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On an Isomonodromy Deformation Equation without the Painlevé Property

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Abstract. We show that the fourth-order nonlinear ODE which controls the pole dynamics in the general solution of equation P_{I}^2 compatible with the KdV equation exhibits two remarkable properties: (1) it governs the isomonodromy deformations of a 2×2 matrix linear ODE with polynomial coefficients, and (2) it does not possess the Painlevé property. We also study the properties of the Riemann–Hilbert problem associated to this ODE and find its large-t asymptotic solution for physically interesting initial data.

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1. INTRODUCTION

The study of relationships between the theory of isomonodromic deformations and the theory of differential equations satisfying the so-called Painlevé property is a well-established branch of the analytic theory of differential equations in complex domain (see below a brief summary of the most important results obtained in this direction). One of the outputs of the present paper suggests that the above mentioned relationship is less straightforward than was traditionally believed. We illustrate the point with an example of a fourth-order ODE for a function $a = a(t)$

$$a'''' + 120(a')^3a'' - 120a'a''t - \tfrac{200}{3}(a')^2 - \tfrac{40}{3}aa'' + \tfrac{200}{9}t = 0. \qquad P_{\mathrm{I}}^{(2,1)}$$

This equation appeared in [42] in the study of pole loci of solutions to a degenerate Garnier system. Our observation is that Eq. $P_{\mathrm{I}}^{(2,1)}$ governs isomonodromic deformations of a certain linear differential operator with polynomial coefficients (see Eq. (1.5) below). However, this equation does not satisfy the Painlevé property, since its general solution has third-order branch points of the form

$$a(t) = a_0 - (t-b)^{1/3} + \mathcal{O}\left((t-b)^{5/3}\right), \quad b\in\mathbb{C} \tag{1.1}$$

(cf. [42]), where the location of the branch points depends on the choice of the solution (the so-called *movable critical singularities*).

In spite of this somewhat surprising phenomenon, the method of isomonodromic deformations proves to be almost as powerful in the study of solutions to the equation $P_{\mathrm{I}}^{(2,1)}$ as in the case of classical Painlevé equations. Namely, it is possible to derive the large-t asymptotics of solutions and, moreover, to describe the branching locus of a given solution in terms of a kind of a spectral problem for a quintic anharmonic oscillator.

Equation $P_{\mathrm{I}}^{(2,1)}$ is of interest on its own. Namely, it describes the behavior of poles of solutions to another fourth-order ODE

$$u_{xxxx} + 10u_x^2 + 20uu_{xx} + 40\big(u^3 - 6tu + 6x\big) = 0 \tag{1.2}$$

usually denoted as P_{I}^2. It is the second member of the so-called P_{I} hierarchy. The coefficients of this equation depend on t as on a parameter. It is well known (see, e.g., [30]) that the equation (1.2) is compatible with the Korteweg–de Vries (KdV) dynamics

$$u_t + uu_x + \tfrac{1}{12}u_{xxx} = 0. \tag{1.3}$$

The Laurent expansion of solutions $u = u(x,t)$ to the system (1.2), (1.3) near a pole $x = a(t)$ has the form

$$u = -\frac{1}{(x-a(t))^2} + \mathcal{O}\left((x-a(t))^2\right), \tag{1.4}$$

where the function $a(t)$ solves[1] Eq. $P_{\mathrm{I}}^{(2,1)}$. In this way, one associates a multivalued solution $a(t)$ to Eq. $P_{\mathrm{I}}^{(2,1)}$ with any solution to Eq. $P_{\mathrm{I}}{}^2$. The branch points (1.1) of $a(t)$ correspond to the triple collisions of the poles of solutions to the $P_{\mathrm{I}}{}^2$. Thus, the above mentioned isomonodromy realization of Eq. $P_{\mathrm{I}}^{(2,1)}$ provides one with a tool for studying the KdV dynamics of poles of solutions to the equation $P_{\mathrm{I}}{}^2$.

Particular solutions to the equation $P_{\mathrm{I}}{}^2$are of interest. We concentrate our attention on one of them, namely, the one that has no poles on the real axis $x \in \mathbb{R}$ (see [3, 11, 12, 5] about the importance of this special solution to Eq. $P_{\mathrm{I}}{}^2$). Such a solution $u(x,t)$ exists [7] for any real t and it is uniquely determined by its asymptotic behavior for large $|x|$. Using the developed techniques along with the isomonodromic description [26] of the special solution to $P_{\mathrm{I}}{}^2$, we arrive at the asymptotic description for large t of poles of this special solution.

Before proceeding to the formulation of main results, let us first briefly recall the basic notions of the present paper, namely, the Painlevé property and the isomonodromic deformations. In the description of the historical framework, we mainly follow the paper [21].

1.1. Painlevé Property

In 1866, L. Fuchs [14] showed that all the singular points of solutions to a linear ODE are among the singularities of its coefficients and thus are independent of the initial conditions. In the nonlinear case, the reasonable problem is to look for ODEs defining the families of functions, called the *general solutions*, which can meromorphically be extended to the universal covering space of a punctured Riemann surface with the punctures determined by the equation. In other words, the problem is to find ODEs whose general solutions are free from branch points and essential singularities depending on the specific choice of the initial data. This property is now called the *Painlevé property*, or the *analytic Painlevé property*, and is obviously shared not only by linear ODEs but also by the ODEs for elliptic functions.

In [15], L. Fuchs started the classification of the first-order ODEs polynomial in u and u', $F(x,u,u') = 0$, with coefficients single-valued in x with respect to the Painlevé property. Poincaré [34] and Painlevé [32] accomplished the analysis without finding new transcendental functions.

In [33], Painlevé revisited the L. Fuchs' idea, extending the program of classification to second-order ODEs of the form $u'' = F(x,u,u')$ with F meromorphic in x and rational in u and u'. In the course of classification of the 2nd order 1st degree ODEs modulo the Möbius transformation [18, 24], 50 equations were found that pass the Painlevé α-test and thus are now called the *Painlevé–Gambier equations*. It occurs that all these equations can be either integrated in terms of the classical linear transcendents or elliptic functions or reduced to one of the six exceptional *classical Painlevé equations* P_{I}–P_{VI}.

1.2. Isomonodromic Deformations

The monodromy group to a linear ODE was first considered by Riemann [37], Schwarz [40] and Poincaré [35]. Apparently, it was L. Fuchs [16] who first set the problem of deformations for the coefficients in a linear equation that leave the monodromy group unchanged. Namely, assuming that solutions of a linear ODE depend on an additional variable, he obtained a system of first-order PDEs that the solutions must satisfy.

A more modern treatment of isomonodromy deformations was developed by R. Fuchs in [17]. He has shown that the monodromy group of a scalar linear ODE with four Fuchsian singularities at $\lambda = 0, 1, \infty, x$ and an apparent singular point at $\lambda = u$ does not depend on the location x of the fourth Fuchsian singular point if the location of the apparent singularity u depends on x according

[1]The connection of Eq. $P_{\mathrm{I}}^{(2,1)}$ with the KdV equation was not considered in [42].

to a nonlinear 2nd order ODE. Later, Fuchs' isomonodromic deformation equation was included as the sixth Painlevé equation P_{VI} into the list of the classical Painlevé equations.

In 1912, Schlesinger [39] generalized the Fuchs' approach, finding the equations of the isomonodromic deformations for arbitrary linear Fuchsian ODEs, while Garnier [19] presented scalar second-order linear ODEs with irregular singular points whose isomonodromic deformations are governed by the lower classical Painlevé equations P_{I}–P_{V}. In 1980, Jimbo, Miwa, and Ueno [25] extended Garnier theory to linear ODEs with generic irregular singularities.

1.3. Painlevé Property of the Isomonodromy Deformation Equations

It is interesting that to the date of the achievements mentioned above, the fact that these equations possess the Painlevé property indeed was not proved rigorously even for the case of the classical Painlevé equations. Thus, the term "Painlevé equation" loosely refers to various equations among which we mention the higher-order ODEs in hierarchies associated with the classical Painlevé equations, the higher-order ODEs in the classifications by Bureau [2] and Cosgrove [8] based on the use of some Painlevé tests, as well as the differential, difference, q-difference, and elliptic-difference equations found in the course of the study of the symmetries and geometry of the classical Painlevé equations, see, e.g., [31, 38].

A general elegant approach to the Painlevé property of the equations of the isomonodromic deformations was presented by Miwa [29] and Malgrange [28] in the early 80s of the last century (recall that the direct proof of the Painlevé property of the classical Painlevé equations appeared even later, see [22]). The approach by Miwa and Malgrange is based on the use of the zero curvature representation and the Riemann–Hilbert correspondence. In fact, they have proved the analytic Painlevé property of the isomonodromic deformation equations for arbitrary linear Fuchsian ODEs and for equations with unbranched irregular singular points. In this respect, we also mention the papers of Inaba and Saito [23], who developed an algebro-geometric approach to the *geometric Painlevé property* of the isomonodromic deformations of the logarithmic and unramified irregular connections, which also implies their analytic Painlevé property.

1.4. Outline of the Paper and Main Results

One can expect that the isomonodromy deformation equations for arbitrary rational connections possess the Painlevé property. However, the naïve induction does not work. Namely, and this is our first result, we show that the ODE ($P_{\mathrm{I}}^{(2,1)}$), which is polynomial in all variables, 4th order, and 1st degree, governs the isomonodromic deformations of the following linear differential equation with polynomial coefficients:

$$\frac{dZ}{d\lambda} = \begin{pmatrix} -\frac{3}{20}a'' & \frac{1}{30}\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 \\ \frac{1}{30}\lambda - \alpha_3 & \frac{3}{20}a'' \end{pmatrix} Z. \tag{1.5}$$

Here $a = a(t)$, $\alpha_k = \alpha_k(t)$, $k = 0, 1, 2, 3$, are some smooth functions.

Proposition 1.1. *The monodromy*[2] *of system* (1.5) *does not depend on* t *if and only if the coefficients have the form*

$$\begin{aligned}
\alpha_3 &= -\frac{1}{10}a' + \alpha_3^0, \quad \alpha_2 = \frac{3}{10}a'^2 - 6\alpha_3^0 a' - t + \alpha_2^0,\\
\alpha_1 &= -\frac{9}{10}a'^3 + 27\alpha_3^0 a'^2 - 3[\alpha_2^0 + 60(\alpha_3^0)^2 - t]a' + a - 40\alpha_3^0 t + \alpha_1^0,\\
\alpha_0 &= -\frac{9}{20}a''' - \frac{54}{5}a'^4 + 432\alpha_3^0 a'^3 + 18\left[t - 330(\alpha_3^0)^2 - \alpha_2^0\right]a'^2,\\
&\quad + 3\left[a - 130\alpha_3^0 t + 10800(\alpha_3^0)^3 + 90\alpha_2^0\alpha_3^0 + \alpha_1^0\right] a' - 30\alpha_3^0\, a\\
&\quad + 2100(\alpha_3^0)^2 t - 54000(\alpha_3^0)^4 - 900\alpha_2^0(\alpha_3^0)^2 - 30\alpha_1^0\alpha_3^0,
\end{aligned}$$

[2] Since the system (1.5) has only an irregular singularity at infinity, here isomonodromicity means independence of t for the Stokes multipliers of the system, see below.

where α_1^0, α_2^0, α_3^0 *are some constants, together with the following ODE for the function* $a = a(t)$:

$$\begin{aligned} a^{IV} &+ \big[120a'^3 - 3600\alpha_3^0 a'^2 + 120\left(\alpha_2^0 + 270(\alpha_3^0)^2 - t\right)a' \\ &- \frac{40}{3}\left(a - 100\alpha_3^0 t + 6300(\alpha_3^0)^3 + 60\alpha_2^0\alpha_3^0 + \alpha_1^0\right)\Big] a'' - \frac{200}{3}a'^2 + \frac{200}{9}t \\ &+ \frac{4000}{3}\alpha_3^0 a' - 6000(\alpha_3^0)^2 - \frac{200}{9}\alpha_2^0 = 0. \end{aligned} \tag{1.6}$$

After a change

$$t \mapsto t + \alpha_2^0 - 30(\alpha_3^0)^2, \quad a \mapsto a + 10\alpha_3^0 t - \alpha_1^0 + 40\alpha_2^0\alpha_3^0 - 300(\alpha_3^0)^3,$$

the equation (1.6) reduces to the equation $P_{\mathrm{I}}^{(2,1)}$.

So, our claim is that the equation ($P_{\mathrm{I}}^{(2,1)}$) describes the isomonodromic deformations of the system (1.5); however, it does not possess the Painlevé property since it is satisfied by a 4-parameter Puiseux series in powers of $(t-b)^{1/3}$. Nevertheless, as we show, this equation can be effectively analyzed using the isomonodromy deformation techniques developed during the last decades. In particular, the ramification points $t = b$ (see Eq. (1.1) above) can be determined by a kind of a spectral problem for the quintic anharmonic oscillator

$$\frac{d^2y}{d\lambda^2} = \tfrac{1}{30}U(\lambda)\,y, \qquad U(\lambda) = \tfrac{1}{30}\lambda^5 - b\lambda^3 + a_0\lambda^2 + (\tfrac{360}{49}b^2 + \tfrac{33}{10}a_9)\lambda - \tfrac{230}{21}a_0 b + \tfrac{143}{30}a_{11}. \tag{1.7}$$

Here a_0, a_9, a_{11} are the coefficients of the Puiseux expansion of $a(t)$ (see Eq. (4.1) below). Namely, the coefficients of the quintic polynomial must be chosen in such a way that the equation (1.7) possesses solutions exponentially decaying on certain contours in the complex plane; for details, see Remark 4.1.

