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Frobenius manifolds and central invariants for the Drinfeld–Sokolov bihamiltonian structures

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Abstract

The Drinfeld–Sokolov construction associates a hierarchy of bihamiltonian integrable systems with every untwisted affine Lie algebra. We compute the complete sets of invariants of the related bihamiltonian structures with respect to the group of Miura-type transformations.

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1. Introduction

The problem of classification of integrable systems of evolutionary PDEs

$$w_t^i = K^i(w; w_x, w_{xx}, \dots), \quad i = 1, \dots, n,$$
$$w = \left(w^1, \dots, w^n\right) \in M^n$$

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was studied by many mathematicians in the last 40 years with the help of various techniques; in such a general setup it remains essentially open, although there are already strong results for many particular subclasses of equations (see, for example, [46,52,38,53,22] and references therein).

Before starting the classification work one has to adopt a definition of complete integrability. For Hamiltonian PDEs

$$K^i(w; w_x, w_{xx}, \dots) = \{w^i(x), H\}, \quad i = 1, \dots, n,$$

with a suitable class of the Poisson brackets $\{\,,\}$ and the Hamiltonians H, one can define integrability, similarly to the finite dimensional case, by assuming existence of a complete family of commuting Hamiltonians (we do not explain here the notion of completeness, see e.g. in [22]). More specific is the class of *bihamiltonian* evolutionary PDEs admitting two different Hamiltonian descriptions

$$K^i(w; w_x, w_{xx}, \dots) = \{w^i(x), H_1\}_1 = \{w^i(x), H_2\}_2$$

with respect to a *compatible pair* of Poisson brackets (see below). Under certain genericity assumptions existence of a bihamiltonian representation ensures complete integrability (see details in [22,17]). Thus, the problem of classification of integrable PDEs reduces to the problem of classification of bihamiltonian structures of a suitable class. Even in this bihamiltonian framework the classification problem is still far from being resolved.

In [22,40,17] we proposed a kind of a perturbative approach to the classification problem considering the subclass of bihamiltonian PDEs admitting a (formal) expansion with respect to a small parameter ϵ

$$\begin{aligned} w_t^i &= A^i_j(w)w_x^j + \epsilon\big[B^i_j(w)w^j_{xx} + C^i_{jk}(w)w^j_x w^k_x\big] \\ &\quad + \epsilon^2\big[D^i_j(w)w^j_{xxx} + E^i_{jk}(w)w^j_x w^k_{xx} + F^i_{jkl}(w)w^j_x w^k_x w^l_x\big] + \cdots, \\ i &= 1, \dots, n \end{aligned} \tag{1.1}$$

(summation over repeated indices will be assumed). Such systems are to be classified with respect to a certain pronilpotent extension of the group of (local) diffeomorphisms of the manifold M^n that we called the *group of Miura-type transformations* (see Section 2 below). In this way we managed to produce a complete set of invariants of the bihamiltonian structures satisfying certain semisimplicity assumptions. The first part of these invariants is a differential-geometric object defined on the manifold M^n called *flat pencil of metrics*; it describes the bihamiltonian structure of the *hydrodynamic limit*

$$w^i_t = A^i_j(w)w^j_x \tag{1.2}$$

of the system (1.1). The second part comes from the deformation theory of these bihamiltonian structures of hydrodynamic type; it consists of n functions of one variable called the *central invariants* of the bihamiltonian structure. The main result of the papers [40,17] says that the flat pencil of metrics along with the collection of central invariants completely characterizes the equivalence class of a semisimple bihamiltonian structure with respect to the group of local Miura-type transformations (for the precise formulation see Theorem 2.12 below). In particular, the systems of bihamiltonian PDEs with all vanishing central invariants are equivalent to the hydrodynamic limit (1.2).

Apart from this trivial case no general results about *existence* of bihamiltonian structures and integrable hierarchies with a given pair

(flat pencil of metrics, collection of central invariants)

is available. The most studied is the class of the so-called *integrable hierarchies of topological type* motivated by the theory of Gromov–Witten invariants. For this class the Poisson pencil comes from a semisimple Frobenius structure on the manifold M^n; all the central invariants are constants equal to each other. Some partial existence results for integrable hierarchies of the topological type will appear elsewhere [23]. So, for the moment we have decided to review the list of known examples of bihamiltonian PDEs of the form (1.1) in the framework of our theory of flat pencils and central invariants.

First examples of such an analysis have been carried out in [40,17]. In the present paper we will consider the flat pencils of metrics and the central invariants for the bihamiltonian hierarchies constructed by V. Drinfeld and V. Sokolov in [13].

The Drinfeld–Sokolov's celebrated paper [13] gives a very simple construction, in terms of the Poisson reduction procedure, of a hierarchy of integrable PDEs associated with a Kac–Moody Lie algebra and a choice of a vertex on the extended Dynkin diagram. In this paper we will only consider the most well-known version of this construction for which the affine Lie algebra is untwisted and the chosen vertex of the Dynkin diagram is c_0 (the one added to the Dynkin diagram of the associated simple Lie algebra). In this case the hierarchy admits a bihamiltonian structure. The importance of this part of the Drinfeld–Sokolov construction became clear after the discovery, due to V. Fateev and S. Lukyanov [26], of the connection of the *second* Poisson structure for the Drinfeld–Sokolov hierarchy with the semiclassical limit $W_{\mathrm{cl}}(\mathfrak{g})$ of the Zamolodchikov's W-algebra [54] (see also [2]). Moreover, according to the conjecture of Drinfeld, proved by B. Feigin and E. Frenkel (see in [28,30]) the classical W-algebra $W_{\mathrm{cl}}(\mathfrak{g})$ arises naturally on the center of the universal enveloping algebra of the affine algebra $\hat{\mathfrak{g}}'$ of the Langlands dual Lie algebra $\mathfrak{g}'$ at the critical level.

In all these theories the *first* Poisson structure of Drinfeld and Sokolov seems to be something superfluous: in the standard definition the classical W-algebra is defined just as the second Poisson structure of Drinfeld and Sokolov. However, in the framework of our differential-geometric classification approach a single Poisson bracket has essentially no invariants: after extension to Miura-type transformations with complex coefficients any two local Poisson brackets of our class are equivalent [32]; see also [9,22].

The main result of this paper is the complete description of the flat pencils of metrics and computation of the central invariants for the Drinfeld–Sokolov bihamiltonian structures for all untwisted affine Lie algebras. We prove that the flat pencils of metrics are obtained from the Frobenius structures on the orbit spaces of the corresponding Weyl groups constructed by one of the authors in [14] via the theory of flat structures of K. Saito et al. [50,49]. The central invariants are proved to be all constants; they are identified with $\frac{1}{48}\times$ the square lengths, with respect to the invariant bilinear form used in the Drinfeld–Sokolov reduction procedure, of the generators in the Cartan subalgebra. In particular, this proves that the Drinfeld–Sokolov integrable hierarchies for the A, D and E series are equivalent, in the sense of Definition 2.3, to an integrable hierarchy of the topological type. The Drinfeld–Sokolov hierarchies associated with nonsimply laced Lie algebras do not belong to the topological type. Their tau-functions do not satisfy the topological recursion relations (see [25] for the B_2 case).

The plan of the paper is as follows: we first recall in the next section the definitions of the bihamiltonian structures, the associated flat pencils of metrics and central invariants. In Section 3 we remind the procedure of the Drinfeld–Sokolov reduction. In Section 4 we formulate the Main Theorem about invariants of the Drinfeld–Sokolov bihamiltonian structures. The proof of this theorem is given in Section 5 for the A_n hierarchies, in Section 6 for the B_n, C_n, D_n hierarchies, and in Section 7 for the hierarchies associated with the exceptional simple Lie algebras. In the final section we give some concluding remarks.

2. Central invariants of semisimple bihamiltonian structures

We study bihamiltonian structures of the following form

$$\{w^i(x), w^j(y)\}_a = \{w^i(x), w^j(y)\}_a^{[0]} + \sum_{k\geqslant 1}\epsilon^k \{w^i(x), w^j(y)\}_a^{[k]},$$

$$\{w^i(x), w^j(y)\}_a^{[k]} = \sum_{l=0}^{k+1} A_{k,l;a}^{ij}\big(w(x); w_x(x), \dots, w^{(l)}(x)\big)\delta^{(k-l+1)}(x-y) \tag{2.1}$$

where $i, j = 1, \dots, n$, $a = 1, 2$. Here $w = (w^1, \dots, w^n) \in M$ for some n-dimensional manifold M. The dependent variables $w^1, \dots, w^n$ will be considered as local coordinates on M. In this paper the manifold M will be assumed to be diffeomorphic to an open ball.

The coefficients $A_{k,l;a}^{ij}$ in (2.1) are homogeneous elements of degree l of the graded ring $\mathcal{B}$ of polynomial functions on the jet bundle of M

$$\mathcal{B} = \varinjlim_k \mathcal{B}_k, \qquad \mathcal{B}_k = \mathcal{C}^\infty(M)\big[w_x, w_{xx}, \dots, w^{(k)}\big], \quad \deg \partial_x^k w^i = k.$$

Antisymmetry and Jacobi identity for both brackets as well as the compatibility condition (see below) are understood as identities for formal power series in ϵ.

Under certain *nondegeneracy assumption* (see below) the leading terms of the Poisson brackets form a bihamiltonian structure of hydrodynamic type. The coefficients of this term will be redenoted as follows

$$\{w^i(x), w^j(y)\}_a^{[0]} = g_a^{ij}\big(w(x)\big)\delta'(x-y) + \Gamma_{k\ a}^{ij}\big(w(x)\big)w_x^k \delta(x-y), \quad a = 1, 2. \tag{2.2}$$

For every $a = 1, 2$ the map $\mathcal{B} \times \mathcal{B} \to \mathcal{B}[\![\epsilon]\!]$ given by the formula

$$(P, Q) \mapsto \frac{\delta P}{\delta w^i(x)} \Pi_a^{ij} \frac{\delta Q}{\delta w^j(x)}, \tag{2.3}$$

$$P = P\big(w; w_x, \dots, w^{(p)}\big), \qquad Q = Q\big(w; w_x, \dots, w^{(q)}\big) \in \mathcal{B},$$

$$\Pi_a^{ij} = g_a^{ij}(w)\partial_x + \Gamma_{k\ a}^{ij}(w)w_x^k + \sum_{k\geqslant 1}\epsilon^k \sum_{l=0}^{k+1} A_{k,l;a}^{ij}\big(w; w_x, \dots, w^{(l)}\big)\partial_x^{k-l+1}$$

induces a Lie algebra structure on the quotient space

$$\bar{\mathcal{B}} := \mathcal{B}[\![\epsilon]\!] / \operatorname{Im} \partial_x \tag{2.4}$$

where

$$\partial_x = \sum_k w^{i,k+1} \frac{\partial}{\partial w^{i,k}}, \quad w^{i,k} := \frac{\partial^k w^i}{\partial x^k}.$$

In the formula (2.3) summation over repeated indices i, j is assumed and

$$\frac{\delta}{\delta w^i(x)} = \frac{\partial}{\partial w^i} - \partial_x \frac{\partial}{\partial w^i_x} + \partial_x^2 \frac{\partial}{\partial w^i_{xx}} - \partial_x^3 \frac{\partial}{\partial w^i_{xxx}} + \cdots$$

is the Euler–Lagrange operator. The class of equivalence in the quotient space (2.4) of any element $P(w; w_x, \ldots; \epsilon) \in \mathcal{B}[\![\epsilon]\!]$ will be denoted by

$$\bar{P} := \int P(w; w_x, \ldots; \epsilon)\, dx \in \bar{\mathcal{B}}$$

and called a *local functional*. According to the above construction the Poisson bracket of two local functionals

$$\bar{P} = \int P(w; w_x, \ldots; \epsilon)\, dx, \qquad \bar{Q} = \int Q(w; w_x, \ldots; \epsilon)\, dx$$

is a local functional given by

$$\{\bar{P}, \bar{Q}\} = \int \frac{\delta P}{\delta w^i(x)} \Pi^{ij} \frac{\delta Q}{\delta w^j(x)}\, dx.$$

Here $\Pi = \Pi_1$ or $\Pi = \Pi_2$. Observe that, if P and Q are two homogeneous differential polynomials of degrees p and q respectively then their bracket (2.3) will be a homogeneous element of the ring $\mathcal{B}[\![\epsilon]\!]$ of formal power series in ϵ of degree $p + q + 1$ if the degree $\deg \epsilon = -1$ is assigned to the indeterminate ϵ. So, for an arbitrary local functional of the degree zero

$$H = \int \sum_{k \geqslant 0} \epsilon^k P_k\big(w; w_x, w_{xx}, \ldots, w^{(k)}\big)\, dx, \quad \deg P_k\big(w; w_x, w_{xx}, \ldots, w^{(k)}\big) = k,$$

the Hamiltonian vector field

$$w^i_t = \big\{w^i(x), H\big\} = \Pi^{ij} \frac{\delta P}{\delta w^j(x)}$$

is a system of evolutionary PDEs of the form (1.1) for any of the two Poisson structures $\Pi^{ij} = \Pi_1^{ij}$ or $\Pi^{ij} = \Pi_2^{ij}$.

By the definition of a bihamiltonian structure, any linear combination with constant coefficients of the two Poisson brackets must be again a Poisson bracket on $\bar{\mathcal{B}}$ (the so-called *compatibility* condition). Due to this property an infinite hierarchy of pairwise commuting systems of PDEs of the form (1.1) can be associated with the bihamiltonian structure (see details in [22]).

In the dispersionless limit $\epsilon \to 0$ Eqs. (1.1) become a system of the first order quasilinear PDEs (1.2). The leading term (2.2) gives a bihamiltonian structure of (1.2). The bihamiltonian structures (2.1) will be considered up to invertible linear transformations with constant coefficients

$$\begin{aligned}
\{\,,\}_1 &\mapsto \kappa_{11}\{\,,\}_1 + \kappa_{12}\{\,,\}_2, \\
\{\,,\}_2 &\mapsto \kappa_{21}\{\,,\}_1 + \kappa_{22}\{\,,\}_2, \\
&\kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} \neq 0.
\end{aligned} \tag{2.5}$$

The dependence of the associated integrable hierarchy on the changes (2.5) is nontrivial; it simplifies if one allows only triangular transformations

$$\begin{aligned}
\{\,,\}_1 &\mapsto \kappa_{11}\{\,,\}_1 \\
\{\,,\}_2 &\mapsto \kappa_{21}\{\,,\}_1 + \kappa_{22}\{\,,\}_2, \\
&\kappa_{11}\kappa_{22} \neq 0.
\end{aligned} \tag{2.6}$$

Definition 2.1. A compatible pair of Poisson brackets (2.1) considered modulo triangular transformations (2.6) is called a Poisson pencil.

The antisymmetry of the Poisson brackets (2.1) gives a system of linear differential constraints for the coefficients. They can be written in a compact form

$$\Pi_a^{ji} = -\left(\Pi_a^{ij}\right)^\dagger, \quad a = 1, 2. \tag{2.7}$$

Here the (formally) adjoint to a scalar differential operator

$$L = \sum_k A_k(x)\partial_x^k$$

is defined by

$$L^\dagger = \sum_k (-\partial_x)^k A_k(x). \tag{2.8}$$

The validity of the Jacobi identity for the pencil of Poisson brackets imposes a system of highly nontrivial nonlinear differential equations for the coefficients.

We will now introduce an equivalence relation in order to formulate the classification problem.

Definition 2.2. A Miura-type transformation is a change of variables of the form

$$w^i \mapsto \tilde{w}^i(w; w_x, w_{xx}, \dots; \epsilon) = F_0^i(w) + \sum_{k\geqslant 1} \epsilon^k F_k^i\left(w; w_x, \dots, w^{(k)}\right) \tag{2.9}$$

where $F_k^i \in \mathcal{B}$ with $\deg F_k^i = k$, and the map $w \mapsto F_0^i(w)$ is a diffeomorphism of M.

All Miura-type transformations form a group $\mathcal{G}(M)$. It acts by automorphisms on the graded ring $\mathcal{B}[[\epsilon]]$. This action commutes with the action of the operator of total x-derivative ∂_x. Therefore the action of the group $\mathcal{G}(M)$ on the Poisson brackets of the form (2.1) is defined. The explicit formula

$$\tilde{\Pi}_a^{kl} = L_i^k \Pi_a^{ij} L_j^{l\,\dagger}, \quad a = 1, 2, \tag{2.10}$$

involves the operator of linearization of (2.9)

$$L_j^i = \sum_m \frac{\partial \tilde{w}^i(w; w_x, w_{xx}, \dots; \epsilon)}{\partial w^{j,m}} \partial_x^m \tag{2.11}$$

and the adjoint operator $L_j^{i\,\dagger}$ (see (2.8)).

Definition 2.3. Two Poisson pencils of the form (2.1) are called equivalent if one can be transformed to another by a combination of a Miura-type transformation and a linear change (2.6).

We will now describe the complete set of invariants of bihamiltonian structures (2.1) under certain nondegeneracy assumptions. Observe first that, at the leading order $\epsilon = 0$ one obtains from (2.10) the tensor law, with respect to the coordinate change $w^i \mapsto F_0^i(w)$, for the $(2, 0)$ symmetric tensors $g_a^{ij}(w)$. So the *nondegeneracy assumption*

$$\det\bigl(g_a^{ij}(w)\bigr) \neq 0 \quad \text{for a generic } w \in M,\ a = 1, 2, \tag{2.12}$$

does not depend on the choice of a coordinate system.

Due to (2.7) each tensor $g_a^{ij}(w)$ for $a = 1, 2$ is symmetric. It defines therefore a symmetric nondegenerate bilinear form on the cotangent bundle T^*M. We will call it a *metric*. The coefficients $\Gamma_{k\ a}^{ij}(w)$ in (2.2) are expressed via the Christoffel coefficients of the metric:

$$\Gamma_k^{ij} = -g^{is}\Gamma_{sk}^j. \tag{2.13}$$

Recall [19] that validity of the Jacobi identity implies vanishing of the curvature of the metric. Thus the signature of the metric $g^{ij}(w)$ is the only local (i.e., $M = B^n =$ a small ball in $\mathbb{R}^n$) invariant of a *single* Poisson bracket of hydrodynamic type (2.2) with respect to the group of local diffeomorphisms. Moreover [32,9], the signature of the metric is the only local invariant of a single Poisson bracket of the form (2.1) with respect to the group $\mathcal{G}(B^n)$. The theory of invariants of bihamiltonian structure is richer.

We will impose in sequel a somewhat stronger assumption for the pair of metrics.

Definition 2.4. We say that the pair of metrics is strongly nondegenerate if for any $\lambda \in \mathbb{C}$ the symmetric matrix $(g_2^{ij}(w) - \lambda g_1^{ij}(w))$ does not degenerate for generic $w \in M$.

In particular the strong nondegeneracy assumption implies that all the roots $\lambda = \lambda(w)$ of the characteristic equation

$$\det\big(g_2^{ij}(w) - \lambda g_1^{ij}(w)\big) = 0 \tag{2.14}$$

are nonconstant functions on M.

We will now add one more assumption requiring that the roots of the characteristic equation (2.14) are pairwise distinct at a generic point of M.

Definition 2.5. A Poisson pencil (2.1) is called *semisimple* at the point $w_0 \in M$ if the roots $\lambda = u^1(w_0), \dots, \lambda = u^n(w_0)$ of the characteristic equation (2.14) are pairwise distinct.

The following statement was proved in [29] (see also [17]).

Lemma 2.6. *Given a strongly nondegenerate pair of metrics $g_a^{ij}(w)$, $a = 1, 2$, satisfying the semisimplicity assumption at a point $w_0 \in M$, then the roots $\lambda = u^1(w), \dots, \lambda = u^n(w)$ of the characteristic equation* (2.14) *define a system of local coordinates near w_0. They are called the* canonical coordinates *of the pencil. In the canonical coordinates the two metrics diagonalize*:

$$g_1^{ij}(u) = f^i(u)\delta_{ij}, \qquad g_2^{ij}(u) = u^i f^i(u)\delta_{ij}, \quad i, j = 1, \dots, n, \tag{2.15}$$

for some functions $f^1(u), \dots, f^n(u)$, $u = (u^1, \dots, u^n) \in M$.

Definition 2.7. A bihamiltonian structure of the form (2.1) is called semisimple if it is semisimple at generic points of M, and the associated pair of metrics (g_1^{ij}, g_2^{ij}) is strongly nondegenerate.

