Simple Lie algebras, Drinfeld–Sokolov hierarchies, and multi-point correlation functions

Marco Bertola, Boris Dubrovin, Di Yang

Abstract

For a simple Lie algebra $\mathfrak{g}$, we derive a simple algorithm for computing logarithmic derivatives of tau-functions of Drinfeld–Sokolov hierarchy of $\mathfrak{g}$-type in terms of $\mathfrak{g}$-valued resolvents. We show, for the topological solution to the lowest-weight-gauge Drinfeld–Sokolov hierarchy of $\mathfrak{g}$-type, the resolvents evaluated at zero satisfy the *topological ODE*.

Contents

1 Introduction

1.1 Simple Lie algebra and Drinfeld–Sokolov hierarchy

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank n, with the Lie bracket denoted by $[\cdot,\cdot]$. Let $\mathrm{ad}: \mathfrak{g} \to \mathrm{gl}(\mathfrak{g})$ be the adjoint representation of $\mathfrak{g}$. We denote by h, $h^\vee$ the Coxeter and dual Coxeter numbers [40] of $\mathfrak{g}$, and $m_1 = 1 < m_2 \leq \ldots \leq m_{n-1} < m_n = h-1$ the exponents. Denote $(\cdot|\cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the *normalized* Cartan–Killing [14] form

$$(x\,|\,y) := \frac{1}{2h^\vee}\mathrm{tr}\,(\mathrm{ad}_x \cdot \mathrm{ad}_y), \qquad \forall\, x, y \in \mathfrak{g}. \tag{1.1.1}$$

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and let $\triangle \subset \mathfrak{h}^*$ be the root system. We choose a set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$. Then $\mathfrak{g}$ has the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\triangle} \mathfrak{g}_\alpha.$$

For any $\alpha \in \triangle$, denote by H_α the unique element in $\mathfrak{h}$ such that $(H_\alpha|X) = \alpha(X), \quad \forall\, X \in \mathfrak{h}$. The normalized Cartan–Killing form induces naturally a bilinear inner product on $\mathfrak{h}^*$:

$$(\alpha|\beta) = (H_\alpha|H_\beta), \quad \forall\, \alpha, \beta \in \mathfrak{h}^*.$$

Denote by $E_i \in \mathfrak{g}_{\alpha_i}$, $F_i \in \mathfrak{g}_{-\alpha_i}$, $H_i = 2H_{\alpha_i}/(\alpha_i|\alpha_i)$ the Weyl generators of $\mathfrak{g}$. They satisfy

$$[E_i, F_i] = H_i\,\delta_{ij}, \quad [H_i, E_j] = A_{ij}\,E_j, \quad [H_i, F_j] = -A_{ij}\,F_j$$

where (A_{ij}) denotes the Cartan matrix associated to $(\mathfrak{g}, \Pi)$, and δ_{ij} is the Kronecker delta. Here and below, free Latin indices take integer values from 1 to n unless otherwise indicated.

Let θ be the highest root w.r.t. Π; recall that $(\theta|\theta) = 2$. We choose $E_{-\theta} \in \mathfrak{g}_{-\theta}$, $E_\theta \in \mathfrak{g}_\theta$, normalized by the conditions $(E_\theta|E_{-\theta}) = 1$ and $\omega(E_{-\theta}) = -E_\theta$, where $\omega : \mathfrak{g} \to \mathfrak{g}$ is the Chevalley involution. Let

$$I_+ := \sum_{i=1}^n E_i \tag{1.1.2}$$

be a principal nilpotent element of $\mathfrak{g}$. Define

$$\Lambda = I_+ + \lambda\,E_{-\theta}. \tag{1.1.3}$$

Denote by $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ the loop algebra of $\mathfrak{g}$. The Lie bracket $[\cdot,\cdot]$ and the Cartan–Killing form $B(\cdot,\cdot)$ extend naturally to $L(\mathfrak{g})$. We have

$$L(\mathfrak{g}) = \mathrm{Ker}\,\mathrm{ad}_\Lambda \oplus \mathrm{Im}\,\mathrm{ad}_\Lambda, \qquad \mathrm{Ker}\,\mathrm{ad}_\Lambda \perp \mathrm{Im}\,\mathrm{ad}_\Lambda. \tag{1.1.4}$$

Recall that the *principal gradation* on $L(\mathfrak{g})$ is defined by

$$\deg\lambda = h, \quad \deg E_i = -\deg F_i = 1, \qquad i = 1, \ldots, n.$$

Observe that

$$\deg\Lambda = 1.$$

This gradation is of course also defined on $\mathfrak{g} = \mathfrak{g} \otimes 1$. With the principal gradation, the loop algebra $L(\mathfrak{g})$ and the simple Lie algebra $\mathfrak{g}$ decompose into direct sums of homogeneous subspaces $L(\mathfrak{g})^j,\ \mathfrak{g}^j,\ j \in \mathbb{Z}:$

$$L(\mathfrak{g}) = \bigoplus_{j\in\mathbb{Z}} L(\mathfrak{g})^j, \qquad \mathfrak{g} = \bigoplus_{j=-(h-1)}^{h-1} \mathfrak{g}^j.$$

We will denote the projection onto the nonnegative subspace by $(\bullet)^+ : L(\mathfrak{g}) \to \sum_{j\geq 0} L(\mathfrak{g})^j$, and onto the negative subspace by $(\bullet)^-$. It is known [39] that $\operatorname{Ker}\operatorname{ad}_\Lambda \subset L(\mathfrak{g})$ admits the following decomposition

$$\operatorname{Ker}\operatorname{ad}_\Lambda = \bigoplus_{j\in E} \mathbb{C}\Lambda_j, \qquad \Lambda_j \in L(\mathfrak{g})^j,\ j \in E,$$
$$[\Lambda_i, \Lambda_j] = 0, \qquad \forall\, i, j \in E.$$

Here, $E := \bigsqcup_{i=1}^n (m_i + h\mathbb{Z})$. We choose normalizations of $\Lambda_j,\ j \in E$ satisfying

$$\Lambda_{m_a+kh} = \Lambda_{m_a}\,\lambda^k, \qquad k \in \mathbb{Z}, \tag{1.1.5}$$
$$(\Lambda_{m_a}\,|\,\Lambda_{m_b}) = h\,\eta_{ab}\,\lambda. \tag{1.1.6}$$

Here and below,

$$\eta_{ab} := \delta_{a+b,n+1}. \tag{1.1.7}$$

Since $\Lambda \in L(\mathfrak{g})^1$, we fix the normalization of Λ_1 such that

$$\Lambda_1 = \Lambda.$$

It is useful to notice that $\Lambda_{m_a},\ a = 1, \ldots, n$ have the form [43]

$$\Lambda_{m_a} = L_{m_a} + \lambda\, K_{m_a-h}, \qquad L_{m_a} \in \mathfrak{g}^{m_a},\ K_{m_a-h} \in \mathfrak{g}^{m_a-h},\ L_{m_a} \neq 0,\ K_{m_a-h} \neq 0.$$

In [19], Drinfeld–Sokolov associate to $\mathfrak{g}$ an integrable hierarchy of Hamiltonian evolutionary PDEs, known as the Drinfeld–Sokolov (DS) hierarchy of $\mathfrak{g}$-type. Let us briefly review their construction in the form suitable for subsequent considerations. Denote by $\mathfrak{b} = \mathfrak{g}^{\leq 0}$ a Borel subalgebra of $\mathfrak{g}$, and $\mathfrak{n} = \mathfrak{g}^{<0}$ a nilpotent subalgebra. Let

$$\mathcal{L} = \partial_x + \Lambda + q(x), \qquad q(x) \in \mathfrak{b}. \tag{1.1.8}$$

Definition 1.1.1. *The basic resolvents $R_a,\ a = 1, \ldots, n$ of $\mathcal{L}$ are defined as the unique solutions to*

$$[\mathcal{L}, R_a] = 0, \quad R_a \in \mathcal{A}^q \otimes \mathfrak{g}((\lambda^{-1})), \tag{1.1.9}$$
$$R_a(\lambda; q, q_x, \ldots) = \Lambda_{m_a} + \text{ lower order terms w.r.t. } \deg, \tag{1.1.10}$$
$$(R_a(\lambda; q, q_x, \ldots)\,|\,R_b(\lambda; q, q_x, \ldots)) = h\,\eta_{ab}\,\lambda. \tag{1.1.11}$$

Here and below, $\mathcal{A}^q$ denotes the ring of differential polynomials in q, namely, an element of $\mathcal{A}^q$ is a polynomial in the entries of $q, q_x, q_{2x}, \ldots$.

Existence and uniqueness of the basic resolvents will be shown in Prop. 2.2.3.

The DS flows for the $\mathfrak{b}$-valued function $q = q(x, \mathbf{T})$, $\mathbf{T} = (T^a_k)^{a=1,\ldots,n}_{k\geq 0}$ are evolution PDEs

$$\frac{\partial q}{\partial T^a_k} = f^a_k(q, q_x, q_{xx}, \ldots) \tag{1.1.12}$$

for some $\mathfrak{b}$-valued differential polynomials f_k^a defined by the following Lax representation

$$\frac{\partial \mathcal{L}}{\partial T_k^a} = \left[(\lambda^k R_a)_+ \,,\, \mathcal{L}\right], \qquad a = 1, \dots, n,\ k \geq 0. \tag{1.1.13}$$

Here $(\bullet)_+$ stands for the polynomial part in λ. These flows are well-defined and pairwise commute [19]; they form the *pre-DS hierarchy*.

Consider transformations of the dependent variable $q(x) \mapsto \tilde{q}(x)$ of the pre-DS hierarchy induced by gauge transformations of the form

$$\mathcal{L} = \partial_x + \Lambda + q(x) \mapsto \widetilde{\mathcal{L}} = e^{\mathrm{ad}_{N(x)}} \mathcal{L} = \partial_x + \Lambda + \tilde{q}(x) \tag{1.1.14}$$

for an arbitrary smooth $\mathfrak{n}$-valued function $N(x)$. A crucial point of the Drinfeld–Sokolov construction is the following statement.

Lemma 1.1.2. *The gauge transformations (1.1.14) are symmetries of the pre-DS flows of (1.1.13). In particular, they map solutions to solutions.*

In our approach the proof of this simple but important statement easily follows by observing that the basic resolvents $\widetilde{R}_a$ of the gauge-transformed operator $\widetilde{\mathcal{L}}$ satisfy

$$\widetilde{R}_a(\lambda; \tilde{q}, \tilde{q}_x, \dots) = e^{\mathrm{ad}_{N(x)}} R_a(\lambda; q, q_x, \dots), \quad a = 1, \dots, n. \tag{1.1.15}$$

The DS hierarchy is obtained from (1.1.13) by considering suitably chosen *gauge invariant* functions q^{can} (see below for more details).

1.2 From resolvents to tau-function

We start from defining tau-functions of an arbitrary solution $q(x, \mathbf{T})$ of the pre-DS hierarchy. Then we verify its independence from the choice of the gauge with respect to the transformations of the form (1.1.14).

Definition 1.2.1. *Define a sequence of functions $\Omega_{a,k;b,\ell} = \Omega_{a,k;b,\ell}(q, q_x, \dots) \in \mathcal{A}^q$, $a, b = 1, \dots, n$, $k, \ell, \geq 0$ by means of the generating function expression below*

$$\sum_{k,\ell \geq 0} \frac{\Omega_{a,k;b,\ell}}{\lambda^{k+1}\mu^{\ell+1}} = \frac{(R_a(\lambda) \,|\, R_b(\mu))}{(\lambda - \mu)^2} - \eta_{ab} \frac{m_a \lambda + m_b \mu}{(\lambda - \mu)^2}. \tag{1.2.1}$$

We call $\Omega_{a,k;b,\ell}$ the two-point correlation functions.

Lemma 1.2.2. *The two-point correlation functions $\Omega_{a,k;b,\ell}$ satisfy the following properties*

$$\Omega_{a,k;b,\ell} \in \mathcal{A}^q, \qquad \Omega_{a,k;b,\ell} = \Omega_{b,\ell;a,k}, \quad \forall\, a, b = 1, \dots, n,\ k, \ell \geq 0, \tag{1.2.2}$$

$$\partial_{T_m^c} \Omega_{a,k;b,\ell} = \partial_{T_k^a} \Omega_{b,\ell;c,m} = \partial_{T_\ell^b} \Omega_{c,m;a,k}, \quad \forall\, a, b, c = 1, \dots, n,\ k, \ell, m \geq 0. \tag{1.2.3}$$

Lemma 1.2.3. *For an arbitrary solution $q(x, \mathbf{T})$ to (1.1.13), there exists $\tau = \tau(x, \mathbf{T})$ such that*

$$\frac{\partial^2 \log \tau}{\partial T_k^a \partial T_\ell^b} = \Omega_{a,k;b,\ell}\left(q(x, \mathbf{T}), q_x(x, \mathbf{T}), \dots\right) \tag{1.2.4}$$

$$\frac{\partial \tau}{\partial x} = -\frac{\partial \tau}{\partial T_0^1}. \tag{1.2.5}$$

The proofs are provided later in the paper.

In view of (1.2.5) we will henceforth identify x with $-T_0^1$ for $\tau(x,\mathbf{T})$. So we will use the short notation $\tau = \tau(\mathbf{T})$. Note that the scalar function $\tau(\mathbf{T})$ advocated for in Lemma 1.2.3 is uniquely determined by the solution $q(x,\mathbf{T})$ only up to a factor of the form

$$\exp\left(d_0 + \sum_{a=1}^{n}\sum_{k\geq 0} d_{a,k}T_k^a\right), \qquad d_0,\, d_{a,k} \text{ are arbitrary constants.} \tag{1.2.6}$$

Definition 1.2.4. *We call $\tau(\mathbf{T})$ the* tau-function *of the solution $q(x,\mathbf{T})$ of the pre-DS hierarchy.*

Definition 1.2.5. *For an arbitrary solution to the pre-DS hierarchy, let $\tau(\mathbf{T})$ be a tau-function of this solution in the sense of Definition 1.2.4. The N***-point correlation functions** *of $\tau(\mathbf{T})$ are defined by*

$$\langle\langle \tau_{a_1k_1}\dots\tau_{a_Nk_N}\rangle\rangle^{DS} = \frac{\partial^N \log\tau}{\partial T_{k_1}^{a_1}\dots\partial T_{k_N}^{a_N}}, \qquad k_1,\dots,k_N\geq 0,\ N\geq 1. \tag{1.2.7}$$

From (1.1.15) it easily follows

Lemma 1.2.6. *The tau-function of a solution to the pre-DS hierarchy is invariant, up to a factor of the form (1.2.6), with respect to the gauge transformations (1.1.14).*

Thus $\tau(\mathbf{T})$ will also be called tau-function of the solution q^{can} of the DS hierarchy corresponding to a gauge-fixed Lax operator. The usual procedure [19] to fix the gauge is by choosing a subspace $\mathcal{V}\subset\mathfrak{b}$ transversal to the adjoint action of the nilpotent subgroup so that $q^{can}(x)$ restricts to a $\mathcal{V}$-valued function (see below).

1.3 Main results

For any $a=1,\dots,n$ introduce the following differential operator depending on a parameter λ

$$\nabla_a(\lambda) = \sum_{k\geq 0}\frac{\partial_{T_k^a}}{\lambda^{k+1}}. \tag{1.3.1}$$

For a given $N\geq 1$ and a collection of integers $a_1,\dots,a_N\in\{1,\dots,n\}$, we define the following generating series of N-point correlations functions by

$$F_{a_1,\dots,a_N}(\lambda_1,\dots,\lambda_N;\mathbf{T}) = \nabla_{a_1}(\lambda_1)\cdots\nabla_{a_N}(\lambda_N)\,\log\tau(\mathbf{T}). \tag{1.3.2}$$

Observe that, for $N\geq 2$ the correlation functions (1.2.7) depend only on the solution $q(x,\mathbf{T})$ of the pre-DS hierarchy. Our goal is to derive an explicit expression for these generating functions for $N\geq 2$ in terms of the defined above basic resolvents.

For any $N\geq 2$ define a cyclic-symmetric N-linear form $B:\mathfrak{g}\times\cdots\times\mathfrak{g}\to\mathbb{C}$ by

$$B(x_1,\dots,x_N) = \operatorname{tr}(\operatorname{ad}_{x_1}\circ\cdots\circ\operatorname{ad}_{x_N}), \qquad \forall\, x_1,\dots,x_N\in\mathfrak{g}. \tag{1.3.3}$$

Theorem 1.3.1. *For an arbitrary solution $q^{can}(\mathbf{T})$ to the DS-hierarchy, let $\tau(\mathbf{T})$ be a tau-function of this solution. Then $\forall\, N\geq 2$, we have*

$$F_{a_1,\dots,a_N}(\lambda_1,\dots,\lambda_N;\mathbf{T}) = -\frac{1}{2\,N\,h^\vee}\sum_{s\in S_N}\frac{B\left(R^{can}_{a_{s_1}}(\lambda_{s_1};\mathbf{T}),\dots,R^{can}_{a_{s_N}}(\lambda_{s_N};\mathbf{T})\right)}{\prod_{j=1}^N(\lambda_{s_j}-\lambda_{s_{j+1}})} -\delta_{N2}\,\eta_{a_1a_2}\frac{m_{a_1}\lambda_1+m_{a_2}\lambda_2}{(\lambda_1-\lambda_2)^2} \tag{1.3.4}$$

where $R_a^{can}(\lambda)$, $a=1,\dots,n$ denote the basic resolvents of $\mathcal{L}^{can} := \partial_x+\Lambda(\lambda)+q^{can}$. In particular, $\forall\, N\geq 2$, $\forall\, a_1,\dots,a_N\in\{1,\dots,n\}$, we have $F_{a_1,\dots,a_N}(\lambda_1,\dots,\lambda_N;\mathbf{T})\in\mathcal{A}^{q^{can}}[[\lambda_1^{-1},\dots,\lambda_N^{-1}]]$.