In Section 2, we discuss the second member of the P_{I} hierarchy, equation ${P_{\mathrm{I}}}^2$, as an equation that describes the isomonodromic solutions to the KdV equation, and find a nonlinear ODE, equation $P_{\mathrm{I}}^{(2,1)}$, that controls its pole dynamics. In Section 3, we derive the above isomonodromic representation for Eq. $P_{\mathrm{I}}^{(2,1)}$presenting a regularization of the linear system for ${P_{\mathrm{I}}}^2$along its singularity locus. In Section 4, we consider the Puiseux series solution for $P_{\mathrm{I}}^{(2,1)}$and the singularity reduction of the corresponding linear system at the branch points. In Section 5, we set the Riemann–Hilbert problems for the previously introduced wave functions and discuss the existence and uniqueness of their solutions. In Section 6, we present the asymptotic analysis of the Riemann–Hilbert problems corresponding to a physically interesting special solution of ${P_{\mathrm{I}}}^2$and $P_{\mathrm{I}}^{(2,1)}$, implementing the steepest-descent method introduced by Deift and Zhou. Our main result in that section is the description of the large-t asymptotics of the singularity locus as a theta-divisor on a modulated elliptic curve,

$$3x_0 + \tfrac{2}{3}\xi_1(x_0) = \tfrac{7}{4}t^{-7/4}\big((n+\tfrac{1}{2})\omega_a(x_0) + (m+\tfrac{1}{2})\omega_b(x_0)\big),$$

where $n \in \mathbb{Z}$, $m \in \mathbb{Z}_+$ and $x_0 = a(t)t^{-3/2}$, the function $\xi_1(x_0)$ is determined by equations (6.6)–(6.8), $\omega_a(x_0)$ and $\omega_b(x_0)$ are the period integrals (6.11) on the elliptic curve (6.10). In Section 7, we find explicitly the pole dynamics of the special solution in a vicinity of the attracting point $x_0^* = \frac{2\sqrt{5}}{9\sqrt{3}}$,

$$\begin{aligned} a^{(m,n)}(t) = \tfrac{2\sqrt{5}}{9\sqrt{3}}\,t^{3/2} &+ t^{-1/4}(m+\tfrac{1}{2})\tfrac{\sqrt[4]{3}}{2\sqrt[4]{5}\sqrt{7}}\ln\big(t^{-7/4}(m+\tfrac{1}{2})\tfrac{3^{11/4}}{2^6 5^{3/4} 7^{5/2} e}\big) \\ &+ i\pi(n+\tfrac{1}{2})\tfrac{\sqrt[4]{3}}{\sqrt[4]{5}\sqrt{7}} + \mathcal{O}(t^{-7/2}\ln^2 t), \end{aligned} \tag{1.8}$$

where $m \in \mathbb{Z}_+$ and $n \in \mathbb{Z}$ enumerate the points of the pole lattice. Thus, asymptotically for large $t \to +\infty$, the poles of the special solution to ${P_{\mathrm{I}}}^2$ never collide. It would be interesting to study their collisions for finite time. In Section 8, we briefly discuss our results and various open problems.

2. LINEAR SYSTEM FOR $P_{\rm I}{}^2$ AND ITS SINGULARITY LOCUS

2.1. Linear System for $P_{\rm I}{}^2$

The 4th-order 1st-degree polynomial ODE,

$$u_{xxxx} + 10u_x^2 + 20uu_{xx} + 40\big(u^3 - 6tu + 6x\big) = 0, \tag{2.1}$$

usually denoted as $P_{\rm I}{}^2$, is the second member of the so-called $P_{\rm I}$ hierarchy. This equation governs the isomonodromy deformations of a linear polynomial ODE described by the system

$$\Psi_\lambda = A\Psi, \quad \Psi_x = B\Psi, \quad \Psi_t = C\Psi, \tag{2.2}$$

where the connection matrices are explicitly written using the generators of $\mathfrak{sl}(2,\mathbb{C})$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$$\begin{aligned} A = \tfrac{1}{240}\big\{&\big[-4u_x\lambda - (12uu_x + u_{xxx})\big]\sigma_3 + \big[8\lambda^2 + 8u\lambda + (12u^2 + 2u_{xx} - 120t)\big]\sigma_+ \\ &+ \big[8\lambda^3 - 8u\lambda^2 - (4u^2 + 2u_{xx} + 120t)\lambda + (16u^3 - 2u_x^2 + 4uu_{xx} + 240x)\big]\sigma_-\big\}, \end{aligned} \tag{2.3a}$$

$$B = \sigma_+ + (\lambda - 2u)\sigma_-, \tag{2.3b}$$

$$C = \tfrac{1}{6}u_x\sigma_3 - \tfrac{1}{3}(\lambda + u)\sigma_+ - \tfrac{1}{3}(\lambda^2 - u\lambda - 2u^2 - \tfrac{1}{2}u_{xx})\sigma_-. \tag{2.3c}$$

Along with (2.1), which is the compatibility condition

$$[\partial_\lambda - A, \partial_x - B] = 0$$

of (2.3a) and (2.3b), the system (2.3) also implies the KdV equation,

$$u_t + uu_x + \tfrac{1}{12}u_{xxx} = 0, \tag{2.4}$$

and equations which follow from (2.1) and (2.4). The solutions of $P_{\rm I}{}^2$(2.1) compatible with the KdV equation (2.4) are called the *isomonodromic* $P_{\rm I}{}^2$ solutions to KdV.

2.2. Laurent Series Solutions to $P_{\rm I}{}^2$ Compatible with the KdV Equation

As is known, equation $P_{\rm I}{}^2$ passes the Painlevé tests as being presented in the list by Cosgrove [8] under the symbol F-V with $y = -u$ and the parameters $\alpha = 240t$, $k = 240$ and $\beta = 0$. See [41] for a proof of the Painlevé property based on the Riemann–Hilbert correspondence.

Below, we are especially interested in the 4-parameter series solution to $P_{\rm I}{}^2$ with the following initial terms (because $P_{\rm I}{}^2$ is polynomial in all its variables, the construction of the complete formal series via a recurrence relation is straightforward),

$$\begin{aligned} &u(x) = -\frac{1}{(x-a)^2} + \sum_{k=0}^{\infty} c_k (x-a)^k, \\ &c_1 = 0, \quad c_2 = 3(c_0^2 - 2t), \quad c_4 = \tfrac{30}{7}a - 10c_0^3 + \tfrac{120}{7}c_0 t, \\ &c_5 = 3 - \tfrac{3}{2}c_0c_3, \quad c_7 = \tfrac{12}{7}(tc_3 - c_0), \quad \ldots, \end{aligned} \tag{2.5}$$

a, c_0, c_3, c_6 are arbitrary.

This 4-parameter series is compatible with the KdV equation (2.4) if the coefficients c_0, c_3, c_6 depend on t in a particular way described by the function $a(t)$,

$$c_0 = a', \quad c_3 = 2a'', \quad c_6 = -\tfrac{1}{3}a''' - 3(a')^4 + 12(a')^2 t - 12t^2, \quad (\ )' = \frac{d}{dt}. \tag{2.6}$$

In its turn, the pole position $a(t)$ must satisfy the equation $P_{\mathrm{I}}^{(2,1)}$,

$$a^{(4)} + 120(a')^3 a'' - 120a'a''t - \tfrac{200}{3}(a')^2 - \tfrac{40}{3}aa'' + \tfrac{200}{9}t = 0. \tag{2.7}$$

Along with the pole dynamics, the function $x = a(t)$ parametrizes the singularity locus for the linear system (2.2), (2.3).

Remark 2.1. Equation $P_{\mathrm{I}}{}^2$ also admits a 3-parameter series solution,

$$\begin{gathered} u(x) = -\frac{3}{(x-\tilde{a})^2} + \sum_{k=2}^{\infty} \tilde{c}_k (x-a)^k, \\ \tilde{c}_2 = -\tfrac{6}{7}t, \quad \tilde{c}_3 = 0, \quad \tilde{c}_4 = -\tfrac{10}{21}, \quad \tilde{c}_5 = -1, \quad \tilde{c}_7 = 0, \quad \tilde{c}_9 = -\tfrac{50}{147}t, \\ \tilde{c}_{10} = \tfrac{2}{13}\big(-\tfrac{50}{1323}\tilde{a}^2 + \tfrac{11}{7}\tilde{c}_6 t + \tfrac{36}{343}t^3\big), \quad \tilde{c}_{11} = -\tfrac{20}{441}\tilde{a}, \quad \ldots, \end{gathered} \tag{2.8}$$

$\tilde{a}, \tilde{c}_6, \tilde{c}_8$ are arbitrary, which, however, is not compatible with the KdV equation (2.4). Clearly it corresponds to the triple collisions of the poles (2.5).

3. SINGULARITY REDUCTION IN THE LINEAR SYSTEM FOR $P_{\mathrm{I}}{}^2$ AND THE ISOMONODROMY PROPERTY OF $P_{\mathrm{I}}^{(2,1)}$

3.1. Singularity Reduction

It is remarkable that the linear system for $P_{\mathrm{I}}{}^2$ can be regularized along the singularity locus $x = a(t)$.

Theorem 3.1. *Let $u(x)$ be the 4-parameter Laurent series solution (2.5) for equation $P_{\mathrm{I}}{}^2$. Then the gauge transformation $Z = R\Psi$ with the gauge matrix*

$$\begin{aligned} R(\lambda) =& 2^{\frac{1}{2}\sigma_3}(\lambda - 2c_0)^{\frac{1}{2}\sigma_3}(x-a)^{-\frac{1}{2}\sigma_3}\tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1) \\ &\times (\lambda - 2c_0)^{-\frac{1}{2}\sigma_3} 2^{\frac{1}{2}\sigma_3}(x-a)^{-\sigma_3}\tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1)(x-a)^{-\frac{1}{2}\sigma_3}, \quad \sigma_1 = \big(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\big), \end{aligned} \tag{3.1}$$

regularizes the connections $\partial_\lambda - A$ and $\partial_x - B$ along the singularity locus $x = a(t)$ and, if

$$c_0 = a' \quad \text{and} \quad c_3 = 2a'',$$

it regularizes also the connection $\partial_t - C$ along the same locus.

Proof. The proof is straightforward by inspection.

Remark 3.1. The above gauge transformation is constructed as a sequence of so-called shearing transformations. Namely, substituting the initial terms of the relevant Laurent series instead of u into the matrix A, one finds the leading order singular term proportional to the sum $-(x-a)^{-4}\sigma_- + (x-a)^{-3}\sigma_3 + (x-a)^{-2}\sigma_+$. Conjugation by $(x-a)^{-\sigma_3/2}$ "shares" the singularity order among the off-diagonal entries. The matrix coefficient of the term of order $\mathcal{O}((x-a)^{-3})$ becomes nilpotent, and its conjugation with $\frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)$ turns it to the normal σ_--form. This brings the lower-order singularities into play, and the procedure can be repeated.

Remark 3.2. As is known, singularity reductions exist also for the linear systems associated with the classical Painlevé equations.

Conjecture 3.1. The singularity reduction exists for an arbitrary isomonodromy system.

3.2. Isomonodromy Representation for $P_{\mathrm{I}}^{(2,1)}$

The linear system for ${P_{\mathrm{I}}}^2$ regularized along its singularity locus $x = a(t)$ yields the linear system for equation $P_{\mathrm{I}}^{(2,1)}$.

Theorem 3.2. *Equation* $P_{\mathrm{I}}^{(2,1)}$ *controls the isomonodromy deformations of a linear matrix ODE with the polynomial coefficients described by the system*

$$Z_\lambda = \mathcal{A}Z, \quad Z_t = \mathcal{C}Z, \tag{3.2}$$

with the coefficient matrices

$$\begin{aligned}
\mathcal{A} = -\tfrac{3}{20}a''\sigma_3 + \big(\tfrac{1}{30}\lambda^4 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0\big)\sigma_+ + \big(\tfrac{1}{30}\lambda - \alpha_3\big)\sigma_-,&\\
\alpha_3 = -\tfrac{1}{10}a', \quad \alpha_2 = \tfrac{3}{10}(a')^2 - t, \quad \alpha_1 = -\tfrac{9}{10}(a')^3 + 3a't + a,&\\
\alpha_0 = 3aa' - \tfrac{27}{4}(a')^4 + \tfrac{9}{5}t(a')^2 + \tfrac{27}{20}c_6 + \tfrac{81}{5}t^2,&\\
\mathcal{C} = \big[-\tfrac{1}{3}\lambda^3 + \beta_2\lambda^2 + \beta_1\lambda + \beta_0\big]\sigma_+ - \tfrac{1}{3}\sigma_-,&\\
\beta_2 = 2a', \quad \beta_1 = 10t - 9(a')^2, \quad \beta_0 = 36(a')^3 - 60a't - 10a.&
\end{aligned} \tag{3.3}$$

Proof. By inspection, the compatibility condition, $\mathcal{A}_t - \mathcal{C}_\lambda + [\mathcal{A}, \mathcal{C}] = 0$, yields

$$c_6 = -\tfrac{1}{3}a''' - 3(a')^4 + 12(a')^2t - 12t^2,$$

and

$$a'''' + 120(a')^3a'' - 120a'a''t - \tfrac{200}{3}(a')^2 - \tfrac{40}{3}aa'' + \tfrac{200}{9}t = 0,$$

i.e., equations for c_6 in (2.6) and $P_{\mathrm{I}}^{(2,1)}$ (2.7).

4. PUISEUX SERIES FOR $a(t)$ AND THE SINGULARITY REDUCTION OF THE LINEAR SYSTEM AT THE BRANCH POINT OF $a(t)$

It is easy to verify that equation $P_{\mathrm{I}}^{(2,1)}$ admits the formal series solution in powers of $(t-b)^{1/3}$ with the following initial terms (again, since the equation is polynomial in all variables, the recurrence relation for its coefficients is straightforward),

$$\begin{aligned}
&a(t) = \sum_{k=0}^{\infty} a_k (t-b)^{k/3},\\
&a_1 = -1, \quad a_2 = a_3 = a_4 = 0, \quad a_5 = -\tfrac{6}{7}b, \quad a_6 = 0,\\
&a_7 = \tfrac{10}{21}a_0, \quad a_8 = -\tfrac{3}{2}, \quad a_{10} = 0, \quad a_{12} = -\tfrac{135}{49}b, \quad \dots,
\end{aligned} \tag{4.1}$$

b, a_0, a_9, a_{11} are arbitrary.

