SISSA

Differential Geometry

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Contents

1 Geometry of Manifolds

1.1 Definition of smooth manifolds

Spaces that locally look like Euclidean spaces are called *manifolds.* Let us give a definition of a smooth manifold.

Definition 1.1.1 *1) An* atlas *on a set M is a collection of*

- *subsets $U_\alpha \subset M$ that cover all M labeled by an at most numerable set of indices $I \ni \alpha$;*
- *for any $\alpha \in I$ a one-to-one map φ_α from U_α to an open domain in the Euclidean space $\mathbb{R}^n$ is given*

$$\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n \tag{1.1.1}$$

The pair $(U_\alpha, \varphi_\alpha)$ is called a *coordinate chart* on M. The Euclidean coordinates in $\mathbb{R}^n$

$$(x_\alpha^1, \dots, x_\alpha^n) \in \varphi_\alpha(U_\alpha) \subset \mathbb{R}^n \tag{1.1.2}$$

define coordinates on the subsets $U_\alpha \subset M$, i.e.,

$$\text{for} \quad P \in U_\alpha \quad \big(x_\alpha^1(P), \dots, x_\alpha^n(P)\big) = \varphi_\alpha(P).$$

2) For any pair of intersecting sets $U_\alpha \cap U_\beta \neq \emptyset$ the domains $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$ and the one-to-one map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \tag{1.1.3}$$

is smooth.

Since the inverse map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth as well, we conclude that the *transition maps* (1.1.3) are all diffeomorphisms.

3) A subset $V \subset M$ is called open *if its intersections with coordinate charts*

$$\varphi_\alpha(V \cap U_\alpha) \subset \mathbb{R}^n$$

are open for all $\alpha \in I$.

This definition provides a structure of *topological space* on M.

A set M equipped with an atlas of coordinate charts with smooth transition maps is called a smooth manifold of dimension n *if it is a* Hausdorff second countable *topological space.*

Recall that a topological space X is called *Hausdorff* if, for any pair of distinct points $P, Q \in X$ there exist disjoint open neighborhoods $U \ni P$, $V \ni Q$, $U \cap V = \emptyset$. It is called *second countable* if one can find a countable collection $\mathcal{B}$ of open subsets of X such that any open $U \subset X$ is a union of subsets from $\mathcal{B}$.

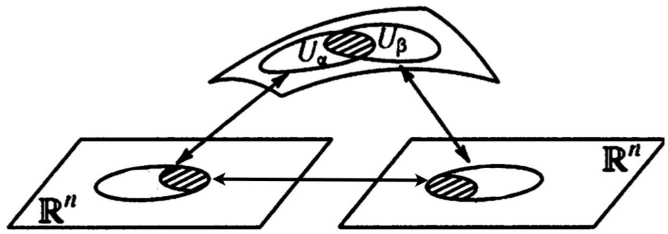

Figure 1: Transition maps on a smooth manifold

Counterexamples. To construct a "non-Hausdorff manifold" take two copies $\mathbb{R}_\pm$ of real line. Denote $x_\pm$ the standard coordinates on these lines. Identify the *negative* points x_- with x_+ on these lines. The resulting set M is covered by two coordinate charts. The points 0_+ and 0_- are distinct; their arbitrary open neighborhoods intersect. To construct a "non-second countable manifold" one can take a disjoint union of an uncountable number of copies of real line.

Example 1.1.2 *The n-dimensional Euclidean space itself, or also any open domain in it, are examples of smooth manifolds.*

Example 1.1.3 *The unit sphere $S^n \subset \mathbb{R}^{n+1}$ is an example of a n-dimensional manifold covered with two coordinate charts. The maps $\pi_\pm$ can be described as stereographic projections of the sphere from the poles $P_\pm = (0, 0, \ldots, \pm 1)$*

$$
\begin{aligned}
&\pi_+ : S^n \setminus P_+ \to \mathbb{R}^n \\
&\pi_+(x^1, \ldots, x^{n+1}) = \left(\frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}} \right) =: (x_+^1, \ldots, x_+^n) \\
&\pi_- : S^n \setminus P_- \to \mathbb{R}^n \\
&\pi_-(x^1, \ldots, x^{n+1}) = \left(\frac{x^1}{1 + x^{n+1}}, \ldots, \frac{x^n}{1 + x^{n+1}} \right) =: (x_-^1, \ldots, x_-^n)
\end{aligned}
\tag{1.1.4}
$$

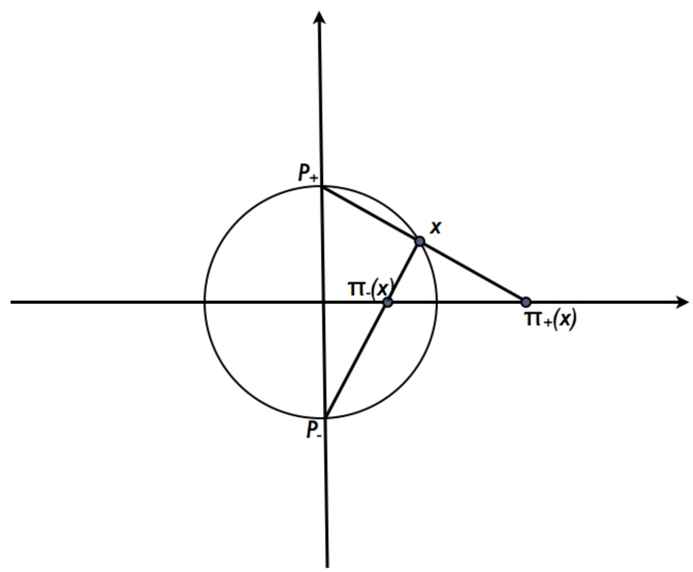

Fig. 9. Stereographic projections on the sphere

The transition maps defined for the points of intersection $S^n \setminus (P_+ \cup P_-)$ *are smooth:*

$$\pi_+ \circ \pi_-^{-1}(x_-^1, \ldots, x_-^n) = \left(\frac{x_-^1}{|x_-|^2}, \ldots, \frac{x_-^n}{|x_-|^2} \right)$$

$$|x_-|^2 = (x_-^1)^2 + \ldots (x_-^n)^2, \quad |x_-| \neq 0.$$

Example 1.1.4 *Points of the* projective space $\mathbb{R}P^n$ *are lines passing through the origin in* $\mathbb{R}^{n+1}$. *Any line can be defined by its* homogeneous coordinates

$$(x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1} \setminus 0$$

considered up to multiplication by a nonzero factor

$$(x^1, \ldots, x^n, x^{n+1}) \sim \lambda(x^1, \ldots, x^n, x^{n+1}), \quad \lambda \neq 0.$$

Denote

$$U_k = \{(x^1, \ldots, x^{n+1} \in \mathbb{R}^{n+1} \,|\, x^k \neq 0\} \subset \mathbb{R}P^n \tag{1.1.5}$$

$k = 1, \ldots, n+1$. *The subsets* $U_1, \ldots, U_{n+1}$ *cover all projective space. The coordinates* $(x_k^1, \ldots, x_k^n)$ *on* U_k *are defined as follows:*

$$\varphi_k(x^1, \ldots, x^{n+1}) = \left(\frac{x^1}{x^k}, \ldots, \frac{x^{n+1}}{x^k} \right) =: (x_k^1, \ldots, x_k^{k-1}, 1, x_k^k, \ldots, x_k^n). \tag{1.1.6}$$

Let us compute the transition maps. On the intersection $U_k \cap U_l$ one has $x^k \neq 0$, $x^l \neq 0$. Let us assume that $k < l$. Then $x_k^{l-1} = \frac{x^l}{x^k} \neq 0$ on the intersection, so

$$\begin{aligned}
x_l^i &= x_k^i \, x_k^{l-1}, \quad i < k \\
x_l^k &= \frac{1}{x_k^{l-1}} \\
x_l^i &= x_k^{i-1} \, x_k^{l-1}, \quad k < i < l \\
x_l^i &= x_k^i \, x_k^{l-1}, \quad l \leq i \leq n.
\end{aligned}$$

This is a smooth map. It is easy to see that also the inverse map is smooth on the intersection.