The partition function. We now consider a particular tau-function that we shall call the *partition function*: it will be denoted by $Z(\mathbf{t})$ where the new time variables $\mathbf{t}$ differ from the original $\mathbf{T}$ by a rescaling (see eq. (1.3.6)). This particular tau-function is uniquely specified up to a multiplicative constant by the following *string equation*

$$\sum_{a=1}^{n}\sum_{k\geq 0} t^a_{k+1}\frac{\partial Z}{\partial t^a_k}+\frac{1}{2}\sum_{a,b=1}^{n}\eta_{ab}\,t^a_0\,t^b_0\,Z=\frac{\partial Z}{\partial t^1_0} \tag{1.3.5}$$

(see details in Section 4.2 below). Here, the time variables t^a_k and T^a_k are related by

$$\frac{\partial}{\partial t^a_k}=c_{a,k}\frac{\partial}{\partial T^a_k},\quad c_{a,k}=\frac{(-1)^k}{\sqrt{-h}^{m_a+hk+1}\,(\frac{m_a}{h})_{k+1}},\qquad a=1,\dots,n,\ k\geq 0 \tag{1.3.6}$$

where $(\cdot)_\ell$ denotes the Pochhammer symbol, i.e. $(y)_\ell:=y(y+1)\cdots(y+\ell-1)$.

Theorem 1.3.2. *Let the subspace* $\mathcal{V}:=\operatorname{Ker}\operatorname{ad}_{I_-}\subset\mathfrak{g}$ *be the lowest weight gauge (see eq. (3.1.1) for the definition of* I_-*), and* $\mathcal{L}^{can}$ *the associated Lax operator. Let* R^{can}_a, $a=1,\dots,n$ *be the basic resolvents of* $\mathcal{L}^{can}$*. For the Drinfeld–Sokolov partition function* Z*, define* $M_a(\lambda)=\lambda^{-\frac{m_a}{h}}R^{can}_a(\lambda;\mathbf{t}=\mathbf{0})$*. Then for any* $a\in\{1,\dots,n\}$*,* $M_a(\lambda)$ *satisfies the topological ODE of* $\mathfrak{g}$*-type*

$$M'=\kappa\,[M,\Lambda],\qquad \kappa=\left(\sqrt{-h}\right)^{-h},\quad {}':=\frac{\mathrm{d}}{\mathrm{d}\lambda}. \tag{1.3.7}$$

Observe that, as $\lambda\to\infty$, the solutions $M_a(\lambda)$ admit the expansions

$$M_a=\lambda^{-\frac{m_a}{h}}\left[\Lambda_{m_a}+\text{lower degree terms w.r.t. deg}\right],\quad a=1,\dots,n.$$

Thus, M_a coincide with the basis of regular solutions to the topological ODE constructed in [8].

1.4 Applications to the FJRW theory

Let $f:\mathbb{C}^m\to\mathbb{C}$ be a quasi-homogeneous polynomial, i.e. there exist positive integers $d,n_1,\dots,n_m$ s.t.

$$f(z^{n_1}x_1,\dots,z^{n_m}x_m)=z^d\,f(x_1,\dots,x_m),\quad \forall\,z\in\mathbb{C}.$$

The weight of x_i is defined to be $q_i=\frac{n_i}{d}$, $i=1,\dots,m$. In general the gradient of f vanishes at the origin and hence the zero level-set $f^{-1}(0)$ is a singular variety and defines a "singularity" in the sense of singularity theory [3]. The function f is called *non-degenerate* if the choice of weights q_i is unique and $x=\mathbf{0}$ is the only singularity of f. Let G_f (or G_{max}) denote the maximal diagonal symmetry group of f, which is the subgroup of $\mathrm{Aut}(f)$ consisting of diagonal matrices γ such that $f(\gamma x)=f(x)$. It is easy to see that the matrix

$$J=\mathrm{diag}(e^{2\pi i q_1},\dots,e^{2\pi i q_m})\in G_f.$$

Let G be a subgroup of G_f containing $\langle J\rangle$. Let n be the dimension of the Fan–Jarvis–Ruan cohomology ring [28] associated to (f,G). Fan–Jarvis–Ruan associate with the pair (f,G) a certain *generalized Witten class*, called the Fan–Jarvis–Ruan–Witten class

$$\Lambda^{f,G}_{g,N}(a_1,\dots,a_N)\in H^*(\overline{\mathcal{M}}_{g,N}),\qquad a_i=1,\dots,n,\ i=1,\dots,N$$

such that incorporation of these cohomological classes to $\overline{\mathcal{M}}_{g,N}$ gives rise to a cohomological filed theory [46, 28]. The FJRW invariants are defined by

$$\langle\tau_{a_1k_1}\cdots\tau_{a_Nk_N}\rangle^{f,G}_g=\int_{\overline{\mathcal{M}}_{g,N}}\psi_1^{k_1}\cdots\psi_N^{k_N}\cdot\Lambda^{f,G}_{g,N}(a_1,\dots,a_N)$$

where ψ_i, $i=1,\dots,N$ are ψ-classes.

Definition 1.4.1. *The partition function $Z^{f,G}$ of FJRW invariants is defined by*

$$Z^{f,G}(\mathbf{t}) = \exp\left(\sum_{g,N\geq 0} \frac{1}{N!} \sum_{a_1,\dots,a_N=1}^{n} \sum_{k_1,\dots,k_N\geq 0} \langle \tau_{a_1 k_1} \dots \tau_{a_N k_N} \rangle_g^{f,G} \, t_{k_1}^{a_1} \cdots t_{k_N}^{a_N} \right).$$

Now we consider an important subclass of singularities, called *simple singularities*. They are classified by the ADE Dynkin diagrams [1, 2]. In particular, we consider

$$A_k: \; f = x^{k+1}, \quad k \geq 1; \qquad D_k: \; f = x^{k-1} + x\, y^2, \quad k \geq 4;$$

$$E_6: \; f = x^3 + y^4; \qquad E_7: \; f = x^3 + x\, y^3; \qquad E_8: \; f = x^3 + y^5.$$

We are also interested in the mirror singularity of D_k [28], denoted by D_k^T:

$$D_k^T: \qquad f = x^{k-1}\, y + y^2, \quad k \geq 4.$$

The maximal diagonal symmetry groups G_f of the above polynomials will be denoted by G_{A_k}, G_{D_k}, $G_{D_k^T}$ and G_{E_n}, $n = 6,\, 7,\, 8$.

Theorem-ADE ([28, 29]). *The following statements hold true*

- *A. The partition function $Z^{A_n,G}(\mathbf{t})$, $n \geq 1$ with $G = \langle J \rangle = G_{A_n}$ is a particular tau-function of the Drinfeld–Sokolov hierarchy of A_n-type satisfying the string equation (1.3.5).*
- *D. The partition function $Z^{D_n,G}(\mathbf{t})$, $n \geq 4$ with n **even** and $G = \langle J \rangle$ is a particular tau-function of the DS hierarchy of D_n-type satisfying (1.3.5).*
- *D'. The partition function $Z^{D_k,G}(\mathbf{t})$, $k \geq 4$ with $G = G_{D_k}$ is a particular tau-function of the DS hierarchy of A_{2k-3}-type satisfying (1.3.5).*
- *D". The partition function $Z^{D_n^T,G}(\mathbf{t})$, $n \geq 4$ with $G = G_{D_n^T}$ is a particular tau-function of the DS hierarchy of D_n-type satisfying (1.3.5).*
- *E. The partition function $Z^{E_n,G}(\mathbf{t})$, $n = 6, 7, 8$, with $G = \langle J \rangle = G_{E_n}$ is a particular tau-function of the DS hierarchy of E_n-type satisfying (1.3.5).*

Summarizing, the partition function $Z^{X_k,G_{X_k}}(\mathbf{t})$ with $X = A, D, D^T$, or E is a particular tau-function of the DS hierarchy of X_k^T-type satisfying (1.3.5).

In the case that $f = x^r$ with $G = \langle J \rangle = G_f$, the FJRW invariants $\langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle_g^{f,G}$ coincide with Witten's r-spin correlators. The statement A of Theorem-ADE justifies Witten's r-spin conjecture [52], which was first proved by Faber–Shadrin–Zvonkine [27]; see "Theorem r-spin" below.

For convenience of the reader let us recall in more details the definition of Witten's r-spin correlators. For a given $N \geq 1$ let $1 \leq a_1, \dots, a_N \leq r$ be integers satisfying the following divisibility condition

$$a_N + \ldots + a_N - N - (2g-2) = m\, r, \quad m \in \mathbb{Z}. \tag{1.4.1}$$

Then for any algebraic curve C of genus g with N marked points $x_1, \ldots, x_N$ there exists a line bundle $\mathcal{T}$ over C such that

$$\mathcal{T}^{\otimes r} = K_C \otimes \mathcal{O}\left((1-a_1)x_1\right) \otimes \ldots \otimes \mathcal{O}\left((1-a_N)x_N\right). \tag{1.4.2}$$

Here K_C is the canonical class of the curve C. Moreover, for a smooth C there are r^{2g} such line bundles. A choice of such an "r-th root" of the bundle (1.4.2) defines a point in a covering of the moduli space. After a suitable compactification this covering is denoted by

$$p : \overline{\mathcal{M}}_{g,N}^{1/r}(a_1, \ldots, a_N) \to \overline{\mathcal{M}}_{g,N}. \tag{1.4.3}$$

For a point $(C, x_1, \ldots, x_N, \mathcal{T})$ in the covering space denote $V = H^1(C, \mathcal{T})$. It defines a vector bundle $\mathcal{V} \to \overline{\mathcal{M}}_{g,N}^{1/r}(a_1, \ldots, a_N)$. Put

$$c_W(a_1, \ldots, a_N) := \frac{1}{r^g} p_* \left(e\left(\mathcal{V}^\vee\right)\right) \in H^{2(m-g+1)}\left(\overline{\mathcal{M}}_{g,N}\right), \qquad a_1, \ldots, a_N = 1, \ldots, r$$

where $e\left(\mathcal{V}^\vee\right)$ is the Euler class of the dual bundle $\mathcal{V}^\vee$. The cohomological class $c_W(a_1, \ldots, a_N)$ is called the *Witten class* [52, 27, 37, 49, 48]. The r-spin intersection numbers are defined by

$$\langle \tau_{a_1 p_1} \ldots \tau_{a_N p_N} \rangle_g^{r-\text{spin}} := \int_{\overline{\mathcal{M}}_{g,N}} c_W(a_1, \ldots, a_N)\, \psi_1^{p_1} \ldots \psi_N^{p_N}. \tag{1.4.4}$$

The numbers $\langle \tau_{a_1 p_1} \ldots \tau_{a_N p_N} \rangle_g^{r-\text{spin}}$ are zero unless

$$\frac{a_1 - 1}{r} + \ldots + \frac{a_N - 1}{r} + \frac{r-2}{r}(g-1) + k_1 + \ldots + k_N = 3g - 3 + N. \tag{1.4.5}$$

The so-called *Vanishing Axiom* conjectured in [37] and proven in [49, 48] tells that the Witten class vanishes if any of a_i, $i = 1, \ldots, N$ reaches r. Hence, below, we only consider the case of $a_1, \ldots, a_N$ belonging to $\{1, \ldots, r-1\}$.

For computing Witten's r-spin correlators, we use the theorems 1.3.1, 1.3.2 for a particular tau-function along with the following result.

Theorem r-spin ([52, 27]). *The partition function of r-spin intersection numbers*

$$Z^{r-spin}(\mathbf{t}) := \exp\left(\sum_{g,N \geq 0} \frac{1}{N!} \sum_{a_1,\ldots,a_N=1}^{n} \sum_{k_1,\ldots,k_N \geq 0} \langle \tau_{a_1 k_1} \ldots \tau_{a_N k_N} \rangle_g^{r-spin} t_{k_1}^{a_1} \cdots t_{k_N}^{a_N} \right)$$

is a particular tau-function of the DS hierarchy of A_n-type, $n = r-1$ satisfying (1.3.5).

In [44], Liu–Ruan–Zhang introduced *cohomological field theories with finite symmetry*, associated with simple singularities and certain symmetry groups, and with a Γ-invariant sector, where Γ is the group of automorphisms of the Dynkin digram. These theories are proved to be related to the DS integrable hierarchies associated to the non-simply laced simple Lie algebras.

Theorem-BCFG ([44]). *The partition function of the Γ-invariant sector of D_{n+1}^T, A_{2n-1}, E_6 FJRW theory with G_{max} is a particular tau-function of the Drinfeld–Sokolov hierarchy of B_n, C_n, F_4-type satisfying (1.3.5); the partition function of the $\mathbb{Z}/3\mathbb{Z}$-invariant sector of $(D_4, \langle J \rangle)$ FJRW theory is a particular tau-function of the Drinfeld–Sokolov hierarchy of G_2-type satisfying (1.3.5).*

Note that the common feature of Theorem-ADE and Theorem-BCFG claims that the partition function of FJRW invariants associated to a simple singularity with a symmetry group (possibly also with an invariant sector) is a tau-function of the DS hierarchy of $\mathfrak{g}$-type, where $\mathfrak{g}$ is a simple Lie algebra. We call these numbers the *FJRW invariants of* $\mathfrak{g}$*-type*, denoted by

$$\langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle_g^{FJRW-\mathfrak{g}}, \text{ or simply by } \langle \tau_{a_1 k_1} \cdots \tau_{a_N k_N} \rangle_g^{\mathfrak{g}}.$$

As before, let n denote the rank of $\mathfrak{g}$. For a given $N \geq 1$ and for a collection of integers $a_1, \ldots, a_N \in \{1, \ldots, n\}$, we define the following generating functions of N-point FJRW invariants of $\mathfrak{g}$-type

$$F^{FJRW}_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N) := (\kappa^{\frac{1}{h+1}} \sqrt{-h})^N \sum_{g,k_1,\ldots,k_N \geq 0} \prod_{\ell=1}^N \frac{(-1)^{k_\ell} \left(\frac{m_{a_\ell}}{h}\right)_{k_\ell+1}}{\left(\kappa^{\frac{1}{h+1}} \lambda_\ell\right)^{\frac{m_{a_\ell}}{h}+k_\ell+1}} \langle \tau_{a_1 k_1} \ldots \tau_{a_N k_N} \rangle_g^{\mathfrak{g}}. \tag{1.4.6}$$

Here $\kappa := \left(\sqrt{-h}\right)^{-h}$.

Combining the results of Theorems 1.3.1 and 1.3.2 with the statements of Theorem-ADE and Theorem-BCFG we arrive at the following formula for the FJRW invariants of $\mathfrak{g}$-type.

Theorem 1.4.2. *Let $\mathfrak{g}$ be a simple Lie algebra and n the rank of $\mathfrak{g}$. Let $M_a = M_a(\lambda)$, $a = 1, \ldots, n$ be the generalized Airy resolvents of $\mathfrak{g}$-type, which are the unique solutions to*

$$M' = [M, \Lambda], \tag{1.4.7}$$

subjected to

$$M_a(\lambda) = \lambda^{-\frac{m_a}{h}} \left[\Lambda_{m_a}(\lambda) + \textit{lower degree terms w.r.t. } \deg\right].$$

Here, h is the Coxeter number and m_a, $a = 1, \ldots, n$ are the exponents of $\mathfrak{g}$. Then the generating functions (1.4.6) for the N-point FJRW invariants of $\mathfrak{g}$-type have the following expressions

$$\frac{\mathrm{d}F^{FJRW}_a}{\mathrm{d}\lambda}(\lambda) = -\frac{1}{2\,h^\vee} B\Big(E_{-\theta}, M_a(\lambda)\Big) + \lambda^{-\frac{h-1}{h}}\,\delta_{a,n}, \quad N = 1, \tag{1.4.8}$$

$$F^{FJRW}_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N) = -\frac{1}{2N\,h^\vee} \sum_{s \in S_N} \frac{B\left(M_{a_{s_1}}(\lambda_{s_1}), \ldots, M_{a_{s_N}}(\lambda_{s_N})\right)}{\prod_{j=1}^N (\lambda_{s_j} - \lambda_{s_{j+1}})}$$
$$-\delta_{N2}\,\eta_{a_1 a_2} \frac{\lambda_1^{-\frac{m_{a_1}}{h}} \lambda_2^{-\frac{m_{a_2}}{h}} (m_{a_1}\lambda_1 + m_{a_2}\lambda_2)}{(\lambda_1 - \lambda_2)^2}, \quad N \geq 2. \tag{1.4.9}$$

Eqs. (1.4.7)–(1.4.9) are equivalent to the proposed formulae in [8] (eq. (4.2.9) of the current paper).

In particular, for given integers $r \geq 2$, $N \geq 1$ and a given collection of indices $a_1, \ldots, a_N$ belonging to $\{1, \ldots, r-1\}$, define

$$F^{r-spin}_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N) := \left(\kappa^{\frac{1}{r+1}} \sqrt{-r}\right)^N \sum_{k_1,\ldots,k_N \geq 0} \prod_{\ell=1}^N \frac{(-1)^{k_\ell} \left(\frac{a_\ell}{r}\right)_{k_\ell+1}}{(\kappa^{\frac{1}{r+1}} \lambda_\ell)^{\frac{a_\ell}{r}+k_\ell+1}} \langle \tau_{a_1 k_1} \ldots \tau_{a_N k_N} \rangle^{r-\mathrm{spin}}. \tag{1.4.10}$$

Here $\kappa = \left(\sqrt{-r}\right)^{-r}$. Note that we have omitted the genus labelling in the notation of correlator, since it can be obtained from the degree-dimension matching (1.4.5).