System (3.2), (3.3) is singular, however, it is regularizable at the branch point $t = b$ of $a(t)$:

Theorem 4.1. *Let* $a(t)$ *be the 4-parameter Puiseux series solution* (4.1) *for equation* $P_{\mathrm{I}}^{(2,1)}$. *Then the gauge transformation* $X = QZ$*, rational in* λ *and* t*, with the gauge matrix*

$$\begin{aligned}
Q(\lambda) =& \sigma_3\big(\tfrac{120}{7}b - \lambda^2\big)^{-\frac{1}{2}\sigma_3}(t-b)^{\frac{1}{6}\sigma_3}\tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)\\
&\times 2^{-\sigma_3}\lambda^{-\frac{1}{2}\sigma_3}\big(\tfrac{120}{7}b - \lambda^2\big)^{\frac{1}{2}\sigma_3}(t-b)^{\frac{1}{3}\sigma_3}\tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)\\
&\times \lambda^{\frac{1}{2}\sigma_3}(t-b)^{\frac{1}{3}\sigma_3}2^{-\frac{1}{2}\sigma_3}\tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)(t-b)^{\frac{1}{2}\sigma_3}
\end{aligned} \tag{4.2}$$

regularizes the connection $\partial_\lambda - \mathcal{A}$ at the branch point $t = b$.

Here σ_1, σ_2, σ_3 are the Pauli matrices, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The new function $X(\lambda)$ satisfies the linear matrix ODE equivalent to the anharmonic oscillator equation (1.7), which does not admit any continuous isomonodromic deformation,

$$X_\lambda X^{-1} = \begin{pmatrix} 0 & \frac{1}{30} \\ U(\lambda) & 0 \end{pmatrix}, \; U(\lambda) = \tfrac{1}{30}\lambda^5 - b\lambda^3 + a_0\lambda^2 + (\tfrac{360}{49}b^2 + \tfrac{33}{10}a_9)\lambda - \tfrac{230}{21}a_0 b + \tfrac{143}{30}a_{11}. \quad (4.3)$$

Remark 4.1. The anharmonic oscillator equation (1.7) can be regarded as a stationary Schrödinger equation. Thus, it is possible to introduce a scattering problem and to map the set of parameters b, a_0, a_9, a_{11} to the set of the scattering data. Indeed, let us introduce the Jost solutions $\psi_\pm^{(k)}$ uniquely determined by the asymptotics

$$\psi_\pm^{(k)} \simeq \lambda^{-\frac{5}{4}\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} e^{\pm\theta}, \quad \theta = \tfrac{1}{105}\lambda^{7/2} - \tfrac{1}{3}t\lambda^{3/2} + x\lambda^{1/2},$$
$$\lambda \to \infty, \quad \arg\lambda = \pi + \tfrac{2\pi}{7}k, \quad k = 0, \pm 1, \pm 2, \pm 3.$$

Here, we define $\lambda^{1/2}$ on the λ complex plane cut along the positive part of the real line and choose its principal branch. Then the incident wave $\psi \simeq \psi_-^{(0)}$ produces several reflected and transmitted waves,

$$\psi = t_k\psi_-^{(k)} + r_k\psi_+^{(k)}, \quad \arg\lambda = \pi + \tfrac{2\pi}{7}k, \quad k = 0, \pm 1, \pm 2, \pm 3,$$

where the coefficients t_k and r_k are called the *transmission* and *reflection coefficients*, respectively. Only 4 of the 14 coefficients are independent. Indeed, there exist six relations of the form $r_{k+1} = r_k$, $t_k = t_{k-1}$, $k = -2, 0, 2$, which come from the preservation of the amplitude of the dominant solution in the sectors $\arg\lambda \in [\pi + \frac{2\pi}{7}k, \pi + \frac{2\pi}{7}(k+1)]$, $k = -3, \dots, 2$; the equation $t_0 = 1$ normalizes the amplitude of the incident wave; the two conditions $t_3 = r_{-3} = 0$ mean that the solution is subdominant as $\lambda \to +\infty$; the last condition $r_3 = it_{-3}$ means the continuity of this subdominant solution across the positive part of the real line,

$$\begin{aligned} &t_3 = r_{-3} = 0, \quad t_0 = t_{-1} = 1, \quad t_2 = t_1 (= 0), \\ &r_1 = r_0 (= i), \quad r_{-1} = r_{-2} (= 0), \quad r_3 = r_2 = it_{-2} = it_{-3} (= i). \end{aligned} \quad (4.4)$$

The four independent scattering coefficients, say, t_1, r_1, t_{-2}, r_{-2}, can be used to reconstruct the parameters b, a_0, a_9, and a_{11} determining the potential $U(\lambda)$.

In (4.4), the numbers in parentheses specify the values of the free transmission and reflection coefficients corresponding to the triple collisions of the poles of the special solution to $P_{\rm I}{}^2$of our interest, see below. Along the oscillatory directions, the wave function corresponding to this special potential is given by the Jost solutions $\psi_-^{(k)}$ $(k = -1, -2, -3)$, $i\psi_+^{(k)}$ $(k = 1, 2, 3)$ and by the sum $\psi_-^{(0)} + i\psi_+^{(0)}$ as $\lambda < 0$. This implies another simple characterization of the potential of our interest. Defining $\lambda^{1/2}$ on the plane cut along the negative part of the real axis, the Schrödinger equation $\psi'' = \frac{1}{30}U(\lambda)\psi$ has the solution with the asymptotics $\psi \simeq \lambda^{-5/4}e^{-\theta}$ uniform in the sector $\arg\lambda \in (-\pi, \pi)$. The existence of a solution with this asymptotics is tantamount to vanishing of four Stokes multipliers $s_{\pm 1} = s_{\pm 2} = 0$ (see below).

Below, however, we characterize $X(\lambda)$ by adopting the Riemann–Hilbert problem instead of the scattering problem.

Remark 4.2. Observe that the successive reduction of the number of the deformation parameters of the λ-equations depend on, namely, for the connection $\partial_\lambda - A$, the space of deformation parameters (t, x) is 2-dimensional, for $\partial_\lambda - \mathcal{A}$, the deformation parameter space reduces to the 1-dimensional space of t, and for $\partial_\lambda - \frac{1}{30}\sigma_+ - U\sigma_-$, it is 0-dimensional.

Remark 4.3. Along with the continuous isomonodromic deformations considered above, it is possible to introduce discrete Poincaré-like isomonodromy mappings from one branch of $a(t)$ to another and from one point of the lattice $(b, a(b))$ to another.

Remark 4.4. It is natural to denote the above-listed connections by the symbols $L_{\rm I}^{(2,2)}$, $L_{\rm I}^{(2,1)}$, and $L_{\rm I}^{(2,0)}$, respectively, to reflect explicitly the number of continuous deformation parameters involved. The equations of the isomonodromic deformations are thus denoted by the symbols $P_{\rm I}^{(2,2)}$ (same as $P_{\rm I}{}^2$), $P_{\rm I}^{(2,1)}$, and $P_{\rm I}^{(2,0)}$ (the last equation is a Poincaré-like mapping of the lattice $(b, a(b))$).

5. RIEMANN–HILBERT PROBLEMS FOR $L_{\rm I}^{(2,2)}$, $L_{\rm I}^{(2,1)}$, AND $L_{\rm I}^{(2,0)}$

The boundary Riemann–Hilbert (RH) problem consists in finding a piecewise holomorphic function by its prescribed analytic properties:

(i) asymptotics at some marked points;

(ii) discontinuity properties across a piecewisely oriented graph.

5.1. Asymptotics of the Canonical Solutions $\Psi_k(\lambda)$, $Z_k(\lambda)$, and $X_k(\lambda)$

Each of the above linear matrix ODEs for Ψ, Z, and X has one irregular singularity at $\lambda = \infty$ and no other singular points. We introduce the formal series solutions that represent the asymptotic behavior of the genuine solutions to these linear ODEs in the interior of particular sectors near $\lambda = \infty$.

Since all the canonical asymptotics differ from each other by the rational left diagonal multiplier, it is convenient to unify the notation and distinguish all three cases introducing a parameter ν,

$$\begin{aligned}
&\Phi_k^{(\nu)}(\lambda) = \lambda^{\frac{\nu}{4}\sigma_3}\tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1)(I+\mathcal{O}(\lambda^{-1/2}))e^{\theta\sigma_3},\\
&\quad \theta = \tfrac{1}{105}\lambda^{7/2} - \tfrac{1}{3}t\lambda^{3/2} + x\lambda^{1/2},\\
&\quad \lambda\to\infty, \quad \arg\lambda \in \left(-\tfrac{3\pi}{7}+\tfrac{2\pi}{7}k, \tfrac{\pi}{7}+\tfrac{2\pi}{7}k\right), \quad k\in\mathbb{Z},\\
&\quad \nu\in\{-1,3,-5\}, \quad \Phi^{(-1)}=\Psi, \quad \Phi^{(3)}=Z, \quad \Phi^{(-5)}=X.
\end{aligned} \tag{5.1}$$

Here, the principal branch of the square root of λ is chosen.

Remark 5.1. Using (5.1) for the asymptotic solutions Z_k and X_k, one has to take into account the reductions $x = a(t)$ for $\nu = 3$ and $t = b$, $x = a(b)$ for $\nu = -5$. However, in what follows, using (5.1) to set the corresponding RH problems, we assume that the exponential θ depends on arbitrary complex deformation parameters t and x because it is not allowed to use the unknown values $a(t)$ and $(b, a(b))$ as the data in the RH problems formulated below. As the result, it is necessary to remember that the solvability domains for the RH problems shrink to the lines $x = a(t)$ for $\nu = 3$ and to a lattice $(t, x) = (b, a(b))$ for $\nu = -5$.

5.2. Stokes Multipliers

The canonical solutions differ from each other by the triangular right matrix multipliers called the *Stokes matrices*,

$$\Phi_{k+1}(\lambda) = \Phi_k(\lambda)S_k, \quad S_{2k-1} = \begin{pmatrix}1 & s_{2k-1}\\ 0 & 1\end{pmatrix}, \quad S_{2k} = \begin{pmatrix}1 & 0\\ s_{2k} & 1\end{pmatrix}, \tag{5.2}$$

moreover

$$s_{k+7} = s_k, \quad s_k + s_{k+2} + s_k s_{k+1} s_{k+2} = -i(1 + s_{k+4}s_{k+5}), \quad k\in\mathbb{Z}. \tag{5.3}$$

Remark 5.2. Since the gauge transformations $R(\lambda)$ and $Q(\lambda)$ are rational, they do not affect the Stokes matrices.

Remark 5.3. The Stokes multipliers s_k are first integrals of the equations $P_{\mathrm{I}}^{(2,2)}$(2.1) and $P_{\mathrm{I}}^{(2,1)}$(2.7).

The solutions of the RH problems below are constructed using collections of canonical solutions in the above-presented sectors so as to have uniform asymptotics in a vicinity of infinity. However, the normalization of the RH problem is a rather subtle thing. To this aim, observe the prolonged canonical asymptotics

$$\frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{\frac{1}{4}\sigma_3}\Psi_k(\lambda)e^{-\theta\sigma_3} =(I+\tfrac{u}{2\lambda}\sigma_1+\mathcal{O}(\lambda^{-3/2}))e^{\lambda^{-1/2}d_1\sigma_3}, \tag{5.4}$$

$$\begin{aligned}\frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{-\frac{3}{4}\sigma_3}Z_k(\lambda)e^{-\theta\sigma_3} &=(I-\tfrac{3a'}{2\lambda}\sigma_1+\tfrac{1}{\lambda^2}(\tfrac{9}{4}(a')^2-\tfrac{15}{2}t)\sigma_1+\mathcal{O}(\lambda^{-5/2}))\\ &\quad\times\exp\{\lambda^{-1/2}d_1\sigma_3+\lambda^{-3/2}d_3\sigma_3+\lambda^{-2}d_4I\},\end{aligned} \tag{5.5}$$

$$\begin{aligned}\frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{\frac{5}{4}\sigma_3}X_k(\lambda)e^{-\theta\sigma_3} &=(I+\tfrac{15b}{2\lambda^2}\sigma_1-\tfrac{15a_0}{2\lambda^3}\sigma_1+\mathcal{O}(\lambda^{-7/2}))\\ &\quad\times\exp\{\lambda^{-1/2}d_1\sigma_3+\lambda^{-3/2}d_3\sigma_3+\lambda^{-5/2}d_5\sigma_3\}.\end{aligned} \tag{5.6}$$

In the exponential factors on the above right-hand sides, the parameters d_j can be expressed in terms of the coefficients of the relevant equation. For instance, d_1 in (5.4) is one of two Hamiltonians of ${P_{\mathrm{I}}}^2$. However, these particular relations are not important for us at this stage.

Now, we have the following problem.