We are now ready to construct invariants of a Poisson pencil with respect to the action of the group of Miura-type transformations. Given a Poisson pencil of the form (2.1) define two matrix-valued formal power series in an indeterminate p with coefficients depending on $w \in M$:

$$\pi_a^{ij}(p; w) := \sum_{k=0}^{\infty} A_{k,0;a}^{ij}(w) p^k, \quad a = 1, 2. \tag{2.16}$$

Recall that $A_{0,0;a}^{ij} = g_a^{ij}$. The antisymmetry (2.7) implies

$$\pi_a^{ji}(-p; w) = \pi_a^{ij}(p; w). \tag{2.17}$$

Lemma 2.8. *Under Miura-type transformations* (2.9) *the matrices* (2.16) *transform in the following way*

$$\tilde\pi_a^{kl}(p; w) = \ell_i^k(p; w)\pi_a^{ij}(p; w)\ell_j^l(-p; w), \quad a = 1, 2, \tag{2.18}$$

where the formal series $\ell_j^i(p; w)$ are defined by the following formula

$$\ell_j^i(p; w) = \sum_{k=0}^{\infty} \frac{\partial F_k^i(w; w_x, \dots, w^{(k)})}{\partial w^{j,k}} p^k. \tag{2.19}$$

Note that, according to the grading rules $\deg F^i_k(w; w_x, \ldots, w^{(k)}) = k$ the coefficients of the series (2.19) are jet-independent.

Proof of the lemma readily follows from the transformation rule (2.10), (2.11).

Denote

$$\mathcal{R}(p, \lambda; w) := \det\bigl(\pi_2^{ij}(p; w) - \lambda \pi_1^{ij}(p; w)\bigr) \tag{2.20}$$

the characteristic polynomial of the pair of matrix-valued power series. According to the lemma the roots $\lambda^1(p; w), \ldots, \lambda^n(p; w)$ of the characteristic polynomial are invariant, up to a permutation, with respect to Miura-type transformations. In general these roots are algebraic functions of p depending on $w \in M$. We will now study the properties of these roots under an additional assumption of semisimplicity.

Lemma 2.9. *Let the pair of metrics (g_1^{ij}, g_2^{ij}) be semisimple at a point $w_0 \in M$, and $\lambda = u^1(w_0), \ldots, \lambda = u^n(w_0)$ be the roots of the characteristic equation*

$$\mathcal{R}(0, \lambda; w_0) = \det\bigl(g_2^{ij}(w_0) - \lambda g_1^{ij}(w_0)\bigr) = 0.$$

Then the roots $\lambda^1(p; w), \ldots, \lambda^n(p; w)$ of the characteristic equation

$$\mathcal{R}(p, \lambda; w) = 0 \tag{2.21}$$

for w sufficiently close to w_0 admit a formal power series expansion in p

$$\lambda^i(p; w) = u^i(w) + \sum_{k=1}^{\infty} \lambda^i_k(w) p^k, \quad i = 1, \ldots, n. \tag{2.22}$$

These power series, considered up to permutations and up to affine transformations

$$\lambda^i(p; w) \mapsto a\lambda^i(p; w) + b, \quad i = 1, \ldots, n,\ a \neq 0,$$

are invariants of the Poisson pencil with respect to the action of the group of Miura-type transformations.

Observe that, due to (2.17), all odd coefficients of the series (2.22) vanish.

In order to avoid inessential complications with signs, let us consider the complex situation assuming the manifold M to be complex analytic and all coefficients of the Poisson brackets and of the Miura-type transformations to be complex analytic functions in w. Then the complete set of local invariants of a semisimple bihamiltonian structures of the form (2.1) consists of

- flat pencil of metrics on M;
- collection of n functions of one variable called *central invariants*.

Flat pencil of metrics on M is, roughly speaking, a pair of (contravariant) metrics $g_1^{ij}(w)$, $g_2^{ij}(w)$ such that, at any point $w \in M$ their arbitrary linear combination

$$a_1 g_1^{ij}(w) + a_2 g_2^{ij}(w)$$

has zero curvature, and the contravariant Christoffel coefficients (2.13) for the above metric have the form of the same linear combination

$$a_1 \Gamma_{k\ 1}^{ij} + a_2 \Gamma_{k\ 2}^{ij}$$

(see details in [16]).

Remark 2.10. A flat pencil of metrics arises on an arbitrary Frobenius manifold according to the following construction [14,15]. Recall that an arbitrary Frobenius manifold is equipped with a flat metric $\langle\, ,\, \rangle$, a product of tangent vectors $(a, b) \mapsto a \cdot b$, and an Euler vector field E. We put

$$(\, ,\,)_1 := \langle\, ,\, \rangle \tag{2.23}$$

and define the second metric[1] on the cotangent bundle from the equation

$$(\omega_1, \omega_2)_2 = i_E \omega_1 \cdot \omega_2 \tag{2.24}$$

that must be valid for an arbitrary pair of 1-forms on the Frobenius manifold. In this formula the identification of tangent and cotangent spaces at every point is done by means of the *first* metric $(\, ,\,)_1$. By means of this identification one defines the product of 1-forms $\omega_1 \cdot \omega_2$ via the product of tangent vectors.

Similarly to Definition 2.3 we give

Definition 2.11. Two flat pencils are called (locally) equivalent if one can be transformed to another by a combination of a (local) diffeomorphism and a linear change (2.5).

The differential geometry problem of local classification of semisimple flat pencils reduces to an integrable system of differential equations (see [17] and references therein).

One thus arrives at the problem of local classification of semisimple bihamiltonian structures of the form (2.1), (2.2) with a *given* flat pencil of metrics (i.e., with the given leading term (2.2)). The theory of *central invariants* gives a parametrization of the infinitesimal deformation space of the bihamiltonian structure (2.2). We will not recall here the underlined cohomological theory [40,17]; we only give the definition and the computational formulae for the central invariants.

Given a semisimple Poisson pencil, consider the roots

$$\lambda^i(p; w) = u^i(w) + \lambda_2^i(w) p^2 + \mathcal{O}\big(p^4\big), \quad i = 1, \ldots, n,$$

of the characteristic equation (2.21). Denote

$$c_i(w) := \frac{1}{3} \frac{\lambda_2^i(w)}{f^i(w)}, \quad i = 1, \ldots, n \tag{2.25}$$

(the coefficient $1/3$ is chosen for a convenience of normalization).

[1] It also appeared in [1] under the guise of the operation of convolution of invariants of reflection groups.

Theorem 2.12. *(See [17].)*

(i) *Each function $c_i(w)$ defined in* (2.25) *depends only on $u^i(w)$,*

$$c_i(w) = c_i\big(u^i(w)\big), \quad i = 1, \dots, n.$$

(ii) *Two semisimple bihamiltonian structures of the form* (2.1) *with the same leading terms $\{\,,\}_a^{[0]}$, $a = 1, 2$, are equivalent* iff *they have the same set of central invariants $c_i(u^i)$, $i = 1, \dots, n$.*

Explicitly, the central invariants can be computed via the coefficients of the Poisson pencil according to the following formula. Denote $P_a^{ij}(u)$ (respectively $Q_a^{ij}(u)$) the components of the tensor $A_{1,0;a}^{ij}(w)$ (respectively $A_{2,0;a}^{ij}(w)$) in the canonical coordinates:

$$\pi_1^{ij}(p; u) = f^i(u)\delta^{ij} + P_1^{ij}(u)p + Q_1^{ij}(u)p^2 + \mathcal{O}\big(p^3\big),$$
$$\pi_2^{ij}(p; u) = u^i f^i(u)\delta^{ij} + P_2^{ij}(u)p + Q_2^{ij}(u)p^2 + \mathcal{O}\big(p^3\big).$$

By using the results of [17], one can find a transformation of the form (2.18) that reduces the two matrix-valued polynomials to the form

$$\tilde{\pi}_1^{ij}(p; u) = f^i(u)\delta^{ij} + \mathcal{O}\big(p^3\big),$$
$$\tilde{\pi}_2^{ij}(p; u) = u^i f^i(u)\delta^{ij} + \mu^i(u)\delta^{ij} p^2 + \mathcal{O}\big(p^3\big).$$

Clearly

$$\lambda_2^i(u) = \frac{\mu^i(u)}{f^i(u)}, \quad i = 1, \dots, n.$$

This yields the following expression for the ith ($i = 1, \dots, n$) central invariant of the semisimple bihamiltonian structure (2.1)

$$c_i\big(u^i\big) = \frac{1}{3(f^i(u))^2}\left[Q_2^{ii}(u) - u^i Q_1^{ii}(u) + \sum_{k\neq i} \frac{(P_2^{ki}(u) - u^i P_1^{ki}(u))^2}{f^k(u)(u^k - u^i)}\right]. \tag{2.26}$$

To compute the central invariants it is sometimes more convenient to use directly the tensor $A_{1,0;a}^{ij}(w)$ and $A_{2,0;a}^{ij}(w)$ in the original coordinates $w^1, \dots, w^n$. To do so, let us denote

$$\Psi(\lambda; w) = \det\big[g_2^{ij}(w) - \lambda g_1^{ij}(w)\big] \tag{2.27}$$

the characteristic polynomial of the pair of metrics g_2^{ij}, g_1^{ij}. In the canonical coordinates $u^1(w), \dots, u^n(w)$ both metrics become diagonal:

$$\sum_{k,l=1}^{n}\left(\frac{\partial\Psi(\lambda;w)}{\partial w^k}\right)_{\lambda=u^i}\left(\frac{\partial\Psi(\lambda;w)}{\partial w^l}\right)_{\lambda=u^j} g_1^{kl}(w)=0,\quad i\neq j,$$
$$\sum_{k,l=1}^{n}\left(\frac{\partial\Psi(\lambda;w)}{\partial w^k}\right)_{\lambda=u^i}\left(\frac{\partial\Psi(\lambda;w)}{\partial w^l}\right)_{\lambda=u^j} g_2^{kl}(w)=0,\quad i\neq j. \tag{2.28}$$

Here we used the implicit function theorem formula

$$\frac{\partial u^i(w)}{\partial w^k}=-\left(\frac{1}{\Psi'(\lambda;w)}\frac{\partial\Psi(\lambda;w)}{\partial w^k}\right)_{\lambda=u^i},\qquad \Psi'(\lambda;w)=\frac{\partial\Psi(\lambda;w)}{\partial\lambda}.$$

Then in the case when the linear in ϵ terms of the bihamiltonian structure (2.1) vanish, the central invariants defined by (2.26) have the following expressions:

$$c_i\left(u^i\right)=\frac{1}{3}\left[\Psi'\left(u^i;w\right)\right]^2\left.\frac{\sum_{k,l=1}^n\left(\frac{\partial\Psi(\lambda;w)}{\partial w^k}\right)\left(\frac{\partial\Psi(\lambda;w)}{\partial w^l}\right)\left(A_{2,0;2}^{kl}(w)-\lambda A_{2,0;1}^{kl}(w)\right)}{\left[\sum_{k,l=1}^n\left(\frac{\partial\Psi(\lambda;w)}{\partial w^k}\right)\left(\frac{\partial\Psi(\lambda;w)}{\partial w^l}\right)g_1^{kl}(w)\right]^2}\right|_{\lambda=u^i}. \tag{2.29}$$

For the general case we obtain a similar but a little bit lengthy formula.

Note that linear transformations (2.5) yield fractional linear transformations of the canonical coordinates

$$u^i\mapsto\frac{\kappa_{21}+u^i\kappa_{22}}{\kappa_{11}+u^i\kappa_{12}},\quad i=1,\dots,n.$$

The transformation law of central invariants is given by

$$c_i\mapsto\Delta^{-1}\left(\kappa_{11}+\kappa_{12}u^i\right)c_i,\quad i=1,\dots,n, \tag{2.30}$$

where $\Delta=\kappa_{11}\kappa_{22}-\kappa_{12}\kappa_{21}$. For the simultaneous rescalings

$$\{\,,\}_a\mapsto\kappa\{\,,\}_a,\quad a=1,2,$$

one has

$$c_i\mapsto\kappa^{-1}c_i,\quad i=1,\dots,n.$$

Observe that the central invariants do not change when rescaling only the *first* Poisson bracket without changing the second one. Because of this the central invariants of a Poisson pencil are well defined up to a common constant factor.

3. The Drinfeld–Sokolov reduction

Before explaining the Drinfeld–Sokolov procedure let us first recall the classical construction of the linear Poisson bracket on the dual space $\mathfrak{g}^*$ to a finite dimensional Lie algebra $\mathfrak{g}$ (the so-called *Lie–Poisson bracket*). It is uniquely defined by the following requirement: given two linear functions a, b on $\mathfrak{g}^*$, $a, b\in\mathfrak{g}$, their Poisson bracket coincides with the commutator in $\mathfrak{g}$:

$$\{a, b\} = [a, b]. \tag{3.1}$$

Choosing a basis in the Lie algebra

$$\mathfrak{g} = \mathrm{span}(e_1, \dots, e_N), \quad [e_i, e_j] = \sum_{k=1}^{N} c_{ij}^k e_k \tag{3.2}$$

one obtains the Lie–Poisson bracket in the associated dual system of coordinates $(x_1, \dots, x_N)$ on $\mathfrak{g}^*$ written in the following form

$$\{x_i, x_j\} = \sum_{k=1}^{N} c_{ij}^k x_k, \quad i, j = 1, \dots, N. \tag{3.3}$$

The Jacobi identity for the linear Poisson bracket (3.3) is equivalent to the Jacobi identity for the Lie algebra (3.2). The Poisson bivector (3.3) will be denoted

$$\pi_{\mathfrak{g}} \in \Lambda^2 T_x \mathfrak{g}^*. \tag{3.4}$$

Linear Hamiltonians

$$H_a(x) = \langle a, x \rangle, \quad a \in \mathfrak{g},\ x \in \mathfrak{g}^*,$$

generate the coadjoint action of the Lie group G associated with $\mathfrak{g}$:

$$\dot{x} = \{x, H_a\} \quad \Leftrightarrow \quad \big\langle b, x(t)\big\rangle = \big\langle e^{-t\,\mathrm{ad}\,a} b, x(0)\big\rangle \quad \text{for any } b \in \mathfrak{g}. \tag{3.5}$$

A simple generalization is given by linear inhomogeneous Poisson bracket

$$\{x_i, x_j\} = \sum_{k=1}^{N} c_{ij}^k x_k + c_{ij}^0. \tag{3.6}$$

It can be interpreted as the Lie–Poisson bracket on the one-dimensional central extension

$$0 \to \mathbb{C}\mathrm{k} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of the Lie algebra by means of the 2-cocycle

$$c^0(e_i, e_j) = c_{ij}^0, \qquad c^0\big([a, b], c\big) + c^0\big([c, a], b\big) + c^0\big([b, c], a\big) = 0. \tag{3.7}$$

Let us now recall the setting of the Marsden–Weinstein Hamiltonian reduction procedure [45,44,47]. Given a Poisson manifold $\mathcal{M}$, a family of Hamiltonians

$$H_1(x), \dots, H_N(x) \in \mathcal{C}^\infty(\mathcal{M})$$

forming an N-dimensional Lie subalgebra $\mathfrak{g}$ in $\mathcal{C}^\infty(\mathcal{M})$

$$\{H_i, H_j\} = \sum_{k=1}^{N} c_{ij}^k H_k(x), \quad c_{ij}^k = \text{const},$$

generates a Poisson action on $\mathcal{M}$ of the connected and simply connected Lie group G associated with $\mathfrak{g}$, assuming that any nontrivial linear combination of the generators $H_1, \dots, H_N$ is not a Casimir of the Poisson bracket on $\mathcal{M}$. The vector-valued function

$$\mathcal{P}(x) = \big(H_1(x), \dots, H_N(x)\big) \in \mathfrak{g}^*$$

is called the *moment map* for the Poisson action. The diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \mathcal{P}\ } & \mathfrak{g}^* \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}^* g} \\ \mathcal{M} & \xrightarrow[\ \mathcal{P}\ ]{} & \mathfrak{g}^* \end{array}$$

is commutative for any $g \in G$.

Given a Hamiltonian $H \in \mathcal{C}^\infty(\mathcal{M})$ invariant with respect to the action of the group G

$$\{H, H_i\} = 0, \quad i = 1, \dots, N,$$

the goal of the reduction procedure is to reduce the order of the Hamiltonian system

$$\dot{x} = \{x, H\} \tag{3.8}$$

i.e., to find a Poisson manifold $(\mathcal{M}^{\text{red}}, \{\,,\}_{\text{red}})$ of a lower dimension and a Hamiltonian $H_{\text{red}} \in \mathcal{C}^\infty(\mathcal{M}^{\text{red}})$ such that problem of integration of the Hamiltonian system (3.8) is reduced to the one for

$$\dot{y} = \{y, H_{\text{red}}\}_{\text{red}}, \quad y \in \mathcal{M}^{\text{red}}.$$

The construction of the reduced space can be given as follows.

Consider a smooth common level surface of the Hamiltonians

$$\mathcal{M}_h := \big\{x \in \mathcal{M} \bigm| H_1(x) = h_1,\ \dots,\ H_N(x) = h_N\big\} = \mathcal{P}^{-1}(h)$$

where

$$h = (h_1, \dots, h_N) \in \mathfrak{g}^*$$

is a regular value of the moment map. Denote $G_h \subset G$ the stabilizer of h with respect to the coadjoint action of G on $\mathfrak{g}^*$. The Lie algebra $\mathfrak{g}_h$ of the stabilizer is the kernel of the map

$$\pi_{\mathfrak{g}} : \mathfrak{g} \simeq T_h^* \mathfrak{g}^* \to T_h \mathfrak{g}^* \simeq \mathfrak{g}^* \tag{3.9}$$

where $\pi_{\mathfrak{g}}$ is the Poisson bivector (3.4) on $\mathfrak{g}^*$. The group G_h acts freely on $\mathcal{M}_h$. Assume this action to be free also on some neighborhood of $\mathcal{M}_h \subset \mathcal{M}$ and that the orbit space $\mathcal{M}_h/G_h$ has a structure of a smooth manifold. Define

$$\mathcal{M}_h^{\mathrm{red}} := \mathcal{M}_h/G_h.$$

We will give a construction of the reduced Poisson bracket on $\mathcal{M}_h^{\mathrm{red}}$ for the simplest case $G_h = G$. In this particular case the Poisson brackets of the generators all vanish on $\mathcal{M}_h$:

$$\{H_i, H_j\}|_{\mathcal{M}_h} = 0, \quad i, j = 1, \ldots, N.$$

Functions on $\mathcal{M}_h^{\mathrm{red}}$ can be identified with G-invariant functions on $\mathcal{M}_h$. For any two G-invariant functions α, β on $\mathcal{M}_h$ denote $\hat{\alpha}$, $\hat{\beta}$ arbitrary extensions of these two functions on a neighborhood of $\mathcal{M}_h$.

Definition 3.1. Under the above assumptions the Poisson bracket on the reduced space $\mathcal{M}_h^{\mathrm{red}} = \mathcal{M}_h/G$ defined by the formula

$$\{\alpha, \beta\}_{\mathrm{red}} := \{\hat{\alpha}, \hat{\beta}\}|_{\mathcal{M}_h} \tag{3.10}$$

is called the reduced Poisson bracket.

It is easy to see that the right-hand side of (3.10) is a G-invariant function on $\mathcal{M}_h$. Moreover, the definition does not depend on the extensions $\hat{\alpha}$, $\hat{\beta}$ of the G-invariant functions α, β.

We now proceed to outline the main steps of the Drinfeld–Sokolov reduction for the case of untwisted affine Lie algebras, the details can be found in [13].