Theorem 1.4.3. *Let $n = r - 1$, $\mathfrak{g} = sl_{n+1}(\mathbb{C})$, $\Lambda = \sum_{i=1}^n E_{i,i+1} + \lambda\, E_{n+1,1}$, and let $M_i = M_i(\lambda)$ be the basis of generalized Airy resolvents of $\mathfrak{g}$-type, uniquely determined by the topological ODE*

$$M' = [M, \Lambda], \tag{1.4.11}$$

subjected to

$$M_a = \lambda^{-\frac{a}{r}} \left[\Lambda^a + \textit{lower degree terms w.r.t.}\ \deg\right].$$

Then the N-point functions (1.4.10) of r-spin intersection numbers have the following expressions

$$\frac{\mathrm{d}F_a^{r-spin}}{\mathrm{d}\lambda}(\lambda) = -(M_a)_{1,n+1}(\lambda) + \lambda^{-\frac{r-1}{r}}\,\delta_{a,n}, \quad N = 1, \tag{1.4.12}$$

$$F_{a_1,\dots,a_N}^{r-spin}(\lambda_1,\dots,\lambda_N) = -\frac{1}{N}\sum_{s\in S_N} \frac{\mathrm{Tr}\left(M_{a_{s_1}}(\lambda_{s_1})\dots M_{a_{s_N}}(\lambda_{s_N})\right)}{\prod_{j=1}^N(\lambda_{s_j}-\lambda_{s_{j+1}})} - \delta_{N2}\,\eta_{a_1a_2}\frac{\lambda_1^{-\frac{a_1}{h}}\lambda_2^{-\frac{a_2}{h}}(a_1\lambda_1+a_2\lambda_2)}{(\lambda_1-\lambda_2)^2}, \qquad N\geq 2. \tag{1.4.13}$$

Example 1.4.4 ($r = 2$)**.** *Witten's 2-spin invariants coincide with intersection numbers of ψ-classes over $\overline{\mathcal{M}}_{g,N}$ [51, 42, 27]. So Thm. 1.4.3 with the choice $r = 2$ recovers the result of [7, 55]:*

$$\sum_{g\geq 0}\sum_{p_1,\dots,p_N\geq 0}\frac{(2p_1+1)!!\cdots(2p_N+1)!!}{2^{2g-2+N}}\int_{\overline{\mathcal{M}}_{g,N}}\psi_1^{p_1}\cdots\psi_N^{p_N}\,\lambda_1^{-\frac{2p_1+3}{2}}\cdots\lambda_N^{-\frac{2p_N+3}{2}}$$

$$= -\frac{1}{N}\sum_{r\in S_N}\frac{\mathrm{Tr}\left(M(\lambda_{r_1})\cdots M(\lambda_{r_N})\right)}{\prod_{j=1}^N(\lambda_{r_j}-\lambda_{r_{j+1}})} - \delta_{N2}\frac{\lambda_1^{-\frac{1}{2}}\lambda_2^{-\frac{1}{2}}(\lambda_1+\lambda_2)}{(\lambda_1-\lambda_2)^2}, \quad N\geq 2$$

where

$$M = \frac{\lambda^{-\frac{1}{2}}}{2}\begin{pmatrix} -\frac{1}{2}\sum_{g=1}^\infty \frac{(6g-5)!!}{96^{g-1}\cdot(g-1)!}\lambda^{-3g+2} & 2\sum_{g=0}^\infty\frac{(6g-1)!!}{96^g\cdot g!}\lambda^{-3g} \\ -2\sum_{g=0}^\infty\frac{6g+1}{6g-1}\frac{(6g-1)!!}{96^g\cdot g!}\lambda^{-3g+1} & \frac{1}{2}\sum_{g=1}^\infty\frac{(6g-5)!!}{96^{g-1}\cdot(g-1)!}\lambda^{-3g+2}\end{pmatrix}.$$

For $N = 1$, it follows easily from (1.4.12) the well-known formula

$$\langle\tau_{3g-2}\rangle_g = \frac{1}{24^g\cdot g!} \quad \textit{for} \quad g\geq 1.$$

Example 1.4.5 ($r = 3$)**.** *We obtain from Theorem 1.4.3 that the only nontrivial one-point correlators have the following explicit expressions*

$$\int_{\overline{\mathcal{M}}_{3m-2,1}} c_W(1)\,\psi_1^{8m-7} = \frac{1}{6^{6m-4}(m-1)!\left(\frac{1}{3}\right)_m}, \quad m\geq 1$$

$$\int_{\overline{\mathcal{M}}_{3m,1}} c_W(2)\,\psi_1^{8m-2} = \frac{1}{6^{6m}m!\left(\frac{2}{3}\right)_m}, \quad m\geq 1.$$

For $N\geq 2$, Witten's 3-spin correlators can be computed from the formulae

$$F_{i_1,\dots,i_N}^{3-spin}(\lambda_1,\dots,\lambda_N) = -\frac{1}{N}\sum_{s\in S_N}\frac{\mathrm{Tr}\left(M_{i_{s_1}}(\lambda_{s_1})\dots M_{i_{s_N}}(\lambda_{s_N})\right)}{\prod_{j=1}^N(\lambda_{s_j}-\lambda_{s_{j+1}})} - \delta_{N2}\,\eta_{i_1i_2}\frac{\lambda_1^{-\frac{i_1}{h}}\lambda_2^{-\frac{i_2}{h}}(i_1\lambda_1+i_2\lambda_2)}{(\lambda_1-\lambda_2)^2}$$

with explicit formulae of $M_a(\lambda)$ given in Appendix A.

Organization of the paper. In Sect. 2 we introduce the definition of tau-function and prove Thm 1.3.1. In Sect. 3 we define the essential series of $\mathfrak{g}$. In Sect. 4, we prove Thm. 1.3.2.

Acknowledgements. We wish to thank Yassir Dinar, Daniele Valeri, Chao-Zhong Wu, Youjin Zhang for helpful discussions. D. Y. is grateful to Youjin Zhang for his advising. The work of M. B. is in part supported by the RGPIN/261229-2011 grant of the Natural Sciences and Engineering Research Council of Canada and by the FQRNT grant "Matrices Aléatoires, Processus Stochastiques et Systèmes Intégrables" (2013–PR–166790). The work of D. Y. was initiated when he was a postdoctoral fellow at SISSA, Trieste; he thanks SISSA for the excellent working conditions.

2 Tau-function of Drinfeld–Sokolov hierarchy

2.1 Fundamental lemma

Let $\mathfrak{g}$ be a simple Lie algebra of rank n, $L(\mathfrak{g})$ its loop algebra. Fix $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by $\rho^\vee \in \mathfrak{h}$ the *Weyl co-vector* of $\mathfrak{g}$, which is uniquely determined by the following equations

$$\alpha_i(\rho^\vee) = 1, \qquad i = 1, \dots, n. \tag{2.1.1}$$

Here $\alpha_i \in \mathfrak{h}^*$, $i = 1, \dots, n$ are simple roots. We define the *principal* grading operator gr on $L(\mathfrak{g})$ by

$$\mathrm{gr} = h\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} + \mathrm{ad}_{\rho^\vee}.$$

It follows that $\deg a = j \in \mathbb{Z}$ *iff* $\mathrm{gr}\, a = j\, a$, $\forall\, a \in \mathcal{L}(\mathfrak{g})$. Decompose

$$L(\mathfrak{g}) = \bigoplus_{j\in\mathbb{Z}} L(\mathfrak{g})^j, \qquad a \in L(\mathfrak{g})^j \;\Leftrightarrow\; \mathrm{gr}\, a = j\, a, \quad j \in \mathbb{Z}.$$

$\forall\, a \in L(\mathfrak{g})$, we will denote the principal decomposition of a by

$$a = \sum_{j\in\mathbb{Z}} a^{[j]}, \qquad a^{[j]} \in L(\mathfrak{g})^j.$$

The following lemma is elementary but it will be frequently used.

Lemma 2.1.1. *Let x, y be any two elements in $\mathfrak{g} = \mathfrak{g}\otimes 1$ satisfying $\mathrm{gr}\, x = k_1\, x,\ \mathrm{gr}\, y = k_2\, y$. If $k_1 + k_2 \neq 0$, then we have $(x\,|\,y) = 0$.*

Proof. Suppose $k_1 \neq 0$. By definition, $\mathrm{gr}\, x = k_1\, x$ implies $[\rho^\vee, x] = k_1\, x$. So we have

$$(x\,|\,y) = \frac{1}{k_1}([\rho^\vee, x]\,|\,y) = -\frac{1}{k_1}(x\,|\,[\rho^\vee, y]) = -\frac{k_2}{k_1}(x\,|\,y) \;\Rightarrow\; \frac{k_1+k_2}{k_1}(x\,|\,y) = 0.$$

The lemma is proved. ☐

Lemma 2.1.2 (fundamental lemma, [19]). *Let $q = q(x)$ be a $\mathfrak{b}$-valued smooth function, where $\mathfrak{b} := g^{\leq 0}$. Let $\mathcal{L} = \partial_x + \Lambda + q(x)$. Then there exists a unique pair (U, H) of the form*

$$U = \sum_{k\geq 1} U^{[-k]}(\lambda; q; q_x, \dots) \in \mathcal{A}^q \otimes \mathrm{Im}\, \mathrm{ad}_\Lambda, \tag{2.1.2}$$

$$H = \sum_{j\in E_+} H^{[-j]}(\lambda; q; q_x, \dots) \in \mathcal{A}^q \otimes \mathrm{Ker}\, \mathrm{ad}_\Lambda, \tag{2.1.3}$$

where Im, Ker *are taken in $\mathfrak{g}((\lambda^{-1}))$, and $E_+ := \{j \geq 0\,|\, j \in E\}$ such that*

$$e^{-\mathrm{ad}_U}\mathcal{L} = \partial_x + \Lambda + H. \tag{2.1.4}$$

Proof. Eq. (2.1.4) is equivalent to

$$e^{-U} \circ \partial_x \circ e^U + e^{-\mathrm{ad}_U} (q + \Lambda) = \partial_x + \Lambda + H.$$

More explicitly this reads

$$\sum_{j=0}^{\infty} \frac{(-\mathrm{ad}_U)^j}{j!} \left(\frac{U_x}{j+1} + q + \Lambda \right) = \Lambda + H. \tag{2.1.5}$$

Comparing components with principal degree $-k$ of both sides of (2.1.5) we obtain

$$H^{[-k]} + \left[U^{[-k-1]}, \Lambda \right] = G_k \left(\lambda; q; U^{[-1]}, \dots, U^{[-k]}; \partial_x(U^{[-1]}), \dots, \partial_x(U^{[-k]}) \right), \qquad k \geq 0. \tag{2.1.6}$$

Here, $G_k \in L(\mathfrak{g})$, $k \geq 0$. Moreover, entries of G_k are polynomials in the entries of

$$q,\, U^{[-1]}, \dots, U^{[-k]},\, \partial_x(U^{[-1]}), \dots, \partial_x(U^{[-k]})$$

whose coefficients are polynomials in λ. The proof proceeds by induction on the principal degree. First, for $k = 0$ eq. (2.1.6) reads

$$H^{[0]} + \left[U^{[-1]}\,,\, \Lambda \right] = q^{[0]}. \tag{2.1.7}$$

Observe that an element $x \in \mathfrak{g}$ has zero principal degree *iff* $x \in \mathfrak{h}$. So $q^{[0]}$ belongs to $\mathfrak{h}$. Let us show that $\mathfrak{h} \subset \mathrm{Im}\, \mathrm{ad}_\Lambda$. This is equivalent to orthogonality

$$(x \,|\, \Lambda_{m_a}) = 0 \quad \text{for any} \quad x \in \mathfrak{h}, \quad a = 1, \dots, n. \tag{2.1.8}$$

Indeed, by Lemma 2.1.1, any element $y \in \mathfrak{g}$ of nonzero principal degree is orthogonal to $\mathfrak{h}$. It remains to recall that any Λ_{m_a} has the form $\Lambda_{m_a} = L_{m_a} + \lambda\, K_{m_a - h}$, where L_{m_a} and K_{m_a-h} belong to $\mathfrak{g}$ and have nonzero principal degree. This proves orthogonality (2.1.8). So we have $H^{[0]} = 0$. Noting that the map $\mathrm{ad}_\Lambda : \mathrm{Im}\, \mathrm{ad}_\Lambda \to \mathrm{Im}\, \mathrm{ad}_\Lambda$ is invertible, and we have

$$U^{[-1]} = \mathrm{ad}_\Lambda^{-1}(q^{[0]}) \in \mathrm{Im}\, \mathrm{ad}_\Lambda. \tag{2.1.9}$$

The second step of the induction clearly follows from eq. (2.1.6) and the decomposition

$$L(\mathfrak{g}) = \mathrm{Ker}\, \mathrm{ad}_\Lambda \oplus \mathrm{Im}\, \mathrm{ad}_\Lambda.$$

The lemma is proved. ☐

Example 2.1.3. *Looking at equation (2.1.5) with principal degree -1, we have*

$$H^{[-1]} - \left[U^{[-2]}, \Lambda \right] = \frac{1}{2} \left[U^{[-1]}, \left[U^{[-1]}, \Lambda \right] \right] + \partial_x(U^{[-1]}) - \left[U^{[-1]}, q^{[0]} \right] + q^{[-1]}.$$

Since $U^{[-2]}$ is assumed to be orthogonal to $\mathrm{Ker}\, \mathrm{ad}_\Lambda$*, this equation uniquely determines $H^{[-1]}$ and $U^{[-2]}$ as indicated in the above proof.*

2.2 $\mathfrak{g}$-valued resolvents

Definition 2.2.1. *Let $q = q(x) \in \mathfrak{b}$. An element $R \in \mathcal{A}^q \otimes \mathfrak{g}((\lambda^{-1}))$ is called a* **resolvent** *of $\mathcal{L}$ if*

$$[\mathcal{L}, R] = 0. \tag{2.2.1}$$

The set of all resolvents of $\mathcal{L}$ is denoted by $\mathcal{M}_\mathcal{L}$, called the resolvent manifold.

Lemma 2.2.2 ([19])**.** *We have*

$$\mathcal{M}_{\mathcal{L}} = e^{\mathrm{ad}_U}\left(\mathrm{Ker}\,\mathrm{ad}_\Lambda\right),$$

where we note that the kernel is taken in $L(\mathfrak{g})$*, namely,* $\mathrm{Ker}\,\mathrm{ad}_\Lambda = \bigoplus_{j\in E}\mathbb{C}\Lambda_j$.

Proof. Lemma 2.1.2 reduces the problem to considering the resolvent manifold of $\partial_x + \Lambda + H$. So, let us look at the following equation for $R_H \in \mathcal{A}^q \otimes \mathfrak{g}((\lambda^{-1}))$:

$$[R_H, \partial_x + \Lambda + H] = 0.$$

Decompose

$$R_H = R_H^{ker} + R_H^{im}, \qquad R_H^{ker} \in \mathcal{A}^q \otimes \mathrm{Ker}\,\mathrm{ad}_\Lambda,\ R_H^{im} \in \mathcal{A}^q \otimes \mathrm{Im}\,\mathrm{ad}_\Lambda.$$

It follows that

$$\frac{\partial R_H^{ker}}{\partial x} + \frac{\partial R_H^{im}}{\partial x} = \left[R_H^{im}, \Lambda + H\right].$$

The RHS of the above equation is in the image of ad_Λ, so we have

$$\frac{\partial R_H^{ker}}{\partial x} = 0, \tag{2.2.2}$$

$$\frac{\partial R_H^{im}}{\partial x} = \left[R_H^{im}, \Lambda + H\right]. \tag{2.2.3}$$

Equation (2.2.2) implies that R_H^{ker} can only depend on λ. The rest is to show that R_H^{im} must vanish. If it does not vanish, then there exists an integer d such that

$$R_H^{im} = \sum_{i=-\infty}^{d} R_H^{im,[i]}, \qquad R_H^{im,[d]} \neq 0.$$

Noting that $\deg H < 0$, then looking at the highest degree term on both sides of eq. (2.2.3) we obtain

$$\left[\Lambda, R_H^{im,[d]}\right] = 0.$$

So we have $R_H^{im,[d]} = 0$. This produces a contradiction. The lemma is proved. ◻

Proposition 2.2.3. $\forall\, a = 1, \dots, n$, *there exists a unique solution to the following system of equations*

$$[\mathcal{L}, R] = 0, \qquad R \in \mathcal{A}^q \otimes \mathfrak{g}((\lambda^{-1})), \tag{2.2.4}$$

$$R(\lambda; q, q_x, \dots) = \Lambda_{m_a} + \text{ lower order terms w.r.t. } \deg, \tag{2.2.5}$$

$$\left(R_a(\lambda; q, q_x, \dots) \mid R_b(\lambda; q, q_x, \dots)\right) = h\, \eta_{ab}\, \lambda. \tag{2.2.6}$$

This unique system of solutions $R_1, \dots R_n$ is called in Sect. 1 the basic resolvents of the operator $\mathcal{L}$.

Proof. The existence follows from the fact that $e^{\mathrm{ad}_U}(\Lambda_{m_a})$ is a solution, where (2.2.6) is due to (1.1.6), and (2.2.5) is due to (2.1.2). The uniqueness follows from Lemma 2.2.2. ◻

Corollary 2.2.4. *Let* U *be defined as in Lemma 2.1.2. Then the basic resolvents* R_a *satisfy*

$$R_a = e^{\mathrm{ad}_U}(\Lambda_{m_a}), \qquad a = 1, \dots, n.$$

Definition 2.2.5. *Define* $P_{m_a+hk} := \lambda^k R_a = e^{\mathrm{ad}_U}(\Lambda_{m_a+hk}), \quad a = 1, \dots, n,\ k \geq 0.$

The pre-DS hierarchy can be written as

$$\frac{\partial \mathcal{L}}{\partial T^a_k} = \left[(P_{m_a+kh})_+ \, , \, \mathcal{L} \right], \qquad a = 1, \ldots, n, \; k \geq 0.$$

As customary in the literature, we will sometimes write T^a_k as T_{m_a+kh}, $a = 1, \ldots, n, k \geq 0$.