Riemann–Hilbert problem 1. For given complex values of the parameters x, t and s_k, $k\in\mathbb{Z}$, satisfying (5.3), writing

$$\theta=\frac{1}{105}\lambda^{7/2}-\frac{1}{3}t\lambda^{3/2}+x\lambda^{1/2}, \tag{5.7}$$

find the piecewise holomorphic 2×2 matrix function $\Phi^{(\nu)}(\lambda)$, $\nu\in\{-1,3,-5\}$, with the following properties:

(1)

$$\lim_{\lambda\to\infty}\frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{-\frac{\nu}{4}\sigma_3}\Phi^{(\nu)}(\lambda)e^{-\theta\sigma_3}=I, \tag{5.8}$$

moreover,

- there is a constant d_1 such that

$$\frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{-\frac{\nu}{4}\sigma_3}\Phi^{(\nu)}(\lambda)e^{-\theta\sigma_3}=I+\lambda^{-1/2}d_1\sigma_3+\mathcal{O}(\lambda^{-1}); \tag{5.9}$$

- if $\nu\in\{3,-5\}$, then there are constants d_1 and d_3 such that

$$\frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{-\frac{3}{4}\sigma_3}2^{\frac{1}{2}\sigma_3}\Phi^{(\nu)}(\lambda)e^{-(\theta+d_1\lambda^{-1/2})\sigma_3}=I+\lambda^{-1}c_1+\lambda^{-3/2}d_3\sigma_3+\mathcal{O}(\lambda^{-2}) \tag{5.10}$$

with some constant matrix c_1;

- if $\nu=-5$, then there are constants d_1, d_3, and d_5 such that

$$\frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\lambda^{-\frac{3}{4}\sigma_3}2^{\frac{1}{2}\sigma_3}\Phi^{(\nu)}(\lambda)e^{-(\theta+d_1\lambda^{-1/2}+d_3\lambda^{-3/2})\sigma_3}=I+\lambda^{-2}c_2+\lambda^{-5/2}d_5\sigma_3+\mathcal{O}(\lambda^{-3}) \tag{5.11}$$

with some constant matrix c_2;

(2) $\|\Phi^{(\nu)}(\lambda)\|<\text{const}$ as $\lambda\to 0$;

(3) on the union of the eight rays $\gamma=\rho\cup\bigl(\cup_{k=1}^{7}\gamma_{k-4}\bigr)$, where $\gamma_k=\bigl\{\lambda\in\mathbb{C}\colon \arg\lambda=\frac{2\pi}{7}k\bigr\}$, $k=-3,-2,\dots,2,3$, and $\rho=\bigl\{\lambda\in\mathbb{C}\colon \arg\lambda=\pi\bigr\}$, all oriented towards infinity, the following jump condition holds true:

$$\Phi_+^{(\nu)}(\lambda)=\Phi_-^{(\nu)}(\lambda)S(\lambda), \tag{5.12}$$

where $\Phi_+^{(\nu)}(\lambda)$ and $\Phi_-^{(\nu)}(\lambda)$ are the limits of $\Phi^{(\nu)}(\lambda)$ on γ from the left and from the right, respectively, and the piecewise constant matrix $S(\lambda)$ is given by equations

$$S(\lambda)\big|_{\lambda\in\gamma_k}=S_k,\quad S_{2k}=I+s_{2k}\sigma_-,\quad S_{2k-1}=I+s_{2k-1}\sigma_+, \tag{5.13a}$$

$$S(\lambda)\big|_{\rho}=i\sigma_1. \tag{5.13b}$$

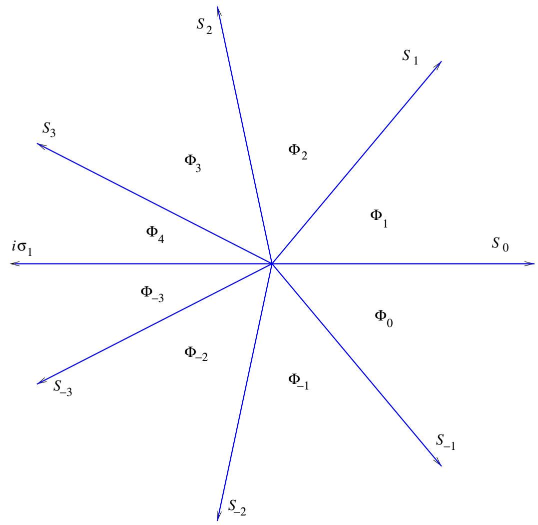

Fig. 5.1. The jump contour γ for the RH problem 1 and canonical solutions $\Phi_j(\lambda)$, $j=-3,-2,\dots,3,4$.

5.2.1. Uniqueness of the solution to the RH problem 1. The solution of the RH problem 1 is unique. Indeed, the scalar function $\det\Phi^{(\nu)}(\lambda)$ is continuous across the set of rays γ and bounded at the origin. Therefore, $\det\Phi^{(\nu)}(\lambda)$ is an entire function, and, taking into account (5.8) and applying the Liouville theorem gives $\det\Phi^{(\nu)}(\lambda)\equiv-1$. Assume for the moment that there are two solutions of the RH problem 1, $\Phi^{(\nu)}(\lambda)$ and $\tilde\Phi^{(\nu)}(\lambda)$. Consider their ratio, $\chi(\lambda)=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\tilde\Phi^{(\nu)}(\lambda)(\Phi^{(\nu)}(\lambda))^{-1}$. It can readily be seen that $\chi(\lambda)$ is continuous across all the rays of the set γ and bounded at the origin, and thus, $\chi(\lambda)$ is an entire function. Using (5.8) gives

$$\lim_{\lambda\to\infty}\lambda^{-\frac{\nu}{4}\sigma_3}\chi(\lambda)\lambda^{\frac{\nu}{4}\sigma_3}=\lim_{\lambda\to\infty}\begin{pmatrix}a&b\lambda^{-\nu/2}\\c\lambda^{\nu/2}&d\end{pmatrix}=I.$$

Since all the entries a,b,c,d are entire functions in λ, the Liouville theorem yields the ambiguity

$$\Phi^{(\nu)}(\lambda)\mapsto\tilde\Phi^{(\nu)}(\lambda)=P^{(\nu)}(\lambda)\Phi^{(\nu)}(\lambda),\tag{5.14}$$
$$P^{(-1)}(\lambda)=I+c_0\sigma_-,\quad P^{(3)}(\lambda)=I+(c_0\lambda+c_1)\sigma_+,\quad P^{(-5)}(\lambda)=I+(c_0\lambda^2+c_1\lambda+c_2)\sigma_-,$$

where c_j are arbitrary constants. However, these ambiguities are eliminated using the asymptotic conditions in (5.9), (5.10), and (5.11).

5.3. Solvability of the RH Problems 1 and the Malgrange Divisor

For given jump matrices, the set of points (x,t) at which the RH problem is not solvable is referred to as the *Malgrange divisor.* It coincides with the zero locus of a Miwa holomorphic τ-function and with the singularity locus of the isomonodromy deformation equation.

In what follows, it is convenient to articulate our assumptions on the singularities and critical points of equations ${P_{\rm I}}^2$and $P_{\rm I}^{(2,1)}$.

Conjecture 5.1. (1) Equation $P_{\rm I}^{(2,2)}$ has no movable singularities except for the movable poles (2.5) satisfying (2.6), (2.7), and their triple collisions (2.8). (2) Equation $P_{\rm I}^{(2,1)}$ has no movable singularities or critical points except for the branch points (4.1).

These assumptions mean that the smooth branches of the Malgrange divisor for the RH problem 1 with $\nu=-1$ (corresponding to $P_{\rm I}^{(2,2)}$) are parameterized by equations $x=a(t)$ and thus coincide

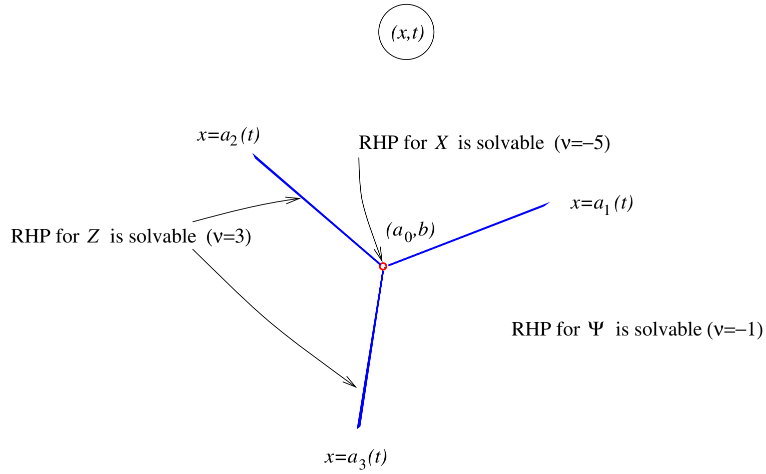

Fig. 5.2. The scheme of the Malgrange divisor and the domains of solvability for the RH problems with $\nu = -1, 3, -5$.

with the solvability set for the RH problem 1 with $\nu = 3$ (corresponding to $P_{\mathrm{I}}^{(2,1)}$). Similarly, the vertices of the Malgrange divisor of the RH problem with $\nu = -1$ correspond to the branch points $t = b$ of $a(t)$, and therefore, coincide with the solvability set of the RH problem with $\nu = -5$.

In other words, Conjecture 5.1 implies the following remarkable properties of the domains of solvability of all three RH problems:

(1) these domains are pairwise disjoint;
(2) all these domains together cover the deformation parameter space $\mathbb{C} \times \mathbb{C} \ni (t, x)$.

5.4. Normalized Inhomogeneous and Homogeneous RH Problems

In this section, we describe another interesting feature of the family of three RH problems. Roughly speaking, any solution of the *inhomogeneous* RH problem with the bigger value of $|\nu|$ allows one to construct infinitely many solutions of the *homogeneous* problems with the smaller value of $|\nu|$.

A precise formulation of this property requires some more accuracy. First of all, let us introduce RH problems that are equivalent to the above ones, nonbranched, and normalized at infinity. To this aim, write

$$\lambda = \zeta^4, \quad \hat{\Phi}_k^{(\nu)}(\zeta) = \tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1)\zeta^{-4\nu\sigma_3}\Phi_k^{(\nu)}(\zeta^4)e^{-\theta(\zeta^4)}, \tag{5.15}$$

and define the following piecewise holomorphic functions,

$$\hat{\Phi}^{(\nu)}(\zeta) = \hat{\Phi}_k^{(\nu)}(\zeta), \quad \arg\zeta \in \left(\tfrac{\pi}{14}(k-1), \tfrac{\pi}{14}k\right), \quad \nu = -1, 3, -5. \tag{5.16}$$

These functions solve the inhomogeneous RH problems on the union of rays

$$\cup_{k=-13}^{14}\ell_k, \quad \ell_k = \{\zeta \in \mathbb{C}\colon \ \arg\zeta = \tfrac{\pi}{14}k\},$$

with a singular point at the origin and normalized to the unit at infinity. The corresponding homogeneous RH problem differs from the inhomogeneous counterpart in the asymptotics of $\Phi^{(\nu)}(\zeta)$ that vanishes as $\zeta \to \infty$.

Slightly abusing our notation, we formulate the inhomogeneous and homogeneous RH problems as follows.

Riemann–Hilbert problem 2. Find a piecewise holomorphic function $\hat{\Phi}_I^{(\nu)}(\zeta)$ ($\hat{\Phi}_0^{(\nu)}(\zeta)$, respectively), $\nu \in \{-1, 3, -5\}$, with the following properties:

(1) $\lim_{\zeta\to\infty} \hat{\Phi}_I^{(\nu)}(\zeta) = I$ ($\lim_{\zeta\to\infty} \hat{\Phi}_0^{(\nu)}(\zeta) = 0$, respectively);

(2) across the rays $\ell_k = \{\zeta \in \mathbb{C}\colon \arg\zeta = \frac{\pi}{14}k\}$, $k = -13, \dots, 13, 14$, all oriented towards infinity, the following jump conditions hold:

$$\hat{\Phi}^{(\nu)}_{+}(\zeta) = \hat{\Phi}^{(\nu)}_{-}(\zeta)e^{\theta\sigma_3}S_k e^{-\theta\sigma_3}, \quad \zeta \in \ell_k, \quad \theta = \tfrac{1}{105}\zeta^{14} - \tfrac{1}{3}t\zeta^6 + x\zeta^2; \tag{5.17}$$

(3) $\|\zeta^{\nu\sigma_3}\frac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\hat{\Phi}^{(\nu)}(\zeta)\| < \text{const} \quad \text{as } \zeta \to 0.$

The following theorem holds.

Theorem 5.1. *Any solution $\hat{\Phi}^{(3)}_I(\zeta)$ to the inhomogeneous RH problem 2 with $\nu = 3$ yields infinitely many solutions $\hat{\Phi}^{(-1)}_0(\zeta)$ to the homogeneous RH problem 2 with $\nu = -1$. Any solution $\hat{\Phi}^{(-5)}_I(\zeta)$ to the inhomogeneous RH problem 2 with $\nu = -5$ yields infinitely many solutions $\hat{\Phi}^{(\nu)}_0(\zeta)$ to the homogeneous RH problem 2 with $\nu = 3, -1$.*

Proof. We prove the first part of the theorem. The proof of the second part is similar.

Consider a solution $\hat{\Phi}^{(3)}_I(\zeta)$ to the RH problem 2 for $\nu = 3$. It has the asymptotics

$$\hat{\Phi}^{(3)}_I(\zeta) = I + \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty,$$
$$\hat{\Phi}^{(3)}_I(\zeta) = \tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\zeta^{-3\sigma_3}\big(M_0 + \mathcal{O}(\zeta^4)\big), \quad \zeta \to 0, \quad \zeta \in \omega_0,$$

where M_0 is a constant matrix, $\det M_0 \neq 0$, and $\omega_0 = \{\zeta \in \mathbb{C}\colon \arg\zeta \in (0, \frac{\pi}{14})\}$. Then

$$\hat{\Phi}^{(-1)}_0(\zeta) = \zeta^{-2}N\hat{\Phi}^{(3)}_I(\zeta), \quad N = \tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix}\tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1), \quad p, q = \text{const},$$

has the same jump properties as $\hat{\Phi}^{(3)}_I(\zeta)$ and the asymptotics of the form

$$\hat{\Phi}^{(-1)}_0(\zeta) = \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty,$$
$$\hat{\Phi}^{(-1)}_0(\zeta) = \tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\zeta^{\sigma_3}\begin{pmatrix} 0 & p \\ 0 & q\zeta^2 \end{pmatrix}\big(M_0 + \mathcal{O}(\zeta^4)\big), \quad \zeta \to 0, \quad \zeta \in \omega_0.$$

Thus, $\hat{\Phi}^{(-1)}_0(\zeta)$ is the solution of the homogeneous RH problem 2 with $\nu = -1$.