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, G the associated connected and simply connected Lie group. Fix a nondegenerate symmetric invariant bilinear form $\langle , \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$. The central extension

$$0 \to \mathbb{C}\mathrm{k} \to \hat{\mathfrak{g}} \to L(\mathfrak{g}) \to 0$$

of the loop algebra $L(\mathfrak{g}) := C^\infty(S^1, \mathfrak{g})$ is defined as the direct sum of vector spaces $\hat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}\mathrm{k}$ equipped with the following Lie bracket

$$\big[q(x) + a\mathrm{k}, p(x) + b\mathrm{k}\big]_{\hat{\mathfrak{g}}} = \big[q(x), p(x)\big] + \omega(q, p)\mathrm{k}.$$

Here the 2-cocycle ω is defined by

$$\omega(q, p) = -\int_{S^1} \big\langle q(x), p'(x)\big\rangle_{\mathfrak{g}}\, dx. \tag{3.11}$$

Denote by $\hat{\mathfrak{g}}^*$ be the space of linear functionals on $\hat{\mathfrak{g}}$ of the following form

$$\ell_{q(x)+a\mathrm{k}}\big[p(x) + b\mathrm{k}\big] = \int_{S^1} \big\langle q(x), p(x)\big\rangle_{\mathfrak{g}}\, dx + ab,$$

where $q(x), p(x) \in L(\mathfrak{g})$, $a, b \in \mathbb{C}$. We identify $\hat{\mathfrak{g}}^*$ with $\hat{\mathfrak{g}}$. Let

$$\mathcal{M} = \{ q(x) + \epsilon \mathrm{k} \mid q(x) \in L(\mathfrak{g}) \} \subset \hat{\mathfrak{g}}^*$$

be the subspace of the linear functionals taking value ϵ at the central element k. Since the central element k is a Casimir w.r.t. the Lie–Poisson structure on $\hat{\mathfrak{g}}^*$, the space $\mathcal{M}$ also possesses a Poisson structure which is uniquely determined by the following condition: the Poisson bracket of two linear functionals

$$H_{a(x)}[q] = \int_{S^1} \langle a(x), q(x) \rangle_{\mathfrak{g}} \, dx, \qquad H_{b(x)}[q] = \int_{S^1} \langle b(x), q(x) \rangle_{\mathfrak{g}} \, dx$$
$$a(x), b(x) \in L(\mathfrak{g}) \tag{3.12}$$

coincides with the Lie bracket in $\hat{\mathfrak{g}}$:

$$\{H_{a(x)}, H_{b(x)}\} = H_{c(x)} + \epsilon \omega(a, b), \quad c(x) = [a(x), b(x)]. \tag{3.13}$$

Here we denote $H[q]$ the value of a functional H on $q(x) + \epsilon \mathrm{k} \in \mathcal{M}$ for brevity. Observe that the above Poisson bracket can also be written in the following form:

$$\begin{aligned} \{H_{a(x)}, H_{b(x)}\}[q] &= \int_{S^1} \left\langle a(x), \left[b(x), \epsilon \frac{d}{dx} + q(x) \right] \right\rangle_{\mathfrak{g}} dx \\ &= -\int_{S^1} \langle a(x), \epsilon b_x(x) + \mathrm{ad}_{q(x)} b(x) \rangle \, dx. \end{aligned} \tag{3.14}$$

Here $[b(x), \epsilon \frac{d}{dx}] = -\epsilon b_x(x)$. Since $\mathcal{M}$ is the level surface of a Casimir, the group $\hat{G} = \exp \hat{\mathfrak{g}}$ also acts on $\mathcal{M}$ as a Poisson action. The space $\mathcal{M}$ can be naturally identified with the following space of first order linear differential operators

$$\mathcal{M} = \left\{ \epsilon \frac{d}{dx} + q(x) \;\middle|\; q(x) \in L(\mathfrak{g}) \right\} \tag{3.15}$$

in such a way that the coadjoint action of $g = \exp(p(x) + b\mathrm{k}) \in \hat{G}$ on $\mathcal{M}$ is given by

$$\mathrm{Ad}^*_g : \epsilon \frac{d}{dx} + q(x) \mapsto \exp(\mathrm{ad}_{p(x)}) \left(\epsilon \frac{d}{dx} + q(x) \right). \tag{3.16}$$

Note that this action does not depend on the central element, so we can regard it as an action of the loop group $L(G) := C^\infty(S^1, G)$ on $\mathcal{M}$.

Remark 3.2. Given a basis $I^1, \dots, I^N$ in $\mathfrak{g}$ such that

$$[I^i, I^j] = c^{ij}_k I^k,$$
$$\langle I^i, I^j \rangle_{\mathfrak{g}} = g^{ij}$$

one obtains a system of coordinates

$$w^i = \langle I^i, \xi \rangle, \quad \xi \in \mathfrak{g}^*,\ i = 1, \dots, N, \tag{3.17}$$

on the dual space $\mathfrak{g}^*$. The Poisson bracket (3.14) can be written in the form

$$\{w^i(x), w^j(y)\} = c_k^{ij} w^k(x)\delta(x-y) - \epsilon g^{ij}\delta'(x-y). \tag{3.18}$$

After division by ϵ, this form of the Poisson bracket is similar to (2.1) but the ϵ-expansion begins with terms of order ϵ^{-1}. These terms will disappear after the reduction.

We choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and denote by Φ the root system corresponding to $\mathfrak{h}$. Let $\Delta = (\alpha_1, \dots, \alpha_n)$ be a base of Φ (where n is the rank of $\mathfrak{g}$), and Φ^+, Φ^- be the positive and negative root systems w.r.t. Δ, then we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- = \left(\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}\right). \tag{3.19}$$

Denote $\mathfrak{b} = \mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}$, $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$ the Borel subalgebras w.r.t. $\mathfrak{h}$, and $\mathfrak{n} = \mathfrak{n}^+$. Let $N \subset G$ be the subgroup of the Lie group G associated with the Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$. The Drinfeld–Sokolov construction can be interpreted as a Hamiltonian reduction procedure w.r.t. the action (3.16) of the loop group $L(N) = C^\infty(S^1, N)$ on $\mathcal{M}$.

By definition, the coadjoint action (3.16) of the subgroup $L(N)$ of $L(G)$ is generated by the linear Hamiltonians of the form $H_{v(x)}$, where $v(x) \in L(\mathfrak{n}) = C^\infty(S^1, \mathfrak{n})$. Therefore the moment map

$$\mathcal{P} : \mathcal{M} \to L(\mathfrak{n})^*$$

associated with the coadjoint action of $L(N)$ is given by the linear functional on $L(\mathfrak{n})$

$$\mathcal{P}\left(\epsilon\frac{d}{dx} + q(x)\right)[v(x)] = \int_{S^1} \langle q(x), v(x)\rangle\, dx. \tag{3.20}$$

Note that the orthogonal complement of $\mathfrak{n}$ w.r.t. the bilinear form $\langle,\rangle_{\mathfrak{g}}$ coincides with $\mathfrak{b}$, so one can identify the dual space $\mathfrak{n}^*$ with the quotient

$$\mathfrak{n}^* = \mathfrak{g}/\mathfrak{b} \simeq \mathfrak{n}^-. \tag{3.21}$$

Thus the moment map (3.20) can be identified with the direct sum projection of the $\mathfrak{g}$-valued function $q(x)$ onto the “lower triangular part”

$$\mathcal{P}\left(\epsilon\frac{d}{dx} + q(x)\right) = \pi^-(q(x)) \tag{3.22}$$

where $\pi^- : \mathfrak{g} \to \mathfrak{n}^-$ is the natural projection.

Now we choose a set of Weyl generators X_i, H_i, Y_i $(i = 1, \dots, n)$ w.r.t. the Cartan decomposition (3.19),

$$X_i \in \mathfrak{g}_{\alpha_i}, \qquad H_i = \alpha_i^\vee \in \mathfrak{h}, \qquad Y_i \in \mathfrak{g}_{-\alpha_i}.$$

Let

$$I = \sum_{i=1}^{n} Y_i \in \mathfrak{n}^- \tag{3.23}$$

be a *principal nilpotent element* (see [36]). Denote

$$\mathcal{M}^I := \mathcal{P}^{-1}(I) = \epsilon \frac{d}{dx} + I + L(\mathfrak{b}) \tag{3.24}$$

the level surface of $\mathcal{P}$ considering I as a constant map $S^1 \to \mathfrak{n}^-$.

From the commutation relations among the Weyl generators it follows that the element $I \in \mathfrak{n}^*$ is invariant with respect to the coadjoint action of $L(\mathfrak{n})$. Therefore the level surface $\mathcal{M}^I$ is invariant with respect to the gauge action of the group $L(N)$. By definition the functionals on the quotient $\mathcal{M}^I/L(N)$ are the gauge invariant functionals on $\mathcal{M}^I$. We will now construct a "system of coordinates" on this quotient space.

According to the theory of simple Lie algebras [36], the map

$$\mathrm{ad}_I : \mathfrak{n} \to \mathfrak{b}$$

is injective. We fix a subspace V of $\mathfrak{b}$ such that

$$\mathfrak{b} = V \oplus [I, \mathfrak{n}], \tag{3.25}$$

so $\dim V = \dim \mathfrak{b} - \dim \mathfrak{n} = n$.

Proposition 3.3. *The Hamiltonian action of the loop group $L(N)$ on $\mathcal{M}^I$ is free, namely, each orbit contains a unique operator of the form*

$$\epsilon \frac{d}{dx} + I + q^{\mathrm{can}}(x) \quad \text{with } q^{\mathrm{can}}(x) \in L(V) = C^\infty\big(S^1, V\big).$$

According to this result of [13] the reduced Poisson manifold can be identified with the space of operators written in the canonical form

$$\mathcal{M}^I/L(N) \simeq \left\{ \epsilon \frac{d}{dx} + I + q^{\mathrm{can}}(x) \;\middle|\; q^{\mathrm{can}}(x) \in L(V) \right\} \tag{3.26}$$

for the given choice of the subspace $V \subset \mathfrak{b}$ of the form (3.25). The functionals on the reduced space $\mathcal{M}^I/L(N)$ can be realized as functionals on $\mathcal{M}^I$ invariant with respect to the gauge action of $L(N)$, they will be called gauge invariant functionals for brevity. We now construct a bihamiltonian structure on the reduced space.

Let us first do the following trivial observation: given an element $\alpha \in \mathfrak{g}$, the formula

$$\{H_{a(x)}, H_{b(x)}\}_\lambda = \frac{1}{\epsilon}\int_{S^1}\left\langle a(x), \left[b(x), \epsilon\frac{d}{dx} + q(x) - \lambda\alpha\right]\right\rangle_{\mathfrak{g}} dx \tag{3.27}$$

(cf. (3.14)) defines a Poisson bracket on $\mathcal{M}$ for an arbitrary λ. Indeed, the translation $q(x) \mapsto q(x) - \lambda\alpha$ for any λ is a Poisson map for a linear Poisson bracket. We obtain thus a Poisson pencil on $\mathcal{M}$.

We now choose α to be a generator of the (one-dimensional) centre of the nilpotent subalgebra,

$$\alpha \in \mathfrak{n}, \quad [\alpha, \mathfrak{n}] = 0. \tag{3.28}$$

A main result of Drinfeld–Sokolov construction is the following proposition:

Proposition 3.4. *Given two gauge invariant functionals $\phi[q]$, $\psi[q]$ on $\mathcal{M}^I$, then for any of their extensions $\hat{\phi}[q]$, $\hat{\psi}[q]$ to $\mathcal{M}$, the functional obtained by restricting the Poisson bracket*

$$\{\hat{\phi}, \hat{\psi}\}_\lambda \tag{3.29}$$

to $\mathcal{M}^I$ is again a gauge invariant functional on $\mathcal{M}^I$.

According to this result, the projection of the Poisson pencil from $\mathcal{M}$ to the reduced space $\mathcal{M}^I/L(N)$ is again a Poisson pencil. In principle this completes the Drinfeld–Sokolov construction, although the explicit realization of the bihamiltonian structure on the reduced space strongly depends on the choice of the subspace V in (3.25). Changing the subspace yields a Miura-type transformation of the resulting bihamiltonian structure. The resulting bihamiltonian structure

$$\{\,,\}_\lambda = \{\,,\}_2 - \lambda\{\,,\}_1 \tag{3.30}$$

is called the Drinfeld–Sokolov bihamiltonian structure associated to the simple Lie algebra $\mathfrak{g}$. The commuting Hamiltonians of the associated integrable hierarchy can be constructed as (formal) spectral invariants of the differential operator

$$\epsilon\frac{d}{dx} + q^{\mathrm{can}}(x) + I - \lambda\alpha. \tag{3.31}$$

In the subsequent sections, we will recall the explicit representations, following [13], of the reduced space and also of the bihamiltonian structures associated to the simple Lie algebras of type A-B-C-D in terms of pseudo-differential operators.

Remark 3.5. There is an alternative approach to the Drinfeld–Sokolov reduction, due to P. Casati and M. Pedroni [8] based on the Marsden–Ratiu reduction. In a recent paper [7] this approach was applied to the G_2 hierarchy.

4. Formulation of main results

As the first result of the present paper, we will identify the dispersionless limit of the Drinfeld–Sokolov bihamiltonian structures with the canonical bihamiltonian structures defined on the jet spaces of the Frobenius manifolds—the orbit spaces of the Weyl groups. To this end, we need first to establish an isomorphism between the reduced manifolds $\mathcal{M}^I/L(N)$ that underline the Drinfeld–Sokolov bihamiltonian structures and the loop spaces of the orbit spaces of the Weyl groups.

Let $\mathfrak{g}$ be a simple Lie algebra. It admits a decomposition w.r.t. the principal gradation

$$\mathfrak{g}=\bigoplus_{1-h\leqslant j\leqslant h-1}\mathfrak{g}^j,\qquad \mathfrak{g}^j=\begin{cases}\mathfrak{h} & \text{if } j=0,\\ \bigoplus_{\alpha\in\Phi,\,\mathrm{ht}\alpha=j}\mathfrak{g}_\alpha & \text{if } j\neq 0,\end{cases}$$

where ht is the height function of roots, h is the Coxeter number of $\mathfrak{g}$ and other notations such as $\mathfrak{h}$ and Φ are defined as in Section 3.

We specify the choice of the complement of the subspace $[I,\mathfrak{n}]$ of $\mathfrak{b}$ that appears in (3.25) so that

$$V=\bigoplus_{j=0}^{h-1}V_j, \tag{4.1}$$

where the subspaces V_j satisfy

$$V_j\subset\mathfrak{b}_j=\mathfrak{b}\cap\mathfrak{g}^j,\qquad \mathfrak{b}_j=V_j\oplus[I,\mathfrak{b}_{j+1}]. \tag{4.2}$$

Note that V_j is not a null space if and only if j is one of the exponents

$$1=m_1\leqslant m_2\leqslant\cdots\leqslant m_n=h-1$$

of the simple Lie algebra $\mathfrak{g}$. For all simple Lie algebras except the ones of D_n type with even n the exponents have multiplicity one, i.e. $\dim V_{m_i}=1$ and the exponents are distinct. For the D_n (with even n) case, the exponents m_i for $i\neq\frac{n}{2},\frac{n}{2}+1$ have multiplicity one, $m_{\frac{n}{2}}=m_{\frac{n}{2}+1}=n-1$ and $\dim V_{n-1}=2$.

To choose a system of local coordinates of the reduced manifold $\mathcal{M}^I/L(N)$ of (3.26), we fix a canonical form

$$\epsilon\frac{d}{dx}+q^{\mathrm{can}}+I\in\mathcal{M}^I/L(N)$$

of the linear operator $\epsilon\frac{d}{dx}+q+I$ under the gauge action of $L(N)$ such that

$$q^{\mathrm{can}}=\sum_{i=1}^{n}w^i\gamma_i\in V. \tag{4.3}$$

Here for the exponent m_i with multiplicity one, γ_i is a basis of the one-dimensional subspace V_{m_i}; for the D_n case with even n, $\gamma_{\frac{n}{2}},\gamma_{\frac{n}{2}+1}$ is a basis of the 2-dimensional subspace V_{n-1}. Then $w^1,\dots,w^n$ form a coordinate system on the space $V\subset\mathfrak{n}$.

Remark 4.1. The subspace $V_{h-1} = \mathfrak{b}_{h-1}$ is determined uniquely since

$$\mathfrak{b}_j = 0 \quad \text{for } j \geqslant h.$$

Recall [6] that $\mathfrak{b}_{h-1}$ coincides with the (one-dimensional) centre of $\mathfrak{n}$. We will choose the basic vector $\gamma_n \in V_{h-1}$ as follows:

$$\gamma_n = \alpha \tag{4.4}$$

where α is the generator of the centre of $\mathfrak{n}$ has been chosen in (3.28) (see also (3.31)).

According to the results of Section 3 there exists a gauge transformation reducing the linear operator $\epsilon\frac{d}{dx} + q + I$ to the canonical form,

$$S^{-1}(x)\left(\epsilon\frac{d}{dx} + q + I\right)S(x) = \epsilon\frac{d}{dx} + q^{\text{can}} + I \tag{4.5}$$

where the function $S(x)$ takes values in the nilpotent group N. The canonical form q^{can} and the reducing gauge transformation $S(x)$ are determined uniquely from the following recursion procedure[2]

$$[I, S_{i+1}] - q_i^{\text{can}} = \sum_{j=1}^{i} S_j q_{i-j}^{\text{can}} - q_i - \sum_{j=1}^{i} q_{i-j} S_j - \epsilon\frac{dS_i}{dx}, \quad i \geqslant 0. \tag{4.6}$$

Here we use decomposition

$$S = 1 + S_1 + S_2 + \cdots \in L(N) \tag{4.7}$$

induced by the principal gradation of $\mathfrak{g}$ since the exponential map

$$\mathfrak{n} \to N$$

is a polynomial isomorphism. As it was proved in [13], the reducing transformation and the canonical form are uniquely determined from the recursion relation. Moreover, they are differential polynomials in q. In particular, the defined above coordinates $w^1, \ldots, w^n$ of q^{can} are certain differential polynomials

$$w^i = w^i\big(q; q_x, \ldots, q^{(h-1)}\big), \quad i = 1, \ldots, n. \tag{4.8}$$

We will now use these differential polynomials for defining a polynomial isomorphism of affine algebraic varieties

$$\mathfrak{h}/W \to V, \tag{4.9}$$

[2] Strictly speaking, the form we write (4.5) and the recursion relation (4.6) uses a matrix realization of the Lie algebra. See [13] for the formulation of the recursion procedure independent of the matrix realization.

where $W = W_{\mathfrak{g}}$ is the Weyl group of the simple Lie algebra $\mathfrak{g}$.

Restricting the differential polynomials $w^i(q; q_x, q_{xx}, \ldots)$ to the Cartan subalgebra

$$q = \xi = \sum_{i=1}^{n} \xi^i \alpha_i^\vee \in C^\infty\big(S^1, \mathfrak{h}\big)$$

we obtain differential polynomials

$$w^1(\xi; \xi_x, \xi_{xx}, \ldots), \ldots, w^n(\xi; \xi_x, \xi_{xx}, \ldots). \tag{4.10}$$

Define polynomial functions on $\mathfrak{h}$ by

$$y^i(\xi) = w^i(\xi; 0, 0, \ldots) \in \mathbb{C}[\mathfrak{h}^*]. \tag{4.11}$$

Lemma 4.2. *The functions $y^i(\xi)$ are W-invariant homogeneous polynomials of degree $m_i + 1$. Moreover, they generate the ring of W-invariant polynomials $\mathbb{C}[\mathfrak{h}^*]^W$.*

Proof. The restriction

$$F(q; q_x, q_{xx}, \ldots) \mapsto F(q; 0, 0, \ldots) =: f(q)$$

of any gauge invariant polynomial function on the differential operators of the form

$$\epsilon \frac{d}{dx} + q$$

yields a polynomial function on $\mathfrak{g}$ invariant w.r.t. adjoint action of the Lie group G. Further restriction onto the Cartan subalgebra establishes an isomorphism

$$S(\mathfrak{g})^G \to S(\mathfrak{h})^W = \mathbb{C}[\mathfrak{h}^*]^W$$

of the ring of Ad-invariant polynomial functions on $\mathfrak{g}$ and the ring of W-invariant polynomial functions on $\mathfrak{h}$, according to Chevalley theorem [6]. Furthermore, the homomorphism

$$S(\mathfrak{g})^G \to S(\mathfrak{b})^N$$

defined by the formula

$$f \mapsto f(I + q), \quad q \in \mathfrak{b},$$

is an isomorphism (see [37, Theorem 1.3]). Finally, according to Theorem 1.2 of [37] the adjoint action of the nilpotent group establishes an isomorphism of affine varieties

$$N \times (I + V) \to I + \mathfrak{b}.$$

Combining these statements we prove that the polynomials $y^1(\xi), \ldots, y^n(\xi)$ generate the ring $\mathbb{C}[\mathfrak{h}^*]^W$.

