$$\begin{aligned} F = {} & \frac{1}{2}\,{t_2}^2 t_5 + \frac{1}{4}\,{t_1}^2 t_2 + \frac{1}{2}\,t_4 t_3 t_2 + \frac{1}{1440}\,{t_4}^5 t_3 - \frac{1}{48}\,{t_4}^2 {t_3}^2 \\ & - \frac{1}{36288}\,{t_4}^8 - \frac{1}{96}\,{t_1}^4 + \frac{1}{2}\,e^{2\,t_5} {t_1}^2 + \frac{1}{6}\,e^{t_5} t_1 {t_4}^4 + \frac{2}{3}\,{t_4}^4 e^{2\,t_5} \\ & + e^{t_5} t_1 t_3 t_4 + t_3 t_4 e^{2\,t_5} + \frac{1}{4}\,e^{4\,t_5} + \frac{1}{48}\,\frac{{t_3}^3}{t_4}. \end{aligned}$$

6. On the Frobenius manifold structures related to the root system of type B_l

For the root system R of type B_l, we also choose the standard base $\alpha_1, \ldots, \alpha_l$ of simple roots as given in [2]. As in [5] we define an indefinite metric $(\ ,\ )^{\sim}$ on $\widetilde{V} = V \oplus \mathbb{R}$ such that $\widetilde{V}$ is the orthogonal direct sum of V and $\mathbb{R}$, V endowed with the W-invariant Euclidean metric

$$(dx_m, dx_n)^{\sim} = \frac{1}{4\pi^2}[(1 - \frac{1}{2}\delta_{n,l})m - \frac{l}{4}\,\delta_{n,l}\delta_{m,l}], \quad 1 \le m \le n \le l \quad (6.1)$$

and $\mathbb{R}$ endowed with the metric

$$(dx_{l+1}, dx_{l+1})^{\sim} = -\frac{1}{4\pi^2 d_k}. \quad (6.2)$$

If the label k of the chosen simple root α_k is less than l, then the numbers d_j are given by

$$d_1 = 1, \ldots, d_k = k, \quad d_{k+1} = \cdots = d_{l-1} = k, \quad d_l = \frac{k}{2}. \quad (6.3)$$

In the case of $k = l$ we have

$$d_j = \frac{j}{2}, \quad j = 1, \ldots, l-1, \quad d_l = \frac{l}{4}. \quad (6.4)$$

The basis of the W_a-invariant Fourier polynomials $y_1(\mathbf{x}), \ldots, y_{l-1}(\mathbf{x}), y_l(\mathbf{x})$ are chosen in the same way as we did for the case of C_l

$$y_j(\mathbf{x}) = \sigma_j(\zeta_1, \cdots, \zeta_l), \quad j = 1, \ldots, l-1, \quad y_l(\mathbf{x}) = \zeta_1^{\frac{1}{2}} \ldots \zeta_l^{\frac{1}{2}} \tag{6.5}$$

with $\zeta_1, \ldots, \zeta_{l-1}$ defined by (2.7) and

$$\zeta_l = e^{2i\pi(x_{l-1}-2x_l)} + e^{-2i\pi(x_{l-1}-2x_l)} + 2. \tag{6.6}$$

So

$$\begin{aligned} \zeta_j^{\frac{1}{2}} &= 2\cos\pi(x_j - x_{j-1}), \quad j \le l-1 \\ \zeta_l^{\frac{1}{2}} &= 2\cos\pi(2x_l - x_{l-1}). \end{aligned}$$

The generators of the ring $\widetilde{W}^{(k)}(B_l)$ have the same form as that of (2.9). It is easy to see that the components of the resulting metric $(g^{ij}(y))$ coincide with those corresponding to the root system of type C_l if we perform the change of coordinates

$$y^j \mapsto \bar{y}^j = y^j, \ y_l(\mathbf{x}) \mapsto \bar{y}^l = (y^l)^2, \ y^{l+1} \mapsto \bar{y}^{l+1} = y^{l+1}, \ j = 1, \ldots, l-1,$$

for $1 \le k \le l-1$ and

$$y^j \mapsto \bar{y}^j = y^j, \ y_l(\mathbf{x}) \mapsto \bar{y}^l = (y^l)^2, \ y^{l+1} \mapsto \bar{y}^{l+1} = \frac{1}{2}y^{l+1}, \ j = 1, \ldots, l-1,$$

for the case when $k = l$. Thus, the Frobenius manifold structure that we obtain in this way from B_l, by fixing the k-th vertex of the corresponding Dynkin diagram, is isomorphic to the one that we obtain from C_l by choosing the k-th vertex of the Dynkin diagram of C_l.

7. Concluding remarks

It remains a challenging problem to understand whether the constructions of the present paper can be generalized to the root systems of the types D_l, E_6, E_7, E_8, F_4, G_2 with respect to the choice of an arbitrary vertex on the Dynkin diagram, as it was suggested in [10] motivating by the results of [11]. Another open problem is to obtain an explicit realization of the integrable hierarchies associated with the Frobenius manifolds of the type $W^{(k)}(R)$. So far this problem was solved only for $k = 1$, $R = A_1$, see [1, 6, 7] for details. We plan to study these problems in subsequent publications.

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