6. LARGE-t ASYMPTOTICS OF A SPECIAL SOLUTION OF EQUATIONS $P_{\mathrm{I}}^{\,2}$AND $P_{\mathrm{I}}^{(2,1)}$

In this section, we construct a large-t asymptotic solution to the RH problem 1 for $\nu = -1$ and $\nu = 3$ corresponding to a special solution of equation $P_{\mathrm{I}}^{\,2}$. This special solution $u(x,t)$ has the asymptotics $u \sim \mp\sqrt[3]{6|x|}$ as $x \to \pm\infty$ and is real and regular on the real line for any $t \in \mathbb{R}$. The physical importance of this solution for $t = 0$ from the point of view of string theory was justified in [3]. In [11, 12], this solution arose in the study of the universality problem of critical behavior of solutions to Hamiltonian perturbations of hyperbolic PDEs.

The above-mentioned properties uniquely distinguish this special solution. In [26], the characterization of this solution in terms of the Stokes multipliers of the associated linear system was found,

$$s_{-2} = s_{-1} = s_1 = s_2 = 0, \quad s_{-3} = s_0 = s_3 = -i. \tag{6.1}$$

6.1. Large-t Asymptotic Spectral Curve for the Special Solution to $P_{\mathrm{I}}^{\,2}$

Assume that $t > 0$ is large and $x \in \mathbb{C}$ is such that

$$x_0 := xt^{-3/2} = \mathcal{O}(1) \quad \text{as} \quad t \to +\infty. \tag{6.2}$$

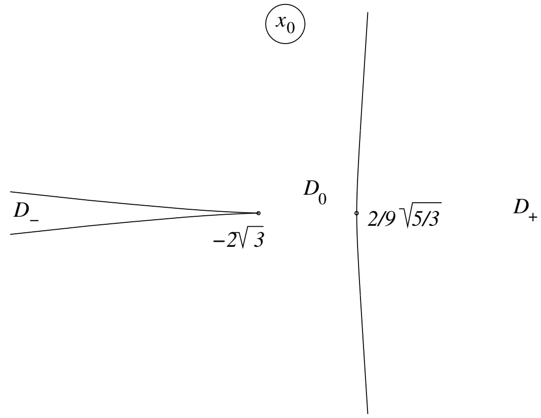

Fig. 6.1. The discriminant set for the special solution of equation ${P_{\rm I}}^2$. In the interior of the domains D_+ and D_-, the asymptotics of the special solution has genus $g = 0$ and, in the domain D_0, it has genus $g = 1$.

Our starting point is the large-t asymptotics of the spectral curve, $\det(\mu - t^{-5/4}A(\lambda)) = 0$, where $\lambda = t^{1/2}\xi$, i.e.,

$$\mu^2 = \tfrac{1}{900}\xi^5 - \tfrac{1}{30}\xi^3 + \tfrac{1}{30}x_0\xi^2 + \tfrac{1}{30}D_1\xi + \tfrac{1}{30}D_0 = \tfrac{1}{900}\prod_{k=1}^{5}(\xi - \xi_k). \tag{6.3}$$

The asymptotic analysis of various degenerate solutions of ${P_{\rm I}}^2$ performed in [20] shows that the topological properties of the asymptotic spectral curve are significantly different in the interior of the domains $D_\pm$ and D_0, see Fig. 6.1. For $x_0 \in D_\pm$, the spectral curve has genus 0,

$$\mu^2 = \tfrac{1}{900}(\xi - \xi_i)^2(\xi - \xi_j)^2(\xi - \xi_k), \tag{6.4}$$

where the double branch points ξ_i, ξ_j and the simple branch point ξ_k satisfy the conditions

$$\xi_{i,j} = -\tfrac{1}{4}\xi_k \pm \sqrt{15 - \tfrac{5}{16}\xi_k^2}, \quad \xi_k^3 - 24\xi_k + 48x_0 = 0,$$

and the ambiguity in the choice of the root of the cubic equation for the simple branch point ξ_k is fixed by demanding that, for $x_0 \in \mathbb{R}$,

$$\operatorname{sgn}(x_0)\,\Re\Big(\int_{\xi_k}^{\xi_{i,j}} \mu(z)\,dz\Big) > 0, \quad x_0 \to \pm\infty.$$

The conditions above are consistent with the quasistationary asymptotic behavior of the special solution to ${P_{\rm I}}^2$(2.1),

$$u(x,t) \simeq t^{1/2}v_0, \quad v_0^3 - 6v_0 + 6x_0 = 0, \quad v_0 \simeq -\sqrt[3]{6x_0}, \quad x_0 \to \pm\infty, \tag{6.5}$$

where the branch of $\sqrt[3]{6x_0}$ that is real on the real line is chosen.

Let us indicate some interesting points in the x_0 complex plane.

At the point $x_0 = -2\sqrt{3} = D_0 \cap D_- \cap \mathbb{R}$, two double branch points of the asymptotic spectral curve coalesce, so that $\xi_{1,2,3,4} = -\sqrt{3}$ and $\xi_5 = 4\sqrt{3}$. Generically, at the asymptotically quadruple branch point, the local solution of the RH problem can be approximated using the Garnier–Jimbo–Miwa Ψ function for the second Painlevé transcendent $P_{\rm II}$[19, 25]. For the RH problem, we are studying, the relevant local approximate solution corresponds to the Hastings–McLeod solution to P_{II}[6].

At any $x_0 \in D_+$ (including boundary points), the spectral curve has one simple and two double branch points. The precise asymptotic location of these branch points can be found for

$$x_0 = \tfrac{2\sqrt{5}}{9\sqrt{3}} = D_0 \cap D_+ \cap \mathbb{R}, \quad \text{where} \quad \xi_3 = -\tfrac{4\sqrt{5}}{\sqrt{3}}, \quad \xi_{1,2} = -\tfrac{\sqrt{5}}{\sqrt{3}}, \quad \text{and} \quad \xi_{4,5} = \tfrac{3\sqrt{5}}{\sqrt{3}}.$$

At this point, as well as at any point of D_+ (including its boundary), the leading-order asymptotic solution of the RH problem can be expressed in elementary functions, cf. [4] for the case of real line.

Observe also the point $x_0 = 2\sqrt{3} \in D_+$, where two double branch points coalesce. The asymptotic branch points corresponding to this point are $\xi_3 = -4\sqrt{3}$ and $\xi_{1,2,4,5} = \sqrt{3}$. Generically, the quadruple degeneration corresponds to the occurrence of P_{II}. However, for the RH problem in question here, the relevant Painlevé function is trivial, and the asymptotic solution to the RH problem remains elementary.

In the interior part of the domain D_0, the large-t asymptotic solution to the RH problem is constructed on the model elliptic curve,

$$\mu^2 = \tfrac{1}{900}(\xi - \xi_1)^2(\xi - \xi_3)(\xi - \xi_4)(\xi - \xi_5), \tag{6.6}$$

where the branch points ξ_j, $j = 3, 4, 5$, are determined by the values of ξ_1 and x_0 as the roots of the cubic equation

$$\xi^3 + 2\xi_1\xi^2 + (3\xi_1^2 - 30)\xi + 4\xi_1^3 - 60\xi_1 + 30x_0 = 0. \tag{6.7}$$

The double branch point ξ_1 is determined as a function of $x_0 \in D_0$ by the system of Boutroux equations, see [27],

$$\Re \int_{\xi_3}^{\xi_4} \mu(z)\, dz = 0, \quad \Re \int_{\xi_4}^{\xi_5} \mu(z)\, dz = 0, \tag{6.8}$$

supplemented by the boundary conditions on $D_\pm \cap D_0$ described above.

Remark 6.1. As x_0 approaches the real segment $(-2\sqrt{3}, \frac{2\sqrt{5}}{9\sqrt{3}})$, all asymptotic branch points become real and satisfy the inequalities $\xi_3 < \xi_1 < \xi_4 < \xi_5$. Thus, the second equation in (6.8) trivializes, while the first of these equations becomes the condition $\int_{\xi_3}^{\xi_4} \mu(z)\, dz = 0$ obtained in [36] in the analysis of the Whitham equations and used in [4] to study the same special solution on the real line.

6.2. Steepest-Descent Analysis of the RH Problem

The strategy of the steepest-descent asymptotic analysis by Deift and Zhou [9, 10] of the RH problem involves several standard steps: (1) a transformation of the jump graph to the steepest-descent directions of a suitable g-function; (2) a construction of local approximate solutions (parametrices); (3) matching all the parametrices into a global parametrix; (4) a proof that the global parametrix approximates indeed the genuine solution to the original RH problem.

Since all the above steps are well explained in the literature (see, e.g., [13]), we omit unnecessary details below.

6.2.1. Transformation of the jump graph to the steepest-descent directions. According to the steepest-descent strategy, we first transform the jump contour for each of the RH problems to the steepest-descent graph for the exponential $\exp\{\int^{\xi} \mu(z)\, dz\}$, see Fig. 6.2.

Observe that, in the special case (6.1), $s_{\pm 2} = s_{\pm 1} = 0$, $s_0 = s_{\pm 3} = -i$, the jump graph depicted on Fig. 5.1 can be transformed to that shown in Fig. 6.3, where

$$\Sigma_- := \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \quad \Sigma_+ := \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}. \tag{6.9}$$

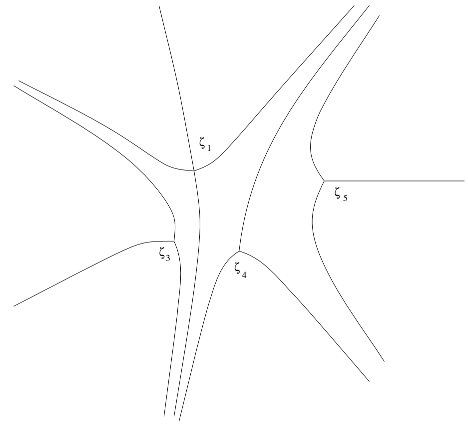

Fig. 6.2. Typical anti-Stokes lines for the special solution of ${P_{\mathrm{I}}}^2$ as $x_0 \in D_0$, $\Im x_0 > 0$.

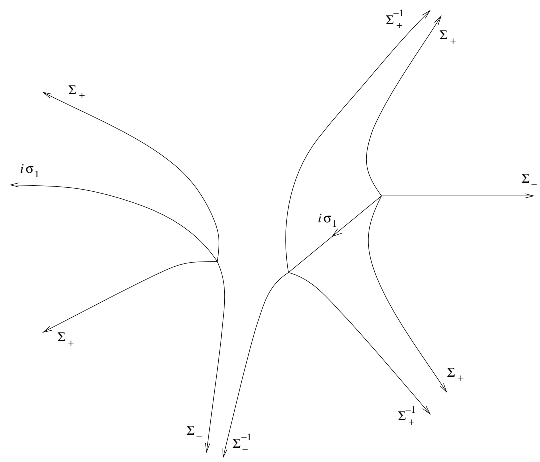

Fig. 6.3. The jump graph for the RH problem in the special case $s_{\pm 2} = s_{\pm 1} = 0$, $s_0 = s_{\pm 3} = -i$.

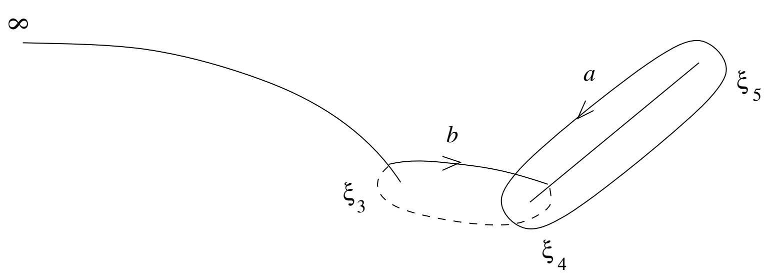

Fig. 6.4. The Riemann surface Γ and the basis of cycles a, b.

Model elliptic curve and Abelian integrals. The large-t asymptotics of $\Psi(\lambda)$ corresponding to the Stokes multipliers (6.1) is constructed on the Riemann surface Γ of the model elliptic curve

$$w^2 = (\xi - \xi_3)(\xi - \xi_4)(\xi - \xi_5), \tag{6.10}$$

glued from two copies of the complex ξ-plane cut along $[\xi_5, \xi_4] \cup [\xi_3, -\infty)$, see Fig. 6.4.

Define the complete elliptic integrals

$$\mathcal{A}, \mathcal{B} = \oint_{a,b} \mu(\xi)\, d\xi = \tfrac{1}{30} \oint_{a,b} (\xi - \xi_1) w(\xi)\, d\xi, \quad \omega_{a,b} = \oint_{a,b} \frac{d\xi}{w(\xi)}, \quad \tau = \frac{\omega_b}{\omega_a}, \quad \Im\tau > 0, \tag{6.11}$$

and Abelian integrals

$$g(\xi)=\int_{\xi_5}^{\xi}\mu(z)\,dz,\quad U(\xi)=\frac{1}{\omega_a}\int_{\xi_5}^{\xi}\frac{dz}{w(z)}. \tag{6.12}$$

We define the integral $g(\xi)$ on the upper sheet of the Riemann surface Γ cut along the sum of intervals $[\xi_5,\xi_4]\cup[\xi_4,\xi_3]\cup[\xi_3,-\infty)$.