Lemma 2.2.6. *$\forall\, i, j \in E_+$, we have*

$$\frac{\partial P_j}{\partial T_i} = [(P_i)_+, P_j], \tag{2.2.7}$$

$$\frac{\partial (P_i)_+}{\partial T_j} - \frac{\partial (P_j)_+}{\partial T_i} + [(P_i)_+, (P_j)_+] = 0. \tag{2.2.8}$$

Proof. Using the fundamental lemma 2.1.2 we have

$$\frac{\partial \mathcal{L}}{\partial T_i} = [(P_i)_+, \mathcal{L}] \quad \Rightarrow \quad \Big[\partial_{T_i} - (P_i)_+, \mathcal{L} \Big] = 0 \quad \Rightarrow \quad \Big[\partial_{T_i} + S_i \, , \, \partial_x + \Lambda + H \Big] = 0$$

where $S_i := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \mathrm{ad}_U^k \left(\frac{\partial U}{\partial T_i} \right) - e^{-\mathrm{ad}_U} \left[(P_i)_+ \right]$. Clearly, S_i takes values in $\mathcal{A}^q \otimes L(\mathfrak{g})$. Decompose

$$S_i = S_i^{ker} + S_i^{im}, \qquad S_i^{ker} \in \mathcal{A}^q \otimes \mathrm{Ker}\, \mathrm{ad}_\Lambda, \quad S_i^{im} \in \mathcal{A}^q \otimes \mathrm{Im}\, \mathrm{ad}_\Lambda.$$

Then we have

$$\frac{\partial H}{\partial T_i} - \frac{\partial S_i}{\partial x} + [S_i, \Lambda + H] = 0 \quad \Rightarrow \quad \begin{cases} \frac{\partial H}{\partial T_i} - \frac{\partial S_i^{ker}}{\partial x} = 0, \\ \frac{\partial S_i^{im}}{\partial x} = [S_i^{im}, \Lambda + H]. \end{cases}$$

Using the same argument as in the proof of Lemma 2.2.2 we find from the above equation for S_i^{im} that S_i^{im} must vanish. So S_i belongs to $\mathcal{A}^q \otimes \mathrm{Ker}\, \mathrm{ad}_\Lambda$. On another hand,

$$\frac{\partial P_j}{\partial T_i} = [(P_i)_+, P_j] \quad \Leftrightarrow \quad [\partial_{T_i} - (P_i)_+ \, , \, P_j] = 0 \quad \Leftrightarrow \quad [\partial_{T_i} - S \, , \, \Lambda_j] = 0.$$

Hence eq. (2.2.7) is proved. Clearly eq. (2.2.7) implies eq. (2.2.8); this is because

$$\text{l.h.s. of eq. (2.2.8)} = [(P_j)_+, P_i]_+ - [(P_i)_+, P_j]_+ + [(P_i)_+, (P_j)_+] = 0.$$

□

Lemma 2.2.7. *$\forall\, a = 1, \ldots, n$ we have*

$$\nabla_a(\lambda)\, R_b(\mu) = \frac{[R_a(\lambda), R_b(\mu)]}{\lambda - \mu} - [Q_a, R_b(\mu)], \qquad Q_a := \mathrm{Coef}(R_a(\lambda), \lambda^1). \tag{2.2.9}$$

Proof. We have

$$\begin{aligned} \nabla_a(\lambda)\, R_b(\mu) &= \sum_{k \geq 0} \frac{\partial_{T^a_k} R_b(\mu)}{\lambda^{k+1}} = \sum_{k \geq 0} \frac{[(\mu^k \, R_a(\mu))_+ \, , \, R_b(\mu)]}{\lambda^{k+1}} \\ &= -\sum_{k \geq 0} \frac{\left[\operatorname*{res}_{\rho=\infty} \frac{\rho^k R_a(\rho)}{\rho - \mu} d\rho \, , \, R_b(\mu) \right]}{\lambda^{k+1}} \\ &= \frac{1}{2\pi\sqrt{-1}} \oint_{|\mu|<|\rho|<|\lambda|} d\rho \, \frac{[R_a(\rho) \, , \, R_b(\mu)]}{(\lambda - \rho)(\rho - \mu)} \\ &= \frac{[R_a(\lambda), R_b(\mu)]}{\lambda - \mu} - \left[\mathrm{Coef}(R_a(\lambda), \lambda^1) \, , \, R_b(\mu) \right]. \end{aligned}$$

□

2.3 Two-point correlation functions

Recall that in Def. 1.2.1, the two-point correlation functions $\Omega_{a,k;b,\ell}$ was defined by

$$\sum_{k,\ell\geq 0} \frac{\Omega_{a,k;b,\ell}}{\lambda^{k+1}\mu^{\ell+1}} = \frac{(R_a(\lambda)\,|\,R_b(\mu))}{(\lambda-\mu)^2} - \eta_{ab}\frac{m_a\lambda + m_b\mu}{(\lambda-\mu)^2}. \tag{2.3.1}$$

Lemma 2.3.1. *Def. 1.2.1, i.e. the above formula (2.3.1) is well-posed.*

Proof. Noting that[1]

$$R_b(\mu) = R_b(\lambda) + R_b'(\lambda)(\mu-\lambda) + (\mu-\lambda)^2\,\partial_\lambda\left(\frac{R_b(\lambda)-R_b(\mu)}{\lambda-\mu}\right) \tag{2.3.2}$$

and using eqs. (1.1.6) we have

$$\frac{(R_a(\lambda)\,|\,R_b(\mu))}{(\lambda-\mu)^2} = \eta_{ab}\,\frac{h\,\lambda}{(\lambda-\mu)^2} - \frac{(R_a(\lambda)\,|\,R_b'(\lambda))}{\lambda-\mu} + \left(R_a(\lambda)\,\middle|\,\partial_\lambda\left(\frac{R_b(\lambda)-R_b(\mu)}{\lambda-\mu}\right)\right).$$

In the above formulae, prime, "$'$", denotes derivative w.r.t. the spectral parameter. Since $R_a(\lambda) = \mathcal{O}(\lambda^1)$, $a=1,\dots,n$, it follows that the third term in the above identity has the form as the l.h.s. of (1.2.1). Therefore it remains to show

$$\eta_{ab}\,\frac{h\,\lambda}{(\lambda-\mu)^2} - \frac{(R_a(\lambda)\,|\,R_b'(\lambda))}{\lambda-\mu} - \eta_{ab}\frac{m_a\lambda+m_b\mu}{(\lambda-\mu)^2}$$

has the form as the l.h.s. of (1.2.1). We will actually prove the above expression vanishes. Indeed,

$$\partial_x\,\big(R_a(\lambda)\,|\,R_b'(\lambda)\big) = \big([R_a(\lambda),\Lambda+q]\,|\,R_b'(\lambda)\big) + \big(R_a(\lambda)\,|\,[R_b'(\lambda),\Lambda+q] + [R_b(\lambda),\Lambda']\big) = 0. \tag{2.3.3}$$

Here we have used the ad-invariance of the Cartan–Killing form and the commutativity between resolvents. Noting that $R_a \in \mathcal{A}^q \otimes \mathfrak{g}((\lambda^{-1}))$, we find that (2.3.3) implies that $(R_a(\lambda)\,|\,R_b'(\lambda))$ does not depend on $q, q_x, q_{2x}, \dots$, i.e. it is just a function of λ. Hence

$$\big(R_a(\lambda)\,|\,R_b'(\lambda)\big) = \big(R_a(\lambda)\,|\,R_b'(\lambda)\big)_{q(x)\equiv 0} = \big(\Lambda_{m_a}\,|\,\Lambda'_{m_b}\big)\,.$$

The second equality uses (2.2.6). To compute $\big(\Lambda_{m_a}\,|\,\Lambda'_{m_b}\big)$, as before, write

$$\Lambda_{m_a} = L_{m_a} + \lambda\,K_{m_a-h}, \quad L_{m_a}\in\mathfrak{g}^{m_a},\ K_{m_a-h}\in\mathfrak{g}^{m_a-h}, \quad a=1,\dots,n.$$

Using Lem. 2.1.1 we have

$$\big(\Lambda_{m_a}\,|\,\Lambda'_{m_b}\big) = (L_{m_a}\,|\,K_{m_b-h})\,.$$

Note that $(\Lambda_{m_a}\,|\,\Lambda_{m_b}) = \eta_{ab}\,h\,\lambda$ implies that

$$(L_{m_a}\,|\,K_{m_b-h}) + (L_{m_b}\,|\,K_{m_a-h}) = \eta_{ab}\,h. \tag{2.3.4}$$

The commutativity $[\Lambda_{m_a},\Lambda_{m_b}]=0$ implies that

$$[K_{m_a-h}, L_{m_b}] + [L_{m_a}, K_{m_b-h}] = 0.$$

[1] We would like to thank Anton Mellit for bringing our attention to the useful formula (2.3.2).

Applying $(\rho^\vee\,|\,\cdot)$ to the above equation and using the ad-invariance of $(\cdot|\cdot)$ we have

$$([\rho^\vee, K_{m_a-h}]\,|\,L_{m_b}) + ([\rho^\vee, L_{m_a}]\,|\,K_{m_b-h}]) = 0 \;\Rightarrow\; (m_a-h)\,(K_{m_a-h}\,|\,L_{m_b}) + m_a\,(L_{m_a}\,|\,K_{m_b-h}]) = 0.$$

Combining eqs. (2.3.4) and the above equation we obtain

$$(L_{m_a}\,|\,K_{m_b-h}) = \eta_{ab}\,m_b, \quad \forall\, a,b = 1,\dots,n. \tag{2.3.5}$$

Hence

$$\eta_{ab}\,\frac{h\,\lambda}{(\lambda-\mu)^2} - \frac{(R_a(\lambda)\,|\,R'_b(\lambda))}{\lambda-\mu} - \eta_{ab}\frac{m_a\lambda+m_b\mu}{(\lambda-\mu)^2} = 0.$$

The lemma is proved. □

Proposition 2.3.2. *The following formulae hold true*

$$\sum_{k\geq 0}\frac{\Omega_{a,k;b,0}}{\lambda^{k+1}} = (R_a(\lambda)\,|\,Q_b) - \eta_{ab}\,m_b, \quad a,b = 1,\dots,n. \tag{2.3.6}$$

In particular, we have

$$\sum_{k\geq 0}\frac{\Omega_{a,k;1,0}}{\lambda^{k+1}} = (R_a(\lambda)\,|\,E_{-\theta}) - \eta_{a1}, \quad a = 1,\dots,n. \tag{2.3.7}$$

Proof. Taking in (2.3.1) the residue w.r.t. μ at $\mu=\infty$ we obtain (2.3.6). Noticing that

$$R_1(\mu) = \mu\,E_{-\theta} + I_+ + \text{ terms with principal degree lower than 1}$$

we must have $Q_1 = \mathrm{Coef}(R_1(\mu),\mu^1) = E_{-\theta}$. This proves (2.3.7). □

2.4 Tau-function: Proof of Lemmata 1.2.2, 1.2.3

We are ready to introduce our definition of tau-function. We begin with the proof of Lemma 1.2.2.

Proof of Lemma 1.2.2. First of all we have

$$\begin{aligned}\sum_{k,\ell\geq 0}\frac{\Omega_{a,k;b,\ell}}{\lambda^{k+1}\mu^{\ell+1}} &= \frac{(R_a(\lambda)\,|\,R_b(\mu))}{(\lambda-\mu)^2} - \eta_{ab}\frac{m_a\lambda+m_b\mu}{(\lambda-\mu)^2} = \frac{(R_b(\mu)\,|\,R_a(\lambda))}{(\mu-\lambda)^2} - \eta_{ba}\frac{m_b\mu+m_a\lambda}{(\mu-\lambda)^2}\\ &= \sum_{k,\ell\geq 0}\frac{\Omega_{b,k;a,\ell}}{\mu^{k+1}\lambda^{\ell+1}} = \sum_{k,\ell\geq 0}\frac{\Omega_{b,\ell;a,k}}{\mu^{\ell+1}\lambda^{k+1}}\end{aligned}$$

where we have used the symmetry property of η_{ab} and $(\cdot|\cdot)$. It follows $\Omega_{a,k;b,\ell} = \Omega_{b,\ell;a,k}$.

Secondly, we have

$$\begin{aligned}\sum_{k,\ell,m\geq 0}\frac{\partial_{T^c_m}\Omega_{a,k;b,\ell}}{\xi^{m+1}\lambda^{k+1}\mu^{\ell+1}} &= \nabla_c(\xi)\sum_{k,\ell\geq 0}\frac{\Omega_{a,k;b,\ell}}{\lambda^{k+1}\mu^{\ell+1}}\\ &= \frac{(\nabla_c(\xi)\,R_a(\lambda)\,|\,R_b(\mu))}{(\lambda-\mu)^2} + \frac{(R_a(\lambda)\,|\,\nabla_c(\xi)\,R_b(\mu))}{(\lambda-\mu)^2}\\ &= \frac{([R_c(\xi),R_a(\lambda)]\,|\,R_b(\mu))}{(\lambda-\mu)^2(\xi-\lambda)} - \frac{([Q_c,R_a(\lambda)]\,|\,R_b(\mu))}{(\lambda-\mu)^2}\\ &\quad+\frac{(R_a(\lambda)\,|\,[R_c(\xi),R_b(\mu)])}{(\lambda-\mu)^2(\xi-\mu)} - \frac{(R_a(\lambda)\,|\,[Q_c,P_b(\mu)])}{(\lambda-\mu)^2}.\end{aligned}$$

Clearly the two terms with negative signs give a zero contribution due to the ad-invariance of the Cartan–Killing form. The remaining two terms simplify to

$$\frac{([R_c(\xi), R_a(\lambda)] \mid R_b(\mu))}{(\lambda-\mu)^2}\left(\frac{1}{\xi-\lambda} - \frac{1}{\xi-\mu}\right) = -\frac{([R_c(\xi), R_a(\lambda)] \mid R_b(\mu))}{(\lambda-\mu)(\mu-\xi)(\xi-\lambda)}.$$

So we have

$$\sum_{k,\ell,m\geq 0} \frac{\partial_{T^c_m}(\Omega_{a,k;b,\ell})}{\xi^{m+1}\lambda^{k+1}\mu^{\ell+1}} = -\frac{([R_c(\xi), R_a(\lambda)] \mid R_b(\mu))}{(\lambda-\mu)(\mu-\xi)(\xi-\lambda)}.$$

This gives also

$$\sum_{k,\ell,m\geq 0} \frac{\partial_{T^a_k}(\Omega_{c,m;b,\ell})}{\lambda^{k+1}\xi^{m+1}\mu^{\ell+1}} = -\frac{([R_a(\lambda), R_c(\xi)] \mid R_b(\mu))}{(\xi-\mu)(\mu-\lambda)(\lambda-\xi)}.$$

Hence

$$\partial_{T^c_m}(\Omega_{a,k;b,\ell}) = \partial_{T^a_k}(\Omega_{c,m;b,\ell}) \tag{2.4.1}$$

due to skew-symmetry of the Lie bracket. The lemma is proved. □

Proof of Lemma 1.2.3. Thirdly, we show the compatibility between (1.2.5) and (1.2.4), namely, to show that

$$\frac{\partial \Omega_{a,k;b,\ell}}{\partial T^{1,0}} = -\frac{\partial \Omega_{a,k;b,\ell}}{\partial x}. \tag{2.4.2}$$

Taking $c = 1, m = 0$ in the already proved identity (2.4.1) we have

$$\partial_{T^a_k}(\Omega_{1,0;b,\ell}) = \partial_{T^1_0}(\Omega_{a,k;b,\ell}).$$

Hence (2.4.2) is equivalent to

$$\frac{\partial \Omega_{1,0;b,\ell}}{\partial T^{a,k}} = -\frac{\partial \Omega_{a,k;b,\ell}}{\partial x}.$$

Now we make a generating function: the above identity is equivalent to

$$\sum_{k,\ell} \frac{\partial \Omega_{1,0;b,\ell}}{\partial T^{a,k}} z^{-k-1} w^{-\ell-1} = -\sum_{k,\ell} \frac{\partial \Omega_{a,k;b,\ell}}{\partial x} z^{-k-1} w^{-\ell-1}.$$

We have

$$\begin{aligned} -RHS &= \frac{B(\partial_x R_a(z), R_b(w))}{(z-w)^2} + \frac{B(R_a(z), \partial_x R_b(w))}{(z-w)^2} \\ &= \frac{B([R_a(z), \Lambda(z)+q] | R_b(w))}{(z-w)^2} + \frac{B(R_a(z), [R_b(w), \Lambda(w)+q])}{(z-w)^2} \\ &= \frac{B(\Lambda(z)+q\,,\, [R_b(w), R_a(z)])}{(z-w)^2} - \frac{B(\Lambda(w)+q\,,\, [R_b(w), R_a(z)])}{(z-w)^2} \\ &= \frac{B(\Lambda(z)-\Lambda(w)\,,\, [R_b(w), R_a(z)])}{(z-w)^2}. \end{aligned}$$

Recall that

$$\Lambda(z) = I_+ + zE_{-\theta}, \qquad \Lambda(w) = I_+ + wE_{-\theta}.$$

So we have

$$-RHS = \frac{B((z-w)E_{-\theta}\,,\, [R_b(w), R_a(z)])}{(z-w)^2} = \frac{B(E_{-\theta}\,,\, [R_b(w), R_a(z)])}{(z-w)}.$$

On another hand, we have

$$\begin{aligned} LHS &= \nabla(z) \sum_l \Omega_{1,0;b,l}\, w^{-l-1} \\ &= \nabla(z)\ [B(E_{-\theta}, R_b(w)) + \text{const}] \\ &= B(E_{-\theta}, \nabla(z)\ [R_b(w)]) \\ &= \frac{B(E_{-\theta}, [R_a(z), R_b(w)])}{z-w} + B(E_{-\theta}, [Q_a, R_b(w)]). \end{aligned}$$

We note that the second term of the last expression must be zero because

$$\deg Q_a + h \le m_a \qquad \Rightarrow \qquad [E_{-\theta}, Q_a] = 0. \tag{2.4.3}$$

The lemma is proved. □

Hence we have arrived at our definition of tau-function, i.e. Def. 1.2.4. In the next subsection, we will prove the gauge invariant property of our definition.

2.5 Gauge invariance

The change of the Lax operator

$$\mathcal{L} = \partial_x + \Lambda + q(x) \quad \mapsto \quad \widetilde{\mathcal{L}} = e^{\mathrm{ad}_{N(x)}}\mathcal{L} = \partial_x + \Lambda + \tilde{q}(x), \qquad N(x) \in \mathfrak{n} \tag{2.5.1}$$

is called a *gauge transformation* $q \mapsto \tilde{q}$. It will also be convenient to deal with the infinitesimal form of (1.1.14), $\widetilde{\mathcal{L}} = \mathcal{L} + \delta\mathcal{L}$,

$$\delta\mathcal{L} = [N(x), \mathcal{L}] = [N(x), q(x) + I_+] - \frac{\partial N(x)}{\partial x}. \tag{2.5.2}$$

Let $\widetilde{R}_a$, $a = 1, \dots, n$ be the basic resolvents of $\widetilde{\mathcal{L}}$. It is not difficult to verify that $\forall\, a = 1, \dots, n$, $\widetilde{R}_a = e^{\mathrm{ad}_{N(x)}} R_a$.