Example 1.1.5 *Given two manifolds M, N of the dimensions n, m respectively one obtains a natural structure of a smooth manifold of dimension $n+m$ on the Cartesian product $M \times N$. Indeed, if $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ and $(V_\beta, \psi_\beta)_{\beta \in J}$ are atlases on these two manifolds then*

$$\begin{aligned}
&(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)_{\alpha \in I,\, \beta \in J} \\
&\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^n \times \mathbb{R}^m = R^{n+m} \\
&(x, y) \mapsto (\varphi_\alpha(x), \psi_\beta(y)).
\end{aligned}$$

is an atlas on $M \times N$. For example, the Cartesian product of two circles $S^1 \times S^1$ is the two-dimensional torus T^2. Representing the circle as the segment $[0, 2\pi]$ with identified endpoints one arrives at a model of the torus by a square with identified sides

$$T^2 = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x,\, y \leq 2\pi, \quad (0, y) \sim (2\pi, y), \quad (x, 0) \sim (x, 2\pi)\}. \tag{1.1.7}$$

In a similar way the Cartesian product of n copies of circles is the n-dimensional torus T^n.

A map

$$f : M \to N \tag{1.1.8}$$

of smooth manifolds with coordinate charts $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ on M and $(V_\beta, \psi_\beta)_{\beta \in J}$ on N of dimensions n and m resp. in local coordinates can be described by m functions of n variables. Namely, given a point $P \in U_\alpha \subset M$ such that $f(P) \in V_\beta \subset N$, in a neighborhood of this point the map is represented by functions

$$\begin{aligned}
&\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \psi_\beta(V_\beta) \\
&y_\beta^1 = f_\beta^1(x_\alpha^1, \dots, x_\alpha^m), \dots, y_\beta^n = f_\beta^n(x_\alpha^1, \dots, x_\alpha^m).
\end{aligned} \tag{1.1.9}$$

Here $(y_\beta^1, \dots, y_\beta^n)$ are coordinates on $V_\beta \subset N$, the n functions f_β^1, $\dots$, f_β^n of variables $(x_\alpha^1, \dots, x_\alpha^m)$ are defined by (1.1.9).

Definition 1.1.6 *The map* (1.1.8) *of smooth manifolds is* smooth *if all its coordinate representations* (1.1.9) *are smooth functions of m variables. In particular, smooth maps $f : M \to \mathbb{R}$ are called* smooth functions *on the manifold M.*

It is easy to check correctness of the definition of smooth maps on the intersections of coordinate charts.

Example 1.1.7 *The space of $n \times n$ matrices $X = (x^i_j)_{1 \le i, j \le n}$ can be identified with a Euclidean space of dimension n^2. The subset of nondegenerate matrices*

$$GL(n) = \{X = (x^i_j) \mid \det X \neq 0\} \tag{1.1.10}$$

is an open domain in $\mathbb{R}^{n^2}$. So $GL(n)$ is a smooth manifold of the same dimension n^2. The product map

$$\begin{aligned} &GL(n) \times GL(n) \to GL(n) \\ &(X, Y) \mapsto XY \end{aligned} \tag{1.1.11}$$

is a smooth map. Indeed, using matrix entries as coordinates on $GL(n)$ we obtain a representation of the map (1.1.11) *by polynomials*

$$(XY)^i_j = \sum_{k=1}^n x^i_k y^k_j, \quad i,\, j = 1, \dots, n. \tag{1.1.12}$$

We leave as an exercise for the reader to verify that the inversion map

$$GL(n) \to GL(n), \quad X \mapsto X^{-1} \tag{1.1.13}$$

is smooth.

Observing that the set of all invertible matrices is a group we arrive at the following general definition.

Definition 1.1.8 *A smooth manifold G is called* Lie group *if a group structure is defined on G*

$$\begin{aligned} &G \times G \to G, \quad (g, h) \mapsto gh \\ &G \to G, \quad g \mapsto g^{-1} \end{aligned} \tag{1.1.14}$$

such that the maps (1.1.14) *are smooth.*

Thus the general linear group $GL(n)$ is an example of a Lie group. Even simpler examples are Euclidean spaces $\mathbb{R}^n$ considered as additive groups. These Lie groups are commutative. Also tori

$$T^n = \mathbb{R}^n / 2\pi\, \mathbb{Z}^n$$

are commutative Lie groups. Observe that these are compact manifolds. The general linear groups are not commutative for $n > 1$.

Definition 1.1.9 *A smooth one-to-one map*

$$f : M \to N$$

of two manifolds is called diffeomorphism *if the inverse map*

$$f^{-1} : N \to M$$

is smooth.Two smooth manifolds M, N are called diffeomorphic *if there exists a diffeomorphism $f : M \to N$.*

It is easy to see that two diffeomorphic manifolds must have equal dimensions. Indeed, the $m \times n$ and $n \times m$ *Jacobi matrices*

$$\left(\frac{\partial y_\beta^k}{\partial x_\alpha^i}\right) \quad \text{and} \quad \left(\frac{\partial x_\alpha^i}{\partial y_\beta^k}\right), \quad 1 \le k \le m, \quad 1 \le i \le n$$

must be mutually inverse, hence $m = n$.

Definition 1.1.10 *Let $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ and $(U'_\beta, \varphi'_\beta)_{\beta \in I'}$ be two atlases on the same space M. They define the* same smooth structure *on M if the identical map $\mathrm{id} : M \to M$ is a diffeomorphism.*

Exercise 1.1.11 *We say that an atlas $(V_\beta, \psi_\beta)_{\beta \in J}$ on M is a* refinement *of another atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ if for any $\beta \in J$ there exists $\alpha(\beta) \in I$ such that $V_\beta \subset U_{\alpha(\beta)}$ and the map $\psi_\beta : V_\beta \to \mathbb{R}^n$ is the restriction of the map $\varphi_{\alpha(\beta)} : U_{\alpha(\beta)} \to \mathbb{R}^n$ onto V_β. Prove that any refinement of an atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$ on a smooth manifold M defines the same smooth structure.*

1.2 Tangent space to a manifold

A *curve* on a manifold M is a smooth map of an interval $(a, b) \in \mathbb{R}$ to M

$$\begin{aligned} \gamma : (a, b) &\to M \\ (a, b) \ni t &\mapsto \gamma(t). \end{aligned} \tag{1.2.1}$$

In local coordinates the curve is represented by $n = \dim M$ smooth functions of one variable

$$t \mapsto \left(x^1(t), \ldots, x^n(t)\right) = x(t).$$

The *velocity vector*

$$\dot{x}(t) = \left(\dot{x}^1(t), \ldots, \dot{x}^n(t)\right) \tag{1.2.2}$$

is tangent to the curve at every point $\left(x^1(t), \ldots, x^n(t)\right)$. Here and below we will use short notations borrowed from classical mechanics

$$\dot{f}(t) = \frac{df(t)}{dt}$$

for the t-derivative of a smooth function $f(t)$. Moreover, the parameter t will sometimes be called ‘time’.