The Boutroux conditions (6.8) imply

$$\mathcal{A},\mathcal{B}\in i\mathbb{R}. \tag{6.13}$$

Observe the following properties of $g(\xi)$ and $U(\xi)$:

(1) as $\xi\to\infty$,

$$\begin{aligned}&U(\xi)=U_\infty+\mathcal{O}(\xi^{-1/2}),\quad U_\infty=-\tfrac{1}{2}\tau,\\&g(\xi)=\vartheta+g_\infty+\mathcal{O}(\xi^{-1/2}),\quad \vartheta=\tfrac{1}{105}\xi^{7/2}-\tfrac{1}{3}\xi^{3/2}+x_0\xi^{1/2},\quad g_\infty=-\tfrac{1}{2}\mathcal{B},\end{aligned} \tag{6.14}$$

(2) $g(\xi)$ and $U(\xi)$ are discontinuous across the polygonal line $[\xi_5,\xi_4]\cup[\xi_4,\xi_3]\cup[\xi_3,-\infty)$ oriented from ξ_5 to infinity, moreover,

$$\begin{aligned}&\xi\in(\xi_5,\xi_4):\quad g_+(\xi)+g_-(\xi)=0,\quad U_+(\xi)+U_-(\xi)=0,\\&\xi\in(\xi_4,\xi_3):\quad g_+(\xi)-g_-(\xi)=-\mathcal{A},\quad U_+(\xi)-U_-(\xi)=-1,\\&\xi\in(\xi_3,\infty):\quad g_+(\xi)+g_-(\xi)=-\mathcal{B},\quad U_+(\xi)+U_-(\xi)=-\tau.\end{aligned} \tag{6.15}$$

6.3. *The "External" Parametrix*

In this subsection, following [13], we solve the permutation RH problem on the segments $[\xi_5,\xi_4]\cup[\xi_3,-\infty)$ whose solution gives a leading-order contribution to the solution of the above RH problem.

Riemann–Hilbert problem 3. For given $t\gg1$ and a complex value of the parameter x_0, find a piecewise holomorphic 2×2 matrix function $\Phi^{(\nu)}(\xi)$ with the following properties:

(1)
$$\lim_{\xi\to\infty}\tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\xi^{-\frac{\nu}{4}\sigma_3}t^{-\frac{\nu}{8}\sigma_3}\Phi^{(\nu)}(\xi)e^{-t^{7/4}\vartheta\sigma_3}=I, \tag{6.16}$$

where

$$\vartheta=\tfrac{1}{105}\xi^{7/2}-\tfrac{1}{3}\xi^{3/2}+x_0\xi^{1/2}; \tag{6.17}$$

(2) across the union of segments $(\xi_5,\xi_4)\cup(\xi_3,-\infty)$ oriented as indicated, the following jump condition holds true:

$$\Phi_+^{(\nu)}(\xi)=\Phi_-^{(\nu)}(\xi)i\sigma_1,\quad \xi\in(\xi_5,\xi_4)\cup(\xi_3,-\infty), \tag{6.18}$$

where $\Phi_+^{(\nu)}(\xi)$ and $\Phi_-^{(\nu)}(\xi)$ are limits of $\Phi^{(\nu)}(\xi)$ on the segments from the left and from the right, respectively, see Fig. 6.3.

We do not impose any conditions on the behavior of $\Phi^{(\nu)}(\xi)$ at the points $\xi=\xi_j$, $j=3,4,5$. As the result, the solution of the model RH problem 3 is determined up to a left rational matrix multiplier with possible poles at $\xi=\xi_j$, $j=3,4,5$, and a certain asymptotics at infinity. Below, we use this rational multiplier to prove or disprove the asymptotic solvability of the original RH problem 1.

β-factor. Consider the principal branches of the functions $\beta_\nu(\xi)$,

$$\beta_{-1}(\xi)=(\xi-\xi_3)^{-1/4}(\xi-\xi_4)^{1/4}(\xi-\xi_5)^{-1/4}, \tag{6.19}$$

$$\beta_3(\xi)=(\xi-\xi_3)^{1/4}(\xi-\xi_4)^{1/4}(\xi-\xi_5)^{1/4} \tag{6.20}$$

both defined on the ξ complex plane with cuts along the polygonal line $[\xi_5,\xi_4]\cup[\xi_4\xi_3]\cup[\xi_3,-\infty)$. These functions solve the scalar RH problems:

(1) $\beta_\nu(\xi)=\xi^{\frac{\nu}{4}}(1+\mathcal{O}(\xi^{-1}))$, as $\xi\to\infty$;

(2) the discontinuity of $\beta_\nu(\xi)$ across the oriented contour $[\xi_5,\xi_4]\cup[\xi_4\xi_3]\cup[\xi_3,-\infty)$ is described by the conditions

$$\begin{aligned} &\xi\in(\xi_5,\xi_4)\colon\quad \beta_{-1}^+(\xi)=i\beta_{-1}^-(\xi),\quad \beta_3^+(\xi)=-i\beta_3^-(\xi),\\ &\xi\in(\xi_4,\xi_3)\colon\quad \beta_{-1}^+(\xi)=\beta_{-1}^-(\xi),\quad \beta_3^+(\xi)=-\beta_3^-(\xi),\\ &\xi\in(\xi_3,-\infty)\colon\quad \beta_{-1}^+(\xi)=i\beta_{-1}^-(\xi),\quad \beta_3^+(\xi)=i\beta_3^-(\xi). \end{aligned}\tag{6.21}$$

The Riemann theta function and the Baker–Akhiezer functions. Introduce the function

$$h_\nu(\xi)=t^{7/4}(g(\xi)-g_\infty)+\delta_\nu(U(\xi)-U_\infty),\quad g_\infty=-\tfrac{1}{2}\mathcal{B},\quad U_\infty=-\tfrac{1}{2}\tau,\tag{6.22}$$

where the parameter δ_ν is defined by

$$\delta_{-1}=-t^{7/4}\mathcal{A},\quad \delta_3=-t^{7/4}\mathcal{A}+i\pi.\tag{6.23}$$

This function $h_\nu(\xi)$ has the following obvious properties:

(1) as $\xi\to+\infty$,

$$h_\nu(\xi)=t^{7/4}\vartheta+\mathcal{O}(\xi^{-1/2});\tag{6.24}$$

(2) $h_\nu(\xi)$ is discontinuous across the polygonal line $[\xi_5,\xi_4]\cup[\xi_4,\xi_3]\cup[\xi_3,-\infty)$ oriented from ξ_5 to infinity, moreover,

$$\begin{aligned} &\xi\in(\xi_5,\xi_4)\colon\quad h_\nu^+(\xi)+h_\nu^-(\xi)=t^{7/4}\mathcal{B}+\delta_\nu\tau,\\ &\xi\in(\xi_4,\xi_3)\colon\quad h_\nu^+(\xi)-h_\nu^-(\xi)=-t^{7/4}\mathcal{A}-\delta_\nu,\\ &\xi\in(\xi_3,\infty)\colon\quad h_\nu^+(\xi)+h_\nu^-(\xi)=0. \end{aligned}\tag{6.25}$$

Using the Riemann theta function, $\Theta(z)=\sum_n e^{\pi i n^2\tau+2\pi i n z}$, define the matrix function $\Phi_\nu^{(BA)}(\xi)$,

$$\Phi_\nu^{(BA)}(\xi)=(\beta_\nu(\xi))^{\sigma_3}\frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\Theta(U(\xi)+V+\phi_\nu)}{\Theta(U(\xi)+\frac{1+\tau}{2})}c_1(\xi) & \frac{\Theta(-U(\xi)+V+\phi_\nu)}{\Theta(-U(\xi)+\frac{1+\tau}{2})}c_1^*(\xi)\\ \frac{\Theta(U(\xi)+V+\phi_\nu-\frac{1+\tau}{2})}{\Theta(U(\xi))}c_2(\xi) & \frac{\Theta(-U(\xi)+V+\phi_\nu-\frac{1+\tau}{2})}{\Theta(-U(\xi))}c_2^*(\xi) \end{pmatrix}e^{h_\nu(\xi)\sigma_3}.\tag{6.26}$$

Here the parameters V, ϕ_ν and the factors $c_j(\xi)$, $c_j^*(\xi)$ are defined by

$$\begin{aligned} &V=-\tfrac{1}{2\pi i}t^{7/4}(\tau\mathcal{A}-\mathcal{B});\quad \phi_{-1}=\tfrac{1+\tau}{2},\quad \phi_3=0;\\ &c_1(\xi)=\tfrac{\Theta(\frac{1}{2})}{\Theta(V+\phi_\nu-\frac{\tau}{2})},\quad c_1^*(\xi)=\tfrac{\Theta(\frac{1}{2}+\tau)}{\Theta(V+\phi_\nu+\frac{\tau}{2})}\quad\text{if}\quad V+\phi_\nu\neq\tfrac{1}{2}+n+m\tau,\\ &c_1(\xi)=\beta_{-1}^{-2}(\xi)\tfrac{\omega_a\Theta(\frac{1}{2})}{2\Theta'(\frac{1+\tau}{2})},\quad c_1^*(\xi)=\beta_{-1}^{-2}(\xi)\tfrac{\omega_a\Theta(\frac{1}{2}+\tau)}{2\Theta'(\frac{1+\tau}{2})}\quad\text{if}\quad V+\phi_\nu=\tfrac{1}{2},\\ &c_2(\xi)=\tfrac{\Theta(\frac{\tau}{2})}{\Theta(V+\phi_\nu-\frac{1}{2}-\tau)},\quad c_2^*(\xi)=\tfrac{\Theta(\frac{\tau}{2})}{\Theta(V+\phi_\nu-\frac{1}{2})}\quad\text{if}\quad V+\phi_\nu\neq\tfrac{\tau}{2}+n+m\tau,\\ &c_2(\xi)=\beta_{-1}^{-2}(\xi)\tfrac{\omega_a\Theta(\frac{\tau}{2})}{2\Theta'(\frac{1+\tau}{2})},\quad c_2^*(\xi)=\beta_{-1}^{-2}(\xi)\tfrac{\omega_a\Theta(\frac{\tau}{2})}{2\Theta'(\frac{1+\tau}{2})}\quad\text{if}\quad V+\phi_\nu=\tfrac{\tau}{2},\qquad n,m\in\mathbb{Z}. \end{aligned}\tag{6.27}$$

It can be shown that $\det\Phi_\nu^{(BA)}(\xi)\equiv-1$; the function $\Phi_\nu^{(BA)}(\xi)$ (6.26) satisfies (6.16) and (6.18) and thus is one of the solutions of the RH problem 3. Any other solution to this RH problem has the form of a product

$$\Psi_\nu^{(BA)}(\xi)=R_\nu(\xi)\Phi_\nu^{(BA)}(\xi),\tag{6.28}$$

where $R_\nu(\xi)$ is rational with poles at $\xi=\xi_j$, $j=3,4,5$, and satisfies the asymptotic condition $R_\nu(\xi)=I+\mathcal{O}(\xi^{-1})$ as $\xi\to\infty$.

6.4. Local Solution of the RH Problem Near the Branch Points $\xi = \xi_j$, $j = 5, 4, 3$

As is well known, near single branch points, the relevant boundary-value problem can be solved using the classical Airy functions, see, e.g., [13].

6.4.1. RH problem for the Airy functions. Define the Wronsky matrix of the Airy functions [1],

$$Z_0(z) = \sqrt{2\pi} e^{-i\pi/4} \begin{pmatrix} v_2(z) & v_1(z) \\ \frac{d}{dz} v_2(z) & \frac{d}{dz} v_1(z) \end{pmatrix} e^{-i\pi\sigma_3/4}, \tag{6.29}$$

where

$$v_1(z) = \mathrm{Ai}\,(z), \quad v_2(z) = e^{i2\pi/3} \mathrm{Ai}\,(e^{i2\pi/3} z). \tag{6.30}$$

Along with $Z_0(z)$, introduce the following auxiliary functions:

$$Z_{-1}(z) = Z_0(z)(\Sigma_+)^{-1},\ Z_1(z) = Z_0(z)\Sigma_-,\ Z_2(z) = Z_1(z)\Sigma_+,\ \Sigma_+ = \left(\begin{smallmatrix} 1 & -i \\ 0 & 1 \end{smallmatrix}\right),\ \Sigma_- = \left(\begin{smallmatrix} 1 & 0 \\ -i & 1 \end{smallmatrix}\right). \tag{6.31}$$

By construction [1],

$$\begin{aligned} Z_j(z) =& z^{-\sigma_3/4} \tfrac{1}{\sqrt{2}} (\sigma_3 + \sigma_1)(I + \mathcal{O}(z^{-\frac{3}{2}})) e^{\frac{2}{3} z^{3/2} \sigma_3}, \\ & z \to \infty, \quad z \in \omega_j = \left\{ z \in \mathbb{C}\colon\ \arg z \in \left(-\pi + \tfrac{2\pi}{3} j, \tfrac{\pi}{3} + \tfrac{2\pi}{3} j\right) \right\}. \end{aligned} \tag{6.32}$$

Assemble the piecewise holomorphic functions $Z^{(j)}(z)$, $j = 5, 4, 3$,

$$Z^{(j)}(z) = \begin{cases} Z_{-1}(z) G_j, & \arg z \in (-\pi, -\frac{2\pi}{3}), \\ Z_0(z) G_j, & \arg z \in (-\frac{2\pi}{3}, 0), \\ Z_1(z) G_j, & \arg z \in (0, \frac{2\pi}{3}), \\ Z_2(z) G_j, & \arg z \in (\frac{2\pi}{3}, \pi), \end{cases} \tag{6.33}$$

where

$$G_5 = G_3 = I, \quad G_4 = i\sigma_3.$$

Observing that the jump matrices of $Z^{(j)}(z)$ coincide with those across the lines emanating from the node points ξ_j, $j = 5, 4, 3$, in Fig. 6.3, we are ready to construct the relevant local parametrices,

$$\Psi_\nu^{(j)}(\xi) = B_j^{(\nu)}(\xi) Z^{(j)}(z^{(j)}(\xi)), \quad j = 5, 4, 3, \quad \nu = -1, 3. \tag{6.34}$$

Here $B_j^{(\nu)}(\xi)$, $j = 5, 4, 3$, are holomorphic in some finite neighborhoods of $\xi = \xi_j$ matrices, and $z = z^{(j)}(\xi)$ are changes of variables biholomorphic in some neighborhoods of $\xi = \xi_j$.