Now let us prove that $\deg y^i(\xi) = m_i + 1$. From the above definition, we know that these functions are determined by the following equation obtained from (4.5) by eliminating d/dx

$$e^{\mathrm{ad}_s}(q + I) = q^{\mathrm{can}} + I, \tag{4.12}$$

where $s \in \mathfrak{n}$, $q \in \mathfrak{b}$, $q^{\mathrm{can}} \in V$ have the decomposition

$$s = \sum_{k=1}^{h-1} s_i, \qquad q = \sum_{k=0}^{h-1} q_k, \qquad q^{\mathrm{can}} = \sum_{i=1}^{h-1} q_i^{\mathrm{can}} \tag{4.13}$$

with $s_k, q_k \in \mathfrak{b}_k$, $q_i^{\mathrm{can}} \in V_i$. Comparing the degree 0 parts of the left and right-hand sides of (4.12), we arrive at

$$\mathrm{ad}_I s_1 = q_0. \tag{4.14}$$

Since the map $\mathrm{ad}_I : \mathfrak{b}_1 \to \mathfrak{b}_0$ is an isomorphism, we have a unique s_1 satisfying the above equation. Restricting to $q_0 = \xi$ we see that s_1 depends linearly on ξ. Continuing this procedure by comparing the degree 1, degree 2 etc. parts, at the ith step we arrive at the equation of the form

$$\mathrm{ad}_I s_i + q_{i-1}^{\mathrm{can}} = F_i \tag{4.15}$$

where $F_i \in \mathfrak{b}_{i-1}$ is a homogeneous polynomial in ξ of degree i. If $i - 1$ is not an exponent, then the above equation has a unique solution with $q_{i-1}^{\mathrm{can}} = 0$ since the map

$$\mathrm{ad}_I : \mathfrak{b}_i \to \mathfrak{b}_{i-1} \tag{4.16}$$

is an isomorphism [36]. So s_i will be a homogeneous polynomial in ξ of degree i. In the case when $i - 1 = m_k$ is an exponent the map (4.16) is only injective. So the solution $s_i \in \mathfrak{b}_i$, $q_{i-1}^{\mathrm{can}} \in V_{i-1}$ of the above Eq. (4.15) exists and is determined uniquely. The degree of homogeneous polynomials $s_i(\xi)$ and $q_{i-1}^{\mathrm{can}}(\xi)$ is equal to

$$\deg s_i(\xi) = \deg q_{i-1}^{\mathrm{can}}(\xi) = \deg F_i(\xi) = i = m_k + 1.$$

Thus the function $y^k(\xi)$ (or $y^k(\xi), y^{k+1}(\xi)$) when m_k has multiplicity one (respectively has multiplicity two) is a homogeneous polynomial of degree $m_k + 1$. In this way we prove that $\deg y^i(\xi) = m_i + 1$ for any $i = 1, \ldots, n$. □

We obtained a ring isomorphism

$$\mathbb{C}[V^*] \to \mathbb{C}[\mathfrak{h}^*]^W.$$

Dualizing we obtain the isomorphism (4.9) of affine algebraic varieties. This induces the isomorphism

$$\left\{\begin{array}{l}\text{gauge invariant differential}\\ \text{polynomials } f(q; q_x, q_{xx}, \dots)\\ \text{on the space of differential}\\ \text{operators } \epsilon\frac{d}{dx}+q+I,\ q(x)\in\mathfrak{b}\end{array}\right\} \to \left\{\begin{array}{l}\text{differential polynomials}\\ \text{on the affine algebraic}\\ \text{variety } \mathfrak{h}/W\end{array}\right\}. \tag{4.17}$$

Recall [14] that the orbit space $M_{\mathfrak{g}} = \mathfrak{h}/W$ carries a natural structure of a polynomial Frobenius manifold. According to (4.17) the Hamiltonians of Drinfeld–Sokolov hierarchy can be realized as polynomial functions, considered modulo total x-derivatives, on the jet space of the Frobenius manifold. We want to compute the Drinfeld–Sokolov bihamiltonian structure in terms of $M_{\mathfrak{g}}$.

Theorem 4.3. *Under the isomorphism* (4.17)*, the Drinfeld–Sokolov bihamiltonian structure associated to an untwisted affine Lie algebra $\hat{\mathfrak{g}}$ is realized as a bihamiltonian structure on the jet space of $M_{\mathfrak{g}}$. Its dispersionless limit coincides with the bihamiltonian structure of hydrodynamic type naturally defined on the jet space of the Frobenius manifold by its flat pencil of metrics defined in Remark* 2.10*.*

Proof. Let us first remind the construction of the flat pencil of metrics on the orbit space $M_{\mathfrak{g}}$. Actually, the construction works uniformly for the orbit space of an arbitrary finite Coxeter group W (in our case $W = W_{\mathfrak{g}}$). For the chosen basis of simple roots $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ denote

$$G_{ab} = \langle \alpha_a^\vee, \alpha_b^\vee \rangle_{\mathfrak{g}}, \quad a, b = 1, \dots, n,$$

the Gram matrix of the invariant bilinear form. Let

$$\left(G^{ab}\right) = (G_{ab})^{-1} \tag{4.18}$$

be the inverse matrix. It gives a (constant) bilinear form on the cotangent bundle $T^*\mathfrak{h}$. The projection of the bilinear form onto the quotient $\mathfrak{h}/W$ defines a bilinear form on $T^*M_{\mathfrak{g}}$ nondegenerate outside the locus $\Delta \subset M_{\mathfrak{g}}$ of singular orbits (the so-called *discriminant* of the Coxeter group W). In order to represent this form in the coordinates let us choose the above constructed system of W-invariant homogeneous polynomials $y^1(\xi), \dots, y^n(\xi)$ generating the ring $\mathbb{C}[\mathfrak{h}^*]^W$. Here $\xi = \xi^a \alpha_a \in \mathfrak{h}$. The polynomial function

$$G^{ab} \frac{\partial y^i(\xi)}{\partial \xi^a} \frac{\partial y^j(\xi)}{\partial \xi^b}$$

is W-invariant for every $i, j = 1, \dots, n$ and, thus, is a polynomial in $y^1, \dots, y^n$. Denote $g_2^{ij}(y)$ these polynomials,

$$g_2^{ij}\left(y(\xi)\right) = G^{ab} \frac{\partial y^i(\xi)}{\partial \xi^a} \frac{\partial y^j(\xi)}{\partial \xi^b}. \tag{4.19}$$

This gives the Gram matrix of the *second* metric on $T^*M_{\mathfrak{g}}$ in the coordinates $y^1, \dots, y^n$. The associated contravariant Christoffel coefficients are polynomials $\Gamma_{k\ 2}^{ij}(y)$ defined from the equations

$$\Gamma_{k\ 2}^{ij}(y)\, dy^k = \frac{\partial y^i}{\partial \xi^a} G^{ab} \frac{\partial^2 y^j}{\partial \xi^b \partial \xi^c}\, d\xi^c. \tag{4.20}$$

To define the first metric, following [49,50], let us assume that the invariant polynomial $y^1(\xi)$ has the maximal degree

$$\deg y^1(\xi) = h.$$

Here h is the Coxeter number of the Lie algebra $\mathfrak{g}$. Put

$$g_1^{ij}(y) := \frac{\partial g_2^{ij}(y)}{\partial y^1}, \qquad \Gamma_{k\ 1}^{ij}(y) := \frac{\partial \Gamma_{k\ 2}^{ij}(y)}{\partial y^1}. \tag{4.21}$$

This is the first metric and the associated contravariant Christoffel coefficients of the flat pencil of metrics (2.23), (2.24) for the Frobenius structure on $M_{\mathfrak{g}}$. The second metric of the pencil depends only on the normalization of the invariant bilinear form. The first metric depends on the choice of the invariant polynomial $y^1(x)$ of the maximal degree. Changing this polynomial yields a rescaling of the first metric; the Frobenius structure will also be rescaled. This rescaling, however, does not change the central invariants (see the end of Section 2).

Define a Poisson bracket for two functionals φ, ψ on $C^\infty(S^1, \mathfrak{h})$ by the formula

$$\{\varphi, \psi\}[\xi] = \int_{S^1} \left\langle \frac{d}{dx} \operatorname{grad}_{\xi(x)}\varphi, \operatorname{grad}_{\xi(x)}\psi \right\rangle_{\mathfrak{g}} dx. \tag{4.22}$$

In terms of the coordinates $\xi^1(x), \dots, \xi^n(x)$, we have

$$\left\{\xi^i(x), \xi^j(y)\right\} = -G^{ij}\delta'(x-y), \quad i, j = 1, \dots, n, \tag{4.23}$$

where (G^{ij}) is defined in (4.18). Then as it is shown in [13], the Miura map

$$\mu : \left(\xi^1, \dots, \xi^n\right) \mapsto \left(w^1(\xi; \xi_x, \xi_{xx}, \dots), \dots, w^n(\xi; \xi_x, \xi_{xx}, \dots)\right)$$

is a Poisson map between $C^\infty(S^1, \mathfrak{h})$ and $\mathcal{M}^I/L(N)$ if the latter is endowed with the second Poisson bracket of the Drinfeld–Sokolov bihamiltonian structure (3.30).

From the above argument and (4.23), we see that the second metric (4.19) defined on the orbit space of W_g coincides, up to a minus sign, with the metric defined on $(I+\mathfrak{b})/N$ by the leading terms of the second Poisson bracket of the Drinfeld–Sokolov bihamiltonian structure associated with the untwisted affine Lie algebra $\hat{\mathfrak{g}}$.

The definition of the first Drinfeld–Sokolov Poisson bracket depends on the choice of the base element α of the one-dimensional center of the nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$, see (3.27), (3.28). We note that $\mathfrak{g}^{m_n} = \mathfrak{g}^{h-1}$ is just the center of $\mathfrak{n}$, so we can take $\gamma_n = \alpha$ in (4.3). Then in terms of the local coordinates $w^1(x), \dots, w^n(x)$ the first Drinfeld–Sokolov Poisson bracket is obtained from the second one by the shifting

$$w^n(x) \mapsto w^n(x) - \lambda, \qquad \partial_x^k w^n(x) \mapsto \partial_x^k w^n(x), \quad k \geqslant 1,$$

and

$$\{\,,\}_2 \mapsto \{\,,\}_2 - \lambda\{\,,\}_1.$$

Thus from the above results it follows the validity of Theorem 4.3. $\square$

Remark 4.4. Let us remind the algorithm of [14] of reconstruction of the Frobenius structure on the orbit space.[3] Let $v^1(\xi), \dots, v^n(\xi)$ be a system of *flat generators* of the ring of W-invariant polynomials in the sense of [49,50]. Geometrically they give a system of flat coordinates for the first metric:

$$\eta^{ij} := \left(dv^i, dv^j\right)_1 = \text{const.}$$

Put

$$g^{ij}(v) := \left(dv^i, dv^j\right)_2.$$

Then there exists an element $F(v)$ of the degree $2h+2$ in the ring of W-invariant polynomials such that

$$\eta^{ik}\eta^{jl}\frac{\partial^2 F(v)}{\partial v^k \partial v^l} = \frac{h}{\deg v^i + \deg v^j - 2} g^{ij}(v). \tag{4.24}$$

The third derivatives

$$c_{ij}^k(v) := \eta^{kl}\frac{\partial^3 F(v)}{\partial v^l \partial v^i \partial v^j}$$

are the structure constants of the multiplication on the tangent space $T_v M_{\mathfrak{g}}$.

Remark 4.5. Relationship of the generalized Drinfeld–Sokolov hierarchies with algebraic Frobenius manifolds is currently under investigation; first results have been obtained in [48,11].

Theorem 4.6. *The suitably ordered central invariants of the Drinfeld–Sokolov bihamiltonian structure for an untwisted affine Lie algebra $\hat{\mathfrak{g}}$ are given by the formula*

$$c_i = \frac{1}{48}\left\langle \alpha_i^\vee, \alpha_i^\vee \right\rangle_{\mathfrak{g}}, \quad i = 1, \dots, n, \tag{4.25}$$

where $\alpha_i^\vee \in \mathfrak{h}$ are the coroots of the simple Lie algebra $\mathfrak{g}$.

In the formula (4.25) we use the same invariant bilinear form as the one used in the definition of the Kac–Moody Lie algebra in Section 3.

If we fix on $\mathfrak{g}$ the so-called *normalized* invariant bilinear form (see [34, §6.2 and Exercise 6.2])

$$\langle a, b \rangle_{\mathfrak{g}} := \frac{1}{2h^\vee}\operatorname{tr}(\operatorname{ad} a \cdot \operatorname{ad} b), \tag{4.26}$$

[3] This construction was extended in [21,24] to the orbit spaces of certain extensions of affine Weyl groups, and in [5] to the orbit spaces of some Jacobi groups. More recently I. Satake [51] extended this construction to the orbit spaces of the reflection groups for elliptic root systems for the so-called case of codimension one.

here $h^\vee$ is the dual Coxeter number, then with the help of the table in §6.7 of [34] one obtains the following values of central invariants, according to Theorem 4.6:

$\mathfrak{g}$	c_1	$\dots$	c_{n-1}	c_n
A_n	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
B_n	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{12}$
C_n	$\frac{1}{12}$	$\dots$	$\frac{1}{12}$	$\frac{1}{24}$
D_n	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
$E_n,\ n=6,7,8$	$\frac{1}{24}$	$\dots$	$\frac{1}{24}$	$\frac{1}{24}$
$F_n,\ n=4$	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{12}$
$G_n,\ n=2$	$\frac{1}{8}$			$\frac{1}{24}$

The "breaking of symmetry" between the central invariants for the nonsimply laced Lie algebras has the following "experimental" explanation. Recall that the central invariants (2.26) are in one-to-one correspondence with the canonical coordinates on the Frobenius manifold, i.e., with the roots $u^1, \dots, u^n$ of the characteristic equation

$$\det\bigl(g_2^{ij}(w) - \lambda\, g_1^{ij}(w)\bigr) = 0. \tag{4.27}$$

It turns out that the characteristic polynomial factorizes in the product of two factors of the degrees p and q, $p+q=n$, where p is the number of long simple roots and q is the number of short simple roots. Such a splitting defines a partition of the set of central invariants in two subsets; the central invariants inside each of the subsets have the same value. For simply laced root systems the characteristic polynomial is irreducible. Recall that the map associating with the point w the collection of the coefficients of the characteristic polynomial (4.27) for the case of simply laced root systems coincides with the *Lyashko–Looijenga map* [42,43], see also [33].

The proof of Theorem 4.6 will be given in Section 5 for the A_n series, in Section 6 for the B_n, C_n, D_n series and in Section 7 for the exceptional cases.

5. The A_n case

We first recall the Drinfeld–Sokolov bihamiltonian structure related to the simple Lie algebra $\mathfrak{g}$ of A_n type. This Lie algebra has the matrix realization $sl(n+1,\mathbb{C})$. We denote by e_{ij} the matrix with 1 at the (i,j)th entry and 0 elsewhere. The Weyl generators of $\mathfrak{g}$ are chosen as

$$X_i = e_{i,i+1}, \qquad Y_i = e_{i+1,i}, \qquad H_i = e_{i,i} - e_{i+1,i+1}, \quad i = 1, \dots, n. \tag{5.1}$$

We use here the invariant bilinear form

$$\langle a, b\rangle_{\mathfrak{g}} = \operatorname{tr}(ab), \tag{5.2}$$

which coincides with the normalized invariant bilinear form (4.26) on $\mathfrak{g}$. The nilpotent subalgebra $\mathfrak{n}$, the Borel subalgebra $\mathfrak{b}$ and the group N are realized as

$$\mathfrak{n} = \{(a_{ij}) \in \mathrm{Mat}(n+1, \mathbb{C}) \mid a_{ij} = 0, \text{ for } i \geqslant j\},$$
$$\mathfrak{b} = \{(a_{ij}) \in \mathrm{Mat}(n+1, \mathbb{C}) \mid a_{ij} = 0, \text{ for } i > j\},$$
$$N = \{(s_{ij}) \in \mathrm{Mat}(n+1, \mathbb{C}) \mid s_{ij} = 0 \text{ for } i > j,\ s_{ii} = 1\}.$$

The element $I \in \mathfrak{g}$ that is introduced in (3.23) now has the expression $\sum_{i=1}^{n} e_{i+1,i}$. We choose the base element $\alpha \in \mathfrak{g}$ of the center of $\mathfrak{n}$, see (3.28), as

$$\alpha = -e_{1,n+1} \in \mathfrak{n}.$$

Let q be an element in $\hat{\mathfrak{b}}$,

$$q = \sum_{i=1}^{n} \sum_{j=i}^{n+1} q_{ij}(x) e_{ij} - \sum_{i=1}^{n} q_{ii}(x) e_{n+1,n+1}.$$

We can choose the coordinate q^{can} on the orbit space (3.26) as [13]

$$q^{\mathrm{can}} = -\big(w_1(x) e_{1,n+1} + w_2(x) e_{2,n+1} + \cdots + w_n(x) e_{n,n+1}\big),$$

where $w_k(x)$ are certain differential polynomials of q_{ij}. Here and henceforth we use lower indices for the variable w instead of upper ones as in (4.3) for the convenience of presentation of relevant formulae. Then the gauge invariant functionals take the following form

$$F = \int_{S^1} f\big(x, w(x), w_x(x), \ldots\big)\, dx. \tag{5.3}$$

The space of the gauge invariant functionals can be described in the following way [13]. Consider the operator

$$\mathcal{L} = \epsilon \frac{d}{dx} + q + I \tag{5.4}$$

as an $(n+1) \times (n+1)$ matrix with entries of differential operators. Let us represent it in the form

$$\mathcal{L} = \begin{pmatrix} \alpha & \beta \\ A & \gamma \end{pmatrix}. \tag{5.5}$$

Here A is an $n \times n$ matrix. We can associate to it a scalar differential operator

$$\Delta(\mathcal{L}) := \beta - \alpha A^{-1} \gamma. \tag{5.6}$$

Define

$$L = -\Delta(\mathcal{L})^{\dagger}, \tag{5.7}$$

where the conjugation of a differential operator is defined as in (2.8). It can be written in the form

$$L = D^{n+1} + w_n(x)D^{n-1} + \cdots + w_2(x)D + w_1(x), \quad D = \epsilon\frac{d}{dx}. \tag{5.8}$$

Gauge invariant functionals on $\mathcal{M}$ will be identified with functionals on the space of Lax operators (5.8). The variational derivative of a gauge invariant functional F w.r.t. L is defined as the following pseudo-differential operator

$$\frac{\delta F}{\delta L} = \sum_{i=1}^{n} D^{-i}\frac{\delta F}{\delta w_i}.$$

It is easy to verify the following identity

$$\delta F = \int \sum_{i=1}^{n} \frac{\delta F}{\delta w_i(x)}\delta w_i\,dx = \operatorname{Tr}\left(\frac{\delta F}{\delta L}\delta L\right) \tag{5.9}$$

where the linear functional Tr on pseudo-differential operators is defined by

$$\operatorname{Tr} A = \int \operatorname{res} A\,dx \in \bar{\mathcal{B}}$$

and the residue of a pseudo-differential operator has the definition

$$\operatorname{res}\left(\sum_{i\leqslant m} f_i D^i\right) = f_{-1}.$$

Recall that, due to the important property of the residue

$$\operatorname{res}(BA) = \operatorname{res}(AB) + \text{total } x\text{-derivative}, \tag{5.10}$$

the formula

$$\operatorname{Tr}(AB) = \int_{S^1} \operatorname{res}(AB)\,dx \in \bar{\mathcal{B}}$$

defines an invariant symmetric inner product between two pseudo-differential operators.

In terms of the gauge invariant functionals F, G, the Drinfeld–Sokolov bihamiltonian structure can be written as

$$\begin{aligned}\{F,G\}_\lambda &= \{F,G\}_2 - \lambda\{F,G\}_1\\ &= \frac{1}{\epsilon}\operatorname{Tr}\left((LY)_+LX - XL(YL)_+ + \frac{1}{n+1}X[L,g_Y]\right) - \lambda\frac{1}{\epsilon}\operatorname{Tr}\big([Y,X]L\big),\end{aligned} \tag{5.11}$$

where $X = \frac{\delta F}{\delta L}$, $Y = \frac{\delta G}{\delta L}$, and the positive part of a pseudo-differential operator $Z = \sum z_i D^i$ is defined by

$$Z_+ = \sum_{i\geqslant 0} z_i D^i.$$

The function g_Y is defined by

$$g_Y = D^{-1}\big(\mathrm{res}[L, Y]\big).$$

Due to (5.10), g_Y is a differential polynomial of the coefficients of the operators L, Y.

In the computation of Poisson brackets of our type it suffices to deal with the linear functionals

$$\ell_X = \int \sum_{i=1}^{n} a_i(x) w_i(x)\, dx, \qquad \ell_Y = \int \sum_{i=1}^{n} b_i(x) w_i(x)\, dx. \tag{5.12}$$

Then the operators $X = \delta\ell_X/\delta L$, $Y = \delta\ell_Y/\delta L$ read

$$X = \sum_{i=1}^{n} D^{-i} a_i(x), \qquad Y = \sum_{i=1}^{n} D^{-i} b_i(x). \tag{5.13}$$

For a pseudo-differential operator $Z = \sum_{i\leqslant m} z_i(x) D^i$, define its symbol as

$$\hat{Z}(x,p) = \sum_{i\leqslant m} z_i(x) p^i.$$

The symbol of the composition of two pseudo-differential operators can be computed by the following well-known formula[4]

$$\begin{aligned} \widehat{Z_1 Z_2}(x,p) &= \hat{Z}_1(x,p) \star \hat{Z}_2(x,p) := e^{\epsilon \frac{\partial^2}{\partial p \partial x'}} \hat{Z}_1(x,p)\hat{Z}_2(x',p')\big|_{x'=x, p'=p} \\ &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \partial_p^k \hat{Z}_1(x,p) \partial_x^k \hat{Z}_2(x,p). \end{aligned} \tag{5.14}$$

Taking the commutator in the leading term one obtains the Poisson bracket on the (x,p)-plane as follows

$$\begin{aligned} f(x,p) \star g(x,p) - g(x,p) \star f(x,p) &= \epsilon\{f,g\} + O\big(\epsilon^2\big), \\ \{f,g\} &:= \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x}. \end{aligned} \tag{5.15}$$

In the sequel we will often omit writing explicitly the x-dependence of the symbol.

