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WDVV Equations and Frobenius Manifolds

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Main Definition

WDVV equations of associativity (after E Witten, R Dijkgraaf, E Verlinde, and H Verlinde) is tantamount to the following problem: find a function $F(v)$ of n variables $v = (v^1, v^2, \ldots, v^n)$ satisfying the conditions [1], [3], and [4] given below. First,

$$\frac{\partial^3 F(v)}{\partial v^1 \partial v^\alpha \partial v^\beta} \equiv \eta_{\alpha\beta} \qquad [1]$$

must be a constant symmetric nondegenerate matrix. Denote $(\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}$ the inverse matrix and introduce the functions

$$c_{\alpha\beta}^{\gamma}(v) = \eta^{\gamma\epsilon} \frac{\partial^3 F(v)}{\partial v^\epsilon \partial v^\alpha \partial v^\beta}, \quad \alpha, \beta, \gamma = 1, \ldots, n \qquad [2]$$

The main condition says that, for arbitrary $v^1, \ldots, v^n$ these functions must be structure constants of an associative algebra, that is, introducing a v-dependent multiplication law in the n-dimensional space by

$$a \cdot b := \left(c_{\alpha\beta}^{1}(v) a^\alpha b^\beta, \ldots, c_{\alpha\beta}^{n}(v) a^\alpha b^\beta\right)$$

one obtains an n-parameter family of n-dimensional associative algebras (these algebras will automatically be also commutative). Spelling out this condition one obtains an overdetermined system of nonlinear PDEs for the function $F(v)$ often also called WDVV associativity equations

$$\frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(v)}{\partial v^\mu \partial v^\gamma \partial v^\delta} = \frac{\partial^3 F(v)}{\partial v^\delta \partial v^\beta \partial v^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(v)}{\partial v^\mu \partial v^\gamma \partial v^\alpha} \qquad [3]$$

for arbitrary $1 \le \alpha, \beta, \gamma, \delta \le n$. (Summation over repeated indices will always be assumed.) The last one is the so-called quasihomogeneity condition

$$EF = (3 - d)F + \tfrac{1}{2} A_{\alpha\beta} v^\alpha v^\beta + B_\alpha v^\alpha + C \qquad [4]$$

where

$$E = \left(a_\beta^\alpha v^\beta + b^\alpha\right) \frac{\partial}{\partial v^\alpha}$$

for some constants a_β^α, b^α satisfying

$$a_1^\alpha = \delta_1^\alpha, \qquad b^1 = 0$$

$A_{\alpha\beta}, B_\alpha, C, d$ are some constants. E is called Euler vector field and d is the charge of the Frobenius manifold.

For $n = 1$ one has $F(v) = (1/6)v^3$. For $n = 2$ one can choose

$$F(u, v) = \tfrac{1}{2} u v^2 + f(u)$$

only the quasihomogeneity [4] makes a constraint for $f(v)$. The first nontrivial case is for $n = 3$. The solution to WDVV is expressed in terms of a function $f = f(x, y)$ in one of the two forms (in the examples all indices are written as lower):

$$\begin{aligned} d \neq 0: \quad & F = \tfrac{1}{2} v_1^2 v_3 + \tfrac{1}{2} v_1 v_2^2 + f(v_2, v_3) \\ & f_{xxy}^2 = f_{yyy} + f_{xxx} f_{xyy} \\ d = 0: \quad & F = \tfrac{1}{6} v_1^3 + v_1 v_2 v_3 + f(v_2, v_3) \\ & f_{xxx} f_{yyy} - f_{xxy} f_{xyy} = 1 \end{aligned} \qquad [5]$$

The function $f(x, y)$ satisfies additional constraint imposed by [4]. Because of this the above PDEs [5] can be reduced (Dubrovin 1992, 1996) to a particular case of the Painlevé-VI equation (*see* Painlevé Equations).

The problem [1], [3], [4] is invariant with respect to linear changes of coordinates preserving the direction of the vector $\partial/\partial v^1$:

$$v^\alpha \mapsto \tilde{v}^\alpha = P_\beta^\alpha v^\beta + Q^\alpha, \ \det(P_\beta^\alpha) \neq 0, \quad P_1^\alpha = \delta_1^\alpha$$

It is also allowed to add to $F(v)$ a polynomial of the degree at most 2. To consider more general non-linear changes of coordinates one has to give a coordinate-free form of the above equations [1], [3], [4]. This gives rise to the notion of Frobenius manifold introduced in Dubrovin (1992).

Recall that a Frobenius algebra is a pair $(A, <,>)$, where A is a commutative associative algebra with a unity e over a field k (we will consider only the cases $k = \mathbb{R}, \mathbb{C}$) and $<,>$ is a k-bilinear symmetric nondegenerate invariant form on A, that is,

$$<x \cdot y, z> = <x, y \cdot z>$$

for arbitrary vectors x, y, z in A.

Definition Frobenius structure $(\cdot, e, <,>, E, d)$ on the manifold M is a structure of a Frobenius algebra on the tangent spaces $T_vM = (A_v, <,>_v)$ depending (smoothly, analytically, etc.) on the point $v \in M$. It must satisfy the following axioms.