6.4.2. Determination of the local change $z = z^{(j)}(\xi)$. This biholomorphic change of variables has to be chosen so as to ensure that the global parametrix, see below, has small enough jumps as $t \to +\infty$, namely, it must satisfy the condition

$$\tfrac{2}{3}(z^{(j)}(\xi))^{3/2} = t^{7/4}(g(\xi) - g(\xi_j)) + \delta_\nu(U(\xi) - U(\xi_j)) + o(1), \quad |\xi - \xi_j| = \text{const}.$$

The biholomorphic condition is satisfied with the choice

$$z^{(j)}(\xi) = t^{7/6} \Big(\tfrac{3}{2} \int_{\hat{\xi}_j}^{\xi} \hat{\mu}(z)\, dz \Big)^{2/3}, \quad j = 5, 4, 3, \tag{6.35}$$

where $\hat{\mu}(\xi)$ has the form (6.3), (6.6) with the branch points $\hat{\xi}_j$,

$$\hat{\mu}^2 = \tfrac{1}{900}\xi^5 - \tfrac{1}{30}\xi^3 + \tfrac{1}{30}x_0\xi^2 + \tfrac{1}{30}\hat{D}_1\xi + \tfrac{1}{30}\hat{D}_0 = \tfrac{1}{900}(\xi - \hat{\xi}_1)^2(\xi - \hat{\xi}_3)(\xi - \hat{\xi}_4)(\xi - \hat{\xi}_5).$$

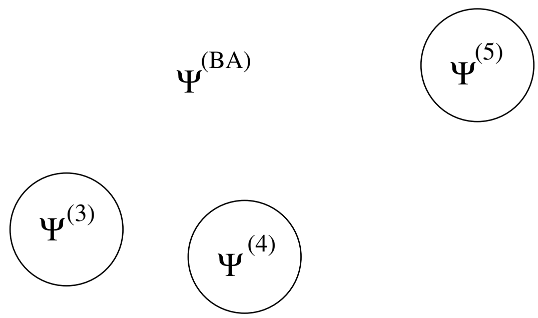

Fig. 6.5. The construction of the global parametrix.

The unique Abelian differential holomorphic on the elliptic curve $\hat{w}^2 = (\xi - \hat{\xi}_3)(\xi - \hat{\xi}_4)(\xi - \hat{\xi}_5)$ is

$$d\hat{U} = \tfrac{2}{\hat{\omega}_a}\Big(\tfrac{\partial}{\partial \hat{D}_1} - \hat{\xi}_1\,\tfrac{\partial}{\partial \hat{D}_0}\Big)\hat{\mu}(\xi)\,d\xi.$$

Thus, the 1-parameter deformation of the model degenerated curve that respects the degeneration is generated by the vector field

$$\tfrac{\partial}{\partial D} = \tfrac{2}{\hat{\omega}_a}\Big(\tfrac{\partial}{\partial \hat{D}_1} - \hat{\xi}_1\,\tfrac{\partial}{\partial \hat{D}_0}\Big). \tag{6.36}$$

Finally, we find the elliptic curve satisfying the asymptotic conditions,

$$\begin{gathered}\mu(\xi) + t^{-7/4}\frac{\delta_\nu}{w(\xi)} = \hat{\mu}(\xi) + \mathcal{O}(t^{-7/2}(\xi-\xi_j)^{-3/2}),\\ \text{for}\quad |\xi-\xi_1| > C_1 t^{-7/8},\quad |\xi-\xi_j| > C_j t^{-7/4},\quad C_j = \text{const},\quad j=3,4,5,\\ \omega_{a,b} = \hat{\omega}_{a,b} + \mathcal{O}(t^{-7/2}),\quad \hat{\tau} = \tau + \mathcal{O}(t^{-7/2}),\\ \mathcal{A} + t^{-7/4}\delta_\nu = \hat{\mathcal{A}} + \mathcal{O}(t^{-7/2}),\quad \mathcal{B} + t^{-7/4}\tau\delta_\nu = \hat{\mathcal{B}} + \mathcal{O}(t^{-7/2}).\end{gathered} \tag{6.37}$$

6.5. The Global Parametrix

The global approximate solution to the RH problem for $\Phi^{(\nu)}(\xi)$, $\nu = -1, 3$, is a piecewise analytic matrix function $\tilde{\Psi}_\nu(\xi)$ defined as follows:

$$\begin{gathered}\tilde{\Psi}_\nu(\xi) = \begin{cases}\Psi_\nu^{(j)}(\xi), & \xi \in C_j,\\ \Psi_\nu^{(BA)}, & \xi \in \mathbb{C}\setminus \cup_j C_j,\end{cases}\\ \Psi_\nu^{(BA)}(\xi) = R_\nu(\xi)\Phi^{(BA)}(\xi),\quad C_j = \{\xi \in \mathbb{C}\colon |\xi - \xi_j| < r\},\\ 0 < r < \tfrac{1}{2}\min_{k\neq j}|\xi_k - \xi_j|,\quad k,j = 3,4,5,\end{gathered} \tag{6.38}$$

see Fig. 6.5.

The exact solution is constructed using the correction function $\chi(\xi)$,

$$\chi(\xi) = \Psi(\xi)\tilde{\Psi}^{-1}(\xi), \tag{6.39}$$

that satisfies the following RH problem:

(1) the limit

$$\lim_{\xi\to\infty}\xi^{1/2}\big(\tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1)\xi^{\frac14\sigma_3}t^{\frac18\sigma_3}\chi(\xi)t^{-\frac18\sigma_3}\xi^{-\frac14\sigma_3}\tfrac{1}{\sqrt{2}}(\sigma_3+\sigma_1) - I\big)$$

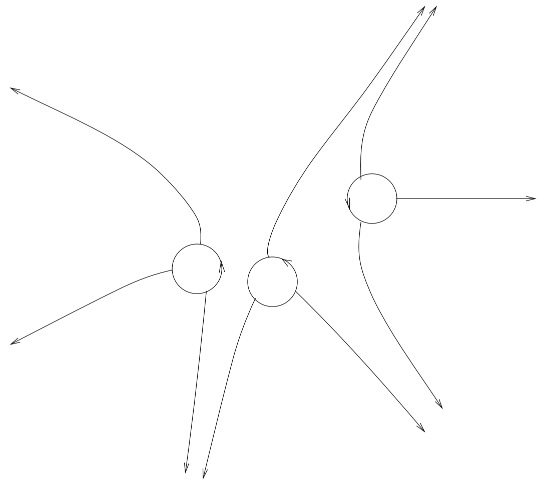

Fig. 6.6. The jump contour γ for the correction function $\chi(\xi)$.

exists and is diagonal;

(2) across the contour γ shown in Fig. 6.6, the jump condition holds,

$$\chi_+(\xi) = \chi_-(\xi)H(\xi), \quad \text{where} \quad H(\xi) = \tilde{\Psi}_-(\xi)S(\xi)\tilde{\Psi}_+^{-1}(\xi).$$

To apply L^2-theory to the latter RH problem [43], all jump matrices $H(\xi)$ across the jump contour γ in Fig. 6.6 must satisfy the estimate $\|H(\xi) - I\|_{L^2(\gamma)\cap C(\gamma)} = o(1)$ as $t \to \infty$. For the infinite tails emanating from the circles, this fact holds because the relevant jumps are uniformly exponentially small.

The jumps across the circles centered at the branch points ξ_j, $j = 3, 4, 5$, can be made small as t is large if one adjusts the rational matrix $R_\nu(\xi)$, $\nu = -1, 3$, and the holomorphic matrices $B_j^{(\nu)}(\xi)$, $\nu = -1, 3$, $j = 5, 4, 3$, in an appropriate way. Omitting the straightforward (but tedious) computations, we present the result,

$$\begin{aligned}
&V - \tfrac{1}{2}, V - \tfrac{\tau}{2}, V - \tfrac{1+\tau}{2} \neq n + m\tau, \quad n, m \in \mathbb{Z}: \\
&\qquad R_{-1}(\xi) = t^{-\frac{1}{8}\sigma_3}\begin{pmatrix} 1 & \frac{1}{\xi-\xi_5}q^{(\infty)} \\ r^{(\infty)} & 1 + \frac{1}{\xi-\xi_5}q^{(\infty)}r^{(\infty)} \end{pmatrix}, \\
&\qquad q^{(\infty)} = -\tfrac{\omega_a}{2}(\xi_5 - \xi_4)\frac{e^{2\pi i V(t)}\Theta(V+\frac{1+\tau}{2})\Theta(V+\frac{\tau}{2})\Theta(\frac{1}{2})\Theta(0)}{\Theta(V+\frac{1}{2})\Theta(V)\Theta(\frac{\tau}{2})\Theta'(\frac{1+\tau}{2})}, \\
&\qquad r^{(\infty)} - q^{(\infty)} = \frac{2\Theta(V)\Theta(V+\frac{1+\tau}{2})\Theta'(\frac{1+\tau}{2})\Theta(0)}{\omega_a\Theta(V+\frac{1}{2})\Theta(V+\frac{\tau}{2})\Theta(\frac{1}{2})\Theta(\frac{\tau}{2})}, \\
&V = \tfrac{1}{2}: \quad R_{-1}(\xi) = t^{-\frac{1}{8}\sigma_3}\left(\frac{\xi - \xi_5}{\xi - \xi_4}\right)^{\sigma_3}, \\
&V = \tfrac{\tau}{2}: \quad R_{-1}(\xi) = t^{-\frac{1}{8}\sigma_3}.
\end{aligned} \tag{6.40}$$

The relevant correction function satisfies the estimates

$$\chi(\xi) - I = \begin{cases} \mathcal{O}(t^{-7/24}), & |\xi| < \text{const}, \\ \mathcal{O}(t^{-7/24}\xi^{-1}), & |\xi| \to \infty, \end{cases} \tag{6.41}$$

in the domain of the parameter $x_0 = xt^{-3/2}$ described by

$$|V - \tfrac{1+\tau}{2} - n - m\tau| > Ct^{-\frac{7}{24}+\epsilon}, \quad n, m \in \mathbb{Z}, \quad \epsilon = \text{const} > 0. \tag{6.42}$$

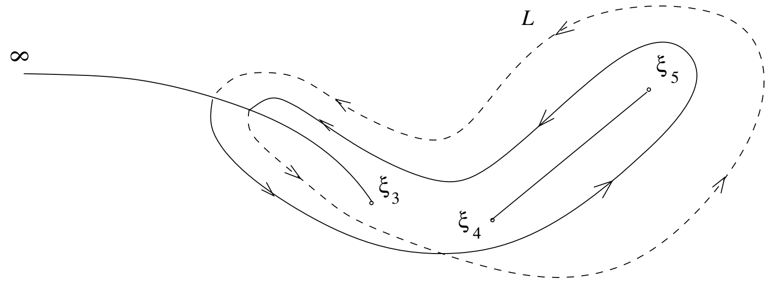

Fig. 7.1. The contour for the canonical dissection of the elliptic curve Γ.

6.6. Large-t Asymptotics of the RH Problem $1(\nu=3)$

In contrast, the computation of the left rational multiplier $R_\nu(\xi)$ for $\nu = 3$ yields

$$R_3(\xi) = t^{\frac{3}{8}\sigma_3} I, \tag{6.43}$$

together with the additional condition

$$\Theta(V) = 0, \tag{6.44}$$

or, equivalently,

$$V = \tfrac{1+\tau}{2} + n + m\tau. \tag{6.45}$$

In this case, the "external" parametrix is elementary,

$$\Phi_3^{(BA)}(\xi) = (\beta_3(\xi))^{\sigma_3} \tfrac{1}{\sqrt{2}}(\sigma_3 + \sigma_1) e^{h_3(\xi)\sigma_3}, \tag{6.46}$$

and the correction function $\chi(\xi)$ satisfies the estimates (6.41).

Remark 6.2. The above computation shows that, as $t \to +\infty$, the RH problem 1 is solvable either with $\nu = -1$ or $\nu = 3$ for each value of the deformation parameter x, $xt^{-3/2} \in D_0$. Thus, there is no room for the solvability of the RH problem with $\nu = -5$. Therefore, at least in the large-t limit, the special solution to ${P_{\rm I}}^2$ has no triple pole collisions and the corresponding solutions to $P_{\rm I}^{(2,1)}$ have no branch points. The relevant Malgrange divisor consists of smooth branches only.