Lemma 2.5.1. *The gauge transformations (1.1.14) are symmetries of the pre-DS hierarchy.*

Proof. We have to prove the commutativity

$$\frac{\partial}{\partial s}\frac{\partial \mathcal{L}}{\partial T} = \frac{\partial}{\partial T}\frac{\partial \mathcal{L}}{\partial s}$$

between the j-th flow of the pre-DS hierarchy

$$\frac{\partial \mathcal{L}}{\partial T_j} = \left[(P_j)_+\ ,\ \mathcal{L}\right], \qquad j \in E_+$$

and the flow given by the infinitesimal gauge transformation

$$\frac{\partial \mathcal{L}}{\partial s} = [N, \mathcal{L}]$$

for some $\mathfrak{n}$-valued function $N = N(x)$. Using (1.1.15) we derive

$$\frac{\partial P_j}{\partial s} = [N, P_j]\,.$$

So, after simple calculations with the help of the Jacobi identity we compute the difference between the mixed derivatives

$$\frac{\partial}{\partial s}\frac{\partial \mathcal{L}}{\partial T} - \frac{\partial}{\partial T}\frac{\partial \mathcal{L}}{\partial s} = \left[[N, P_j]_+ - \left[N, (P_j)_+\right], \mathcal{L}\right] = 0.$$

□

The two-point correlation functions $\widetilde{\Omega}_{a,k;b,\ell}$, $k,\ell \geq 0$ associated to $\widetilde{\mathcal{L}}$ are defined by

$$\sum_{k,\ell\geq 0} \frac{\widetilde{\Omega}_{a,k;b,\ell}}{\lambda^{k+1}\mu^{\ell+1}} = \frac{\left(\widetilde{R}_a(\lambda)\,|\,\widetilde{R}_b(\mu)\right)}{(\lambda-\mu)^2} - \eta_{ab}\frac{m_a\lambda + m_b\mu}{(\lambda-\mu)^2}. \tag{2.5.3}$$

Lemma 2.5.2. *$\forall\, a,b \in \{1,\dots,n\}$, $k,\ell \geq 0$, we have $\widetilde{\Omega}_{a,k;b,\ell} = \Omega_{a,k;b,\ell}$.*

Proof. $\left(\widetilde{R}_a(\lambda)\,|\,\widetilde{R}_b(\mu)\right) = \left(e^{\mathrm{ad}_{N(x)}} R_a(\lambda)\,|\,e^{\mathrm{ad}_{N(x)}} R_a(\mu)\right) = \left(R_a(\lambda)\,|\,R_b(\mu)\right).$ □

In a similar way one can easily prove that $\forall\, N \geq 2$ the correlation functions $\langle\langle \tau_{a_1k_1}\dots\tau_{a_Nk_N}\rangle\rangle^{DS}$ are gauge invariant.

Now we are ready to prove Lemma 1.2.6.

Proof of Lemma 1.2.6. The lemma easily can be proved by applying Lem. 2.5.2 and Def. 1.2.4. □

Due to Lemma 1.2.6 it is clear that $\forall\, N \geq 3$ the correlation functions $\langle\langle \tau_{a_1k_1}\dots\tau_{a_Nk_N}\rangle\rangle^{DS}$ are gauge invariant.

2.6 Gauge fixing and Drinfeld–Sokolov hierarchy

We consider in this section a particular family of gauges [19, 5, 23].

Definition 2.6.1. *A linear subspace $\mathcal{V} \subset \mathfrak{b}$ is called a gauge of DS-type if $\mathfrak{b} = \mathcal{V} \oplus [I_+, \mathfrak{n}]$.*

Let $\mathcal{V}$ be a gauge of DS-type. The fact that $\mathrm{ad}_{I_+} : \mathfrak{n} \to \mathfrak{b}$ is injective implies $\dim_{\mathbb{C}} \mathcal{V} = n$. Write

$$\mathcal{V} = \bigoplus_{j=-(h-1)}^{0} \mathcal{V}^j, \qquad \mathcal{V}^j \subset \mathfrak{g}^j.$$

Denote $\mathfrak{b}^j = \mathfrak{b}\cap\mathfrak{g}^j$. We have $\mathfrak{b}^j = \mathcal{V}^j \oplus [I_+, \mathfrak{b}^{j-1}]$, $j = -(h-1),\dots,0$. Clearly, $\mathcal{V}^{-(h-1)} = \mathbb{C}E_{-\theta}$. Noticing that for $j = -(h-1),\dots,0$, the dimension $\dim \mathfrak{b}^j$ can be different from $\dim \mathfrak{b}^{j-1}$ *iff* $-j$ is an exponent of $\mathfrak{g}$ [47, 19], we find that $\mathcal{V}^j$ is a null space unless $(-j)$ is an exponent. Thus

$$\mathcal{V} = \bigoplus_{a=1}^{n} V_a, \quad \dim_{\mathbb{C}} V_a = 1$$

where non-zero elements in V_a have principal degree $-m_a$. We now take a basis $\{X^1,\dots,X^n\}$ of $\mathcal{V}$ satisfying $\deg X^a = -m_a$. It has been proved in [19] that for any Lax operator $\mathcal{L} = \partial_x + \Lambda + q(x)$, there exists a *unique* $\mathfrak{n}$-valued function $N^{can}(x)$ such that

$$e^{\mathrm{ad}_{N^{can}(x)}}\mathcal{L} = \partial_x + \Lambda + q^{can}(x) =: \mathcal{L}^{can}, \quad \text{for some } \mathcal{V}\text{-valued function } q^{can}. \tag{2.6.1}$$

Write $q^{can} = \sum_{a=1}^n w_a\, X^a = (w_1,\dots,w_n)$. The DS-flows of q^{can}, or say of w_a, can be written as

$$\frac{\partial q^{can}}{\partial T^a_k} = \left[\left(\lambda^k R^{can}_a\right)_+, \mathcal{L}\right] + \left[\frac{\partial e^{N^{can}}}{\partial T^a_k} e^{-N^{can}}, \mathcal{L}\right]. \tag{2.6.2}$$

A priori the RHS of (2.6.2) has a dependence in q, as we can see from the second term that it contains flow of components of $\mathfrak{n}$. However, Lem. 2.5.1 says that the gauge transformation is a symmetry of the pre-DS hierarchy. So RHS of (2.6.2) depends only on q^{can}, i.e. w_a, $a = 1,\dots,n$ satisfy equations of the form

$$\frac{\partial w_a}{\partial T^b_k} = G_{a,b,k}\left(q^{can}, q^{can}_x, q^{can}_{xx}, \dots\right), \qquad k \geq 0. \tag{2.6.3}$$

Definition 2.6.2. *Equations (2.6.3) are called the* ***DS hierarchy of $\mathfrak{g}$-type*** *associated to $\mathcal{V}$.*

Let R_a^{can} be the basic resolvents of $\mathcal{L}^{can}$, and $\Omega^{can}_{a,k;b,\ell}$ the two-point correlations functions of $\mathcal{L}^{can}$, i.e.

$$\sum_{k,\ell\geq 0} \frac{\Omega^{can}_{a,k;b,\ell}}{\lambda^{k+1}\mu^{\ell+1}} = \frac{(R_a^{can}(\lambda)\,|\,R_b^{can}(\mu))}{(\lambda-\mu)^2} - \eta_{ab}\frac{m_a\lambda+m_b\mu}{(\lambda-\mu)^2}. \tag{2.6.4}$$

Corollary 2.6.3. *Let $\tau(\mathbf{T})$ be a tau-function of the DS hierarchy. The following formulae hold true*

$$\frac{\partial^2 \log\tau}{\partial T_k^a \partial T_\ell^b} = \Omega^{can}_{a,k;b,\ell}, \qquad \forall\, a,b=1,\dots,n,\ k,\ell\geq 0.$$

Proof. By gauge invariance of two-point correlation functions. □

We also call $\tau(\mathbf{T})$ a tau-function of the solution $q^{can}(\mathbf{T}) = (w^1(\mathbf{T}),\dots,w^n(\mathbf{T}))$.

2.7 Proof of Theorem 1.3.1

The proof will be almost identical to the proof for the case $\mathfrak{g}=A_1$ case [7]. Let $\mathcal{V}$ be any gauge of DS-type. Fix $X^1,\dots,X^n$ a basis of $\mathcal{V}$ satisfying deg $X^a=-m_a$.

Lemma 2.7.1. *Let $\mathcal{L}^{can}=\partial_x+\Lambda+q^{can}$, $q^{can}=\sum_{a=1}^n w_a X^a$. For every $a=1,\dots,n$ a solution to*

$$\begin{aligned}
&[\mathcal{L}^{can}, R^{can}]=0, \qquad R^{can}\in\mathcal{A}^w\otimes L(\mathfrak{g}),\\
&R^{can}(\lambda;w;w_x,w_{2x},\dots)=\Lambda_{m_a}+\ \textit{lower order terms w.r.t.}\ \deg,\\
&R^{can}(\lambda;0;0,\dots,0)=\Lambda_{m_a}
\end{aligned} \tag{2.7.1}$$

exists and is unique. Here $w=(w_1,\dots,w_n)$.

Proof. The lemma is a particular case of Prop. 2.2.3. □

Proof. of Thm. 1.3.1. For any permutation $s=[s_1,\dots,s_p]\in S_p$, $p\geq 2$, define

$$P(s) := -\prod_{j=1}^p \frac{1}{\lambda_{s_j}-\lambda_{s_{j+1}}}, \qquad \lambda_{s_{p+1}}\equiv\lambda_{s_1}.$$

We first prove the generating formula of multi-point correlation functions of a solution of the pre-DS hierarchy, then we use the ad-invariance of B for the gauge-fixed case.

Let $\mathcal{L}=\partial_x+\Lambda+q(x)$, $q(x)\in\mathfrak{b}$ be a linear operator, R_a the basic resolvents of $\mathcal{L}$. For an arbitrary solution $q(x,\mathbf{T})$ to the pre-DS hierarchy (1.1.13), let $\tau(\mathbf{T})$ be the corresponding tau-function, and $F_{a_1,\dots,a_N}(\mathbf{T})$, $N\geq 1$ the generating series of N-point correlations functions of $\tau(\mathbf{T})$.

We now use mathematical induction to prove formula (1.3.4) with R^{can} replaced by R. For $N=2$, the formula is true by definition. Suppose it is true for $N=p\,(p\geq 2)$, then for $N=p+1$, we have

$$\begin{aligned}
&F_{\alpha_1,\dots,\alpha_{p+1}}(\lambda_1,\dots,\lambda_{p+1};\mathbf{T}) = \nabla_{\alpha_{p+1}}(\lambda_{p+1})\,F_{\alpha_1,\dots,\alpha_p}(\lambda_1,\dots,\lambda_p;\mathbf{T})\\
=\ & -\frac{1}{2\,h^\vee\,p}\,\nabla_{\alpha_{p+1}}(\lambda_{p+1})\sum_{s\in S_p}\frac{B\left(R_{\alpha_{s_1}}(\lambda_{s_1}),\dots,R_{\alpha_{s_p}}(\lambda_{s_p})\right)}{\prod_{j=1}^p(\lambda_{s_j}-\lambda_{s_{j+1}})}\\
=\ & -\frac{1}{2\,h^\vee\,p}\sum_{s\in S_p}\sum_{q=1}^p\frac{B\left(R_{\alpha_{s_1}}(\lambda_{s_1}),\dots,\left[\frac{R_{\alpha_{p+1}}(\lambda_{p+1})}{\lambda_{p+1}-\lambda_{s_q}}+Q_{\alpha_{p+1}},R_{\alpha_{s_q}}(\lambda_{s_q})\right],\dots,R_{\alpha_{s_p}}(\lambda_{s_p})\right)}{\prod_{j=1}^p(\lambda_{s_j}-\lambda_{s_{j+1}})}.
\end{aligned}$$

Recall that the elements $Q_\alpha \in \mathfrak{g}$ were defined in eq. (2.2.9). Now we observe that the terms containing the commutator with $Q_{\alpha_{p+1}}$ sum up to zero due to the ad–invariance of B, namely due to the formula

$$\sum_{q=1}^{p} (X_1, \dots, [A, X_q], X_{q+1}, \dots, X_p) = 0 \ , \quad \forall\, X_1, \dots, X_p, A \in \mathfrak{g}.$$

Thus we are left with

$$\begin{aligned}
= \;& \frac{1}{2\,h^\vee\,p} \sum_{s\in S_p} P(s) \sum_{q=1}^{p} \Bigg(\frac{B\left(R_{\alpha_{s_1}}(\lambda_{s_1}), \dots, R_{\alpha_{s_{q-1}}}(\lambda_{s_{q-1}}), R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_q}}(\lambda_{s_q}), \dots, R_{\alpha_{s_p}}(\lambda_{s_p})\right)}{\lambda_{p+1} - \lambda_{s_q}} \\
& - \frac{B\left(R_{\alpha_{s_1}}(\lambda_{s_1}), \dots, R_{\alpha_{s_{q-1}}}(\lambda_{s_{q-1}}), R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_q}}(\lambda_{s_q}), \dots, R_{\alpha_{s_p}}(\lambda_{s_p})\right)}{\lambda_{p+1} - \lambda_{s_{q-1}}} \Bigg) \\
= \;& \frac{1}{2\,h^\vee\,p} \sum_{s\in S_p} P(s) \sum_{q=1}^{p} (\lambda_{s_q} - \lambda_{s_{q-1}}) \\
& \qquad \frac{B\left(R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_q}}(\lambda_{s_q}), \dots, R_{\alpha_{s_p}}(\lambda_{s_p}), R_{\alpha_{s_1}}(\lambda_{s_1}), \dots, R_{\alpha_{s_{q-1}}}(\lambda_{s_{q-1}})\right)}{(\lambda_{p+1} - \lambda_{s_q})(\lambda_{p+1} - \lambda_{s_{q-1}})} \\
= \;& \frac{1}{2\,h^\vee\,p} \sum_{q=1}^{p} \sum_{s\in S_p} P([p+1, s_q, \dots, s_p, s_1, \dots, s_{p-1}]) \\
& \qquad B\left(R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_q}}(\lambda_{s_q}), \dots, R_{\alpha_{s_p}}(\lambda_{s_p}), R_{\alpha_{s_1}}(\lambda_{s_1}), \dots, R_{\alpha_{s_{q-1}}}(\lambda_{s_{q-1}})\right) \\
= \;& \frac{1}{2\,h^\vee} \sum_{s\in S_p} P([p+1, s])\, B\left(R_{\alpha_{p+1}}(\lambda_{p+1}), R_{\alpha_{s_1}}(\lambda_{s_1}), \dots, R_{\alpha_{s_p}}(\lambda_{s_p})\right).
\end{aligned}$$

For any gauge $\mathcal{V}$ of DS-type, there exists a unique $\mathfrak{n}$-valued smooth function $N(x)$ such that

$$e^{\mathrm{ad}_{N(x)}} \mathcal{L} = \mathcal{L}^{can}.$$

Observing that $\widetilde{R}_a = e^{\mathrm{ad}_{N(x)}} R_a$ and using the Ad-invariance of B we obtain

$$F_{a_1,\dots,a_N}(\lambda_1, \dots, \lambda_N; \mathbf{T}) = -\sum_{s\in S_N} \frac{B\left(R^{can}_{a_{s_1}}(\lambda_{s_1}), \dots, R^{can}_{a_{s_N}}(\lambda_{s_N})\right)}{2\,N\,h^\vee \prod_{j=1}^{N} (\lambda_{s_j} - \lambda_{s_{j+1}})} - \delta_{N2}\,\eta_{a_1 a_2} \frac{m_{a_1}\,\lambda_1 + m_{a_2}\,\lambda_2}{(\lambda_1 - \lambda_2)^2}.$$

Finally, $F_{a_1,\dots,a_N}(\lambda_1, \dots, \lambda_N; \mathbf{T}) \in \mathcal{A}^{q^{can}}[[\lambda_1^{-1}, \dots, \lambda_N^{-1}]]$ due to Lem. 2.7.1. The theorem is proved. ☐

Corollary 2.7.2. *For an arbitrary solution q^{can} to the DS hierarchy of $\mathfrak{g}$-type associated to $\mathcal{V}$ let τ be a tau-function of this solution. The following formulae hold true*

$$\sum_{k\geq 0} \frac{\langle\langle \tau_{a,k} \tau_{b,0} \rangle\rangle^{DS}}{\lambda^{k+1}} = (R^{can}_a(\lambda) \,|\, Q^{can}_b) - \eta_{ab}\, m_b, \quad a, b = 1, \dots, n. \tag{2.7.2}$$

In particular, we have

$$\sum_{k\geq 0} \frac{\langle\langle \tau_{a,k} \tau_{1,0} \rangle\rangle^{DS}}{\lambda^{k+1}} = (R^{can}_a(\lambda) \,|\, E_{-\theta}) - \eta_{a1}, \quad a = 1, \dots, n. \tag{2.7.3}$$

Proof. Taking in (1.3.4) with $N = 2$ the residue w.r.t. μ at $\mu = \infty$ we obtain (2.7.2). To show (2.7.3), we only need to notice that for $b = 1$, $\mathrm{Coeff}(R_1^{can}(\mu), \mu^1) = E_{-\theta}$. Indeed,

$$R_1^{can}(\mu) = \lambda\, E_{-\theta} + I_+ + \ldots$$

Here, the dots denote terms with principal degree lower than 1 which contain no more λ^1-power. □

More explicitly, let (U^{can}, H^{can}) be the unique pair associated to $\mathcal{L}^{can}$. Note that

$$R_a^{can} = e^{\mathrm{ad}_{U^{can}}}\, \Lambda_{m_a}. \tag{2.7.4}$$

Also note that Eq. (2.1.2) implies that U^{can} must have the following decomposition

$$U^{can} = \sum_{k\geq 0} U_{-k}^{can}\, \lambda^{-k}, \qquad U_0^{can} \in \mathfrak{n},\, U_{-k}^{can} \in \mathfrak{g},\, k \geq 1.$$

Hence we have

$$Q_b^{can} = \ \mathrm{Coeff}(R_b^{can}(\mu), \mu^1) = e^{\mathrm{ad}_{U_0^{can}}}\, K_{m_b - h}, \quad b = 1, \ldots, n. \tag{2.7.5}$$

Before ending this section, we consider taking a faithful irreducible matrix realization π of $\mathfrak{g}$. Let χ be the unique constant satisfying

$$(a|b) = \chi\, \mathrm{Tr}(\pi(a)\pi(b)), \qquad \forall\, a, b \in \mathfrak{g}. \tag{2.7.6}$$

For simplicity we will write $\pi(a)$ just as a, for $a \in \mathfrak{g}$. Similarly as Thm. 1.3.1 we have

Proposition 2.7.3. *Let $\mathcal{V}$ be a gauge of DS-type, $\mathcal{L}^{can}$ the gauge fixed Lax operator (2.6.1), and R_a^{can}, $a = 1, \ldots, n$ the basic resolvents of $\mathcal{L}^{can}$. For an arbitrary solution $q^{can}(\mathbf{T})$ to the DS hierarchy associated to $\mathcal{V}$, we have*

$$F_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N; \mathbf{T}) = -\frac{1}{\chi \cdot N} \sum_{s \in S_N} \frac{\mathrm{Tr}\; R_{a_{s_1}}^{can}(\lambda_{s_1}) \cdots R_{a_{s_N}}^{can}(\lambda_{s_N})}{\prod_{j=1}^{N}(\lambda_{s_j} - \lambda_{s_{j+1}})} - \delta_{N2}\, \eta_{a_1 a_2} \frac{m_{a_1}\, \lambda_1 + m_{a_2}\, \lambda_2}{(\lambda_1 - \lambda_2)^2}. \tag{2.7.7}$$

Remark 2.7.4. *The r.h.s. of (1.3.4) and the r.h.s. of (2.7.7) coincide. However, this does not mean the summands coincide with each other.*

2.8 An algorithm for writing the DS-hierarchy

Let $\mathcal{V}$ be any gauge of DS-type, $\{X^1, \ldots, X^n\}$ a basis of $\mathcal{V}$ s.t. $\deg X^a = -m_a$ and let

$$\mathcal{L}^{can} = \partial_x + \Lambda + q^{can}(x), \qquad q^{can}(x) = \sum_{a=1}^{n} w_a(x)\, X^a.$$

Recall that there exists a unique $\mathfrak{n}$-valued function $N^{can}(x)$ s.t.

$$e^{\mathrm{ad}_{N^{can}}} \mathcal{L} = \mathcal{L}^{can}.$$

Denote by R_a^{can}, $a = 1, \ldots, n$ the basic resolvents of $\mathcal{L}^{can}$. The corresponding DS-hierarchy will be defined as in (2.6.2). Although we know that RHS of (2.6.2) depends only on $q^{can}, q_x^{can}, \ldots$, the second term of RHS of (2.6.2) contains evolution of general components in $\mathfrak{n}$.