Example 1.2.1 *Choosing n arbitrary real numbers $a^1, \ldots, a^n$ one obtains a curve*

$$x^i(t) = a^i t, \quad i = 1, \ldots, n \tag{1.2.3}$$

with a prescribed velocity vector

$$\dot{x}(t) = \left(a^1, \ldots, a^n\right). \tag{1.2.4}$$

Let $P \in M$ be a point of a n-dimensional manifold M. We want to define the *tangent space* $T_P M$ consisting of all tangent vectors of curves passing through P.

Definition 1.2.2 *1) Two curves $\gamma_1(t) = \left(x_1^1(t), \ldots, x_1^n(t)\right)$ and $\gamma_2(t) = \left(x_2^1(t), \ldots, x_2^n(t)\right)$ on M passing through $P \in M$ at $t = 0$ are called* equivalent *if their velocity vectors at this point coincide*

$$\left(\dot{x}_1^1(0), \ldots, \dot{x}_1^n(0)\right) = \left(\dot{x}_2^1(0), \ldots, \dot{x}_2^n(0)\right). \tag{1.2.5}$$

2) Class of equivalence of curves passing through P is called tangent vector to the manifold *at the point P.*

3) The set of all tangent vectors at the point $P \in M$ is called the tangent space $T_P M$ *to the manifold at this point.*

We will show now that the tangent space $T_P M$ at any point P of an n-dimensional manifold M is isomorphic to the Euclidean space $\mathbb{R}^n$. To this end we first prove

Lemma 1.2.3 *The equivalence relation between curves passing through a given point of the manifold does not depend on the choice of local coordinates.*

Proof: After a change of local coordinates

$$x^{i'} = x^{i'}(x^1, \ldots, x^n), \quad i = 1, \ldots, n \tag{1.2.6}$$

the curve $\left(x^1(t), \ldots, x^n(t)\right)$ will be represented by n smooth functions

$$x^{i'}(t) = x^{i'}\left(x^1(t), \ldots, x^n(t)\right), \quad i' = 1, \ldots, n.$$

The velocity vector of this curve in new coordinates can be computed by applying the chain rule

$$\dot{x}^{i'}(t) = \frac{\partial x^{i'}}{\partial x^i}\dot{x}^i(t), \quad i' = 1, \ldots, n \tag{1.2.7}$$

(warning: in this formula summation in the repeated index i but not in i'. The indices i and i' are independent.). Thus, for a given pair of two curves $x_1^i(t)$ and $x_2^i(t)$ with coinciding velocity vectors $\dot{x}_1^i(0) = \dot{x}_2^i(0)$, $i = 1, \ldots, n$ at the point $P = \left(x_1^1(0), \ldots, x_1^n(0)\right) = \left(x_2^1(0), \ldots, x_2^n(0)\right)$ their velocity vectors in new coordinates will also coincide,

$$\dot{x}_1^{i'}(0) = \frac{\partial x^{i'}(P)}{\partial x^i}\dot{x}_1^i(0) = \frac{\partial x^{i'}(P)}{\partial x^i}\dot{x}_2^i(0) = \dot{x}_2^{i'}(0), \quad i = 1, \ldots, n.$$

□

Using this Lemma, and also in view of Example 1.2.1 one arrives at

Corollary 1.2.4 *Any system of local coordinates on a neighborhood of a point P on a n-dimensional manifold M establishes an isomorphism*

$$T_P M \simeq \mathbb{R}^n.$$

Proof: Indeed, in local coordinates near P any tangent vector at the point P is defined by n numbers $\left(\dot{x}^1(0), \dots, \dot{x}^n(0)\right)$ that can take arbitrary values. $\square$

The transformation rule (1.2.7) can be used for an alternative definition of tangent vectors.

Definition 1.2.5 *A tangent vector at the point P of an n-dimensional manifold M is a correspondence that associates an n-tuple of real numbers $(v_\alpha^1, \dots, v_\alpha^n)$ with any coordinate chart $U_\alpha \subset M$ containing P. In another coordinate chart $U_\beta \subset M$ containing P the same vector is described by another n-tuple $(v_\beta^1, \dots, v_\beta^n)$. It is required that the two n-tuples are related by the transformation law*

$$v_\beta^i = \frac{\partial x_\beta^i(P)}{\partial x_\alpha^j} v_\alpha^j, \quad i = 1, \dots, n. \tag{1.2.8}$$

Using matrix notations one can rewrite the transformation rule (1.2.8) as the result of multiplication by the Jacobi matrix

$$\begin{pmatrix} v_\beta^1 \\ \vdots \\ v_\beta^n \end{pmatrix} = \begin{pmatrix} \partial x_\beta^1/\partial x_\alpha^1 & \dots & \partial x_\beta^1/\partial x_\alpha^n \\ \vdots & \dots & \vdots \\ \partial x_\beta^n/\partial x_\alpha^1 & \dots & \partial x_\beta^n/\partial x_\alpha^n \end{pmatrix}_P \begin{pmatrix} v_\alpha^1 \\ \vdots \\ v_\alpha^n \end{pmatrix}. \tag{1.2.9}$$

Recall that the Jacobi matrix of the transition functions must not degenerate at the point $P \in U_\alpha \cap U_\beta$

$$\det \left(\frac{\partial x_\beta^i(P)}{\partial x_\alpha^j} \right) \neq 0. \tag{1.2.10}$$

Example 1.2.6 *For $M = \mathbb{R}^n$ the tangent space can be naturally identified with the space $\mathbb{R}^n$ itself. Same for manifolds realized as open domains in $\mathbb{R}^n$.*

Given a manifold M one can construct the set of all tangent vectors

$$TM = \{(x, v) \,|\, x \in M, \ v \in T_x M\}. \tag{1.2.11}$$

Exercise 1.2.7 *Introduce on TM a structure of $2n$-dimensional smooth manifold, where $n = \dim M$.*

The manifold TM is called the *total space of tangent bundle* on M.

Exercise 1.2.8 *Prove that the total space of tangent bundle to the circle $M = S^1$ is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.*

Let $f : M \to N$ be a smooth map of manifolds of dimensions n and m respectively. It maps smooth curves $x(t)$ passing through a point $P \in M$ to smooth curves $f(x(t))$ passing through the point $f(P) \in N$.