7. ASYMPTOTIC DISTRIBUTION OF POLES

To compute the large-t asymptotic distribution of poles of the special solution to ${P_{\rm I}}^2$, we use the phase shift given in (6.45) and the definition of V in (6.27),

$$\tfrac{1}{2\pi i} t^{7/4}(\tau\mathcal{A} - \mathcal{B}) = n + \tfrac{1}{2} + (m + \tfrac{1}{2})\tau. \tag{7.1}$$

Using the canonical dissection of the Riemann surface, the difference $\omega_b\mathcal{A} - \omega_a\mathcal{B}$ is expressed in terms of a single contour integral over the contour $\mathcal{L}$ depicted in Fig. 7.1,

$$\omega_b\mathcal{A} - \omega_a\mathcal{B} = \omega_a \oint_{\mathcal{L}} U(z)\mu(z)\,dz.$$

Inflating the contour $\mathcal{L}$, we transform it to a contour encircling the infinite point, which is a branch point of the curve. Then, expanding the integrand at infinity and using the residue theorem, we find

$$\tau\mathcal{A} - \mathcal{B} = \oint_{\mathcal{L}} (U(z) - U_\infty)\mu(z)\,dz = -\tfrac{8\pi i}{7\omega_a}(3x_0 + \tfrac{2}{3}\xi_1), \tag{7.2}$$

and the distribution formula for poles (7.1) yields

$$3x_0 + \tfrac{2}{3}\xi_1 = \tfrac{7}{4} t^{-7/4}\big((n + \tfrac{1}{2})\omega_a + (m + \tfrac{1}{2})\omega_b\big). \tag{7.3}$$

Since ξ_1 is determined by x_0 via the Boutroux equations (6.8), equation (7.3) determines the position of the pole, $x_0^{(n,m)} = x_{n,m}t^{-3/2}$, as a transcendent function of two integers (n, m). As $t \to +\infty$, the particular pole $x_0^{(n,m)}$ approaches the attracting point satisfying the equation

$$\xi_1(x_0) = -\tfrac{9}{2}x_0. \tag{7.4}$$

7.1. Quasistationary Solutions of Equation $P_{\rm I}^{(2,1)}$

Although it is not known how to solve the transcendent equation (7.4), the problem of finding the attractor to the pole distribution is significantly simplified by observing that the pole attractors correspond to the quasistationary solutions of equation $P_{\rm I}^{(2,1)}$,

$$a(t) = a_\infty t^{3/2}(1 + \mathcal{O}(t^{-\epsilon})), \quad \epsilon > 0. \tag{7.5}$$

Substituting (7.5) into $P_{\rm I}^{(2,1)}$(2.7), we find an algebraic equation for the parameter a_∞,

$$a_\infty^4 - \tfrac{236}{243}a_\infty^2 + \tfrac{160}{2187} = 0, \tag{7.6}$$

with the roots $a_\infty \in \{\pm\frac{2\sqrt{5}}{9\sqrt{3}}, \pm\frac{2\sqrt{2}}{3}\}$. However, only one of the roots, namely,

$$a_\infty = \tfrac{2\sqrt{5}}{9\sqrt{3}}, \tag{7.7}$$

is consistent with the above properties of the large-t asymptotic spectral curve for the special solution of ${P_{\rm I}}^2$. The linearization of equation $P_{\rm I}^{(2,1)}$ at the 0-parameter power-series solution with the coefficient (7.7) of the leading-order term has four linearly independent solutions. Two of them are exponential, $\sim \exp[\pm i\frac{2\sqrt{2}}{3}(\frac{5}{3})^{3/4}\frac{4}{7}t^{7/4}]$, and we set them aside. Two other solutions of the linearized equation, $\sim t^{-1/4}$ and $\sim t^{-1/4}\ln t$, are relevant to our quasistationary behavior of the poles. Using them, we form the 2-parameter series

$$\begin{aligned}
&a(t) = t^{3/2}\sum_{k=0}^{\infty}(t^{-\frac{7}{4}}\ln(t^{-\frac{7}{4}}))^k a_k(t), \quad a_k(t) = \sum_{l=0}^{\infty} a_{kl}t^{-\frac{7}{4}l}, \quad a_{00} = \tfrac{2\sqrt{5}}{9\sqrt{3}},\\
&\quad a_{10}, a_{01} \in \mathbb{C} \quad \text{are arbitrary},\\
&a_{20} = -\tfrac{3\sqrt{15}}{784}a_{10}^2, \quad a_{11} = -\tfrac{3\sqrt{15}}{392}(a_{01} - 2a_{10})a_{10},\\
&a_{02} = \tfrac{3\sqrt{3}(\sqrt{15} - 50a_{01}^2 + 200a_{01}a_{10} + 3130a_{10}^2)}{7840\sqrt{5}},\\
&a_{30} = \tfrac{1467}{307328}a_{10}^3, \quad a_{21} = \tfrac{27}{307328}(163a_{01} + 861a_{10})a_{10}^2,\\
&a_{12} = \tfrac{27}{9834496}(1907\tfrac{\sqrt{3}}{\sqrt{5}} + 5216a_{01}^2 + 55104a_{01}a_{10} + 233392a_{10}^2)a_{10},\\
&a_{03} = \tfrac{9}{3073280}(1630a_{01}^3 + 25830a_{01}^2a_{10} + 218805a_{01}a_{10}^2 + 788362a_{10}^3)\\
&\qquad + \tfrac{9\sqrt{3}}{49172480\sqrt{5}}(28605a_{01} + 258143a_{10}), \ldots
\end{aligned} \tag{7.8}$$

Let us relate the free parameters a_{01}, a_{10} in (7.8) to the integers n and m in (7.3). Recall that, along the boundary ∂D_+, two branch points ξ_4 and ξ_5 of the asymptotic spectral curve coalesce. Namely, if $t \to +\infty$, then the limiting values corresponding to the attracting point are

$$x_0^* = \tfrac{2\sqrt{5}}{9\sqrt{3}}, \quad \xi_3^* = -\tfrac{4\sqrt{5}}{\sqrt{3}}, \quad \xi_{1,2}^* = -\tfrac{\sqrt{5}}{\sqrt{3}}, \quad \xi_{4,5}^* = \sqrt{15}. \tag{7.9}$$

The asymptotic behavior of the branch points in the model elliptic spectral curve compatible with the expansion (7.8) is given by

$$\begin{aligned}
\xi_1 = \xi_2 &= -\tfrac{\sqrt{5}}{\sqrt{3}} + \tfrac{3}{4}(a_{01} + 7a_{10} + a_{10}\ln(t^{-7/4}))t^{-7/4} + \mathcal{O}(t^{-7/2}\ln^2 t),\\
\xi_3 &= -\tfrac{4\sqrt{5}}{\sqrt{3}} - \tfrac{6}{7}\big(a_{01} + 4a_{10} + a_{10}\ln(t^{-7/4})\big)t^{-7/4} + \mathcal{O}(t^{-7/2}\ln^2 t),\\
\xi_{4,5} &= \sqrt{15} \mp \sqrt{6}\sqrt[4]{15}\sqrt{a_{10}}\,t^{-7/8}\\
&\quad - \tfrac{9}{28}\big(a_{01} + 11a_{10} + a_{10}\ln(t^{-7/4})\big)t^{-7/4} + \mathcal{O}(t^{-21/8}\ln^2 t).
\end{aligned} \tag{7.10}$$

The asymptotic behavior of the periods $\omega_{a,b}$, as $\xi_5 - \xi_4 \to 0$, has the forms

$$\begin{aligned}
\omega_a &= \tfrac{2\pi i}{\sqrt{\xi_4 - \xi_3}}\big(1 - \tfrac{\xi_5 - \xi_4}{4(\xi_4 - \xi_3)} + \mathcal{O}((\xi_5 - \xi_4)^2)\big),\\
\omega_b &= \tfrac{2}{\sqrt{\xi_4 - \xi_3}}\ln\tfrac{\xi_5 - \xi_4}{16(\xi_4 - \xi_3)} + \mathcal{O}((\xi_5 - \xi_4)\ln(\xi_5 - \xi_4)), \quad \xi_5 - \xi_4 \to 0.
\end{aligned} \tag{7.11}$$

Thus, for large t,

$$\begin{aligned}
\omega_a &= i\pi\tfrac{2\sqrt[4]{3}}{\sqrt{7}\sqrt[4]{5}}(1 + \mathcal{O}(t^{-7/8})),\\
\omega_b &= -\tfrac{\sqrt[4]{3}\sqrt{7}}{4\sqrt[4]{5}}\ln t + \tfrac{2\sqrt[4]{3}}{\sqrt{7}\sqrt[4]{5}}\ln\tfrac{3^{3/2}\sqrt{-a_{10}}}{2^{3/2}7^{3/2}\sqrt[4]{15}} + \mathcal{O}(t^{-7/8}\ln t), \quad t \to +\infty.
\end{aligned} \tag{7.12}$$

Finally, the coefficients a_{10} and a_{01} determining the asymptotic series for $a(t)$ (7.8),

$$a(t) = \tfrac{2\sqrt{5}}{9\sqrt{3}}t^{3/2} + t^{-1/4}\big(a_{01} + a_{10}\ln(t^{-7/4})\big) + \cdots,$$

follow, using the asymptotic formula (7.3),

$$\begin{aligned}
a_{10}\ln(t^{-7/4}) + a_{01} + \cdots &= \big(-\tfrac{2\sqrt{5}}{9\sqrt{3}} - \tfrac{2}{9}\xi_1\big)t^{7/4} + \tfrac{7}{12}((n + \tfrac{1}{2})\omega_a + (m + \tfrac{1}{2})\omega_b)\\
&= -\tfrac{1}{6}a_{10}\ln(t^{-7/4}) - \tfrac{1}{6}a_{01} - \tfrac{7}{6}a_{10}\\
&\quad + \tfrac{7}{12}(n + \tfrac{1}{2})i\pi\tfrac{2\sqrt[4]{3}}{\sqrt{7}\sqrt[4]{5}} + \tfrac{7}{12}(m + \tfrac{1}{2})\tfrac{\sqrt[4]{3}}{\sqrt[4]{5}\sqrt{7}}\big[\ln(t^{-7/4}) + \ln\tfrac{3^3 a_{10}}{2^5 7^2\sqrt{15}}\big] + \cdots
\end{aligned} \tag{7.13}$$

Equating coefficients at $\ln(t^{-7/4})$, we find $a_{10} = a_{10}^{(m,n)}$,

$$a_{10}^{(m,n)} = (m + \tfrac{1}{2})\tfrac{\sqrt[4]{3}}{2\sqrt[4]{5}\sqrt{7}}, \quad m \in \mathbb{Z}_+, \tag{7.14}$$

while the constant terms yield the family of values of $a_{01} = a_{01}^{(m,n)}$,

$$a_{01}^{(m,n)} = (m + \tfrac{1}{2})\tfrac{\sqrt[4]{3}}{2\sqrt[4]{5}\sqrt{7}}\ln\big[(m + \tfrac{1}{2})\tfrac{3^{11/4}}{2^6 5^{3/4} 7^{5/2} e}\big] + i\pi(n + \tfrac{1}{2})\tfrac{\sqrt[4]{3}}{\sqrt[4]{5}\sqrt{7}}, \qquad m \in \mathbb{Z}_+, \quad n \in \mathbb{Z}. \tag{7.15}$$

Formulas (7.14) and (7.15) with (7.8) yield the asymptotic formula (1.10) for $a^{(m,n)}(t)$, which implies that the poles of the special solution to $P_{\mathrm{I}}{}^2$ in the vicinity of the attracting point $x_0^* = \frac{2\sqrt{5}}{9\sqrt{3}}$ form a regular lattice with slowly modulated intervals and a boundary formed by the line of poles corresponding to the values $m = 0$ and $n \in \mathbb{Z}$. In particular, the interval between the two rightmost vertical lines of poles is given by

$$a^{(0,n)}(t) - a^{(1,n)}(t) = -t^{-1/4}\tfrac{\sqrt[4]{3}}{2\sqrt[4]{5}\sqrt{7}}\ln\big[t^{-7/4}\tfrac{3^{17/4}}{2^7 5^{3/4} 7^{5/2} e}\big] + \mathcal{O}(t^{-2}\ln^2 t).$$

Note that the boundary ∂D_+ formally corresponds to $m = -\frac{1}{2}$, $n \in \mathbb{R}$, and the distance between the first vertical line of poles and ∂D_+ is

$$a^*(t) - a^{(0,n)}(t) = -t^{-1/4}\tfrac{\sqrt[4]{3}}{4\sqrt[4]{5}\sqrt{7}}\ln\big[t^{-7/4}\tfrac{3^{11/4}}{2^7 5^{3/4} 7^{5/2} e}\big] + \mathcal{O}(t^{-2}\ln^2 t).$$

8. PROBLEMS AND PERSPECTIVES

We have presented and studied equation $P_{\mathrm{I}}^{(2,1)}$, which is, to our best knowledge, the first example of a differential equation that controls the isomonodromy deformations of a linear ODE with rational coefficients and does not possess the Painlevé property. At the first glance, its existence contradicts the theorem by Miwa and Malgrange. However, this is not the case. The absence of the Painlevé property in $P_{\mathrm{I}}^{(2,1)}$ is related to the fact that the domain of solvability of the corresponding RH problem in the 2-dimensional complex space with the coordinates (t, x) is restricted to the Malgrange divisor of ${P_{\mathrm{I}}}^2$, i.e., to the set of complex lines $(t, a(t))$, which may intersect. Actually, this fact provides us with important information on the nontrivial analytic structure of the Malgrange divisor for ${P_{\mathrm{I}}}^2$ which forms a Riemann surface with infinitely many sheets and third-order branch points.

Along with this, the discovery of equation $P_{\mathrm{I}}^{(2,1)}$ provides us with a new wide field of research. For instance, it is interesting to explore the possibility of existence of similar equations associated with other isomonodromic solutions of KdV or other integrable PDEs like the nonlinear Schrödinger or Pohlmeyer–Lund–Regge equations.

Other interesting problem not discussed above is the structure on $P_{\mathrm{I}}^{(2,1)}$ induced by the singularity reduction of the Hamiltonian structure on ${P_{\mathrm{I}}}^2$(and, in the case of a successful extension of the singularity reduction methodology to the hierarchies associated with other Painlevé equations, the structures induced by the Weyl symmetries).

Finally, we mention the problem of characterizing initial data to ${P_{\mathrm{I}}}^2$ (and other isomonodromy deformation equations) whose singularity reductions do not have branch points. In the ${P_{\mathrm{I}}}^2$ case, we conjecture that its special solution considered above does not have merging poles for any t, and therefore, the relevant solution of $P_{\mathrm{I}}^{(2,1)}$ does not have branch points. If this conjecture is true, it can serve as another characteristic property of this special solution.

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