So the following question is under consideration:

For any given gauge $\mathcal{V}$, can we write down the DS-hierarchy for q^{can} using only the information of R_a^{can}?

Let us give a positive answer to this question by using our definition of tau-function.

1. Compute the basic resolvents R_a^{can}, $a = 1, \ldots, n$.
2. Calculate the Miura transformation $w_a \mapsto r_a$ from eq. (2.7.3). Recall that the normal coordinates are defined by $r_a := \langle\langle \tau_{a,0}\tau_{1,0}\rangle\rangle^{DS}$.
3. Calculate $\langle\langle \tau_{b,k}\tau_{a,0}\rangle\rangle^{DS}$ from eqs. (2.7.2). Note that the DS-flows for the normal coordinates r_a are

$$\frac{\partial r_a}{\partial T^{b,k}} = -\partial_x \langle\langle \tau_{b,k}\tau_{a,0}\rangle\rangle^{DS}, \qquad a, b = 1, \ldots, n,\ k \geq 0. \tag{2.8.1}$$

   The r.h.s of eqs. (2.8.1) are differential polynomials in w. Substituting $w_a \mapsto r_a$ in the r.h.s. of eqs. (2.8.1) we obtain the DS hierarchy for r_a.
4. Substitute the inverse Miura transformation to the DS hierarchy for r_a we obtain the DS hierarchy.

3 Computational aspect of resolvents

3.1 The lowest weight gauge

Recall that there is a particular choice of a gauge of DS-type [5], called the *lowest weight gauge*. Let us review its construction. Write the Weyl co-vector as $\rho^\vee = \sum_{i=1}^n x_i\, H_i,\ x_i \in \mathbb{C}$ and define

$$I_- = 2\sum_{i=1}^n x_i\, F_i. \tag{3.1.1}$$

Then $I_+, I_-, \rho^\vee$ generate an $sl_2(\mathbb{C})$ Lie subalgebra of $\mathfrak{g}$:

$$[\rho^\vee, I_+] = I_+, \quad [\rho^\vee, I_-] = -I_-, \quad [I_+, I_-] = 2\rho^\vee. \tag{3.1.2}$$

According to [43, 5] there exist elements $\gamma^1, \ldots, \gamma^n \in \mathfrak{g}$ such that

$$\operatorname{Ker} \operatorname{ad}_{I_-} = \operatorname{Span}_{\mathbb{C}}\{\gamma^1, \ldots, \gamma^n\}, \quad [\rho^\vee, \gamma^i] = -m_i\, \gamma^i.$$

Since $\gamma^n \in \mathbb{C}E_{-\theta}$ we could and will normalize it to be

$$\gamma^n = E_{-\theta}. \tag{3.1.3}$$

The subspace $\operatorname{Ker} \operatorname{ad}_{I_-} \subset \mathfrak{b}$ is a gauge of DS-type, which is called the lowest weight gauge. Denote by

$$\mathcal{L}^{can} = \partial_x + \Lambda + q^{can}(x)$$

the gauge fixed Lax operator associated to $\operatorname{Ker} \operatorname{ad}_{I_-}$, where $q^{can}(x) := \sum_{a=1}^n u_a(x)\, \gamma^a$.

Definition 3.1.1. *The functions u_a, $a = 1, \ldots, n$ are called the* **lowest weight coordinates***.*

3.2 Extended principal gradation

Definition 3.2.1. *Define the* **extended principal degree** *by the following degree assignments*

$$\deg^e \partial_x = 1, \quad \deg^e \lambda = h, \tag{3.2.1}$$

$$\deg^e u_i = m_i + 1, \tag{3.2.2}$$

$$\deg^e E_i = 1, \quad \deg^e F_i = -1, \quad i = 1, \ldots, n. \tag{3.2.3}$$

It is easy to see that, if $a \in L(\mathfrak{g})^j$ then $\deg^e a = \deg a = j$. Namely, the extended principal degree coincides with the principal degree for any loop algebra element. In particular,

$$\deg^e \gamma^i = -m_i, \quad \deg^e \operatorname{ad}_{I_+}^j \gamma^i = -m_i + j, \qquad j = 0, \ldots, 2m_i. \tag{3.2.4}$$

Lemma 3.2.2. *For the gauge-fixed Lax operator $\mathcal{L}^{can}$, we have $\deg^e \mathcal{L}^{can} = 1$.*

Let (U^{can}, H^{can}) be the unique pair associated to $\mathcal{L}^{can}$, and R_a^{can} the basic resolvents.

Lemma 3.2.3. *The following formulae hold true*

$$\deg^e U^{can} = 0, \quad \deg^e H^{can} = 1, \quad \deg^e R_a^{can} = m_a, \qquad a = 1, \ldots, n. \tag{3.2.5}$$

Proof. By using the recursion procedure (2.1.6) and by the mathematical induction. ☐

Corollary 3.2.4. *The N-point ($N \geq 2$) generating series of correlation functions $F_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N; \mathbf{T})$ are homogenous of degree $-Nh + \sum_{\ell=1}^N m_{a_\ell}$ w.r.t. the extended principal gradation.*

3.3 Essential series of the Drinfeld–Sokolov hierarchy

Recall that the simple Lie algebra $\mathfrak{g}$ admits the lowest weight decomposition [5]

$$\mathfrak{g} = \bigoplus_{a=1}^n \mathfrak{L}^a, \qquad \mathfrak{L}^a = \operatorname{Span}_{\mathbb{C}}\{\gamma^a, \operatorname{ad}_{I_+} \gamma^a, \ldots, \operatorname{ad}_{I_+}^{2m_a} \gamma^a\}$$

where each $\mathfrak{L}^a$ is an $sl_2(\mathbb{C})$-module w.r.t. the $sl_2(\mathbb{C})$ Lie subalgebra generated by $I_+, I_-, 2\rho^\vee$, called a lowest weight module. It is then clear that any $\mathfrak{g}$-valued function $R(\lambda)$ can be uniquely written as

$$R(\lambda) = \sum_{a=1}^n \sum_{m=0}^{2m_a} K_{am}(\lambda) \operatorname{ad}_{I_+}^m \gamma^a.$$

Theorem 3.3.1. *Let $\mathcal{L}^{can} = \partial_x + \Lambda + q^{can} = \partial_x + \Lambda + \sum_{a=1}^n u_a \gamma^a$ be a Lax operator associated to the lowest weight gauge. Let $R^{can} \in \mathcal{A}^u \otimes \mathfrak{g}((\lambda^{-1}))$ be any resolvent of $\mathcal{L}^{can}$. Write*

$$R^{can} = \sum_{i=1}^n \mathcal{R}_i \operatorname{ad}_{I_+}^{2m_i} \gamma^i + \sum_{i=1}^n \sum_{m=0}^{2m_i - 1} K_{im} \operatorname{ad}_{I_+}^m \gamma^i. \tag{3.3.1}$$

We have 1) $\forall\, i \in \{1, \ldots, n\}$, $m \in \{0, 1, \ldots, 2m_i - 1\}$, K_{im} has the following expression

$$K_{im} = \sum_{j=1}^n \sum_{\ell=0}^{2m_i - m} \left(s_{i,\ell,0}^j + \lambda\, s_{i,\ell,1}^j \right) \partial_x^\ell (\mathcal{R}_j),$$

where the coefficients $s_{i,\ell,0}^j, s_{i,\ell,1}^j$ belong to $\mathcal{A}^u$, and they do not depend on the choice of the resolvent.

2) The ODE $[\mathcal{L}^{can}, R^{can}] = 0$ is equivalent to n scalar linear ODEs for $\mathcal{R}_1, \ldots, \mathcal{R}_n$.

3) The following formulae hold true for the degrees of the coefficients (3.3.1) of the basic resolvents

$$\deg^e \mathcal{R}_{a;\,i} = m_a - m_i, \quad \deg^e K_{a;\,im} = m_a + m_i - m, \quad i, a = 1, \ldots, n;\ m = 0, \ldots, 2m_i - 1. \tag{3.3.2}$$

Proof of Thm. 3.3.1 Write

$$R^{can}(\lambda; u; u_x, \dots) = \sum_{i=1}^{n} \sum_{m=0}^{2m_i} K_{im}(\lambda; u; u_x, \dots) \, \mathrm{ad}_{I_+}^m \gamma^i, \qquad K_{i,2m_i} := \mathcal{R}_i.$$

Substituting the above expressions into (2.2.4) we obtain

$$\sum_{i=1}^{n} \sum_{m=0}^{2m_i} \frac{\partial K_{im}}{\partial x} \mathrm{ad}_{I_+}^m \gamma^i + \sum_{i=1}^{n} \sum_{m=1}^{2m_i} K_{i,m-1} \, \mathrm{ad}_{I_+}^m \gamma^i + \left[\lambda \, \gamma^n + \sum_{\ell=1}^{n} u_\ell \gamma^\ell, \, \sum_{i=1}^{n} \sum_{m=0}^{2m_i} K_{im} \, \mathrm{ad}_{I_+}^m \gamma^i \right] = 0. \quad (3.3.3)$$

Introduce the lowest weight structure constants $c_{\ell ijs}^m$ by

$$[\gamma^\ell, \, \mathrm{ad}_{I^+}^m \gamma^i] = \sum_{j=1}^{n} \sum_{s=0}^{2m_j} c_{\ell ijs}^m \, \mathrm{ad}_{I^+}^s \gamma^j, \qquad i, \ell = 1, \dots, n, \; m = 0, \dots, 2m_i. \quad (3.3.4)$$

Substituting (3.3.4) into (3.3.3) we obtain

$$\sum_{i=1}^{n} \sum_{m=0}^{2m_i} \frac{\partial K_{im}}{\partial x} \mathrm{ad}_{I_+}^m \gamma^i + \sum_{i=1}^{n} \sum_{m=1}^{2m_i} K_{i,m-1} \, \mathrm{ad}_{I_+}^m \gamma^i + \sum_{\ell=1}^{n} \sum_{i=1}^{n} \sum_{m=0}^{2m_i} \sum_{j=1}^{n} \sum_{s=0}^{2m_j} \widetilde{u_\ell} \, K_{im} \, c_{\ell ijs}^m \, \mathrm{ad}_{I^+}^s \gamma^j = 0 \quad (3.3.5)$$

where $\widetilde{u_\ell} = u_\ell + \lambda \, \delta_{\ell,n}$. It follows that

$$K_{j,s-1} + \frac{\partial K_{js}}{\partial x} + \sum_{\ell=1}^{n} \sum_{i=1}^{n} \sum_{m=0}^{2m_i} \widetilde{u_\ell} \, K_{im} \, c_{\ell ijs}^m = 0, \qquad j = 1, \dots, n, \; s = 0, \dots, 2m_j. \quad (3.3.6)$$

Here $K_{j,-1} := 0$. Noting that the structure constant $c_{\ell ijs}^m$ are zero unless

$$0 \le m = m_i + m_\ell + s - m_j \le 2m_i. \quad (3.3.7)$$

Hence we obtain

$$K_{j,s-1} = -\frac{\partial K_{js}}{\partial x} - \sum_{\substack{\ell, \, i=1 \\ m_i \ge |m_\ell + s - m_j|}}^{n} \widetilde{u_\ell} \cdot K_{i, m_i + m_\ell + s - m_j} \, c_{\ell ijs}^{m_i + m_\ell + s - m_j}, \quad j = 1, \dots, n, \; s = 0, \dots, 2m_j. \quad (3.3.8)$$

Define an ordering for pairs of integers $\{(j, s) \mid j = 1, \dots, n, \; s = 0, \dots, 2m_j\}$: we say $(j_1, s_1) > (j_2, s_2)$, if $s_1 > s_2$, or $s_1 = s_2$ and $j_1 < j_2$. Noting that $K_{i,2m_i} := \mathcal{R}_i$ we can use (3.3.8) to solve out $K_{j,s-1}$ in terms of $\mathcal{R}_j$ and their x-derivatives starting from the largest pair $(j, s-1) = (n, 2m_n - 1)$ to the smallest pair $(j, s-1) = (n, 0)$. This proves Part 1) of the theorem.

Taking $s = 0$ in (3.3.8) we obtain the system of ODEs for $\mathcal{R}_1, \dots, \mathcal{R}_n$, which proves Part 2).

Formulae (3.3.2) follow from Lemma 3.2.3 and eq.(3.3.1), which proves Part 3). □

Definition 3.3.2. *We call $\mathcal{R}_{a;1}, \dots, \mathcal{R}_{a;n}$ the essential series of the DS hierarchy of the $\mathfrak{g}$-type.*

Using the same argument as in [8], the essential series $\mathcal{R}_{a;a}$ never vanishes.

Definition 3.3.3. *We call $\mathcal{R}_{a;a}$ the fundamental series of the DS hierarchy.*

4 Proof of Theorem 1.3.2

4.1 Relation between normal coordinates and lowest weight coordinates

The concept of normal coordinates was introduced in [26]; see also [24].

Definition 4.1.1. *We call $r_a := \langle\langle \tau_{a,0}\tau_{1,0}\rangle\rangle^{DS}$ the normal coordinates of the DS hierarchy.*

Recall that

$$\Lambda_{m_a}(\lambda) = L_{m_a} + \lambda\, K_{m_a-h}, \qquad L_{m_a} \in \mathfrak{g}^{m_a},\ K_{m_a-h} \in \mathfrak{g}^{m_a-h}.$$

Using the commutativity between $\Lambda_{m_1}, \ldots, \Lambda_{m_n}$ along with the normalization (1.1.6) we have

$$[L_{m_a}, L_{m_b}] = 0, \quad [K_{m_a-h}, K_{m_b-h}] = 0, \tag{4.1.1}$$

$$[K_{m_a-h}, L_{m_b}] + [L_{m_a}, K_{m_b-h}] = 0 \tag{4.1.2}$$

and

$$(L_{m_a} \,|\, K_{m_b-h}) = \eta_{ab}\, m_b, \quad \forall\, a, b = 1, \ldots, n. \tag{4.1.3}$$

Note that $L_{m_1} = I_+$, we have in particular

$$[I_+, L_{m_a}] = 0\ , \quad \forall\, a = 1, \ldots, n. \tag{4.1.4}$$

Therefore the elements L_{m_a} are the highest weight vectors of the lowest weight module $\mathcal{L}^a$, i.e.

$$L_{m_a} = const \cdot \mathrm{ad}_{I_+}^{2m_a} \gamma^a, \quad const \neq 0.$$

Lemma 4.1.2. *The lowest weight vectors γ^a can be normalized such that*

$$(\gamma^a \,|\, L_{m_a}) = 1. \tag{4.1.5}$$

Proof. We know that different irreducible representations of $sl_2(\mathbb{C})$ are orthogonal w.r.t. to $(\cdot|\cdot)$ and, hence, the nondegeneracy of $(\cdot|\cdot)$ implies the nondegeneracy of its restriction to each irreducible representation. Note that

$$(\, \gamma^a \,|\, \mathrm{ad}_{I_-}^k L_{m_a}) = -(I_- \,|\, [\gamma^a, \mathrm{ad}_{I_-}^{k-1} L_{m_a}]) = 0, \quad \forall\, k \in \{1, \ldots, 2m_a\}.$$

So $(\gamma^a \,|\, L_{m_a}) \neq 0$ since otherwise we obtain a contradiction with the nondegeneracy of $(\cdot|\cdot)$. Hence for $a = 1, \ldots, n-1$, we can normalize γ^a such that $(\gamma^a \,|\, L_{m_a}) = 1$. Particular consideration must be addressed for γ^n, since we have already defined $\gamma^n = E_{-\theta}$. Taking in (4.1.3) $a = n$, $b = 1$ we obtain

$$(L_{m_n} \,|\, K_{m_1-h}) = 1 \ \Rightarrow\ (L_{m_n} \,|\, E_{-\theta}) = 1,$$

which finishes the proof. □

From now on we fix a choice of $\gamma^1, \ldots, \gamma^n$ satisfying (4.1.5). Then Lemmata 2.1.1, 4.1.2 imply

$$(\gamma^a \,|\, L_{m_b}) = \delta^a_b. \tag{4.1.6}$$

Note that for D_n with n even, eq. (4.1.6) is valid with a suitable choice of $\gamma^{n/2}$, $\gamma^{n/2+1}$.

According to Cor. 3.2.4 and Thm. 1.3.1, $\langle\langle \tau_{a,k}\tau_{1,0}\rangle\rangle$ are differential polynomials in u, homogeneous of degree

$$m_a + 1 + kh$$

w.r.t. to $\deg^e$. In particular, we have

$$\deg^e r_a = m_a + 1, \qquad a = 1, \ldots, n.$$

We arrive at

Lemma 4.1.3. *There exists a Miura transformation $u \to r$ of the form*

$$r_a = c_a\, u_a + P_a[u_1, \ldots, u_{a-1}], \qquad a = 1, \ldots, n \tag{4.1.7}$$

for some non-zero constants c_a, where P_a are differential polynomials in $u_1, \ldots, u_{a-1}$ satisfying

$$\deg^e P_a[u_1, \ldots, u_{a-1}] = m_a + 1. \tag{4.1.8}$$

Remark 4.1.4. *For D_n with n even, Lemma 4.1.3 is valid with a suitable choice of $\gamma^{n/2}$, $\gamma^{n/2+1}$.*

Remark 4.1.5. *The inverse Miura transformation has the form*

$$u_a = c_a^{-1}\, r_a + \widetilde{P}_a([r_1, \ldots, r_{a-1}]), \tag{4.1.9}$$

thanks to the triangular nature of the transformation (4.1.7).