Definition 1.2.9 *The* induced map *of tangent spaces*

$$f_* : T_P M \to T_{f(P)} N \tag{1.2.12}$$

is defined by

$$T_P M \ni \dot{x}(0) \mapsto f_*(\dot{x}(0)) := \frac{d}{dt} f(x(t))_{t=0} \in T_{f(P)} N. \tag{1.2.13}$$

Lemma 1.2.10 *Let $(U, (x^1, \dots, x^n))$ be a coordinate chart near the point $P \in M$ and $(V, (y^1, \dots, y^m))$ be a coordinate chart near the point $f(P) \in N$. Let the smooth map in the local coordinates have the form*

$$x = (x^1, \dots, x^n) \mapsto f(x) = (y^1(x), \dots, y^m(x)). \tag{1.2.14}$$

In these coordinates the induced map $f_* : T_P M \to T_{f(P)} N$ is a linear map defined by the $m \times n$ Jacobi matrix

$$v = (v^1, \dots, v^n) \mapsto f_*(v) = \left(\frac{\partial y^i(P)}{\partial x^j} v^j \right) \tag{1.2.15}$$

or, in an equivalent matrix form

$$\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \mapsto f_*(v) = \begin{pmatrix} \partial y^1/\partial x^1 & \dots & \partial y^1/\partial x^n \\ \vdots & \dots & \vdots \\ \partial y^m/\partial x^1 & \dots & \partial y^m/\partial x^n \end{pmatrix}_P \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}. \tag{1.2.16}$$

Proof: Applying the chain rule to the computation of the velocity vector of the curve $f(x(t))$ one obtains

$$\frac{d}{dt} y^i(x(t)) = \frac{\partial y^i}{\partial x^j} \frac{dx^j(t)}{dt}, \quad i = 1, \dots, n.$$

□

Example 1.2.11 *For a smooth function*

$$f : M \to \mathbb{R}$$

on a manifold M the induced map is a linear function on the tangent space at any point

$$\begin{aligned} & f_* : T_P M \to \mathbb{R} \\ & v = (v^1, \dots, v^n) \mapsto f_*(v) = \frac{\partial f}{\partial x^1} v^1 + \dots + \frac{\partial f}{\partial x^n} v^n. \end{aligned} \tag{1.2.17}$$

This linear map coincides with the differential of the function f

$$f_* = df(x)$$

i.e., with the principal linear part of the increment of the function in the direction of the vector:

$$f(x + t\, v) - f(x) = t f_*(v) + \mathcal{O}(t^2). \tag{1.2.18}$$

Also in general the induced map of linear spaces is often called the *differential* of the map $f : M \to N$

$$df(x) : T_x M \to T_{f(x)} N. \tag{1.2.19}$$

We will now define a dual object: the so-called *cotangent space* $T_P^* M$ to a n-dimensional manifold at a given point $P \in M$. Elements of this space are called *covectors*; in local coordinates they are described by values of gradients of smooth functions at this point:

$$\left(\frac{\partial f(P)}{\partial x^1}, \dots, \frac{\partial f(P)}{\partial x^n} \right). \tag{1.2.20}$$

Similarly to Definition 1.2.2 one can give

Definition 1.2.12 *Two smooth functions f, g on M are called* equivalent *at the point $P \in M$ if their differentials coincide at the point P. A class of equivalence of smooth function at a point $P \in M$ is called a* covector *at this point. The set $T_P^* M$ of all covectors at a given point $P \in M$ is called the* cotangent space *at this point.*

In local coordinates covectors can be described by n-tuples of real numbers $(\omega_1, \dots, \omega_n)$. However, the transformation law of covectors is different from the one for vectors: namely, if $(\omega_1^\alpha, \dots, \omega_n^\alpha)$ and $(\omega_1^\beta, \dots, \omega_n^\beta)$ are components of the same covector at a point $P \in U_\alpha \cap U_\beta$ in two coordinate charts U_α and U_β respectively then

$$\omega_i^\alpha = \frac{\partial x_\beta^j(P)}{\partial x_\alpha^i} \omega_j^\beta, \quad i = 1, \dots, n. \tag{1.2.21}$$

Indeed, this formula can be easily derived by applying the chain rule to the partial derivatives of a function f

$$\omega_i^\alpha = \frac{\partial f(P)}{\partial x_\alpha^i}, \quad \omega_i^\beta = \frac{\partial f(P)}{\partial x_\beta^i}. \tag{1.2.22}$$

One can actually *define* covectors at a given point with n-tuples of real numbers for any coordinate chart near this point; these n-tuples must transform according to the rule (1.2.21) when passing from one coordinate chart to another one. Observe that the transformation rule (1.2.22) for covectors is different from the one (1.2.8) for vectors. In order to make the comparison more clear let us rewrite (1.2.22) in matrix form

$$(\omega_1^\alpha, \dots, \omega_n^\alpha) = \left(\omega_1^\beta, \dots, \omega_n^\beta \right) \left(\begin{array}{ccc} \partial x_\beta^1 / \partial x_\alpha^1 & \dots & \partial x_\beta^1 / \partial x_\alpha^n \\ \vdots & \dots & \vdots \\ \partial x_\beta^n / \partial x_\alpha^1 & \dots & \partial x_\beta^n / \partial x_\alpha^n \end{array} \right)_P . \tag{1.2.23}$$

That is, change of components of a vector from a chart U_α to U_β is obtained by multiplication by the Jacobi matrix

$$J = \left(\frac{\partial x_\beta^i}{\partial x_\alpha^j} \right)_P$$

while a similar change of components of a covector is given by multiplication by $\left(J^{-1} \right)^T$.

Like above one can prove that the cotangent space $T_P^* M$ on a n-dimensional manifold M is a n-dimensional linear space. The differentials dx_α^1, …, dx_α^n of local coordinate functions x_α^1, …, x_α^n define a basis in the cotangent space $T_P^* M$ at every point P inside the chart U_α.

Lemma 1.2.13 *There is a natural duality between tangent and cotangent spaces at the same point via the following nondegenerate pairing*

$$T^*_P M \times T_P M \to \mathbb{R} \tag{1.2.24}$$

$$(\omega, v) \mapsto \omega_i v^i = (\omega_1, \ldots, \omega_n) \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix}.$$

Proof: Let $x(t)$ be a smooth curve such that $x(0) = P$, $\dot{x}(0) = v$; let $f(x)$ be a smooth function such that $df(P) = \omega$. Then

$$\frac{d}{dt} f(x(t))_{t=0} = \left(\frac{\partial f(x(t))}{\partial x^i} \dot{x}^i(t) \right)_{t=0} = \omega_i v^i.$$

Since the left hand side of this equation does not depend on the choice of representatives of the vector v and covector ω, the right hand side is well defined and, in particular, it does not depend on the choice of local coordinates. □

Exercise 1.2.14 *Prove directly by using the transformation laws* (1.2.8) *and* (1.2.22) *that the sum $\omega_i v^i$ does not depend on the choice of a coordinate chart.*

Because of Lemma 1.2.13 the cotangent space $T^*_P M$ can be naturally identified with the dual space to the tangent one

$$T^*_P M = Hom(T_P M, \mathbb{R}).$$

In the same way one can identify

$$T_P M = Hom(T^*_P M, \mathbb{R}).$$

Any smooth map of manifolds $f : M \to M$ defines a *pullback* of cotangent spaces

$$f^* : T^*_{f(P)} N \to T^*_P M. \tag{1.2.25}$$

By definition the value of the pullback of a covector $\omega \in T^*_{f(P)} N$ on a vector $v \in T_P M$ is equal to the value of ω on the vector $f_*(v)$

$$(f^*(\omega), v) = (\omega, f_*(v)). \tag{1.2.26}$$

In local coordinates the pullback is written via the same Jacobi matrix by multiplication of row-vectors

$$\omega = (\omega_1, \ldots, m) \mapsto f^*(\omega) = \left(\frac{\partial y^i(P)}{\partial x^j} \omega_i \right) \tag{1.2.27}$$

or, equivalently

$$\omega = (\omega_1, \ldots, \omega_m) \mapsto f^*(\omega) = (\omega_1, \ldots, \omega_m) \begin{pmatrix} \partial y^1/\partial x^1 & \ldots & \partial y^1/\partial x^n \\ \vdots & \ldots & \vdots \\ \partial y^m/\partial x^1 & \ldots & \partial y^m/\partial x^n \end{pmatrix}_P . \tag{1.2.28}$$

Like the above construction of the manifold TM of tangent vectors to M we define the *total space of cotangent bundle* to M by

$$T^*M = \{(x, \omega) \,|\, x \in M,\ \omega \in T^*M_x\}. \tag{1.2.29}$$

Exercise 1.2.15 *Introduce on T^*M a structure of $2n$-dimensional smooth manifold, where $n = \dim M$.*

1.3 Vector fields

So far all vectors and covectors were attached to a given point of a manifold. Now we consider vector and covector *fields*.