FM1. The curvature of the metric $<,>_v$ on M (not necessarily positive definite) vanishes. Denote ∇ the Levi-Civita connection for the metric. The unity vector field e must be flat, $\nabla e = 0$.

FM2. Let c be the 3-tensor $c(x, y, z) := <x \cdot y, z>$, $x, y, z \in T_vM$. The 4-tensor $(\nabla_w c)(x, y, z)$ must be symmetric in $x, y, z, w \in T_vM$.

FM3. A linear vector field $E \in \mathrm{Vect}(M)$ (called Euler vector field) must be fixed on M, that is, $\nabla\nabla E = 0$, such that

$$\mathrm{Lie}_E(x \cdot y) - \mathrm{Lie}_E x \cdot y - x \cdot \mathrm{Lie}_E y = x \cdot y$$
$$\mathrm{Lie}_E <,> = (2 - d) <,>$$

for some number $d \in k$ called "charge."

The last condition (also called quasihomogeneity) means that the derivations $Q_{\mathrm{Func}(M)} := E$, $Q_{\mathrm{Vect}(M)} := \mathrm{id} + \mathrm{ad}_E$ define on the space $\mathrm{Vect}(M)$ of vector fields on M a structure of graded Frobenius algebra over the graded ring of functions $\mathrm{Func}(M)$.

Flatness of the metric $<,>$ implies local existence of a system of flat coordinates $v^1, \ldots, v^n$ on M. Usually, they are chosen in such a way that

$$e = \frac{\partial}{\partial v^1}$$

is the unity vector field. In such coordinates, the problem of local classification of Frobenius manifolds reduces to the WDVV associativity equations [1], [3], [4]. Namely, $\eta_{\alpha\beta}$ is the constant Gram matrix of the metric in these coordinates

$$\eta_{\alpha\beta} := \left\langle \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta} \right\rangle$$

The structure constants of the Frobenius algebra $A_v = T_vM$

$$\frac{\partial}{\partial v^\alpha} \cdot \frac{\partial}{\partial v^\beta} = c_{\alpha\beta}^\gamma(v) \frac{\partial}{\partial v^\gamma} \qquad [6]$$

can be locally represented by third derivatives [2] of a function $F(v)$ satisfying [1], [3], [4]. The function $F(v)$ is called "potential" of the Frobenius manifold. It is defined up to adding of an at most quadratic polynomial in $v^1, \ldots, v^n$.

A generalization of the above definition to the case of Frobenius supermanifolds can be found in Manin (1999). For the more general class of the so-called F-manifolds, the requirement of the existence of a flat invariant metric has been relaxed.

Deformed Flat Connection

One of the main geometrical structures of the theory of Frobenius manifolds is the deformed flat connection. This is a symmetric affine connection on $M \times \mathbb{C}^*$ defined by the following formulas:

$$\begin{aligned}
&\tilde\nabla_x y = \nabla_x y + z x \cdot y, \quad x, y \in TM, z \in \mathbf{C}^* \\
&\tilde\nabla_{\mathrm{d}/\mathrm{d}z} y = \partial_z y + E \cdot y - \frac{1}{z}\mathcal{V} y \\
&\tilde\nabla_x \frac{\mathrm{d}}{\mathrm{d}z} = \tilde\nabla_{\mathrm{d}/\mathrm{d}z} \frac{\mathrm{d}}{\mathrm{d}z} = 0
\end{aligned} \qquad [7]$$

where, as above, ∇ is the Levi-Civita connection for the metric $<,>$ and

$$\mathcal{V} := \frac{2-d}{2} - \nabla E \qquad [8]$$

is an operator on the tangent bundle TM antisymmetric with respect to $<,>$,

$$<\mathcal{V}x, y> = -<x, \mathcal{V}y>$$

Observe that the unity vector field e is an eigenvector of this operator with the eigenvalue

$$\mathcal{V}e = -\frac{d}{2} e$$

The connection $\tilde\nabla = \tilde\nabla(z)$ is not metric but it satisfies

$$\nabla <x, y> = <\tilde\nabla(-z)x, y> + <x, \tilde\nabla(z)y>$$
$$x, y \in TM$$

for any $z \in \mathbb{C}^*$. As it was discovered in Dubrovin (1992), vanishing of the curvature of the connection $\tilde\nabla$ is essentially equivalent to the axioms of Frobenius manifold.

Definition A "deformed flat function" $f(v;z)$ on a domain in $M \times \mathbb{C}^*$ is defined by the requirement of horizontality of the differential df

$$\tilde{\nabla} df = 0 \qquad [9]$$

Due to vanishing of the curvature of $\tilde{\nabla}$ locally there exist n independent deformed flat functions $f_1(v;z), \ldots, f_n(v;z)$ such that their differentials, together with the flat 1-form dz, span the cotangent plane $T^*_{(v;z)}(M \times \mathbb{C}^*)$. They will be called "deformed flat coordinates." The global analytic properties of deformed flat coordinates can be derived, for the case of semisimple Frobenius manifolds, from the results of the section "Moduli of semisimple Frobenius manifolds" discussed later.