The symbol of the positive part of a pseudo-differential operator can be computed by Cauchy integral formula

$$\widehat{Z_+}(p) = \big(\hat{Z}(p)\big)_+ = \oint \frac{dq}{2\pi i} \frac{\hat{Z}(q)}{q-p} \tag{5.16}$$

[4] Warning: we use here the symbol $\star$ that usually arises in the quantization of Poisson brackets. However our "star product" is different from the standard one.

where the integration is taken along the circle of radius $|q| > |p|$.

Let

$$\lambda(x, p) = p^{n+1} + w_n(x)p^{n-1} + \cdots + w_2(x)p + w_1(x) = \hat{L}$$

be the symbol of the Lax operator (5.8).

Theorem 5.1.

(i) *The dispersionless limit of the A_n Drinfeld–Sokolov bihamiltonian structure is given by the following formulae*

$$\{\lambda(x,p), \lambda(y,q)\}_1 = \frac{\lambda'(p) - \lambda'(q)}{p-q}\delta'(x-y) + \left[\frac{\lambda_x(p) - \lambda_x(q)}{(p-q)^2} - \frac{\lambda'_x(q)}{p-q}\right]\delta(x-y), \tag{5.17}$$

$$\begin{aligned}\{\lambda(x,p), \lambda(y,q)\}_2 = {} & \left(\frac{\lambda'(p)\lambda(q) - \lambda'(q)\lambda(p)}{p-q} + \frac{1}{n+1}\lambda'(p)\lambda'(q)\right)\delta'(x-y) \\ & + \left[\frac{\lambda_x(p)\lambda(q) - \lambda_x(q)\lambda(p)}{(p-q)^2} + \frac{\lambda_x(q)\lambda'(p) - \lambda'_x(q)\lambda(p)}{p-q}\right. \\ & \left. + \frac{1}{n+1}\lambda'(p)\lambda'_x(q)\right]\delta(x-y).\end{aligned} \tag{5.18}$$

(ii) *The central invariants of the bihamiltonian structure are equal to*

$$c_1 = c_2 = \cdots = c_n = \frac{1}{24}.$$

Before proceeding to the proof let us explain the notations in the formulae (5.17)–(5.18). In the left-hand sides we simply write the generating polynomials for the matrices $\{w_i(x), w_j(y)\}_{1,2}$ of Poisson brackets, i.e.,

$$\{\lambda(x,p), \lambda(y,q)\}_{1,2} = \sum_{i,j=1}^{n} \{w_i(x), w_j(y)\}_{1,2}\, p^{i-1}q^{j-1}.$$

In the right-hand sides we denote $\lambda(p) \equiv \lambda(x,p)$,

$$\lambda'(p) = \frac{\partial}{\partial p}\lambda(x,p), \qquad \lambda_x(p) = \partial_x\lambda(x,p).$$

Same for the terms depending on q, i.e. $\lambda(q) \equiv \lambda(x,q)$, $\lambda'(q) = \frac{\partial}{\partial q}\lambda(x,q)$ etc.

In particular the coefficients of $\delta'(x-y)$ in the formulae (5.17) and (5.18) give expressions for the generating functions of the pair of flat metrics on the orbit space $M_{\mathfrak{g}}$, $\mathfrak{g} = A_n$, described in Theorem 4.3, i.e.

$$\big(d\lambda(p), d\lambda(q)\big)_1 \equiv \sum_{i,j=1}^{n} (dw_i, dw_j)_1 p^{i-1} q^{j-1} = \frac{\lambda'(p) - \lambda'(q)}{p-q}, \tag{5.19}$$

$$\begin{aligned}\big(d\lambda(p), d\lambda(q)\big)_2 &\equiv \sum_{i,j=1}^{n} (dw_i, dw_j)_2 p^{i-1} q^{j-1} \\ &= \frac{\lambda'(p)\lambda(q) - \lambda'(q)\lambda(p)}{p-q} + \frac{1}{n+1}\lambda'(p)\lambda'(q).\end{aligned} \tag{5.20}$$

These formulae for the metrics were already found in [50] (observe that the sign of the second metric (the coefficients of $\delta'(x-y)$ of (5.18)) is opposite to the one given in Proposition 2.4.2 of [50]), although their relationships with the Drinfeld–Sokolov brackets were not discussed. Similarly, the generating functions for the Christoffel coefficients for the two metrics can be recovered from the $\delta(x-y)$ term in (5.17), (5.18). This generating function was found in [4] by a straightforward computation of the Christoffel coefficients.

Proof of Theorem 5.1. Let us introduce the symbols

$$f(p) = \sum_{i=1}^{n} \frac{a_i(x)}{p^i}, \qquad g(p) = \sum_{i=1}^{n} \frac{b_i(x)}{p^i}. \tag{5.21}$$

They are related to the symbols of the operators (5.13) via

$$\hat{X}(p) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \partial_p^k \partial_x^k f(p), \qquad \hat{Y}(p) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \partial_p^k \partial_x^k g(p). \tag{5.22}$$

We begin with the calculation of the leading term of the first Poisson bracket. Due to (5.15) one obtains

$$\{\ell_X, \ell_Y\}_1 = \int \operatorname{res}\big(\big\{g(x,p), f(x,p)\big\}\lambda(x,p)\big)\,dx + O(\epsilon).$$

Here res of a symbol is just the coefficient of p^{-1}. Integrating by parts one rewrites

$$\int \operatorname{res}\big(\{g, f\}\lambda\big)\,dx = \int \operatorname{res}\big(f\{\lambda, g\}\big)\,dx.$$

As the series f contains only negative powers of p, one can replace the series $\{\lambda, g\}$ by its positive part

$$\{\lambda, g\}_+ = \oint \frac{dq}{2\pi i} \frac{\lambda'(q) g_x(q) - \lambda_x(q) g'(q)}{q-p}.$$

Integrating by parts in q and inserting two zero terms

$$-\oint \frac{dq}{2\pi i} \frac{\lambda'(p)}{q-p} g_x(q) = 0, \qquad \oint \frac{dq}{2\pi i} \frac{\lambda_x(p)}{(q-p)^2} g(q) = 0$$

one obtains the following expression for the leading term of the first Poisson bracket

$$\{\ell_X, \ell_Y\}_1 = \int dx \oint \frac{dp}{2\pi i} \oint \frac{dq}{2\pi i} \biggl[f(p) \frac{\lambda'(p) - \lambda'(q)}{p-q} g_x(q) + \left(\frac{\lambda_x(p) - \lambda_x(q)}{(p-q)^2} - \frac{\lambda'_x(q)}{p-q} \right) f(p) g(q) \biggr] + O(\epsilon).$$

This gives the formula (5.17). Note that the rational functions

$$\frac{\lambda'(p) - \lambda'(q)}{p-q}$$

and

$$\frac{\lambda_x(p) - \lambda_x(q)}{(p-q)^2} - \frac{\lambda'_x(q)}{p-q}$$

have no singularity on the diagonal, so the order of the loop integrals is inessential.

A similar computation proves also the formula (5.18).

Let us proceed to computing the higher order corrections. Note that what we want to compute is just four tensors $P_a^{ij}(w)$, $Q_a^{ij}(w)$ ($a = 1, 2$) independent of the jet coordinates (see (2.26)). So through the computation we can omit all the derivatives of w_i w.r.t. x, i.e. we can treat w_i as constants. By using this assumption, one can obtain

$$g_Y = \oint \frac{dq}{2\pi i} \sum_{k=1}^{\infty} \frac{\epsilon^{k-1}}{k!} \partial_q^k \lambda(q) \partial_x^{k-1} \hat{Y}(q). \tag{5.23}$$

By substituting the formulae (5.14), (5.16), (5.22), (5.23) into the formula (5.11), we can obtain

$$\{\ell_X, \ell_Y\}_a = \int dx \oint \frac{dp}{2\pi i} \oint \frac{dq}{2\pi i} \times \sum_{k,i,s,j,t \geqslant 0} \partial_p^i \partial_x^s f(p) \tilde{A}_{a,k,i,s,j,t}(p,q,x) \epsilon^{k+s+t-1} \partial_q^j \partial_x^t g(q), \quad a = 1, 2.$$

After few integration by parts, the above equation reduces to the following one

$$\{\ell_X, \ell_Y\}_a = \int dx \oint \frac{dp}{2\pi i} \oint \frac{dq}{2\pi i} \sum_{k,s \geqslant 0} f(p) A_{a,k,s}(p,q,x) \epsilon^{k+s-1} \partial_x^s g(q). \tag{5.24}$$

We already know the coefficients

$$A_{1,0,1} = \frac{\lambda'(p) - \lambda'(q)}{p-q}$$

and

$$A_{2,0,1}=\frac{\lambda'(p)\lambda(q)-\lambda'(q)\lambda(p)}{p-q}+\frac{1}{n+1}\lambda'(p)\lambda'(q).$$

The subsequent coefficients $A_{a,0,2}$, $A_{a,0,3}$ $(a=1,2)$ read

$$\begin{aligned}
A_{1,0,2}&=\frac{\lambda'(q)-\lambda'(p)}{(q-p)^2}-\frac{\lambda''(q)+\lambda''(p)}{2(q-p)},\\
A_{1,0,3}&=\frac{\lambda'(q)-\lambda'(p)}{(q-p)^3}-\frac{\lambda''(q)+\lambda''(p)}{2(q-p)^2}+\frac{\lambda'''(q)-\lambda'''(p)}{6(q-p)},\\
A_{2,0,2}&=\frac{\lambda'(q)\lambda(p)-\lambda(q)\lambda'(p)}{(q-p)^2}-\frac{\lambda''(q)\lambda(p)-2\lambda'(q)\lambda'(p)+\lambda(q)\lambda''(p)}{2(q-p)}\\
&\quad-\frac{\lambda''(q)\lambda'(p)-\lambda'(q)\lambda''(p)}{2(n+1)},\\
A_{2,0,3}&=\frac{\lambda'(q)\lambda(p)-\lambda(q)\lambda'(p)}{(q-p)^3}-\frac{\lambda''(q)\lambda(p)-2\lambda'(q)\lambda'(p)+\lambda(q)\lambda''(p)}{2(q-p)^2}\\
&\quad+\frac{\lambda'''(q)\lambda(p)-3\lambda''(q)\lambda'(p)+3\lambda'(q)\lambda''(p)-\lambda(q)\lambda'''(p)}{6(q-p)}\\
&\quad+\frac{2\lambda'''(q)\lambda'(p)-3\lambda''(q)\lambda''(p)+2\lambda'(q)\lambda'''(p)}{12(n+1)}.
\end{aligned}\tag{5.25}$$

Now we introduce two complex numbers P, Q such that $|P|<|p|$, $|Q|<|q|$, and define the functions $f(p)$, $g(p)$ as

$$f(p)=\frac{1}{p-P}\delta(x-y)=\sum_{i=1}^{\infty}\frac{P^{i-1}}{p^i}\delta(x-y),\qquad g(p)=\frac{1}{q-Q}\delta(x-z).$$

Here, unlike the form given in (5.21), we allow the symbols $f(p)$, $g(p)$ to contain terms of the form $\frac{1}{p^i}$ with $i>n$. However, it is easy to see that these additional terms do not affect the Poisson bracket (5.24).

It follows then that

$$\begin{aligned}
\ell_X&=\lambda(y,P)-P^{n+1}=w_n(y)P^{n-1}+\cdots+w_2(y)P+w_1(y),\\
\ell_Y&=\lambda(z,Q)-Q^{n+1}=w_n(z)Q^{n-1}+\cdots+w_2(z)Q+w_1(z),
\end{aligned}$$

and the formula (5.24) reads

$$\begin{aligned}
\{\lambda(y,P),\lambda(z,Q)\}_a&=\sum_{k,s\geqslant 0}\epsilon^{k+s-1}\delta^{(s)}(y-z)\left[\oint\frac{dp}{2\pi i}\oint\frac{dq}{2\pi i}\frac{A_{a,k,s}(p,q,y)}{(p-P)(q-Q)}\right]\\
&=\sum_{k,s\geqslant 0}\epsilon^{k+s-1}A_{a,k,s}(P,Q,y)\delta^{(s)}(y-z).
\end{aligned}\tag{5.26}$$

Let $r_1, \dots, r_n$ be the critical points of the polynomial $\lambda(p)$, i.e., the roots of $\lambda'(r) = 0$. Assuming them to be pairwise distinct, we have

$$A_{1,0,1}(r_i, r_j, x) = \delta_{ij}\lambda''(x, r_i), \qquad A_{2,0,1}(r_i, r_j, x) = \delta_{ij}\lambda(x, r_i)\lambda''(x, r_i).$$

This shows that the critical *values* $u^i = \lambda(r_i)$ are the canonical coordinates of the bihamiltonian structure (5.26). Then the quantities in the formula (2.26) read

$$f^i = \lambda''(r_i),$$
$$Q_1^{ii} = \frac{1}{12}\lambda^{(4)}(r_i), \qquad Q_2^{ii} = \frac{1}{12}\lambda(r_i)\lambda^{(4)}(r_i) + \frac{n}{n+1}\frac{\lambda''(r_i)^2}{4},$$
$$P_1^{ki} = \frac{\lambda''(r_k) + \lambda''(r_i)}{2(r_k - r_i)}, \qquad P_2^{ki} = \frac{\lambda''(r_k)\lambda(r_i) + \lambda(r_k)\lambda''(r_i)}{2(r_k - r_i)}.$$

Thus the central invariants read

$$\begin{aligned} c_i &= \frac{1}{3\lambda''(r_i)^2}\left(\frac{n}{n+1}\frac{\lambda''(r_i)^2}{4} + \sum_{k\neq i}\frac{(\lambda(r_k) - \lambda(r_i))\lambda''(r_i)^2}{4\lambda''(r_k)(r_k - r_i)^2}\right) \\ &= \frac{1}{12}\left(\frac{n}{n+1} + \sum_{k\neq i}\frac{(\lambda(r_k) - \lambda(r_i))}{\lambda''(r_k)(r_k - r_i)^2}\right) = \frac{1}{12}\left(\frac{n}{n+1} + \frac{1-n}{2(n+1)}\right) \\ &= \frac{1}{24}. \end{aligned}$$

Here the third equality is obtained by applying the residue theorem to the meromorphic function

$$m(q) = \frac{\lambda(q) - \lambda(r_i)}{\lambda'(q)(q - r_i)^2}.$$

The theorem is proved. □

6. The B_n, C_n and D_n cases

The simple Lie algebras of type B_n, C_n and D_n can be realized as matrix Lie algebras $o(2n+1)$, $sp(2n)$ and $o(2n)$. The details of these realizations are omitted here, see Appendix 1 of [13]. Note that the Weyl generators X_i, Y_i, H_i we choose here correspond respectively to $Y_i, X_i, -H_i$ of [13]. We begin with the following scalar differential operators satisfying certain symmetry/antisymmetry conditions:

$$B_n: \quad L = D^{2n+1} + \sum_{i=1}^{n} w_i(x)D^{2i-1} + \sum_{i=1}^{n} v_i(x)D^{2i-2}, \quad L + L^\dagger = 0, \tag{6.1}$$

$$C_n: \quad L = D^{2n} + \sum_{i=1}^{n} w_i(x)D^{2i-2} + \sum_{i=2}^{n} v_i(x)D^{2i-3}, \quad L = L^\dagger, \tag{6.2}$$

$$D_n\colon\quad L = D^{2n-1} + \sum_{i=2}^{n} w_i(x) D^{2i-3} + \sum_{i=2}^{n} v_i(x) D^{2i-4} + \rho(x) D^{-1} \rho(x), \qquad L + L^\dagger = 0. \tag{6.3}$$

Here $L^\dagger$ is the adjoint operator (2.8), the coefficients $v_i(x)$ are linear combinations of derivatives of $w_i(x)$ uniquely determined by the symmetry/antisymmetry conditions. We assume $w_1(x) = \rho^2(x)$ for the D_n case.

As for the A_n case, the above scalar (pseudo) differential operators can also be derived from the differential operator $\mathcal{L}$ of the form (5.4). In the present cases, the matrices q are upper triangular ones belonging to $\mathfrak{o}(2n+1)$, $\mathfrak{sp}(2n)$ and $\mathfrak{o}(2n)$ respectively. The matrices I are given respectively by

$$I = \sum_{i=1}^{n} (e_{i+1,i} + e_{2n+2-i,2n+1-i}), \qquad I = \sum_{i=1}^{n-1} (e_{i+1,i} + e_{2n+1-i,2n-i}) + e_{n+1,n}$$

and

$$I = \sum_{i=1}^{n-1} (e_{i+1,i} + e_{2n+1-i,2n-i}) + \frac{1}{2}(e_{n+1,n-1} + e_{n+2,n}).$$

The scalar differential operators L are given by $-\Delta(\mathcal{L})^\dagger$, where the operator Δ is defined as in (5.6).

The variational derivative of a functional of L w.r.t. L is now defined as

$$\frac{\delta F}{\delta L} = \frac{1}{2} \sum_{i=1}^{n} \left(D^{-2i+\nu} \frac{\delta F}{\delta w_i(x)} + \frac{\delta F}{\delta w_i(x)} D^{-2i+\nu} \right), \tag{6.4}$$

where $\nu = 0, 1, 2$ for the B_n, C_n and D_n cases respectively. This definition ensures the validity of (5.9).

In order to have a uniform expression of the Drinfeld–Sokolov second Hamiltonian structures for the three types of simple Lie algebras, we fix in this section the invariant bilinear form on $\mathfrak{g}$ by

$$\langle a, b \rangle_{\mathfrak{g}} = \operatorname{tr}(ab). \tag{6.5}$$

Let us note that the normalized invariant bilinear form defined in (4.26) for the simple Lie algebras of type B_n, C_n, D_n have the expressions

$$\frac{1}{2}\operatorname{tr}(ab), \qquad \operatorname{tr}(ab), \qquad \frac{1}{2}\operatorname{tr}(ab) \tag{6.6}$$

respectively. With the above fixed invariant bilinear form, the second Hamiltonian structures for the three types of simple Lie algebras have a uniform expression

$$\{F, G\}_2 = \frac{1}{\epsilon} \operatorname{Tr}\bigl[(LY)_+ LX - XL(YL)_+\bigr], \tag{6.7}$$

while the first ones are defined as the Lie derivatives of the second ones along the coordinate w_i, where $i = 1$ for B_n, C_n and $i = 2$ for D_n,

$$\{F, G\}_2(w_i, \ldots) - \lambda\{F, G\}_1(w_i, \ldots) = \{F, G\}_2(w_i - \lambda, \ldots).$$

Explicitly,

$$B_n\colon \quad \{F, G\}_1 = \frac{1}{\epsilon} \operatorname{Tr} L(YDX - XDY), \tag{6.8}$$

$$C_n\colon \quad \{F, G\}_1 = \frac{1}{\epsilon} \operatorname{Tr} L(YX - XY), \tag{6.9}$$

$$D_n\colon \quad \{F, G\}_1 = \frac{1}{\epsilon} \operatorname{Tr} L(X_+DY_+ - Y_+DX_+ + Y_-DX_- - X_-DY_-). \tag{6.10}$$

Let us now describe the main result of this section. Let

$$\lambda_B(p) = p^{2n+1} + \sum_{i=1}^{n} w_i(x) p^{2i-1}, \tag{6.11}$$

$$\lambda_C(p) = p^{2n} + \sum_{i=1}^{n} w_i(x) p^{2i-2}, \tag{6.12}$$

$$\lambda_D(p) = p^{2n-1} + \sum_{i=2}^{n} w_i(x) p^{2i-3} + \frac{w_1(x)}{p} \tag{6.13}$$

be the $\epsilon = 0$ limits of the symbols of the Lax operators (6.1)–(6.3). Introduce

$$\begin{aligned} \Lambda_B(P) = \Lambda_C(P) &= P^n + w_n(x)P^{n-1} + \cdots + w_1(x), \\ \Lambda_D(P) &= P^{n-1} + w_n(x)P^{n-2} + \cdots + w_2(x) + \frac{w_1(x)}{P} \end{aligned} \tag{6.14}$$

by the following substitution:

$$\begin{aligned} \lambda_B(p) &= p\Lambda_B\bigl(p^2\bigr), \\ \lambda_C(p) &= \Lambda_C\bigl(p^2\bigr), \\ \lambda_D(p) &= p\Lambda_D\bigl(p^2\bigr). \end{aligned} \tag{6.15}$$

Theorem 6.1.