Definition 1.3.1 *A* smooth vector field *on a manifold M is a vector $v(x) \in T_xM$ at any point $x \in M$ depending smoothly on the point x.*

Smooth dependence on the point means that, in a coordinate chart the components $v^1(x)$, $\dots$, $v^n(x)$ are smooth functions of local coordinates. We leave it as an exercise for the reader to verify independence of this definition from the choice of local coordinates.

With a smooth vector field $v(x) = \big(v^1(x), \dots, v^n(x)\big)$ on a manifold M one can associate a *dynamical system* on M represented by a system of $n = \dim M$ autonomous ordinary differential equations (ODEs)

$$\left.\begin{array}{ccc} \dot{x}^1 & = & v^1(x) \\ \dot{x}^2 & = & v^2(x) \\ \dots & \dots & \dots \\ \dot{x}^n & = & v^n(x) \end{array}\right\} \tag{1.3.1}$$

Here and below we will often use the notation for the time derivative $\dot{x} = \frac{dx}{dt}$ borrowed from classical mechanics. In this way the dynamical system will read

$$\dot{x} = v(x).$$

A *solution* $x(t) = \big(x^1(t), \dots, x^n(t)\big)$, $t \in (a, b) \subset \mathbb{R}$ to the dynamical system (1.3.1) is a collection of smooth functions $x^1(t)$, $\dots$, $x^n(t)$ satisfying

$$\frac{dx^i(t)}{dt} = v^i\left(x^1(t), \dots, x^n(t)\right), \quad i = 1, \dots, n.$$

It defines an *integral curve* of the vector field v, i.e., a smooth map

$$\gamma : (a, b) \ni t \mapsto \big(x^1(t), \dots, x^n(t)\big) = \gamma(t) \in M \tag{1.3.2}$$

such that the velocity vector of the curve coincides with the values of the vector field at the points of the curve.

Remark 1.3.2 *A time-dependent system of ODEs*

$$\frac{dx}{dt} = v(t, x), \quad x \in M \tag{1.3.3}$$

can be interpreted as a dynamical system on $\mathbb{R} \times M \ni (t, x)$

$$\left.\begin{array}{ccc} \frac{dt}{d\tau} & = & 1 \\ & & \\ \frac{dx}{d\tau} & = & v(t, x). \end{array}\right\} \tag{1.3.4}$$

According to the theory of ordinary differential equations for any point $x_0 \in M$ there exists an integral curve $x(t)$ defined for sufficiently small $|t|$ passing through this point:

$$x(t = 0) = x_0.$$

The curve is uniquely determined by the initial condition. Another useful result is the *rectification theorem*. It says that, near a point $x_0 \in M$ such that $v(x_0) \neq 0$ there exists a system of local coordinates $(x^1, \ldots, x^n)$ such that the vector field in these coordinates has the following form

$$v(x) = (1, 0, \ldots, 0). \tag{1.3.5}$$

In these coordinates the integral curves of the vector field are obtained by translations along the first coordinate

$$x(t) = \left(x_0^1 + t, x_0^2, \ldots, x_0^n\right).$$

For vector fields on *compact* manifolds the following important statement holds true.

Theorem 1.3.3 *Any integral curve $x(t)$ of a smooth vector field defined on a smooth compact manifold can be extended to all values of the parameter $t \in \mathbb{R}$.*

Using the theorem about smooth dependence of solutions on the initial data one can easily prove that the map

$$\begin{aligned} &g_t : M \to M \\ &g_t(x_0) = x(t) \quad \text{where} \quad \dot{x} = v(x), \quad x(0) = x_0 \end{aligned} \tag{1.3.6}$$

for every $t \in \mathbb{R}$ is a diffeomorphism. In the particular case $t = 0$ the diffeomorphism (1.3.6) is the identity map.

Exercise 1.3.4 *Prove that the diffeomorphisms g_t generated by an arbitrary smooth vector field on a compact manifold M form a* one-parameter group*, i.e.,*

$$\begin{aligned} &g_s \circ g_t = g_{s+t} \quad \forall\, s,\, t \in \mathbb{R} \\ &g_0 = \mathrm{id} \\ &g_t^{-1} = g_{-t}. \end{aligned} \tag{1.3.7}$$

Example 1.3.5 *For a* linear vector field *on the n-dimensional Euclidean space*

$$v(x) = Ax \tag{1.3.8}$$

defined by a constant $n \times n$ matrix A the solution to the associated system of linear differential equations

$$\dot{x} = Ax \tag{1.3.9}$$

with the initial datum

$$x(0) = x_0$$

can be expressed via matrix exponential function

$$x(t) = e^{t\,A} x_0 \tag{1.3.10}$$
$$e^{t\,A} = \mathrm{id} + \frac{tA}{1!} + \frac{t^2 A^2}{2!} + \dots .$$

In this particular case the equations (1.3.9) *from the definition of a one-parameter group of diffeomorphisms follow from the well known property of the matrix exponential function*

$$e^{A+B} = e^A e^B \quad \textit{if the matrices commute,} \quad BA = AB. \tag{1.3.11}$$

In a more general case of a smooth vector field $v(x)$ on a non-compact manifold M defines a one-parameter group g_t of *local diffeomorphisms*. That means that, for any point $x_0 \in M$ there exists an open neighborhood $U \ni x_0$ and a number $\epsilon > 0$ such that the integral curve $x(t)$ of the vector field with the initial data $x(0) = x_0$ exists on U for $|t| < \epsilon$. The map $g_t : U \to U$ on any sufficiently small open subset $U \subset M$ is defined for sufficiently small $|t|$ in the same way as in (1.3.6). It satisfies (1.3.7) if $|t| < \epsilon$, $|s| < \epsilon$, $|s+t| < \epsilon$ where ϵ is as above.

Exercise 1.3.6 *Prove that the matrix $e^{t\,A}$ is orthogonal if A is an antisymmetric matrix. Derive that the linear vector field* (1.3.8) *is tangent to the spheres $|x|^2 = R^2$ if the matrix A is antisymmetric.*

To a vector field $v(x)$ one can associate a *differential operator* on smooth functions

$$v : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad f \mapsto v\,f$$
$$v\,f(x) = v^i(x) \frac{\partial f(x)}{\partial x^i}. \tag{1.3.12}$$

Due to the formula

$$f(x + t\,v(x)) - f(x) = t\,v\,f(x) + \mathcal{O}(t^2) \tag{1.3.13}$$

the operator (1.3.12) coincides with the derivative of the function f along the vector $v(x)$.

Theorem 1.3.7 *For any smooth vector field $v(x)$ the operator* (1.3.12) *posseses the following properties:*

$$\text{(1.3.14)}$$

- linearity $\quad v\,(\alpha f + \beta g) = \alpha\, v f + \beta\, v g, \quad f,\, g \in \mathcal{C}^\infty(M), \quad \alpha,\, \beta \in \mathbb{R}$
- Leibnitz identity $\quad v\,(fg) = (v f)\, g + f\, v g.$

Conversely, any operator satisfying these two properties coincides with the derivative along a smooth vector field.