One can relax the definition of Frobenius manifold dropping the last axiom FM3. The potential $F(v)$ in this case satisfies [1] and [3] but not [4]. In this case, the deformed flat connection $\tilde{\nabla}$ is just a family of affine flat connections on M depending on the parameter $z \in \mathbb{C}$ given by the first line in [7]. The curvature and torsion of this family of connections vanishes identically in z. The deformed flat functions of $\tilde{\nabla}$ defined as in [9] can be chosen in the form of power series in z. The flatness equations written in the flat coordinates on M yield a recursion equation for the coefficients of these power series

$$\tilde{\nabla} df = 0, \quad f = \sum_{p \geq 0} \theta_p(v) z^p$$
$$\partial_\lambda \partial_\mu f = z c^\nu_{\lambda\mu}(v) \partial_\nu f$$

$$\begin{aligned} &\partial_\lambda \partial_\mu \theta_0(v) = 0 \\ &\partial_\lambda \partial_\mu \theta_{p+1}(v) = c^\nu_{\lambda\mu}(v) \partial_\nu \theta_p(v) \end{aligned} \quad p \geq 0 \qquad [10]$$

Thus, $f(v;0)$ is just an affine linear function of the flat coordinates $v^1, \ldots, v^n$; the dependence on z can be considered as a deformation of the affine structure. This motivates the name "deformed flat coordinates." The coefficients of the expansions of the deformed flat coordinates are the leading terms of the ε-expansion of the Hamiltonian densities of the integrable hierarchies associated with the Frobenius manifolds (see below).

Intersection Form of a Frobenius Manifold

Another important geometric structure on M is the intersection form of the Frobenius manifold. It is a symmetric bilinear form on the cotangent bundle T^*M defined by the formula

$$(\omega_1, \omega_2) = i_E \omega_1 \cdot \omega_2, \quad \omega_1, \omega_2 \in T^*M \qquad [11]$$

Here the multiplication law on the cotangent planes is defined by means of the isomorphism.

$$<,>: TM \to T^*M$$

The discriminant $\Sigma \subset M$ is a proper analytic (for an analytic M) subset where the intersection form degenerates. One can introduce a new metric on the open subset $M \backslash \Sigma$ taking the inverse of the intersection form. A remarkable result of the theory of Frobenius manifolds is vanishing of the curvature of this new metric. Moreover, the new flat metric together with the following new multiplication:

$$x * y := x \cdot y \cdot E^{-1}$$

defines on $M \backslash \Sigma$ a structure of an almost-dual Frobenius manifold (Dubrovin 2004). In the original flat coordinates $v^1, \ldots, v^n$ the coordinate expressions for the new metric and for the associated Levi-Civita connection ∇^*, called the Gauss–Manin connection, read

$$\begin{aligned} g^{\alpha\beta}(v) &:= (dv^\alpha, dv^\beta) = E^\gamma(v) c^{\alpha\beta}_\gamma(v) \\ \nabla^{*\alpha}\, dv^\beta &= \Gamma^{\alpha\beta}_\gamma(v)\, dv^\gamma \\ \Gamma^{\alpha\beta}_\gamma(v) &:= -g^{\alpha\nu}(v)\Gamma^\beta_{\nu\gamma}(v) = c^{\alpha\epsilon}_\gamma(v)\left(\frac{1}{2} - \mathcal{V}\right)^\beta_\epsilon \end{aligned} \qquad [12]$$

The pair (,) and < , > of bilinear forms on T^*M possesses the following property crucial for understanding the relationships between Frobenius manifolds and integrable systems: they form a flat pencil. That means that on the complement to the subset

$$\Sigma_\lambda := \left\{ v \in M \,|\, \det\left(g^{\alpha\beta}(v) - \lambda \eta^{\alpha\beta}\right) = 0 \right\}$$

The inverse to the bilinear form

$$(,)_\lambda := (,) - \lambda <,> \qquad [13]$$

defines a metric with vanishing curvature. Flat functions $p = p(v;\lambda)$ for the flat metric are determined from the system

$$(\nabla^* - \lambda \nabla)\, dp = 0 \qquad [14]$$

They are called "periods" of the Frobenius manifold. The periods $p(v;\lambda)$ are related to the deformed flat functions $f(v;z)$ by the suitably regularized Laplace-type integral transform

$$p(v;\lambda) = \int_0^\infty e^{-\lambda z} f(v;z) \frac{dz}{\sqrt{z}} \qquad [15]$$

Choosing a system of n independent periods, one obtains a system of flat coordinates $p^1(v;\lambda), \ldots, p^n(v;\lambda)$ for the metric $(,)_\lambda$ on $M \backslash \Sigma_\lambda$,

$$\left(dp^i(v;\lambda), dp^j(v;\lambda)\right)_\lambda = G^{ij} \qquad [16]$$

for some constant nondegenerate matrix G^{ij}.