Lemma 4.1.6. *The constants c_a in Lemma 4.1.3 have the following explicit expressions*

$$c_a = -\frac{m_a}{h}. \tag{4.1.10}$$

Proof. Fix $a \in \{1, \ldots, n\}$. We are to compute $r_a|_{u_1,\ldots,u_{a-1}\equiv 0}$. Assume $u_1 \equiv 0, \ldots, u_{a-1} \equiv 0$. Looking at equation (2.1.5) for the pair (U, H) we obtain

$$U^{[-1]} = \ldots = U^{[-m_a]} = 0 = H^{[-1]} = \ldots = H^{[1-m_a]}.$$

The first nontrivial equation arises from the component of principal degree $-m_a$ in (2.1.5):

$$H^{[-m_a]} + \left[U^{[-m_a-1]}, \Lambda\right] = u_a\, \gamma^a \qquad \text{(no summation in } a). \tag{4.1.11}$$

Let us decompose the elements $H^{[-m_a]}, U^{[-m_a-1]}$ as follows

$$\begin{aligned} H^{[-m_a]} &= \frac{g_a(x)}{\lambda}\, \Lambda_{h-m_a} = g_a(x)\, K_{-m_a} + \frac{g_a(x)}{\lambda}\, L_{h-m_a}, \quad a = 1, \ldots, n, \\ U^{[-m_a-1]} &= \frac{1}{\lambda} Y_{h-m_a-1} + W_{-m_a-1}, \qquad a = 1, \ldots, n-1, \\ U^{[-m_n-1]} &= \frac{1}{\lambda} Y_0. \end{aligned}$$

Substituting these expressions in (4.1.11) and comparing the coefficients of powers of λ we obtain

$$\lambda^{-1}: \quad g_a(x)\, L_{h-m_a} + [Y_{h-m_a-1}, I_+] = 0, \tag{4.1.12}$$

$$\lambda^{0}: \quad g_a(x)\, K_{-m_a} + [Y_{h-m_a-1}, E_{-\theta}] + [W_{-m_a-1}, I_+] = u_a \gamma^a, \tag{4.1.13}$$

$$\lambda^{1}: \quad [W_{-m_a-1}, E_{-\theta}] = 0 \qquad \text{(automatic!)}. \tag{4.1.14}$$

Since L_{h-m_a} is the highest weight vector of the irreducible $sl_2(\mathbb{C})$-module $\mathcal{L}^{n+1-a}$, the solution to eq. (4.1.12) is

$$Y_{h-m_a-1} = \frac{g_a(x)}{2(h-m_a)} [I_-, L_{h-m_a}] + f(x)\, L_{h-m_a-1}$$

for some function $f(x)$ which is a differential polynomial in u. So we have

$$\begin{aligned} [Y_{h-m_a-1}, E_{-\theta}] &= \frac{g_a(x)}{2(h-m_a)}\, [I_-, [L_{h-m_a}, E_{-\theta}]] + f(x)\, [L_{h-m_a-1}, E_{-\theta}] \\ &\overset{(4.1.2)}{=} \frac{g_a(x)}{2(h-m_a)} [I_-, [I_+, K_{-m_a}]] + f(x)\, [L_{h-m_a-1}, E_{-\theta}]. \end{aligned} \tag{4.1.15}$$

Plugging (4.1.15) into (4.1.13) we find

$$g_a(x)K_{-m_a} + \frac{g_a(x)}{2(h-m_a)}[I_-,[I_+,K_{-m_a}]] + [W_{-m_a-1},I_+] + f(x)\,[L_{h-m_a-1},E_{-\theta}] = u_a\,\gamma^a.$$

Employing the Jacobi identity we obtain

$$g_a(x)\,\frac{h}{h-m_a}\,K_{-m_a} + \left[I_+\,,\,\frac{g_a(x)}{2(h-m_a)}[I_-,K_{-m_a}] - W_{-m_a-1}\right] + f(x)\,[L_{h-m_a-1},E_{-\theta}] = u_a\gamma^a.$$

Taking the inner products of both sides of the above equation with L_{m_a} we have

$$\left(L_{m_a}\middle|\,\frac{h\,g_a(x)}{h-m_a}\,K_{-m_a} + \left[I_+\,,\,\frac{g_a(x)\,[I_-,K_{-m_a}]}{2(h-m_a)} - W_{-m_a-1}\right] + f(x)\,[L_{h-m_a-1},E_{-\theta}]\right) = u_a\,\left(L_{m_a}|\gamma^a\right). \tag{4.1.16}$$

Noticing that L_{m_a} is a highest weight vector of the $sl_2(\mathbb{C})$-module $\mathcal{L}^a$, i.e.

$$[L_{m_a}\,,\,I_+]=0, \qquad [L_{m_a},L_{h-m_a-1}]=0,$$

and using (4.1.3), (4.1.5) we obtain

$$g_a(x) = \frac{h-m_a}{h\cdot(L_{m_a}\,|\,K_{-m_a})}\,(L_{m_a}\,|\,\gamma^a)\,u_a(x) = \frac{1}{h}\,u_a(x).$$

Using Def. 4.1.1 and eq. (2.7.3) we have

$$-r_a = \operatorname*{res}_{\lambda=\infty}\left(e^U\Lambda_{m_a}e^{-U}\,\middle|\,E_{-\theta}\right) = \operatorname*{res}_{\lambda=\infty}\left(\Lambda_{m_a}(\lambda)\,\middle|\,E_{-\theta} - [U(\lambda),E_{-\theta}] + \frac{1}{2}[U(\lambda),[U(\lambda),E_{-\theta}]] + \dots\right).$$

The only possible contribution to the residue comes from the terms of principal degree $-h-m_a$ and the first one in the series is easily seen to be residueless

$$\operatorname*{res}_{\lambda=\infty}(\Lambda_{m_a}(\lambda)|E_{-\theta})\,d\lambda = 0.$$

Note that we have already shown that U has the form

$$U = U^{[-m_a-1]} + \sum_{j\le -m_a-2} U^{[j]}.$$

Therefore only the very next term $-(\Lambda_{m_a}(\lambda)\,|\,[U(\lambda),E_{-\theta}])$ can contribute to the residue. Thus

$$r_a = \operatorname*{res}_{\lambda=\infty}\,(\,\Lambda_{m_a}(\lambda)\,|\,[U(\lambda),E_{-\theta}]\,) = \operatorname*{res}_{\lambda=\infty}\left(\,\Lambda_{m_a}(\lambda)\,|\,[U^{[-m_a-1]}(\lambda),E_{-\theta}]\right). \tag{4.1.17}$$

Now substituting

$$\Lambda_{m_a}(\lambda) = \lambda\,K_{m_a-h} + L_{m_a}, \quad U^{[-m_a-1]} = \frac{1}{\lambda}\,Y_{h-m_a-1} + W_{-m_a-1} \tag{4.1.18}$$

in (4.1.17) we obtain

$$\begin{aligned}
-\,r_a(x) &= \left(L_{m_a}\middle|[Y_{h-m_a-1},E_{-\theta}]\right) = \left(L_{m_a}\middle|\left[\frac{g_a(x)}{2(h-m_a)}[I_-,L_{h-m_a}] + f(x)\,L_{h-m_a-1},E_{-\theta}\right]\right)\\
&= \frac{g_a(x)}{2(h-m_a)}\,(\,L_{m_a}\,|\,[[E_{-\theta}\,,\,L_{h-m_a}]\,,\,I_-]\,)\\
&= \frac{g_a(x)}{2(h-m_a)}\,(\,L_{m_a}\,|\,[[K_{-m_a}\,,\,I_+]\,,\,I_-]\,) = \frac{g_a(x)}{2(h-m_a)}\,(\,[I_+\,,\,[I_-\,,\,L_{m_a}]]\,|\,K_{-m_a}\,)\\
&= g_a(x)\,\frac{m_a}{h-m_a}\,(\,L_{m_a}\,|\,K_{-m_a}\,) = \frac{m_a}{h}\,u_a(x).
\end{aligned}$$

The lemma is proved. □

Remark 4.1.7. *For the A_n case, a similar lemma on relations between normal coordinates and Wronskian-gauge coordinates was obtained e.g. in [9]; see Lemma 3.1 therein.*

4.2 Partition function and topological ODE

The partition function of the DS hierarchy of $\mathfrak{g}$-type is a particular tau-function specified (up to a constant factor) by the string equation (1.3.5). The compatibility between the string equation and the DS hierarchy follows from the fact that the flow $\partial_{s_{-1}}$ defined by

$$\partial_{s_{-1}}\tau := \sum_{a=1}^{n}\sum_{k\geq 0} t^a_{k+1}\frac{\partial \tau}{\partial t^a_k} + \frac{1}{2}\sum_{a,b=1}^{n} \eta_{ab}\, t^a_0\, t^b_0 \tau - \frac{\partial \tau}{\partial t^1_0}$$

is an additional symmetry of the DS hierarchy.

The function $u = u(\mathbf{T}) = u(\mathbf{t})$ associated to $Z(\mathbf{t})$ is called the topological solution to the lowest-weight-gauge DS hierarchy, and $r = r(\mathbf{t}) = r(\mathbf{T})$ the topological solution in normal coordinates.

Lemma 4.2.1. *The normal coordinates associated to the partition function Z satisfy*

$$r_a(\mathbf{t})|_{t^a_k=\delta^a_1\,\delta_{k,0}\,t^1_0} = -\delta_{a,n}\frac{h-1}{h\cdot\kappa}\, t^1_0, \qquad \kappa := \sqrt{-h}^{\,-h}. \tag{4.2.1}$$

Proof. By applying the t^a_0-derivative on both sides of eq. (1.3.5) we have

$$\frac{\partial^2 \log Z}{\partial t^1_0\,\partial t^a_0}\Big|_{t^a_k=\delta^a_1\,\delta_{k,0}\,t^1_0} = \delta_{a,n}\, t^1_0.$$

Hence from (1.3.6) we obtain

$$\frac{\partial^2 \log Z}{\partial T^1_0\,\partial T^a_0}\Big|_{t^a_k=\delta^a_1\,\delta_{k,0}\,t^1_0} = -\delta_{a,n}\frac{h-1}{h}\sqrt{-h}^{\,h}\, t^1_0.$$

The lemma is proved. ☐

Lemma 4.2.2. *The topological solution to the lowest-weight-gauge DS hierarchy of $\mathfrak{g}$-type satisfies*

$$u_a(\mathbf{t})|_{t^a_k=\delta^a_1\,\delta_{k,0}\,t^1_0} = \delta_{a,n}\frac{1}{\kappa}\, t^1_0. \tag{4.2.2}$$

Proof. By applying Lemma 4.1.3, Lemma 4.1.6 and Lemma 4.2.1. ☐

Topological ODE of $\mathfrak{g}$-type. Let $u = u(\mathbf{T}) = u(\mathbf{t})$ be the topological solution to the lowest-weight-gauge DS hierarchy. Note that

$$t^1_0 = -T^1_0 = x.$$

Define

$$M_a(\lambda, x) = \lambda^{-\frac{m_a}{h}}\, R^{can}_a|_{t^b_k=x\,\delta^b_1\,\delta_{k,0}}, \qquad a=1,\dots,n;$$

then we have

$$\Big[\partial_x + \Lambda + \frac{x}{\kappa}\gamma^n\,,\, M_a(\lambda,x)\Big] = 0. \tag{4.2.3}$$

Noting that $\gamma^n = E_{-\theta}$ we have

$$\partial_x\,(M_a) + \Big[I_+ + \Big(\lambda + \frac{x}{\kappa}\Big)\, E_{-\theta}\,,\, M_a\Big] = 0, \qquad a=1,\dots,n. \tag{4.2.4}$$

Lemma 4.2.3 (Key Lemma)**.** *The following formulae hold true*

$$\partial_x (M_a) = \frac{1}{\kappa} \partial_\lambda (M_a), \qquad a = 1, \dots, n. \tag{4.2.5}$$

Proof. Consider the transformation of independent variables $(\lambda, x) \to (s, x)$ defined by

$$s = \lambda + \frac{x}{\kappa}, \qquad x = x.$$

Then we have

$$\frac{1}{\kappa} \frac{dM_a}{ds} + [I_+ + s\, E_{-\theta}\,,\, M_a] = 0, \qquad a = 1, \dots, n. \tag{4.2.6}$$

Note that eq. (4.2.6) for M_a is the topological ODE of $\mathfrak{g}$-type [8]. At $s = \infty$, M_a is a regular solution satisfying that

$$M_a = s^{-\frac{m_a}{h}}\, \Lambda_{m_a}(s) + \text{lower order terms w.r.t. } \deg^p. \tag{4.2.7}$$

According to the uniqueness part of Thm. 1.2 in [7] we have

$$M_a(s, x) = M_a(s). \tag{4.2.8}$$

The lemma is proved. □

Proof. of Thm. 1.3.2. Note that $M_a(\lambda) = M_a(\lambda; x = 0)$. Substituting eq. (4.2.5) into eq. (4.2.4), and then taking $x = 0$. we obtain

$$[L, M_a(\lambda)] = 0, \qquad L := \partial_\lambda + \kappa\, \Lambda,$$
$$M_a(\lambda) = \lambda^{-\frac{m_a}{h}} \left[\Lambda_{m_a}(\lambda) + \text{lower order terms w.r.t. } \deg^p\right].$$

The theorem is proved. □

Proof. of Thm. 1.4.2. By Thm-ADE, Thm-BCFG, Thm. 1.3.1, and by Thm. 1.3.2 we obtain

$$(\kappa \sqrt{-h})^N \sum_{g, k_1, \dots, k_N \geq 0} (-1)^{k_1 + \dots + k_N} \prod_{\ell=1}^N \frac{\left(\frac{m_{i_\ell}}{h}\right)_{k_\ell + 1}}{\left(\kappa\, \tilde\lambda_\ell\right)^{\frac{m_{i_\ell}}{h} + k_\ell + 1}} \langle \tau_{i_1 k_1} \dots \tau_{i_N k_N} \rangle_g^{\mathfrak{g}}$$
$$= \; -\frac{1}{2N\, h^\vee} \sum_{s \in S_N} \frac{B\left(\tilde M_{i_{s_1}}(\tilde\lambda_{s_1}), \dots, \tilde M_{i_{s_N}}(\tilde\lambda_{s_N})\right)}{\prod_{j=1}^N (\tilde\lambda_{s_j} - \tilde\lambda_{s_{j+1}})}$$
$$-\delta_{N2}\, \eta_{i_1 i_2} \frac{\tilde\lambda_1^{-\frac{m_{i_1}}{h}} \tilde\lambda_2^{-\frac{m_{i_2}}{h}} (m_{i_1} \tilde\lambda_1 + m_{i_2} \tilde\lambda_2)}{(\tilde\lambda_1 - \tilde\lambda_2)^2}, \quad N \geq 2. \tag{4.2.9}$$

where $\tilde M_a = \tilde M_a(\tilde\lambda)$, $a = 1, \dots, n$ are the unique solutions to

$$\frac{\mathrm{d}\tilde M}{\mathrm{d}\tilde\lambda} = \kappa\, [\tilde M, \Lambda(\tilde\lambda)], \qquad \kappa = \left(\sqrt{-h}\right)^{-h},$$
$$\tilde M_a(\tilde\lambda) = \tilde\lambda^{-\frac{m_a}{h}} \left[\Lambda_{m_a}(\tilde\lambda) + \text{lower degree terms w.r.t. } \deg\right].$$

Now consider the following conjugation of $\tilde M_a$ together with a rescaling in $\tilde\lambda$:

$$M_a(\lambda) = \sigma^{\rho^\vee}\, \tilde M_a(\tilde\lambda)\, \sigma^{-\rho^\vee},$$
$$\lambda = \sigma^{-h}\, \tilde\lambda$$

where $\sigma := \kappa^{-\frac{1}{h+1}}$. It is straightforward to check that

$$\frac{\mathrm{d}M}{\mathrm{d}\lambda} = [M, \Lambda(\lambda)],$$
$$M_a(\lambda) = \lambda^{-\frac{m_a}{h}} \left[\Lambda_{m_a}(\lambda) + \text{lower degree terms w.r.t. deg}\right].$$

Combining with (4.2.9), this proves the validity of the formula (1.4.9). To prove formula (1.4.8), one further needs to observe the following identity obtained from the string equation (1.3.5)

$$\langle \tau_{a,k+1}\tau_{1,0}\rangle^{FJRW-\mathfrak{g}} = \langle \tau_{ak}\rangle^{FJRW-\mathfrak{g}}, \quad a = 1, \ldots, n,\ k \geq 0.$$

The rest of proving (1.4.8) follows from the identity (2.7.3) and the above conjugation of $\tilde{M}_a$ with the rescaling in $\tilde{\lambda}$. □

Proof of Thm. 1.4.3. The theorem is a particular case of Thm. 1.4.2 with the particular realization of A_n Lie algebra being consistent with normalization of flows suggested by Witten [52]. □

Example 4.2.4 (Rationality of Witten's r-spin intersection numbers.)**.** *It is known that Witten's r-spin intersection numbers are non-negative rational numbers. Let us verify the rationality through (1.4.12) and (1.4.13). Indeed, our definition of N-point r-spin correlators reads*

$$\begin{aligned} F^{r-spin}_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N) &= \left(\kappa^{\frac{1}{r+1}}\sqrt{-r}\right)^N \sum_{k_1,\ldots,k_N\geq 0} \prod_{\ell=1}^N \frac{(-1)^{k_\ell}\left(\frac{a_\ell}{r}\right)_{k_\ell+1}}{(\kappa^{\frac{1}{r+1}}\lambda_\ell)^{\frac{a_\ell}{r}+k_\ell+1}} \langle \tau_{a_1k_1}\cdots\tau_{a_Nk_N}\rangle^{r-\text{spin}} \\ &= \sum_{g\geq 0}(-r)^{g-1+N} \sum_{k_1,\ldots,k_N\geq 0} \prod_{\ell=1}^N \frac{(-1)^{k_\ell}\left(\frac{a_\ell}{r}\right)_{k_\ell+1}}{\lambda_\ell^{\frac{a_\ell}{r}+k_\ell+1}} \langle \tau_{a_1k_1}\cdots\tau_{a_Nk_N}\rangle_g^{r-\text{spin}} \end{aligned}$$

where we have used $\kappa = \sqrt{-r}^{-r}$ and the dimension-degree matching (1.4.5). Clearly, all the coefficients are rational. On the other hand, the r.h.s. of (1.4.12) or of (1.4.13) belongs to $\mathbb{Q}[[\lambda_1^{-1/r}, \ldots, \lambda_N^{-1/r}]]$ as our regular solutions $M_a(\lambda)$, $a = 1, \ldots, n$ to the topological ODEs of $sl_n(\mathbb{C})$-type (1.4.11) are of rational coefficients. The rationality of r-spin correlators is verified.