Proof: The first part of the theorem follows from an easy computation. Let us now prove the converse statement. We will begin with the case $M = \mathbb{R}^n$. Let $f \mapsto Af$ be a linear operator on the space $\mathcal{C}^\infty(\mathbb{R}^n)$ satisfying Leibnitz identity. Define functions

$$v^i(x) := Ax^i, \quad i = 1, \dots, n \tag{1.3.15}$$

and consider the linear differential operator

$$\tilde{A} = v^i(x)\frac{\partial}{\partial x^i}.$$

By construction

$$\tilde{A} f = A f \tag{1.3.16}$$

for any linear function

$$f(x) = a_i x^i + b.$$

Applying Leibnitz identity one proves (1.3.16) for any polynomial function $f(x)$. Since any smooth function can be approximated by polynomials the equallity (1.3.16) holds true for any smooth function f.

Example 1.3.8 *Given a system of local coordinates* $(x^1, \dots, x^n)$ *on an open domain* $U \subset M$*, one defines* $n = \dim M$ *smooth vector fields on* U

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \tag{1.3.17}$$

given by unit tangent vectors of the coordinate lines. Clearly these vector fields form a basis in $T_x M$ *at every point* $x \in U$*. Any vector field* $v(x)$ *is represented as a linear combination of the basic vector fields*

$$v(x) = v^1(x)\frac{\partial}{\partial x^1} + \dots + v^n(x)\frac{\partial}{\partial x^n}. \tag{1.3.18}$$

The basis does depend on the choice of local coordinates.

Exercise 1.3.9 *A function* $f \in \mathcal{C}^\infty(M)$ *is called* first integral *of a vector field* v *if*

$$v f \equiv 0. \tag{1.3.19}$$

Prove that any first integral takes constant values on integral curves of the vector field. Prove that the vector field v *is tangent to the level surface of any first integral of this vector field.*

The identification between vector fields and first order linear differential operators allows us to introduce an important operation of *Lie bracket* of two vector fields. The definition is based on the following

Lemma 1.3.10 *The commutator* $[A, B] := A\,B - B\,A$ *of two first order linear differential operators*

$$A = v^i(x)\frac{\partial}{\partial x^i}, \quad B = w^j(x)\frac{\partial}{\partial x^j}$$

is again a first order linear differential operator given by the formula

$$[A, B] = \left(v^i(x)\frac{\partial w^k(x)}{\partial x^i} - w^i(x)\frac{\partial v^k(x)}{\partial x^i} \right)\frac{\partial}{\partial x^k}. \tag{1.3.20}$$

Proof: For an arbitrary smooth function $f = f(x)$ one has

$$A\,B\,f = v^i(x)\frac{\partial}{\partial x^i}\left(w^j(x)\frac{\partial f(x)}{\partial x^j}\right) = v^i(x)\frac{\partial w^j(x)}{\partial x^i}\frac{\partial f(x)}{\partial x^j} + v^i(x)w^j(x)\frac{\partial^2 f(x)}{\partial x^i \partial x^j}.$$

Since the second derivative $\partial^2 f/\partial x^i \partial x^j$ of a smooth function f is symmetric in i, j, one has

$$v^i(x)w^j(x)\frac{\partial^2 f(x)}{\partial x^i \partial x^j} = w^i(x)v^j(x)\frac{\partial^2 f(x)}{\partial x^i \partial x^j}.$$

Thus

$$[A, B]\,f = \left(v^i(x)\frac{\partial w^j(x)}{\partial x^i} - w^i(x)\frac{\partial v^j(x)}{\partial x^i}\right)\frac{\partial f(x)}{\partial x^j}.$$

□

Definition 1.3.11 *The* Lie bracket[1] *of two vector fields v and w is the vector field $[v, w]$ with the components*

$$[v, w]^k = v^i(x)\frac{\partial w^k(x)}{\partial x^i} - w^i(x)\frac{\partial v^k(x)}{\partial x^i}, \quad k = 1, \dots, n. \tag{1.3.21}$$

Independence of the above definition from the choice of local coordinates easily follows from Theorem 1.3.7 and Lemma 1.3.10.

Example 1.3.12 *The basic vector fields* (1.3.17) *commute pairwise*

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0, \quad i,\, j = 1, \dots, n.$$

Example 1.3.13 *The commutator of two linear vector fields*

$$v(x) = Ax, \quad w(x) = Bx$$

on a Euclidean space $\mathbb{R}^n$, where A, $B \in Mat(n, \mathbb{R})$ is again a linear vector field

$$[v, w](x) = -[A, B]x. \tag{1.3.22}$$

Here

$$[A, B] = AB - BA$$

is the matrix commutator.

The commutator of vector fields is a bilinear antisymmetric operation

$$\begin{aligned} &[\alpha u + \beta v, w] = \alpha[u, w] + \beta[v, w], \quad [u, \alpha v + \beta w] = \alpha[u, v] + \beta[u, w] \\ &[v, u] = -[u, v] \\ &u,\, v,\, w \in Vect(M), \quad \alpha,\, \beta \in \mathbb{R}. \end{aligned} \tag{1.3.23}$$

[1] Also often called *commutator*

Lemma 1.3.14 *For any three vector fields u, v, w the* Jacobi identity *holds true*

$$[[u,v],w]+[[w,u],v]+[[v,w],u]=0. \tag{1.3.24}$$

Proof: By definition the action of the double commutator on a smooth function f is equal to

$$[[u,v],w]f=[u,v](wf)-w([u,v]f)=u(v(wf))-v(u(wf))-w(u(vf))+w(v(uf)).$$

Adding to this expression two more terms

$$w(u(vf))-u(w(vf))-v(w(uf))+v(u(wf)) \quad \text{and} \quad v(w(uf))-w(v(uf))-u(v(wf))+u(w(vf))$$

obtained by cyclic permutations of u, v, w one arrives at the proof of the Jacobi identity.
□

Definition 1.3.15 *A linear space equipped with an antisymmetric bilinear operation satisfying Jacobi identity* (1.3.24) *is called* Lie algebra.

We obtain a structure of Lie algebra on the space of smooth vector fields $Vect(M)$.

Exercise 1.3.16 *Let M be a submanifold in a manifold N. Prove that vector fields on N tangent to M form a Lie subalgebra in $Vect(N)$.*

Exercise 1.3.17 *Prove that linear vector fields $v_A(x)=Ax$ (see* (1.3.8)*) form a Lie subalgebra in $Vect(\mathbb{R}^n)$. Prove that the map*

$$v_A \mapsto -A$$

establishes an isomorphism of this Lie subalgebra with the Lie algebra of matrices with respect to the matrix commutator $[A,B]=AB-BA$.

We will now show that pairwise commuting vector fields on a manifold define an action of an abelian group.