(i) *The dispersionless limits of the Drinfeld–Sokolov bihamiltonian structures associated to the simple Lie algebras of type B_n, C_n, and D_n have the following uniform expression*

$$\begin{aligned}\{\Lambda(x,P),\Lambda(y,Q)\}_1 &= 2\frac{P\Lambda'(P)-Q\Lambda'(Q)}{P-Q}\delta'(x-y)\\ &\quad+\left[\frac{P+Q}{(P-Q)^2}\big(\Lambda_x(P)-\Lambda_x(Q)\big)-2\frac{Q\Lambda_x'(Q)}{P-Q}\right]\delta(x-y),\end{aligned} \tag{6.16}$$

$$\begin{aligned}\{\Lambda(x,P),\Lambda(y,Q)\}_2 &= 2\frac{P\Lambda'(P)\Lambda(Q)-Q\Lambda'(Q)\Lambda(P)}{P-Q}\delta'(x-y)\\ &\quad+\left[\frac{P+Q}{(P-Q)^2}\big(\Lambda_x(P)\Lambda(Q)-\Lambda_x(Q)\Lambda(P)\big)\right.\\ &\quad\left.+2\frac{P\Lambda'(P)\Lambda_x(Q)-Q\Lambda_x'(Q)\Lambda(P)}{P-Q}\right]\delta(x-y),\end{aligned} \tag{6.17}$$

where $\Lambda(x,P)=\Lambda_B$, Λ_C, *or* Λ_D *respectively.*

(ii) *The central invariants of the Drinfeld–Sokolov bihamiltonian structures read*

$$B_n:\quad c_1=\cdots=c_{n-1}=\frac{1}{12},\quad c_n=\frac{1}{6}, \tag{6.18}$$

$$C_n:\quad c_1=\cdots=c_{n-1}=\frac{1}{12},\quad c_n=\frac{1}{24}, \tag{6.19}$$

$$D_n:\quad c_1=c_2=\cdots=c_n=\frac{1}{12}. \tag{6.20}$$

Note that the rescaling

$$\langle,\rangle_{\mathfrak g}\mapsto\kappa\langle,\rangle_{\mathfrak g}$$

of the invariant bilinear form on $\mathfrak g$ yields the rescaling of the central invariants (2.26) of the related Drinfeld–Sokolov bihamiltonian structure

$$c_i\mapsto\kappa c_i,\quad i=1,\dots,n.$$

So from the definition of the normalized bilinear form (4.26) and (5.2), (6.5), (6.6) and Theorems 5.1, 6.1 it follows the validity of Theorem 4.6 for the cases A_n, B_n, C_n, D_n. Before proceeding to the proof of the theorem let us explain the rule of labeling of the central invariants for the B_n and C_n cases. The reader may remember that the labeling of the central invariants is in one-to-one correspondence with labeling of the canonical coordinates. It will be shown below that the canonical coordinates of the bihamiltonian structure of (6.16), (6.17) for the B_n and C_n cases are defined as follows:

$$\begin{aligned}&u^i=\Lambda(r_i),\quad i=1,\dots,n,\\ &\frac{d}{dP}\Lambda(P)\big|_{P=r_i}=0,\quad i=1,\dots,n-1,\ r_n=0.\end{aligned} \tag{6.21}$$

In these cases $r_n=0$ is always a critical point of $\Lambda(p^2)$. The associated critical value $u^n=\Lambda(0)$ "breaks the symmetry" between the canonical coordinates; the corresponding central invariant c_n differs from others.

Proof of Theorem 6.1. The derivation of the dispersionless Poisson structures (6.16), (6.17) follows the lines of the proof of Theorem 5.1. We will omit this part of the proof, and proceed directly to computation of the central invariants.

Some part of the computation can be done uniformly for all the three types of Lie algebras. To this end we introduce the symbol

$$\lambda(p) = p^{2n+1-\nu} + \sum_{i=1}^{n} w_i(x) p^{2i-1-\nu} \tag{6.22}$$

and also

$$f(p) = \sum_{i\geqslant 1} \frac{a_i(x)}{p^{2i-\nu}}, \qquad g(p) = \sum_{i\geqslant 1} \frac{b_i(x)}{p^{2i-\nu}} \tag{6.23}$$

(we plan to still use the linear functionals (5.12)). Recall that $\nu = 0, 1, 2$ for B_n, C_n and D_n respectively. The symbols of the pseudo-differential operators X and Y read

$$\hat{X}(p) = f(p) + \frac{1}{2}\sum_{k\geqslant 1} \frac{\epsilon^k}{k!} \partial_p^k \partial_x^k f(p), \qquad \hat{Y}(p) = g(p) + \frac{1}{2}\sum_{k\geqslant 1} \frac{\epsilon^k}{k!} \partial_p^k \partial_x^k g(p). \tag{6.24}$$

We omit the derivatives of w_i w.r.t. x in $\hat{L}(p)$ just like in the previous section.

By using the same method used in the proof of Theorem 5.1, we obtain the coefficients $A_{2,0,1}$, $A_{2,0,2}$ and $A_{2,0,3}$ in the expansion (5.24)

$$\begin{aligned} A_{2,0,1} &= \frac{\lambda'(q)\lambda(p) - \lambda'(p)\lambda(q)}{q-p}, \qquad A_{2,0,2} = 0, \\ A_{2,0,3} &= \frac{\lambda'(q)\lambda(p) - \lambda'(p)\lambda(q)}{2(q-p)^3} - \frac{\lambda''(q)\lambda(p) - 2\lambda'(q)\lambda'(p) + \lambda''(p)\lambda(q)}{4(q-p)^2} \\ &\quad + \frac{\lambda'(q)\lambda''(p) - \lambda'(p)\lambda''(q)}{4(q-p)} + \frac{\lambda'''(q)\lambda(p) - \lambda'''(p)\lambda(q)}{6(q-p)}. \end{aligned} \tag{6.25}$$

Now let P, Q be two complex numbers such that $|P| < |p|^2$ and $|Q| < |q|^2$. Define the functions $a_i(x)$, $b_i(x)$ as in (6.23) from the following expansions

$$f(p) = \frac{p^\nu}{p^2 - P}\delta(x-y) = \sum_{k=1}^{\infty} \frac{P^{k-1}}{p^{2k-\nu}}\delta(x-y), \qquad g(q) = \frac{q^\nu}{q^2 - Q}\delta(x-z).$$

Then $\ell_X = \Lambda(y, P) - P^n$, $\ell_Y = \Lambda(z, Q) - Q^n$, where

$$\Lambda(y, P) = P^n + w_n(y)P^{n-1} + \cdots + w_1(y). \tag{6.26}$$

The second Poisson bracket between the linear functionals now reads

$$\{\Lambda(y,P),\Lambda(z,Q)\}_2=\sum_{k,s\geqslant 0}\epsilon^{k+s-1}\delta^{(s)}(y-z)\left[\oint\frac{dp}{2\pi i}\oint\frac{dp}{2\pi i}\frac{(pq)^\nu A_{2,k,s}(p,q,y)}{(p^2-P)(q^2-Q)}\right].$$

Denote by $R_{2,1}$ the coefficient of $\epsilon^0\delta'(y-z)$. It is easy to obtain

$$R_{2,1}=2\frac{P\Lambda'(P)\Lambda(Q)-Q\Lambda'(Q)\Lambda(P)}{P-Q}. \tag{6.27}$$

Here $\Lambda(P)=\Lambda(y,P)$, $\Lambda(Q)=\Lambda(y,Q)$, and the primes stand for differentiations w.r.t. P or Q. Then by definition one can obtain the coefficient of $\epsilon^0\delta'(y-z)$ in $\{\Lambda(y,P),\Lambda(z,Q)\}_1$ denoted by $R_{1,1}$

$$B_n, C_n:\quad R_{1,1}=2\frac{P\Lambda'(P)-Q\Lambda'(Q)}{P-Q}, \tag{6.28}$$

$$D_n:\quad R_{1,1}=2\frac{PQ(\Lambda'(P)-\Lambda'(Q))+P\Lambda(Q)-Q\Lambda(P)}{P-Q}. \tag{6.29}$$

Denote the coefficients of $\epsilon^2\delta'''(y-z)$ in $\{\Lambda(y,P),\Lambda(z,Q)\}_\alpha$ by $R_{\alpha,3}$. After a lengthy computation, we obtain

$$\begin{aligned}B_n:\quad R_{2,3}={}&\frac{(P+Q)^2(\Lambda'(P)\Lambda(Q)-\Lambda'(Q)\Lambda(P))}{(P-Q)^3}+4\frac{P^2\Lambda'''(P)\Lambda(Q)-Q^2\Lambda'''(Q)\Lambda(P)}{3(P-Q)}\\&+2\frac{PQ(\Lambda'(P)\Lambda''(Q)-\Lambda'(Q)\Lambda''(P))}{P-Q}+2\frac{P\Lambda''(P)\Lambda(Q)-Q\Lambda''(Q)\Lambda(P)}{P-Q}\\&+3\Lambda'(P)\Lambda'(Q)-2\frac{PQ(\Lambda''(P)\Lambda(Q)-2\Lambda'(P)\Lambda'(Q)+\Lambda(P)\Lambda''(Q))}{(P-Q)^2}.\end{aligned} \tag{6.30}$$

$$\begin{aligned}R_{1,3}={}&\frac{(P+Q)^2(\Lambda'(P)-\Lambda'(Q))}{(P-Q)^3}+4\frac{P^2\Lambda'''(P)-Q^2\Lambda'''(Q)}{3(P-Q)}\\&+2\frac{P\Lambda''(P)-Q\Lambda''(Q)}{P-Q}-2\frac{PQ(\Lambda''(P)+\Lambda''(Q))}{(P-Q)^2}.\end{aligned} \tag{6.31}$$

$$\begin{aligned}C_n:\quad R_{2,3}={}&\frac{(P^2+6PQ+Q^2)(\Lambda'(P)\Lambda(Q)-\Lambda'(Q)\Lambda(P))}{2(P-Q)^3}\\&+4\frac{P^2\Lambda'''(P)\Lambda(Q)-Q^2\Lambda'''(Q)\Lambda(P)}{3(P-Q)}+2\frac{PQ(\Lambda'(P)\Lambda''(Q)-\Lambda'(Q)\Lambda''(P))}{P-Q}\\&+\frac{P\Lambda''(P)\Lambda(Q)-Q\Lambda''(Q)\Lambda(P)}{P-Q}+\Lambda'(P)\Lambda'(Q)\\&-2\frac{PQ(\Lambda''(P)\Lambda(Q)-2\Lambda'(P)\Lambda'(Q)+\Lambda(P)\Lambda''(Q))}{(P-Q)^2}.\end{aligned} \tag{6.32}$$

$$R_{1,3}=\frac{(P^2+6PQ+Q^2)(\Lambda'(P)-\Lambda'(Q))}{2(P-Q)^3}+4\frac{P^2\Lambda'''(P)-Q^2\Lambda'''(Q)}{3(P-Q)}$$

$$+\frac{P\Lambda''(P)-Q\Lambda''(Q)}{P-Q}-2\frac{PQ(\Lambda''(P)+\Lambda''(Q))}{(P-Q)^2}. \tag{6.33}$$

$$\begin{aligned} D_n\colon\quad R_{2,3}={}&4\frac{PQ(\Lambda'(P)\Lambda(Q)-\Lambda'(Q)\Lambda(P))}{(P-Q)^3}\\ &+4\frac{P^2\Lambda'''(P)\Lambda(Q)-Q^2\Lambda'''(Q)\Lambda(P)}{3(P-Q)}+2\frac{PQ(\Lambda'(P)\Lambda''(Q)-\Lambda'(Q)\Lambda''(P))}{P-Q}\\ &-\Lambda'(P)\Lambda'(Q)-2\frac{PQ(\Lambda''(P)\Lambda(Q)-2\Lambda'(P)\Lambda'(Q)+\Lambda(P)\Lambda''(Q))}{(P-Q)^2}\\ &+\frac{P^2\Lambda'(P)\Lambda(Q)-Q^2\Lambda'(Q)\Lambda(P)}{PQ(P-Q)}-\Lambda(0)\frac{P\Lambda'(P)+Q\Lambda'(Q)}{PQ}. \end{aligned} \tag{6.34}$$

$$\begin{aligned} R_{1,3}={}&\frac{4PQ(P\Lambda'''(P)-Q\Lambda'''(Q))}{3(P-Q)}-2\frac{PQ(P\Lambda''(P)+Q\Lambda''(Q))}{(P-Q)^2}\\ &+\frac{4PQ(P^2\tilde\Lambda'(P)-Q^2\tilde\Lambda'(Q))}{(P-Q)^3}+\frac{PQ(\tilde\Lambda'(P)-\tilde\Lambda'(Q))}{P-Q}-\frac{\Lambda(0)(P+Q)}{PQ}, \end{aligned} \tag{6.35}$$

where

$$\tilde\Lambda(P)=\Lambda(P)/P. \tag{6.36}$$

Now we begin to compute the central invariants for the B_n, C_n cases. The formulae (6.27) (6.28) show that in these two cases we have the same dispersionless limit, so the corresponding Drinfeld–Sokolov bihamiltonian structures have the same canonical coordinates. Let $r_1,\dots,r_n$ be defined as in (6.21). Then we have $u^n=w_1$ and $u^1,\dots,u^{n-1}$ are the critical values of $\Lambda(P)$. From the formulae (6.27) and (6.28), one can see that $u^1,\dots,u^n$ can serve as the canonical coordinates of the Drinfeld–Sokolov bihamiltonian structures of B_n and C_n type. Following the notations in (2.26), we have

$$B_n\colon\quad f^i=2r_i\Lambda''(r_i),\qquad f^n=2\Lambda'(0);$$

$$Q_1^{ii}=3\Lambda''(r_i)+\frac{14}{3}r_i\Lambda'''(r_i)+r_i^2\Lambda''''(r_i),\qquad Q_1^{nn}=3\Lambda''(0);$$

$$Q_2^{ii}=r_i^2\Lambda''(r_i)^2+\Lambda(r_i)Q_1^{ii},\qquad Q_2^{nn}=2\Lambda'(0)^2+3\Lambda(0)\Lambda''(0);$$

$$c_i=\frac{Q_2^{ii}-\Lambda(r_i)Q_1^{ii}}{3(f^i)^2}=\frac{1}{12},\qquad c_n=\frac{Q_2^{11}-\Lambda(0)Q_1^{11}}{3(f^1)^2}=\frac{1}{6}.$$

$$C_n\colon\quad f^i=2r_i\Lambda''(r_i),\qquad f^n=2\Lambda'(0);$$

$$Q_1^{ii}=3\Lambda''(r_i)+\frac{11}{3}r_i\Lambda'''(r_i)+r_i^2\Lambda''''(r_i),\qquad Q_1^{nn}=\frac{3}{2}\Lambda''(0);$$

$$Q_2^{ii}=r_i^2\Lambda''(r_i)^2+\Lambda(r_i)Q_1^{ii},\qquad Q_2^{nn}=\frac{1}{2}\Lambda'(0)^2+\frac{3}{2}\Lambda(0)\Lambda''(0);$$

$$c_i = \frac{Q_2^{ii} - \Lambda(r_i)Q_1^{ii}}{3(f^i)^2} = \frac{1}{12}, \qquad c_n = \frac{Q_2^{11} - \Lambda(0)Q_1^{11}}{3(f^1)^2} = \frac{1}{24}.$$

Here $i = 1, \ldots, n$.

To compute the central invariants for the D_n case, we first rewrite the two Poisson brackets in terms of the symbol $\tilde{\Lambda}$ defined in (6.36). Let $\tilde{R}_{\alpha,k}$ be obtained from $R_{\alpha,k}$ of (6.30), (6.31) with Λ replaced by $\tilde{\Lambda}$. Denote by $S_{\alpha,k}$ the coefficients of $\epsilon^{k-1}\delta^{(k)}(y-z)$ in $\{\tilde{\Lambda}(P,y), \tilde{\Lambda}(Q,z)\}_\alpha$. Then we have $S_{\alpha,1} = \tilde{R}_{\alpha,1}$, and

$$S_{2,3} = \tilde{R}_{2,3} - \Lambda(0)\frac{P\Lambda'(P) + Q\Lambda'(Q)}{P^2Q^2}, \qquad S_{1,3} = \tilde{R}_{2,3} - \frac{\Lambda(0)(P+Q)}{P^2Q^2}.$$

Let $r_1, \ldots, r_n$ be the critical point of $\tilde{\Lambda}(P)$, and $u^1, \ldots, u^n$ be the corresponding critical values, they can serve as the canonical coordinates of the Drinfeld–Sokolov bihamiltonian structure in the D_n case. So we have

$$D_n\colon \quad f^i = 2r_i\tilde{\Lambda}''(r_i), \qquad Q_1^{ii} = 3\tilde{\Lambda}''(r_i) + \frac{14}{3}r_i\tilde{\Lambda}'''(r_i) + r_i^2\tilde{\Lambda}''''(r_i) - \frac{2\Lambda(0)}{r_i^3},$$

$$Q_2^{ii} = \tilde{\Lambda}(r_i)Q_1^{ii} + r_i^2\tilde{\Lambda}''(r_i)^2, \qquad c_i = \frac{Q_2^{ii} - \tilde{\Lambda}(r_i)Q_1^{ii}}{3(f^i)^2} = \frac{1}{12}.$$

The theorem is proved. □

7. The exceptional cases

In this section we will use the approach of [3] based on the Dirac reduction procedure [12] to compute the Drinfeld–Sokolov bihamiltonian structures associated to the exceptional Lie algebras,[5] and then proceed to calculating the central invariants. Let us consider the Poisson bracket $\pi_{\mathfrak{g}}(I)$ on $\mathfrak{g}^*$ evaluated at the point I as a skew symmetric bilinear form on

$$\mathfrak{g} \simeq T_I^*\mathfrak{g}^*$$

(cf. (3.9)). The stabilizer $\operatorname{Ker}\operatorname{ad}_I$ of I coincides with the kernel of this bilinear form. The quotient

$$\mathfrak{g}/\operatorname{Ker}\operatorname{ad}_I$$

acquires a symplectic structure induced by $\pi_{\mathfrak{g}}(I)$. The projection

$$\mathfrak{n} \hookrightarrow \mathfrak{g} \to \mathfrak{g}/\operatorname{Ker}\operatorname{ad}_I$$

realizes the nilpotent subalgebra $\mathfrak{n}$ as a Lagrangian subspace in the quotient. Let

$$\mathfrak{n}_{\text{dual}} \subset \mathfrak{h} \oplus \mathfrak{n}^- \tag{7.1}$$

[5] The hierarchies associated with the simply laced exceptional root systems have been systematically treated by V. Kac and M. Wakimoto in [35]. They did not consider however the bihamiltonian structure of the exceptional hierarchies.

be a pullback of a complementary Lagrangian subspace of the image of $\mathfrak{n}$ such that

$$\mathfrak{g} = \operatorname{Ker}\operatorname{ad}_I \oplus \mathfrak{n} \oplus \mathfrak{n}_{\mathrm{dual}}. \tag{7.2}$$

A choice of $\mathfrak{n}_{\mathrm{dual}}$ specifies the transversal subspace $V \subset \mathfrak{b}$ of (3.25) by the equation

$$\langle b, q^{\mathrm{can}}\rangle_{\mathfrak{g}} = 0 \quad \forall b \in \mathfrak{n}_{\mathrm{dual}},\ q^{\mathrm{can}} \in V. \tag{7.3}$$

One can unify constraints (3.24) and (7.3) by considering a system of equations for $q \in \mathfrak{g}$:

$$\begin{aligned} \langle a, q\rangle_{\mathfrak{g}} &= \langle a, I\rangle_{\mathfrak{g}} \quad \forall a \in \mathfrak{n}, \\ \langle b, q\rangle_{\mathfrak{g}} &= 0 \quad \forall b \in \mathfrak{n}_{\mathrm{dual}}. \end{aligned} \tag{7.4}$$

The solution

$$q = I + q^{\mathrm{can}}$$

determines the transversal slice V. The reduced Poisson bracket on q^{can}-valued loops can be obtained as follows. Let us choose a basis

$$f_1, \ldots, f_{2m} \in \mathfrak{n} \oplus \mathfrak{n}_{\mathrm{dual}}, \quad 2m = 2\dim\mathfrak{n} = \dim\mathfrak{g} - n.$$

Introduce two $2m \times 2m$ matrices

$$\begin{aligned} P &= (P_{ab}), \qquad P_{ab} = -\langle I + q^{\mathrm{can}}, [f_a, f_b]\rangle_{\mathfrak{g}}, \\ Q &= (Q_{ab}), \qquad Q_{ab} = \langle f_a, f_b\rangle_{\mathfrak{g}}. \end{aligned} \tag{7.5}$$

By construction of $\mathfrak{n}_{\mathrm{dual}}$ the matrix

$$P|_{q^{\mathrm{can}}=0} = \pi_{\mathfrak{g}}(I)|_{\mathfrak{n}\oplus\mathfrak{n}_{\mathrm{dual}}}$$

does not degenerate. Consider matrix differential operator

$$M := P + Q\epsilon\partial_x \tag{7.6}$$

with coefficients depending on q^{can} (via P). Note that the matrix of pairwise Poisson brackets of the constraints (7.4) is equal to

$$\{\langle f_a, q(x)\rangle_{\mathfrak{g}}, \langle f_b, q(y)\rangle_{\mathfrak{g}}\} = -\frac{1}{\epsilon} M_{ab}\delta(x-y).$$

The following statement was proved in [27].