A 3-spin

The matrices $M_i(\lambda)$, $i = 1, 2$ for the Witten's 3-spin invariants have the following explicit expressions. Denote $M_i(\lambda) = (M_i(\lambda)^a_b)_{a,b=1,\ldots,3}$. Then we have

$$(M_1)^1_1 = \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{4}{3})}{108^g\, g!\, \Gamma(g+\frac{1}{3})} \lambda^{-\frac{24g+4}{3}} - \frac{1}{72} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{16}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+16}{3}}$$

$$(M_1)^1_2 = -\sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{1}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+1}{3}} + \frac{1}{24} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{13}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+13}{3}}$$

$$(M_1)^1_3 = -\frac{1}{12} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{10}{3})}{108^{3g}\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+10}{3}}$$

$$(M_1)^2_1 = \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{7}{3})}{108^g\, g!\, \Gamma(g+\frac{1}{3})} \lambda^{-\frac{24g+7}{3}} - \frac{1}{12} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{10}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+7}{3}} - \frac{1}{72} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{19}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+19}{3}}$$

$$(M_1)^2_2 = \frac{1}{36} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{16}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+16}{3}}$$

$$(M_1)^2_3 = -\sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{1}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+1}{3}} - \frac{1}{24} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{13}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+13}{3}}$$

$$(M_1)^3_1 = -\frac{1}{72} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{22}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+22}{3}} - \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{1}{3})}{108^g\, g!\, \Gamma(g+\frac{1}{3})} \lambda^{-\frac{24g-2}{3}}$$

$$(M_1)^3_2 = \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{7}{3})}{108^g\, g!\, \Gamma(g+\frac{1}{3})} \lambda^{-\frac{24g+7}{3}} - \frac{1}{12} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{10}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+7}{3}} + \frac{1}{72} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{19}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+19}{3}}$$

$$(M_1)^3_3 = -\sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{4}{3})}{108^g\, g!\, \Gamma(g+\frac{1}{3})} \lambda^{-\frac{24g+4}{3}} - \frac{1}{72} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{16}{3})}{108^g\, g!\, \Gamma(g+\frac{4}{3})} \lambda^{-\frac{24g+16}{3}}$$

and

$$(M_2)^1_1 = -\frac{1}{6} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{8}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+8}{3}} - \frac{1}{144} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{20}{3})}{108^g\, g!\, \Gamma(g+\frac{5}{3})} \lambda^{-\frac{24g+20}{3}}$$

$$(M_2)^1_2 = \frac{1}{144} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{17}{3})}{108^g\, g!\, \Gamma(g+\frac{5}{3})} \lambda^{-\frac{24g+17}{3}} + \frac{1}{2} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{5}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+5}{3}}$$

$$(M_2)^1_3 = -\sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{2}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+2}{3}}$$

$$(M_2)^2_1 = -\frac{1}{144} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{23}{3})}{108^g\, g!\, \Gamma(g+\frac{5}{3})} \lambda^{-\frac{24g+23}{3}} - \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{2}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g-1}{3}} + \frac{1}{6} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{11}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+11}{3}}$$

$$(M_2)^2_2 = \frac{1}{3} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{8}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+8}{3}}$$

$$(M_2)^2_3 = \frac{1}{144} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{17}{3})}{108^g\, g!\, \Gamma(g+\frac{5}{3})} \lambda^{-\frac{24g+17}{3}} - \frac{1}{2} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{5}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+5}{3}}$$

$$(M_3)^3_1 = -\frac{1}{6} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{14}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+14}{3}} + \frac{1}{144} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{17}{3})}{108^g\, g!\, \Gamma(g+\frac{5}{3})} \lambda^{-\frac{24g+14}{3}}$$

$$(M_3)^3_2 = -\frac{1}{144} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{23}{3})}{108^g\, g!\, \Gamma(g+\frac{5}{3})} \lambda^{-\frac{24g+23}{3}} - \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{2}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g-1}{3}} - \frac{1}{6} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{11}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+11}{3}}$$

$$(M_3)^3_3 = -\frac{1}{6} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{8}{3})}{108^g\, g!\, \Gamma(g+\frac{2}{3})} \lambda^{-\frac{24g+8}{3}} + \frac{1}{144} \sum_{g\geq 0} \frac{(-1)^g 3^{6g} \Gamma(8g+\frac{20}{3})}{108^g\, g!\, \Gamma(g+\frac{5}{3})} \lambda^{-\frac{24g+20}{3}}.$$

B Remark on tau-functions

Let us recall a consistent gauge slice introduced by Hollowood–Miramontes (HM) [35]. It is proven in [35, 36] that for any smooth function $q(x) \in \mathfrak{b}$, there exists

$$V(x) = \sum_{k\geq 0} \frac{V_k(x)}{\lambda^k} \in L(\mathfrak{g})_{\leq 0}, \quad V_k(x) \in \mathfrak{g}$$

such that

$$e^{-\mathrm{ad}_V}\mathcal{L} = \partial_x + \Lambda, \qquad \mathcal{L} = \partial_x + \Lambda + q. \tag{B.0.1}$$

Note that the functions $V_k(x)$ in general are not differential polynomials in q [35, 53]. The HM gauge is characterized by

$$V_0 = 0, \qquad \text{i.e. } V \in L(\mathfrak{g})_{<0} = \frac{\mathfrak{g}}{\lambda} \oplus \mathcal{O}(\lambda^{-2}). \tag{B.0.2}$$

It is straightforward to derive from eq. (B.0.1) an infinite sequence of equations

$$q(x) = [V_1, E_{-\theta}],$$
$$\partial_x(V_1) + [V_1, I_+] + \frac{1}{2}[V_1, [V_1, E_{-\theta}]] = -[V_2, E_{-\theta}],$$

etc. Existence of the HM gauge has been proved by Hollowood and Miramontes in [35]. For the DS hierarchy associated to the HM gauge, T_0^1 can be identified with $-x$ [35].

So now we **assume** $V_0 = 0$ and denote $\Phi = e^V$. Let

$$\mathcal{L}^{HM} = \partial_x + \Lambda + q^{HM}$$

and let R_a^{HM} be the basic resolvents of $\mathcal{L}^{HM}$. Define

$$w = \exp(V)\exp(-\xi), \qquad \text{with } \xi := -\sum_{a=1}^{n}\sum_{k\geq 0} T_k^a\,\Lambda_{m_a+kh}.$$

Recall that w is called the wave function associated to the HM gauge,

Lemma B.0.1 ([35, 36])**.** *Denote $\Phi = e^V$. The DS hierarchy of the HM gauge can be viewed as the compatibility between the linear flows*

$$w_{T_k^a} = \left(\lambda^k\,R_a^{HM}\right)_+ w. \tag{B.0.3}$$

Definition B.0.2 (Cafasso–Wu, [13])**.** *For an arbitrary solution q^{HM} to the DS hierarchy associated to the HM gauge, the tau-function τ^{CW} of this solution is defined by*

$$\frac{\partial \log \tau^{CW}}{\partial T_k^a} = -\operatorname*{res}_{\lambda=\infty} \lambda^k \left(\Phi^{-1}(\lambda;\mathbf{T})\,\Phi_\lambda(\lambda;\mathbf{T}) \,\middle|\, \Lambda_{m_a}(\lambda)\right) \mathrm{d}\lambda, \quad a = 1,\dots,n,\ k\geq 0. \tag{B.0.4}$$

We leave as an exercise to the readers to prove the following

Proposition B.0.3. *Up to a factor of the form (1.2.6), τ coincides with τ^{CW}.*

Remark B.0.4. *Eq. (2.3.4) uniquely determines τ^{CW} of q^{HM} only up to a constant; however, the freedom (1.2.6) for τ^{CW} of q^{HM} also exists, because of the non-locality of $V(x)$.*

References

[1] Arnold, V. I. (1972). Normal forms of functions near degenerate critical points, Weyl groups A_k, D_k, E_k and Lagrange singularities. Funct. Anal. Appl., **6**, N4, 254 − 272.

[2] Arnold, V. I. (1976). Local normal forms of functions. Inventiones mathematicae, **35** (1), 87–109.

[3] Arnold, V. I., Varchenko, A. N., Gusein-Zade, S. M. (2012). Singularities of Differentiable Maps: Volume II Monodromy and Asymptotic Integrals (Vol. **83**). Springer Science Business Media.

[4] Bakalov, B., Milanov, T. (2013). $\mathcal{W}$-constraints for the total descendant potential of a simple singularity. Compositio Mathematica, **149** (05), 840–888.

[5] Balog, J., Fehér, L., O'Raifeartaigh, L., Forgacs, P., Wipf, A. (1990). Toda theory and W-algebra from a gauged WZNW point of view. Annals of Physics, **203** (1), 76–136.

[6] Balogh, F., Yang, D., Zhou, J. Explicit formula for Witten's r-spin partition function. *to appear.*

[7] Bertola, M., Dubrovin, B., Yang, D. (2016). Correlation functions of the KdV hierarchy and applications to intersection numbers over $\overline{\mathcal{M}}_{g,n}$. Physica D: Nonlinear Phenomena, **327**, 30–57.

[8] Bertola, M., Dubrovin, B., Yang, D. (2015). Simple Lie algebra and topological ODEs. Preprint arXiv: 1508.03750.

[9] Bertola, M., Yang, D. (2015). The partition function of the extended r-reduced Kadomtsev–Petviashvili hierarchy. J. Phys. A: Math. Theor., **48** (2015), 195205.

[10] Bouchard, V., Eynard, B. (2016). Reconstructing WKB from topological recursion. Preprint arXiv: 1606.04498.

[11] Buryak, A. (2016). Open Intersection Numbers and the Wave Function of the KdV Hierarchy. Moscow Mathematical Journal, **16** (1), 27–44.

[12] Cafasso, M., Wu, C.-Z. (2015). Tau functions and the limit of block Toeplitz determinants. IMRN, **2015** (20), 10339–10366.

[13] Cafasso, M., Wu, C.-Z. (2015). Borodin–Okounkov formula, string equation and topological solutions of Drinfeld-Sokolov hierarchies. Preprint arXiv: 1505.00556v2.

[14] Cartan, É. (1894). Sur la structure des groupes de transformations finis et continus (Vol. 826). Nony.

[15] De Groot, M. F., Hollowood, T. J., Miramontes, J. L. (1992). Generalized Drinfel'd–Sokolov hierarchies. Comm. Math. Phys., **145** (1), 57–84.

[16] Dickey, L. A. (1981). Integrable nonlinear equations and Liouville's theorem, I. Comm. Math. Phys., **82** (3), 345–360.

[17] Dickey, L. A. (1981). Integrable nonlinear equations and Liouville's theorem, II. Comm. Math. Phys., **82** (3), 361–375.

[18] Dickey, L. A. (2003). Soliton Equations and Hamiltonian Systems (Vol. 26). World Scientific.

[19] Drinfeld, V. G., Sokolov, V. V. (1985). Lie algebras and equations of Korteweg–de Vries type, J. Math. Sci. **30** (2), 1975–2036. Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki (Noveishie Dostizheniya) **24** (1984), 81–180.

[20] Dubrovin, B. (1996). Geometry of 2D topological field theories. In "Integrable Systems and Quantum Groups" (Montecatini Terme, 1993), Editors: Francaviglia, M., Greco, S.. Springer Lecture Notes in Math., **1620**, 120–348.

[21] Dubrovin, B. (1999). Painlevé transcendents in two-dimensional topological field theory. In The Painlevé property (pp. 287–412). Springer, New York.

[22] Dubrovin, B. (2014). Gromov–Witten invariants and integrable hierarchies of topological type. In Topology, Geometry, Integrable Systems, and Mathematical Physics: Novikov's Seminar 2012–2014, vol. **234**, American Mathematical Soc..

[23] Dubrovin, B., Liu, S.-Q., Zhang, Y. (2008). Frobenius manifolds and central invariants for the Drinfeld–Sokolov biHamiltonian structures. Advances in Mathematics, **219** (3), 780–837.

[24] Dubrovin, B., Liu, S.-Q., Yang, D., Zhang, Y. (2016). Hodge integrals and tau-symmetric integrable hierarchies of Hamiltonian evolutionary PDEs. Advances in Mathematics, **293**, 382–435.

[25] Dubrovin, B., Zhang, Y. (1998). Bihamiltonian hierarchies in 2D topological field theory at one-loop approximation. Comm. Math. Phys., **198** (2), 311–361.

[26] Dubrovin, B., Zhang, Y. (2001). Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants. Preprint arXiv: math/0108160.

[27] Faber, C., Shadrin, S., Zvonkine, D. (2010). Tautological relations and the r-spin Witten conjecture. In Annales scientifiques de l'École normale supérieure (Vol. 43, No. 4, pp. 621–658). Elsevier.

[28] Fan, H., Jarvis, T., Ruan, Y. (2013). The Witten equation, mirror symmetry, and quantum singularity theory. Annals of Mathematics, **178** (1), 1–106.

[29] Fan, H., Francis, A., Jarvis, T., Merrell, E., Ruan, Y. (2016). Witten's D_4 Integrable Hierarchies Conjecture, Chinese Annals of Mathematics, **37** (2), 175–192.

[30] Frenkel, E., Givental, A., Milanov, T. (2010). Soliton equations, vertex operators, and simple singularities. Functional analysis and other mathematics, **3** (1), 47–63.

[31] Gel'fand, I. M., Dikii, L. A. (1977). The resolvent and Hamiltonian systems. Functional Analysis and Its Applications, **11** (2), 93–105.

[32] Givental, A. (2001). Gromov–Witten invariants and quantization of quadratic Hamiltonians. Mosc. Math. J, **1** (4), 551–568.

[33] Givental, A. (2003). A_n–Singularities and nKdV Hierarchies. Moscow Mathematical Journal, **3** (2), 475–505.

[34] Givental, A., Milanov, T. (2005). Simple singularities and integrable hierarchies. In The breadth of symplectic and Poisson geometry (pp. 173–201). Birkhäuser Boston.

[35] Hollowood, T., Miramontes, J. L. (1993). Tau-functions and generalized integrable hierarchies. Comm. Math. Phys., **157** (1), 99–117.

[36] Hollowood, T. J., Miramontes, J., Guillen, J. S. (1994). Additional symmetries of generalized integrable hierarchies. Journal of Physics A: Mathematical and General, **27** (13), 4629.

[37] Jarvis, T. J., Kimura, T., Vaintrob, A. (2001). Moduli spaces of higher spin curves and integrable hierarchies. Compositio Mathematica, **126** (2), 157–212.

[38] Jimbo, M., Miwa, T., Ueno K. (1981). Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function. Physica D, **2** (2), 306–352.

[39] Kac, V. G. (1978). Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function and the very strange formula. Advances in Mathematics, **30** (2), 85–136.

[40] Kac, V. G. (1994). Infinite-dimensional Lie algebras (Vol. 44). Cambridge University Press.

[41] Kac, V. G., Wakimoto, M. (1989). Exceptional hierarchies of soliton equations. In Proceedings of symposia in pure mathematics (Vol. 49, p. 191).

[42] Kontsevich, M. (1992). Intersection theory on the moduli space of curves and the matrix Airy function. Comm. Math. Phys., **147** (1), 1–23.

[43] Kostant, B. (1959). The Principal Three-Dimensional Subgroup and the Betti Numbers of a Complex Simple Lie Group. American Journal of Mathematics, 973–1032.

[44] Liu, S.-Q., Ruan, Y., Zhang, Y. (2014). BCFG Drinfeld–Sokolov Hierarchies and FJRW-Theory. Inventiones Mathematicae. doi: 10.1007/s00222-014-0559-3.

[45] Liu, S.-Q., Yang, D., Zhang, Y. (2013). Uniqueness Theorem of $\mathcal{W}$-Constraints for Simple Singularities. Letters in Mathematical Physics, **103** (12), 1329–1345.

[46] Manin, Y. I. (1999). Frobenius manifolds, quantum cohomology, and moduli spaces (Vol. 47). AMS.

[47] Varadarajan, V. S. (2013). Lie groups, Lie algebras, and their representations (Vol. 102). Springer Science and Business Media.

[48] Polishchuk, A. (2004). Witten's top Chern class on the moduli space of higher spin curves. In Frobenius manifolds (pp. 253–264). Vieweg + Teubner Verlag.

[49] Polishchuk, A., Vaintrob, A. (2001). Algebraic construction of Witten's top Chern class. In Advances in Algebraic Geometry motivated by Physics, E. Previato, ed., 229–250. AMS.

[50] Saito, K. (1993). On a linear structure of the quotient variety by a finite reflexion group. Publications of the Research Institute for Mathematical Sciences, **29** (4): 535–579.

[51] Witten, E. (1991). Two-dimensional gravity and intersection theory on moduli space. Surveys in Diff. Geom., **1**, 243–310.

[52] Witten, E. (1993). Algebraic geometry associated with matrix models of two-dimensional gravity (pp. 235–269). Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX.

[53] Wu, C.-Z. (2012). Tau functions and Virasoro symmetries for Drinfeld-Sokolov hierarchies. Preprint arXiv: 1203.5750.

[54] Zhou, J. (2013). Solution of W-Constraints for R-Spin Intersection Numbers. Preprint arXiv: 1305.6991.

[55] Zhou, J. (2015). On absolute N-point function associated with Gelfand–Dickey polynomials. unpublished.

Marco Bertola

SISSA, via Bonomea 265, Trieste 34136, Italy

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve W., Montréal, Québec, H3G 1M8, Canada

marco.bertola@{sissa.it, concordia.ca}

Boris Dubrovin
SISSA, via Bonomea 265, Trieste 34136, Italy
dubrovin@sissa.it

Di Yang
Max-Planck-Institut für Mathematik, Vivatsgasse 7, Bonn 53111, Germany
diyang@mpim-bonn.mpq.de