Lemma 1.3.18 *Given two vector fields v, w on a manifold M, consider two systems of ODEs*

$$\frac{dx}{dt}=v(x), \quad \frac{dx}{ds}=w(x). \tag{1.3.25}$$

The common solution $x(t,s)$ to these two systems with an arbitrary initial data $x(0,0)=x_0\in M$ exists for sufficiently small $|t|$, $|s|$ iff *the vector fields commute,*

$$[v,w]=0.$$

Proof: By definition the common solution must satisfy

$$\frac{\partial x(t,s)}{\partial t} = v\left(x(t,s)\right), \quad \frac{\partial x(t,s)}{\partial s} = w\left(x(t,s)\right).$$

Computing the mixed derivative in two different ways one obtains

$$\frac{\partial^2 x^k(t,s)}{\partial t\, \partial s} = \frac{\partial}{\partial t}\frac{\partial x^k(t,s)}{\partial s} = \frac{\partial v^k(x(t,s))}{\partial t} = \frac{\partial v^k(x(t,s))}{\partial x^i}\frac{\partial x^i(t,s)}{\partial t} = \frac{\partial v^k(x(t,s))}{\partial x^i} w^i(x(t,s))$$

and

$$\frac{\partial^2 x^k(t,s)}{\partial s\, \partial t} = \frac{\partial}{\partial s}\frac{\partial x^k(t,s)}{\partial t} = \cdots = \frac{\partial w^k(x(t,s))}{\partial x^i} v^i(x(t,s)).$$

Due to the symmetry of mixed derivatives in $t \leftrightarrow s$ one has

$$\frac{\partial v^k(x(t,s))}{\partial x^i} w^i(x(t,s)) - \frac{\partial w^k(x(t,s))}{\partial x^i} v^i(x(t,s)) = 0.$$

Setting $t = s = 0$ one concludes that

$$[v,w]|_{x_0} = 0.$$

Since the point $x_0 \in M$ is arbitrary it follows vanishing of the Lie bracket $[v,w]$.

Let us prove the converse statement. First, if the initial point x_0 is a stationary point for both of the vector field, i.e., $v(x_0) = w(x_0) = 0$ then the common solution to (1.3.25) has the form $x(t,s) \equiv x_0$. Consider now the case where, say, the vector field w does not vanish at x_0. In that case one can do a local change of coordinates on a neighborhood of x_0 such that the vector field w becomes a shift along one of coordinates. Let us use the same notations for the new system of coordinates, such that

$$w(x) = \frac{\partial}{\partial x^1}.$$

Vanishing of the Lie bracket $[v,w] = 0$ then implies that the vector field v does not depend on x^1

$$v = v\left(x^2, \dots, x^n\right).$$

For sufficiently small $|t|$ denote $\bar{x}(t)$ the solution to the system $d\bar{x}/dt = v(\bar{x})$ with the initial data $\bar{x}(0) = x_0$. Define vector valued function $x(t,s) = \left(x^1(t,s), \dots, x^n(t,s)\right)$ by the formula

$$x^1(t,s) = \bar{x}^1(t) + s, \quad x^i(t,s) = \bar{x}^i(t) \quad \text{for} \quad i \geq 2.$$

The function satisfies the first equation

$$\frac{\partial}{\partial t} x(t,s) = \frac{\partial}{\partial t}\bar{x}(t) = v\left(\bar{x}(t)\right) = v\left(x(t,s)\right)$$

as the right hand side does not depend on the first coordinate. It does obviously satisfy also the second equation

$$\frac{\partial x^i(t,s)}{\partial s} = \delta_1^i = w^i.$$

It satisfies the initial condition $x(0,0) = x_0$. $\square$

Exercise 1.3.19 *Let v, w be two vector fields on a manifold M. Denote*

$$g_t : M \to M, \quad h_s : M \to M$$

the one-parameter group of diffeomorphisms generated by these vector fields. Prove that the diffeomorphisms g_t and h_s commute for all sufficiently small t, $s \in \mathbb{R}$ iff *the vector fields commute:*

$$g_t \circ h_s = h_s \circ g_t \Leftrightarrow [v, w] = 0.$$

Exercise 1.3.20 *Prove the following version of Lemma 1.3.18 for systems of non-autonomous ODEs: two systems of the form*

$$\frac{\partial x}{\partial t} = v(t, s, x), \quad \frac{\partial x}{\partial s} = w(t, s, x), \quad x \in M \tag{1.3.26}$$

admit, for sufficiently small $|t|$, $|s|$ a (unique) common solution $x = x(t, s)$ with an arbitrary initial data $x(0, 0) = x_0 \in M$ iff *the (t, s)-dependent vector fields v, w satisfy*

$$\frac{\partial v}{\partial s} - \frac{\partial w}{\partial t} = [v, w]. \tag{1.3.27}$$

Exercise 1.3.21 *For the particular case of a pair of systems of linear ODEs*

$$\frac{\partial x}{\partial t} = A\, x, \quad \frac{\partial x}{\partial s} = B\, x \tag{1.3.28}$$

$$A = A(t, s), \quad B = B(t, s) \quad \text{are smooth functions with values in} \quad GL(n, \mathbb{R})$$

the conditions of compatibility read

$$\frac{\partial A}{\partial s} - \frac{\partial B}{\partial t} + [A, B] = 0 \tag{1.3.29}$$

(the so-called zero curvature equations*).*

Let us now consider the covector fields on smooth manifolds, i.e., a covector $\omega(x) \in T_x^* M$ defined at any point $x \in M$ smoothly depending on the point. At the points of any coordinate chart $(U, (x^1, \dots, x^n))$ on M one has n covectors

$$dx^1,\, dx^2, \dots, dx^n \in T_x^* M, \quad x \in U \subset M \tag{1.3.30}$$

defined as differentials of the coordinate functions. The values of these covectors on the basic vectors can be easily computed from the definition of differential:

$$\left(dx^i, \frac{\partial}{\partial x^j} \right) = \delta_j^i. \tag{1.3.31}$$

So, at every point $x \in U$ the covectors dx^1, $dx^2, \dots, dx^n$ define a basis in $T_x^* M$ dual to the basis

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^n}$$

in the tangent space $T_x M$. If $(\omega_1(x), \dots, \omega_n(x))$ are the components of the covector $\omega(x)$ in the coordinates $(x^1, \dots, x^n)$ then the decomposition of the covector with respect to the basis (1.3.30) reads

$$\omega = \omega_1(x) dx^1 + \omega_2(x) dx^2 + \dots + \omega_n(x) dx^n \equiv \omega_i(x) dx^i. \tag{1.3.32}$$

Such expressions are called *differential 1-forms.*

Example 1.3.22 *A 1-form on the line is an expression $\omega = f(x)dx$. It can be integrated over any segment of the line*

$$\int_a^b \omega = \int_a^b f(x)dx. \tag{1.3.33}$$

More generally for any smooth curve in the manifold M

$$x = x(t), \quad a \le t \le b$$

1.4 Smooth functions on manifolds, partitions of unity.

One of the main structures associated with a smooth manifold M is the space $\mathcal{C}^\infty(M)$ of smooth functions on M. This is a linear space with respect to obvious operations of sum of functions and multiplication of functions by real constants. Moreover, it is an *algebra*, i.e., the product of functions satisfies the properties

$$\begin{aligned} &f(gh) = (fg)h \quad \forall\, f,\, g,\, h \in \mathcal{C}^\infty(M) \quad \text{(associativity)} \\ &(\alpha\, f + \beta\, g)h = \alpha\, fh + \beta\, gh, \quad \forall\, f,\, g,\, h \in \mathcal{C}^\infty(M), \quad \forall\, \alpha,\, \beta \in \mathbb{R}. \end{aligned}$$

Clearly this algebra is commutative

$$gf = fg$$

and has a unity $f \equiv 1$.