Lemma 7.1. *The inverse M^{-1} to* (7.6) *is a matrix-valued differential operator of finite order with coefficients depending polynomially on $q^{\mathrm{can}}, q^{\mathrm{can}}_x, \ldots$.*

Let

$$\gamma^1, \dots, \gamma^n \in \operatorname{Ker} \operatorname{ad} I \tag{7.7}$$

be a basis in the centralizer of I. Recall [36] that this centralizer is a commutative subalgebra in $\mathfrak{n}^-$ having generators only in the degrees $-m_1, \dots, -m_n$; the number of generators in the degree $-m_k$ is equal to the multiplicity of the exponent m_k. The linear functions of $q^{\mathrm{can}} \in V$ given by

$$w^i = \langle \gamma^i, q^{\mathrm{can}} \rangle_{\mathfrak{g}}, \quad i = 1, \dots, n, \tag{7.8}$$

define a system of coordinates on V. Denote $\gamma_1, \dots, \gamma_n$ the dual basis in V,

$$\langle \gamma^i, \gamma_j \rangle_{\mathfrak{g}} = \delta^i_j, \qquad \langle f_a, \gamma_i \rangle_{\mathfrak{g}} = 0, \quad i, j = 1, \dots, n,\ a = 1, \dots, 2m, \tag{7.9}$$

so

$$q^{\mathrm{can}} = \sum_{i=1}^n w^i \gamma_i. \tag{7.10}$$

Introduce the $n \times 2m$ matrix differential operator

$$N = \left(N^i_a\right) = \left(R^i_a + S^i_a \epsilon \partial_x\right), \qquad N^i_a = \epsilon \langle \gamma^i, f_a \rangle_{\mathfrak{g}} \partial_x - \langle q^{\mathrm{can}}, [\gamma^i, f_a] \rangle_{\mathfrak{g}}. \tag{7.11}$$

Denote $N^\dagger$ the matrix of (formally) adjoint differential operators,

$$\left(N^\dagger\right)^a_i = N^{i\,\dagger}_a, \quad i = 1, \dots, n,\ a = 1, \dots, 2m. \tag{7.12}$$

Then the matrix of the second reduced Poisson bracket is given by the formula

$$\left\{w^i(x), w^j(y)\right\}^{\mathrm{red}}_2 = -\frac{1}{\epsilon}\left(N M^{-1} N^\dagger\right)^{ij} \delta(x-y). \tag{7.13}$$

The first reduced bracket is given by a similar formula

$$\left\{w^i(x), w^j(y)\right\}^{\mathrm{red}}_1 = \frac{1}{\epsilon}\left(N M^{-1} \tilde M M^{-1} N^\dagger + \tilde N M^{-1} N^\dagger + N M^{-1} \tilde N^\dagger\right)^{ij} \delta(x-y) \tag{7.14}$$

where the $n \times 2m$ and $2m \times 2m$ matrices $\tilde N^i_a$ and $\tilde M_{ab}$ respectively are defined as follows:

$$\tilde N^i_a = \langle \alpha, [\gamma^i, f_a] \rangle_{\mathfrak{g}}, \qquad \tilde M_{ab} = \langle \alpha, [f_a, f_b] \rangle_{\mathfrak{g}}, \tag{7.15}$$

where $\alpha \in \mathfrak{n}$ is the generator of the center of $\mathfrak{n}$ chosen above (see (3.28)). We will see below that the terms of order ϵ^{-1} disappear from (7.13), (7.14).

Let us now explain how we compute the Frobenius structure and the central invariants using the formula (7.13). For the second metric g_2^{ij} one obtains

$$\big(g_2^{ij}\big(q^{\mathrm{can}}\big)\big) = RP^{-1}QP^{-1}R^T - SP^{-1}R^T + RP^{-1}S^T \tag{7.16}$$

where R^T, S^T denotes their transposed matrices. The matrices $(A^{ij}_{1,0;2})$ and $(A^{ij}_{2,0;2})$ have the following form:

$$\begin{aligned}\big(A^{ij}_{1,0;2}\big(q^{\mathrm{can}}\big)\big) &= -RP^{-1}QP^{-1}QP^{-1}R^T - RP^{-1}QP^{-1}S^T \\ &\quad + SP^{-1}QP^{-1}R^T + SP^{-1}S^T,\end{aligned} \tag{7.17}$$

$$\begin{aligned}\big(A^{ij}_{2,0;2}\big(q^{\mathrm{can}}\big)\big) &= RP^{-1}QP^{-1}QP^{-1}QP^{-1}R^T - RP^{-1}QP^{-1}QP^{-1}S^T \\ &\quad + SP^{-1}QP^{-1}QP^{-1}R^T + SP^{-1}QP^{-1}S^T,\end{aligned} \tag{7.18}$$

where the matrices $R = (R^i_a)$, $S = (S^i_a)$ are defined in (7.11). Doing the shift

$$q^{\mathrm{can}} \mapsto q^{\mathrm{can}} + \lambda\alpha, \quad \alpha \in \{\text{the center of } \mathfrak{n}\} \tag{7.19}$$

one obtains in (7.16)–(7.18) linear functions in λ. The coefficients of λ of these functions give the matrices g_1^{ij}, $A^{ij}_{1,0;1}(q^{\mathrm{can}})$ and $A^{ij}_{2,0;1}(q^{\mathrm{can}})$ respectively.

The dual bases $\gamma^i \in \operatorname{Ker}\operatorname{ad} I$ and $\gamma_i \in V$ can be chosen as follows. According to [36] the triple

$$I_- := I, \qquad \rho = \sum_{i=1}^n \omega_i, \qquad I_+ = \sum_{i=1}^n a_i X_i \tag{7.20}$$

defines an embedding of the sl_2 Lie algebra into $\mathfrak{g}$,

$$[I_+, I_-] = 2\rho, \quad [\rho, I_\pm] = \pm I_\pm. \tag{7.21}$$

Here $\omega_1, \ldots, \omega_n \in \mathfrak{h}$ are the fundamental weights, i.e. the basis dual to the basis of simple roots, and the integer coefficients $a_1, \ldots, a_n$ are defined from the decomposition

$$2\rho = \sum_{i=1}^n a_i H_i. \tag{7.22}$$

We put

$$V := \operatorname{Ker}\operatorname{ad} I_+. \tag{7.23}$$

We choose

$$\begin{aligned}&\gamma_i \in \operatorname{Ker}\operatorname{ad} I_+ \cap \mathfrak{g}^{m_i}, \quad i = 1, \ldots, n, \\ &\gamma^i \in \operatorname{Ker}\operatorname{ad} I_- \cap \mathfrak{g}^{-m_i}, \quad i = 1, \ldots, n.\end{aligned} \tag{7.24}$$

For all exceptional Lie algebras the vectors γ_i and γ^i are determined uniquely up to normalization. We can normalize them in such a way that

$$\langle \gamma^i, \gamma_j \rangle_{\mathfrak{g}} = \delta^i_j.$$

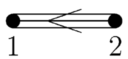
Lemma 7.2. *For the exceptional simple Lie algebras of type G_2, F_4, E_6, E_7, E_8, the central invariants of the corresponding Drinfeld–Sokolov bihamiltonian structures coincide with the values listed in the table that is given at the end of Section* 4.

Proof. The lemma can be proved by a straightforward computation by using the formula (2.26) for the Lie algebras of G_2 and F_4 types. For the E type cases we can use the formula (2.29) and the implicit function theorem to compute the central invariants, however the computations become very involved; so we use a different method based on a comparison of the Drinfeld–Sokolov bihamiltonian structure with the one obtained in [20] (see below).

Since the central invariants do not depend on the choice of α in (7.19), in what follows we will fix $\alpha = \gamma_n$.

We first illustrate the procedure by considering the G_2 case[6] in detail.

Let X_i, H_i, Y_i $(i = 1, 2)$ be a set of Weyl generators of the simple Lie algebra $\mathfrak{g}$ of G_2 type, whose Dynkin diagram is labeled as follows



We define a Chevalley basis of $\mathfrak{g}$

$$X_3 = -[X_1, X_2], \qquad Y_3 = [Y_1, Y_2],$$
$$X_4 = -[X_1, X_3]/2, \qquad Y_4 = [Y_1, Y_3]/2,$$
$$X_5 = -[X_1, X_4]/3, \qquad Y_5 = [Y_1, Y_4]/3,$$
$$X_6 = -[X_2, X_5], \qquad Y_6 = [Y_2, Y_5].$$

The normalized invariant bilinear form is given by

$$\langle X_1, Y_1\rangle_{\mathfrak{g}} = \langle X_3, Y_3\rangle_{\mathfrak{g}} = \langle X_4, Y_4\rangle_{\mathfrak{g}} = 3,$$
$$\langle X_2, Y_2\rangle_{\mathfrak{g}} = \langle X_5, Y_5\rangle_{\mathfrak{g}} = \langle X_6, Y_6\rangle_{\mathfrak{g}} = 1,$$
$$\langle H_1, H_1\rangle_{\mathfrak{g}} = 6, \qquad \langle H_1, H_2\rangle_{\mathfrak{g}} = -3, \qquad \langle H_2, H_2\rangle_{\mathfrak{g}} = 2. \tag{7.25}$$

The elements ρ, I_+ read

$$\rho = 3H_1 + 5H_2, \qquad I_+ = 6X_1 + 10X_2.$$

We choose a basis of $\operatorname{Ker} \operatorname{ad} I_+$

$$\gamma_1 = \frac{3}{5}X_1 + X_2, \qquad \gamma_2 = X_6.$$

Then we can obtain the result of the Dirac reduction:

[6] Explicit formulae for the G_2 bihamiltonian structure were obtained in the original paper [13]. In [27] they have been rederived using the Dirac reduction procedures.

$$\left(g_2^{ij}\right)=\begin{pmatrix} -\frac{5w_1}{7} & -\frac{15w_2}{7} \\ -\frac{15w_2}{7} & -\frac{768w_1^5}{875}-\frac{1144}{525}w_2w_1^2 \end{pmatrix}, \qquad \left(g_1^{ij}\right)=\begin{pmatrix} 0 & -\frac{15}{7} \\ -\frac{15}{7} & -\frac{1144w_1^2}{525} \end{pmatrix},$$

$$\left(A_{2,0,2}^{ij}\right)=\begin{pmatrix} \frac{25}{14} & 0 \\ 0 & \frac{42152w_1^4}{13125}+\frac{62w_2w_1}{21} \end{pmatrix}, \qquad \left(A_{2,0,1}^{ij}\right)=\begin{pmatrix} 0 & 0 \\ 0 & \frac{62w_1}{21} \end{pmatrix},$$

and $A_{1,0,2}^{ij}=A_{1,0,1}^{ij}=0$.

If we introduce the flat coordinates

$$t_1=w_2-\frac{572w_1^3}{3375}, \qquad t_2=-\frac{7w_1}{15},$$

the above metrics are just the flat pencil defined by the following Frobenius manifold

$$F=\frac{1}{2}t_1^2t_2+\frac{24}{35}t_2^7, \qquad E=t_1\frac{\partial}{\partial t_1}+\frac{1}{3}t_2\frac{\partial}{\partial t_2}.$$

In the flat coordinates, we have

$$\left(g_2^{ij}\right)=\begin{pmatrix} 48t_2^5 & t_1 \\ t_1 & \frac{t_2}{3} \end{pmatrix}, \qquad \left(g_1^{ij}\right)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\left(A_{2,0,2}^{ij}\right)=\begin{pmatrix} 88t_2^4-\frac{310t_1t_2}{49} & \frac{286t_2^2}{147} \\ \frac{286t_2^2}{147} & \frac{7}{18} \end{pmatrix}, \qquad \left(A_{2,0,1}^{ij}\right)=\begin{pmatrix} -\frac{310t_2}{49} & 0 \\ 0 & 0 \end{pmatrix}.$$

The canonical coordinates are

$$u^1=t_1+4t_2^3, \qquad u^2=t_1-4t_2^3,$$

from which we can compute the quantities appeared in the formula (2.26)

$$f^1=24t_2^2, \qquad f^2=-24t_2^2,$$

$$Q_2^{11}=\frac{9344t_2^4}{49}-\frac{310t_1t_2}{49}, \qquad Q_2^{22}=\frac{4768t_2^4}{49}-\frac{310t_1t_2}{49},$$

$$Q_1^{11}=-\frac{310t_2}{49}, \qquad Q_1^{22}=-\frac{310t_2}{49}.$$

So the central invariants are given by

$$c_1=\frac{Q_2^{11}-u^1Q_1^{11}}{3(f^1)^2}=\frac{1}{8}, \qquad c_2=\frac{Q_2^{22}-u^2Q_1^{22}}{3(f^2)^2}=\frac{1}{24}.$$

The F_4 case

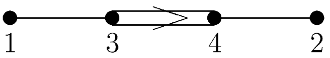
The root system of type F_4 contains 24 positive roots, it is not convenient to define the Chevalley basis explicitly, so we use an alternative way below to describe this basis.

The simple Lie algebra of type F_4 has a 26-dimensional matrix realization [31], whose Weyl generators are

$$\begin{aligned}
X_1 &= e_{1,2} + e_{6,8} + e_{7,10} + e_{9,12} + 2e_{11,13} + e_{11,14} \\
&\quad + e_{13,16} + e_{15,18} + e_{17,20} + e_{19,21} + e_{25,26}, \\
X_2 &= e_{4,5} + e_{6,7} + e_{8,10} - e_{17,19} - e_{20,21} - e_{22,23}, \\
X_3 &= e_{2,3} - e_{4,6} - e_{5,7} - e_{9,11} - e_{12,13} - 2e_{12,14} \\
&\quad - e_{14,15} - e_{16,18} + e_{20,22} + e_{21,23} - e_{24,25}, \\
X_4 &= e_{3,4} - e_{7,9} - e_{10,12} + e_{15,17} + e_{18,20} + e_{23,24},
\end{aligned}$$

$$\begin{aligned}
Y_1 &= e_{2,1} + e_{8,6} + e_{10,7} + e_{12,9} + e_{13,11} + 2e_{16,13} \\
&\quad + e_{16,14} + e_{18,15} + e_{20,17} + e_{21,19} + e_{26,25}, \\
Y_2 &= e_{5,4} + e_{7,6} + e_{10,8} - e_{19,17} - e_{21,20} - e_{23,22}, \\
Y_3 &= e_{3,2} - e_{6,4} - e_{7,5} - e_{11,9} - e_{14,12} - e_{15,13} \\
&\quad - 2e_{15,14} - e_{18,16} + e_{22,20} + e_{23,21} - e_{25,24}, \\
Y_4 &= e_{4,3} - e_{9,7} - e_{12,10} + e_{17,15} + e_{20,18} + e_{24,23}.
\end{aligned}$$

These generators correspond to the following labels on the Dynkin diagram



The normalized Killing form can be computed by the following formula

$$\langle A, B\rangle_{\mathfrak{g}} = \frac{1}{6}\operatorname{tr}(AB).$$

Let α_i be the simple root corresponding to X_i, $i = 1, \ldots, 4$. For any positive root $\beta \in \Phi^+$ of the form

$$\beta = \sum_{i=1}^{4} n_i \alpha_i, \quad \text{where } n_i \geqslant 0,\ i = 1, \ldots, 4,$$

we define $X_\beta = X_{n_1,\ldots,n_4}$ (respectively $Y_\beta = Y_{n_1,\ldots,n_4}$) to be the matrix in the root space $\mathfrak{g}_\beta$ (respectively $\mathfrak{g}_{-\beta}$) such that the first nonzero element of the first nonzero row (respectively column) is equal to 1. Since $\dim \mathfrak{g}_{\pm\beta} = 1$, X_β, Y_β are fixed in this way uniquely. By a straightforward calculation, one can show that

$$\left\{H_i, X_\beta, Y_\beta \;\middle|\; i = 1, \ldots, 4,\ \beta \in \Phi^+\right\}$$

form a Chevalley basis, and the element ρ is given by

$$\rho = \frac{1}{2}\sum_{\beta\in\Phi^+} [X_\beta, Y_\beta].$$

The element I_+ now reads

$$I_+ = 16X_1 + 22X_2 + 30X_3 + 42X_4.$$

We fix a basis $\{\gamma_i\}_{i=1}^4$ of $V = \operatorname{Ker}\operatorname{ad} I_+$ as follows:

$$\gamma_1 = X_{0001} + \frac{5}{7}X_{0010} + \frac{11}{21}X_{0100} + \frac{8X_{1000}}{21}, \qquad \gamma_4 = X_{2243},$$
$$\gamma_2 = X_{0122} - \frac{8}{21}X_{1121} + \frac{128}{231}X_{2021}, \qquad \gamma_3 = X_{2122} + \frac{15}{8}X_{1132}.$$

By using the formulae given at the beginning of the present section, we can compute the reduced Poisson brackets w.r.t. the above basis. To present the result, we introduce the following flat coordinates

$$t_1 = w_4 - \frac{762841w_1^6}{49009212} - \frac{129973w_2w_1^3}{259308} - \frac{2783w_3w_1^2}{3528} - \frac{56741w_2^2}{142296},$$
$$t_2 = \frac{1781w_1^4}{64827} + \frac{34w_2w_1}{231} + w_3, \qquad t_3 = \frac{4199w_1^3}{63504} + \frac{4199w_2}{3696}, \qquad t_4 = -\frac{13w_1}{42}.$$

Then the two metrics given by the coefficients of the leading terms of the reduced Poisson brackets correspond to the flat pencil of metric of the Frobenius manifolds with potential

$$F = \frac{1}{2}t_1^2t_4 + t_1t_2t_3 + \frac{20736t_4^{13}}{143} + \frac{82944t_3^2t_4^7}{2527} + \frac{1083}{20}t_2^2t_4^5 + \frac{288}{19}t_2t_3^2t_4^3 + \frac{27648t_3^4t_4}{130321} + \frac{6859t_2^3t_4}{1152},$$

its Euler vector field is

$$E = \sum_{i=1}^4 E^i\frac{\partial}{\partial t_1} = t_1\frac{\partial}{\partial t_1} + \frac{2t_2}{3}\frac{\partial}{\partial t_2} + \frac{t_3}{2}\frac{\partial}{\partial t_3} + \frac{t_4}{6}\frac{\partial}{\partial t_4}.$$

In the coordinates t_i, the first metric g_1^{ij} given by the coefficients of the leading terms of the first Poisson structure has the standard expression [16]

$$\left(g_1^{ij}\right) = (\eta_{ij})^{-1}, \quad \eta_{ij} = \partial_{t_i}\partial_{t_j}\,Lie_e\,F, \tag{7.26}$$

where the unity vector field e is given by

$$e = \frac{\partial}{\partial t_1}. \tag{7.27}$$

The second metric g_2^{ij} satisfies the formula

$$g_2^{ij}(t) = \sum_{m=1}^{n} E^m c_m^{ij}(t), \quad \text{with } c_m^{ij}(t) = g_1^{ik} g_1^{jl} \partial_{t_m} \partial_{t_k} \partial_{t_l} F. \tag{7.28}$$

Here $n = 4$.

The coefficients $A_{1,0;a}^{ij}(a = 1, 2)$ of the reduced Poisson brackets are equal to zero. The coefficients $A_{2,0;2}^{ij}$ read

$$\begin{aligned}
A_{2,0,2}^{11} ={}& 238464 t_4^{10} - \frac{79854336 t_3 t_4^7}{4693} + \frac{362769128 t_2 t_4^6}{37349} + \frac{82248768000 t_3^2 t_4^4}{13482989} \\
&+ \frac{65740256 t_1 t_4^4}{371293} - \frac{286440}{247} t_2 t_3 t_4^3 + \frac{6443534125 t_2^2 t_4^2}{2689128} - \frac{53236224 t_3^3 t_4}{1694173} \\
&- \frac{4015872 t_1 t_3 t_4}{54587} + \frac{1656}{19} t_2 t_3^2 + \frac{443 t_1 t_2}{26}, \\
A_{2,0,2}^{12} ={}& -\frac{15818112 t_4^8}{4693} + \frac{42634554624 t_3 t_4^5}{13482989} - \frac{6453151372 t_2 t_4^4}{21163701} - \frac{51777792 t_3^2 t_4^2}{1694173} \\
&- \frac{28255104 t_1 t_4^2}{709631} + \frac{7349328 t_2 t_3 t_4}{54587} + \frac{153 t_2^2}{13}, \\
A_{2,0,2}^{13} ={}& \frac{204693422 t_4^7}{37349} - \frac{5205718984 t_3 t_4^4}{7054567} + \frac{9722937545 t_2 t_4^3}{5378256} + \frac{3133152 t_3^2 t_4}{54587} \\
&+ \frac{79 t_1 t_4}{4} - \frac{3611 t_2 t_3}{312}, \\
A_{2,0,2}^{14} ={}& \frac{16435064 t_4^5}{1113879} + \frac{14507020 t_3 t_4^2}{709631} - \frac{2783 t_2 t_4}{312}, \\
A_{2,0,2}^{22} ={}& \frac{13824 t_4^6}{19} - \frac{3170304 t_3 t_4^3}{89167} + \frac{4883336 t_2 t_4^2}{125229} + \frac{13824 t_3^2}{6859} + \frac{2400 t_1}{4693}, \\
A_{2,0,2}^{23} ={}& -\frac{2508 t_4^5}{13} + \frac{197596 t_3 t_4^2}{2197} + \frac{817 t_2 t_4}{312}, \\
A_{2,0,2}^{24} ={}& \frac{39412 t_4^3}{6591} - \frac{56 t_3}{247}, \qquad A_{2,0,2}^{34} = -\frac{2261 t_4^2}{624}, \qquad A_{2,0,2}^{44} = \frac{13}{24},
\end{aligned}$$

and the coefficients $A_{2,0;1}^{ij}$ are given by

$$A_{2,0;1}^{ij}(t) = \frac{\partial}{\partial t_1} A_{2,0;2}^{ij}(t). \tag{7.29}$$

Now we begin to compute the central invariants. We first find the canonical coordinates from the characteristic equation $\det(g_2^{ij} - \lambda g_1^{ij}) = 0$. The roots can be represented in the form

$$u_{\mu_1,\mu_2} = \left(t_1 + \frac{288}{19} t_3 t_4^3\right) + \mu_1 \left(\frac{57}{2} t_2 t_4^2 + \frac{288}{361} t_3^2\right)$$

$$+\mu_2\frac{(361t_2+\mu_1 576t_3t_4+2736t_4^4)^{\frac{3}{2}}}{228\sqrt{57}},$$

where $\mu_1,\mu_2=\pm 1$. We number them in the way that

$$u^1=u_{++},\qquad u^2=u_{+-},\qquad u^3=u_{-+},\qquad u^4=u_{--}.$$

We then compute the metrics g_1, g_2 and the functions $A_{2,0;1}$, $A_{2,0;1}$ in the canonical coordinates. After a straightforward computation, we obtain the central invariants from the formula (2.26), they read

$$\{c_1,c_2,c_3,c_4\}=\left\{\frac{1}{24},\frac{1}{24},\frac{1}{12},\frac{1}{12}\right\},$$

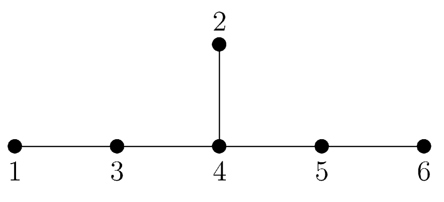
which proves the lemma for the F_4 case.

The E_6 case

The proof of the lemma for the simple Lie algebras of E types are similar to that of the F_4 case. We take E_6 for example. It has a 27-dimensional matrix realization [31], the Weyl generators are realized as

$$X_1=e_{6,7}+e_{8,9}+e_{10,11}+e_{12,14}+e_{15,17}+e_{26,27},$$
$$X_2=e_{4,5}+e_{6,8}+e_{7,9}-e_{18,20}-e_{21,22}-e_{23,24},$$
$$X_3=e_{4,6}+e_{5,8}+e_{11,13}+e_{14,16}+e_{17,19}+e_{25,26},$$
$$X_4=e_{3,4}-e_{8,10}-e_{9,11}-e_{16,18}-e_{19,21}+e_{24,25},$$
$$X_5=e_{2,3}-e_{10,12}-e_{11,14}-e_{13,16}+e_{21,23}+e_{22,24},$$
$$X_6=e_{1,2}+e_{12,15}+e_{14,17}+e_{16,19}+e_{18,21}+e_{20,22},$$

and $Y_i={X_i}^T$, $i=1,\dots,6$. The Dynkin diagram for these generators are given by



The normalized Killing form is

$$\langle A,B\rangle_{\mathfrak g}=\frac{1}{6}\,\mathrm{tr}(AB).$$

The Chevalley basis is defined in the same way as we did above for the F_4 case. The element I_+ reads

$$I_+=16X_1+22X_2+30X_3+42X_4+30X_5+16X_6.$$

The basis $\{\gamma_i\}_{i=1}^6$ of $V = \operatorname{Ker} \operatorname{ad} I_+$ is chosen as

$$
\begin{aligned}
\gamma_1 &= X_1 + \frac{11}{8}X_2 + \frac{15}{8}X_3 + \frac{21}{8}X_4 + \frac{15}{8}X_5 + X_6,\\
\gamma_2 &= X_{001111} - \frac{11}{15}X_{010111} + X_{101110} + \frac{11}{15}X_{111100},\\
\gamma_3 &= X_{011111} - \frac{21}{8}X_{011210} - \frac{16}{11}X_{101111} - X_{111110},\\
\gamma_4 &= X_{011221} + \frac{8}{15}X_{111211} + X_{112210},\\
\gamma_5 &= X_{111221} + X_{112211},\\
\gamma_6 &= X_{122321}.
\end{aligned}
$$

The flat coordinates have the expressions

$$
\begin{aligned}
t_1 &= \frac{5339887w_1^6}{84934656} + \frac{129973w_3w_1^3}{442368} - \frac{2783w_4w_1^2}{77760}\\
&\quad + \frac{1679w_2^2w_1}{24300} + \frac{56741w_3^2}{1672704} + \frac{w_6}{81},\\
t_2 &= \frac{4}{15}w_2w_1^2 + \frac{2w_5}{27}, \qquad t_3 = \frac{33839w_1^4}{147456} + \frac{2261w_3w_1}{12672} - \frac{38w_4}{405},\\
t_4 &= \frac{1547w_1^3}{9216} + \frac{221w_3}{528}, \qquad t_5 = \frac{52w_2}{135}, \qquad t_6 = \frac{13w_1}{8}.
\end{aligned}
$$

The potential of the corresponding Frobenius manifold is given by

$$
\begin{aligned}
F = -\frac{3^8}{2}\bigg(&\frac{1}{2}t_1^2t_6 + t_1t_2t_5 + t_1t_3t_4 + \frac{t_6^{13}}{185328} + \frac{1}{576}t_5^2t_6^8 + \frac{1}{252}t_4^2t_6^7\\
&+ \frac{1}{60}t_3^2t_6^5 + \frac{1}{24}t_4t_5^2t_6^5 + \frac{1}{24}t_2^2t_6^4 + \frac{1}{24}t_3t_5^2t_6^4 + \frac{1}{24}t_5^4t_6^3 + \frac{1}{6}t_3t_4^2t_6^3\\
&+ \frac{1}{6}t_2t_4t_5t_6^3 + \frac{1}{4}t_4^2t_5^2t_6^2 + \frac{1}{2}t_2t_3t_5t_6^2 + \frac{1}{12}t_4^4t_6 + \frac{1}{6}t_3^3t_6 + \frac{1}{6}t_2t_5^3t_6\\
&+ \frac{1}{2}t_3t_4t_5^2t_6 + \frac{1}{2}t_2^2t_4t_6 + \frac{1}{12}t_4t_5^4 + \frac{1}{4}t_3^2t_5^2 + \frac{1}{2}t_2^2t_3 + \frac{1}{2}t_2t_4^2t_5\bigg). \qquad (7.30)
\end{aligned}
$$

Note that the function $-\frac{2}{3^8}F(t)$ was obtained as polynomial solutions of the WDVV equations associated to the root systems of type E_6 by P. Di Francesco et al. in [10]. Polynomial solutions to the WDVV equations associated to the root systems of type E_7 and E_8 are also computed in [10].

The Euler vector field and the unity vector field have the forms

$$E=\sum_{k=1}^{6}E^k\frac{\partial}{\partial t_k}=t_1\frac{\partial}{\partial t_1}+\frac{3}{4}t_2\frac{\partial}{\partial t_2}+\frac{2}{3}t_3\frac{\partial}{\partial t_3}+\frac{1}{2}t_4\frac{\partial}{\partial t_4}+\frac{5}{12}t_5\frac{\partial}{\partial t_5}+\frac{1}{6}t_6\frac{\partial}{\partial t_6}, \tag{7.31}$$

$$e=\frac{1}{81}\frac{\partial}{\partial t_1}. \tag{7.32}$$

The two flat metrics g_1, g_2 are expressed by the formulae given in (7.26), (7.28). We will not write down the explicit expression of the functions $A^{ij}_{2,0;1}$, $A^{ij}_{2,0;2}$, since in this case they are quite long. As a consequence of this fact, the computation of the central invariants by using the formula (2.26) becomes rather tedious. To avoid this complexity, we employ an alternative way to prove the result that the central invariants of the Drinfeld–Sokolov bihamiltonian structure related to the E_6 (also for E_7, E_8) type simple Lie algebra are equal to $\frac{1}{24}$.

Our approach is to establish, through an appropriate Miura-type transformation, a relationship of the present bihamiltonian structure to the one defined by a semisimple Frobenius manifold via the formulae of Theorems 1 and 2 of [20]. Then the needed result follows if we can prove that the central invariants of the bihamiltonian structure given by Theorems 1 and 2 of [20] are equal to $\frac{1}{24}$. This fact can be proved by using properties of a semisimple Frobenius manifold. In fact, by using the formulae (3.9), (3.14), (3.15), (5.24) of [20] we can express the functions f^i, Q^{ii}_1, Q^{ii}_2, P^{ki}_1, P^{ki}_2 that appear in (2.26) as follows:

$$f^i=\frac{1}{\psi_{i1}^2},\qquad P^{ki}_1=P^{ki}_2=0,$$

$$Q^{ii}_1=\frac{1}{12}\sum_{j=1}^{n}\left(\frac{\gamma_{ij}}{\psi_{i1}^3\psi_{j1}}+\frac{\gamma_{ij}\psi_{j1}}{\psi_{i1}^5}\right),$$

$$Q^{ii}_2=\frac{1}{24}\left[\frac{1}{3\psi_{i1}^4}+2\sum_{j=1}^{n}\left(\frac{u^i\gamma_{ij}}{\psi_{i1}^3\psi_{j1}}+\frac{u^i\gamma_{ij}\psi_{j1}}{\psi_{i1}^5}\right)\right].$$

Here n is the dimension of the semisimple Frobenius manifold, $u^1,\dots,u^n$ are its canonical coordinates, the functions γ_{ij} are the rotation coefficients of the flat metric of the Frobenius manifold, and the functions ψ_{i1} are defined by (4.5) of [20]. By plugging the above expressions into the formula (2.26) we immediately obtain the result $c_i=\frac{1}{24}$, $i=1,\dots,n$.

Now let us assume that the needed Miura-type transformation has the form

$$\tilde{t}_i=t_i-\epsilon^2\left(\sum_m K^i_m t_{m,xx}+\sum_{k,l}M^i_{kl}t_{k,x}t_{l,x}\right),\quad i=1,\dots,6,$$

A straightforward computation shows that there is a unique choice of the $(1,1)$ tensor K^i_j with the following nonzero components:

$$K^1_3=\frac{92t_6}{247},\qquad K^1_4=\frac{1172287t_6^2}{2016846},\qquad K^1_5=\frac{3197t_5}{4056},\qquad K^4_6=\frac{17t_6}{26},$$

$$K^2_5=\frac{460t_6}{507},\qquad K^2_6=\frac{502t_5}{507},\qquad K^3_4=\frac{19}{39},\qquad K^3_6=\frac{47120t_6^2}{59319},$$

$$K_6^1 = \frac{2054383 t_6^4}{60149466} + \frac{7521 t_4 t_6}{5746} + \frac{115 t_3}{247}.$$

such that in the new coordinates, the coefficients of $\epsilon^2\delta'''(x-y)$ of our reduced bihamiltonian structure can be expressed, in terms of the potential $F(\tilde{t}) = F(t)|_{t\mapsto\tilde{t}}$ given in (7.30), by the following formulae of Theorems 1 and 2 of [20]:

$$A^{ij}_{2,0;1}(\tilde{t}) = \frac{1}{12}\partial_{t_k}\left(g_1^{kl} c_l^{ij}\right), \tag{7.33}$$

$$A^{ij}_{2,0;2}(\tilde{t}) = \frac{1}{12}\left(\partial_{t_k}\left(g_2^{kl} c_l^{ij}\right) + \frac{1}{2} c_l^{kl} c_k^{ij}\right), \tag{7.34}$$

where $g_1^{ij}(\tilde{t})$, $g_2^{ij}(\tilde{t})$, $c_k^{ij}(\tilde{t})$ are defined as in (7.26), (7.28) by using the function $F(t)$ and the vector fields (7.31), (7.32), and then replacing t by $\tilde{t}$.

Since in the present case the central invariants are determined by the coefficients of $\epsilon^2\delta'''(x-y)$ of the bihamiltonian structure, the above Miura-type transformation (with arbitrary chosen functions M^i_{kl}) already establishes the fact that all the central invariants of the bihamiltonian structure that we are considering are equal to $\frac{1}{24}$.

For the simple Lie algebra of type E_7, E_8, we give the relevant data in the Appendices A and B of the preprint version [18] of the present paper. The notations in [18] are in agreement with that of the above E_6 case. We thus complete the proof of the lemma. □

8. Conclusion

In this paper, we compute the central invariants of the bihamiltonian structures of Drinfeld–Sokolov reduction related to the affine Kac–Moody algebras of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $G_2^{(1)}$, $F_4^{(1)}$, $E_{6,7,8}^{(1)}$ with the standard gradation which is given by the vertex c_0 in the (extended) Dynkin diagram. Our result is based on a case by case computation. It is interesting to give a case independent derivation of the central invariants of the Drinfeld–Sokolov bihamiltonian structures. One of the main difficulty lies in the fact that the central invariants depend in general explicitly on the canonical coordinates, although in our present cases all the invariants we obtained are constants. Examples of bihamiltonian structures that possess nonconstant central invariants can be found in [40]. The results of [22] suggest that constancy of the central invariants is related to existence of a *tau-structure* of the hierarchy.

For the standard gradations defined by another vertex, Drinfeld and Sokolov did not give the bihamiltonian structures. We point out that the generalized KdV equations for other standard gradations do possess bihamiltonian structures of the form (6.7), but in general these bihamiltonian structures have infinite many terms. This is because all these equations are related to the generalized mKdV equations through a Miura-type transformation, while these transformations are invertible in the formal power series sense. This fact has an immediate corollary that the central invariants of these bihamiltonian structures are the same with the ones we have computed.

The generalized KdV equations related to the *twisted* affine Lie algebras seem not to possess a bihamiltonian structure. We give here a counterexample [39].

Let us consider the generalized KdV equation related to $A_2^{(2)}$ equipped with the standard gradation defined by the vertex c_0. The simplest integrable equation reads [13]

$$w_t = 5w^2 w_x + 5\epsilon^2 (w_x w_{xx} + w w_{xxx}) + \epsilon^4 w_{xxxxx}. \tag{8.1}$$

Proposition 8.1. *Eq.* (8.1) *possesses only one local Hamiltonian structure found in* [13]

$$w_t = \{w(x), H\}, \qquad H = \int (w^3 - 3w_x^2)\, dx,$$

$$\{w(x), w(y)\} = 2w(x)\delta'(x-y) + w_x \delta(x-y) + \frac{\epsilon^2}{2}\delta'''(x-y). \tag{8.2}$$

The proof was obtained in [39] following the scheme of [41]. Let us give here the sketch of the proof. First, we construct the so-called *quasitriviality transformation* $v \mapsto w$,

$$\begin{aligned}
w = {} & v + \frac{\epsilon^2}{4}\partial_x^2(\log v_1 - \log v) \\
& + \epsilon^4 \partial_x^2 \left(\frac{32 v_2}{v^2} - \frac{27 v_1^2}{v^3} - \frac{12 v_3}{v v_1} + \frac{5 v_4}{v_1^2} + \frac{11 v_2^2}{v v_1^2} - \frac{21 v_3\, v_2}{v_1^3} + \frac{16 v_2^3}{v_1^4} \right) \\
& + \epsilon^6 \partial_x^2 \left[\frac{7533 v_1^4}{4480 v^6} - \frac{43081 v_2 v_1^2}{13440 v^5} + \frac{2063 v_3 v_1}{1920 v^4} + \frac{2077 v_2^2}{3360 v^4} - \frac{619 v_4}{2240 v^3} \right. \\
& + \frac{1}{v_1}\left(\frac{239 v_2 v_3}{1344 v^3} + \frac{41 v_5}{672 v^2} \right) - \frac{1}{v_1^2}\left(\frac{157 v_2^3}{1920\, v^3} + \frac{1549 v_4 v_2}{6720 v^2} + \frac{13 v_3^2}{80 v^2} + \frac{7 v_6}{640 v} \right) \\
& + \frac{1}{v_1^3}\left(\frac{8753 v_3 v_2^2}{13440\, v^2} + \frac{383 v_5 v_2}{4480 v} + \frac{689 v_3 v_4}{4480 v} + \frac{v_7}{384} \right) \\
& - \frac{1}{v_1^4}\left(\frac{4303 v_2^4}{13440 v^2} + \frac{185 v_4 v_2^2}{448 v} + \frac{2607 v_3^2 v_2}{4480 v} + \frac{21 v_2 v_6}{640} + \frac{103 v_4^2}{2240} + \frac{159 v_3 v_5}{2240} \right) \\
& + \frac{1}{v_1^5}\left(\frac{9343 v_3 v_2^3}{6720 v} + \frac{1059 v_5 v_2^2}{4480} + \frac{3819 v_3 v_4 v_2}{4480} + \frac{177 v_3^3}{896} \right) \\
& \left. - \frac{1}{v_1^6}\left(\frac{131 v_2^5}{210 v} + \frac{83}{70} v_4 v_2^3 + \frac{2241}{896} v_3^2 v_2^2 \right) + \frac{59}{14}\frac{v_2^4 v_3}{v_1^7} - \frac{5}{3}\frac{v_2^6}{v_1^8} \right] + O(\epsilon^8)
\end{aligned} \tag{8.3}$$

transforming any monotone solution of the dispersionless equation

$$v_t = 5v^2 v_x$$

to a solution of (8.1). In this long formula we denote the jet coordinates by $v_1 = v_x$, $v_2 = v_{xx}$ etc. According to [39,41], any local Hamiltonian structure of (8.1) with coefficients depending *polynomially* on the jet coordinates $w_x, w_{xx}, \ldots$ must be obtained from some dispersionless Hamiltonian structure of the form

$$\{v(x), v(y)\} = \varphi(v(x))\delta'(x-y) + \frac{1}{2}\varphi'(v(x)) v_x(x)\delta(x-y)$$

by applying the quasitriviality transformation (8.3). The unknown function $\varphi(v)$ has to be chosen in such a way to ensure cancellation of all the jet dependent denominators in the transformed bracket

$$\{w(x), w(y)\} = \varphi(w)\delta'(x-y) + \frac{1}{2}\varphi'(w)w_x\delta(x-y) + \epsilon^2 Z_2 + \epsilon^4 Z_4 + \epsilon^6 Z_6 + \cdots. \quad (8.4)$$

Here Z_2 is a polynomial for any $\varphi(v)$, while Z_4 contains the following term

$$-\frac{3}{160}\frac{w_{xx}^4}{w^2 w_x^4}\left[3w^2\varphi''(w) - 2w\varphi'(w) + 2\varphi(w)\right]\delta'(x-y).$$

So, to ensure Z_4 is a polynomial, we must have

$$\varphi(w) = c_1 w + c_2 w^{\frac{2}{3}}$$

for some constants c_1 and c_2. Next, Z_6 contains the following term

$$\frac{5c_2 w_{xx}^5}{432 w^{10/3} w_x^4}\delta'(x-y),$$

which implies $c_2 = 0$. So we have $\varphi(w) = c_1 w$, by taking $c_1 = 2$, we obtain the Hamiltonian structure (8.2). The proposition is proved.

In a similar way we have analyzed another example of an integrable scalar equation associated with $A_2^{(2)}$. It would be interesting to prove in general that the Drinfeld–Sokolov hierarchies associated with twisted Kac–Moody Lie algebras never admit a local bihamiltonian structure.

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