Example 1.4.1 *For $M = \mathbb{R}^n$ the algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ coincides with the algebra of smooth functions of n variables. For the case $M = D \subset \mathbb{R}^n$ of an open domain in Euclidean space the space $\mathcal{C}^\infty(\mathbb{R}^n)$ coincides with the algebra of smooth functions of n variables defined on D.*

Example 1.4.2 *The space of smooth 2π-periodic functions*

$$f(x + 2\pi) = f(x)$$

can be identified with functions on the circle $\mathcal{C}^\infty(S^1)$. In a similar way smooth functions on the n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ (n factors) can be realized by smooth functions of n variables 2π-periodic in each variable

$$f(x_1 + 2\pi m_1, x_2 + 2\pi m_2, \dots, x_n + 2\pi m_n) = f(x_1, x_2, \dots, x_n), \quad m_1,\, m_2, \dots, m_n \in \mathbb{Z}^n.$$

Example 1.4.3 *Smooth functions on the projective space $\mathbb{R}P^n$ can be identified with smooth homogeneous functions on $\mathbb{R}^{n+1} \setminus 0$*

$$f(\lambda\, x) = f(x) \quad \forall\, \lambda \neq 0.$$

One can also define a structure of a topological space on the space of smooth functions. Roughly speaking the convergence of a sequence of smooth functions in $\mathcal{C}^\infty(M)$ is defined as the uniform convergence on compact subsets in M of the functions together with their partial derivatives of all orders. The operations defined above give continuous maps of the topological vector spaces

$$\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M).$$

We will not enter into details of these constructions here (they require to use a Riemannian metric on M that will be defined later).

A smooth map of manifolds $f : M \to N$ induces the *pullback* homomorphism of algebras of smooth functions

$$\begin{gathered} f^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M) \\ \mathcal{C}^\infty(N) \ni g \mapsto g \circ f \in \mathcal{C}^\infty(M). \end{gathered} \tag{1.4.1}$$

In particular for every coordinate chart (U, φ) on M the pullback induced by the inclusion $U \hookrightarrow M$ together with the φ^{-1} map induces a *restriction homomorphism*

$$\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\varphi(U)). \tag{1.4.2}$$

Restricting a smooth function from an n-dimensional manifold M to a coordinate chart one obtains a smooth function of n variables. An important point of the theory of smooth manifolds is the possibility to extend to the entire manifold the functions defined locally. To this end one has to construct a sufficiently rich list of $\mathcal{C}^\infty$-smooth functions.

Let us give a list of useful examples of such $\mathcal{C}^\infty$-smooth functions.

1) The function

$$q(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{1.4.3}$$

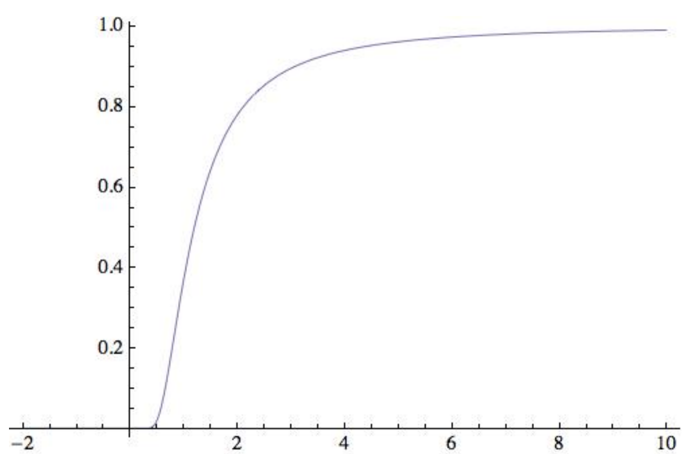

Fig. 10. Graph of the function (1.4.3)

is $\mathcal{C}^\infty$-smooth. All its derivatives vanish at the origin.

2) The $\mathcal{C}^\infty$-smooth function

$$r(x) = \begin{cases} e^{-\frac{1}{(1-x^2)^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \tag{1.4.4}$$

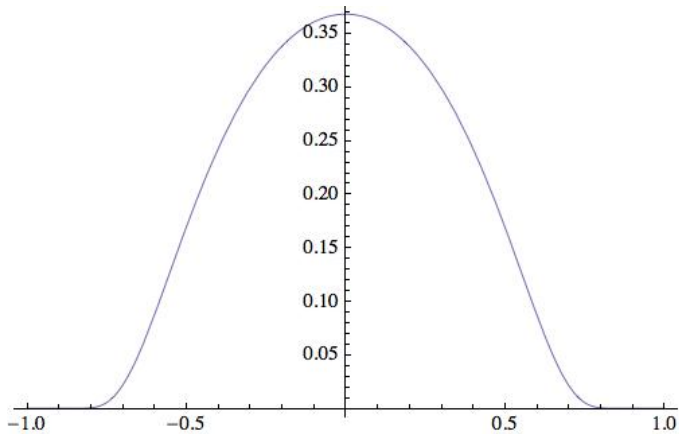

Fig. 11. Graph of the function (1.4.4)

is positive on $(-1, 1)$ and vanishes outside this interval. All its derivatives vanish at $x = \pm 1$.

3) The $\mathcal{C}^\infty$-smooth monotone function

$$p(x) = \frac{\int_{-\infty}^{x} r(x)\,dx}{\int_{-\infty}^{\infty} r(x)\,dx} \tag{1.4.5}$$

is equal to 0 for $x \leq -1$, to 1 for $x \geq 1$, and smoothly interpolates between 0 and 1 in the interval $(-1, 1)$.

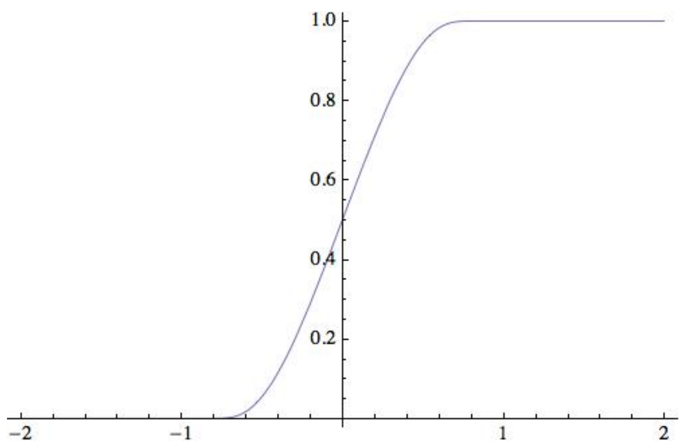

Fig. 12. Graph of the function (1.4.5)

The function

$$p\left(\frac{x_1 + x_2 - 2x}{x_2 - x_1}\right)$$

for arbitrary $x_1 < x_2$ is equal to 1 for $x < x_1$, to 0 for $x > x_2$ and smoothly interpolates between 1 and 0 on the interval (x_1, x_2).

4) For two positive numbers $0 < r < R$ the function

$$P_{r,R}(x) = p\left(\frac{r + R - 2|x|}{R - r}\right), \quad x = (x^1, \dots, x^n), \quad |x| = \sqrt{(x^1)^2 + \cdots + (x^n)^2} \tag{1.4.6}$$

is a $\mathcal{C}^\infty$-smooth function of n variables satisfying

$$\begin{array}{l} P_{r,R}(x) = 0, \quad |x| \geq R \\ P_{r,R}(x) = 1, \quad |x| \leq r \\ 0 < P_{r,R}(x) < 1, \quad r < |x| < R. \end{array} \tag{1.4.7}$$

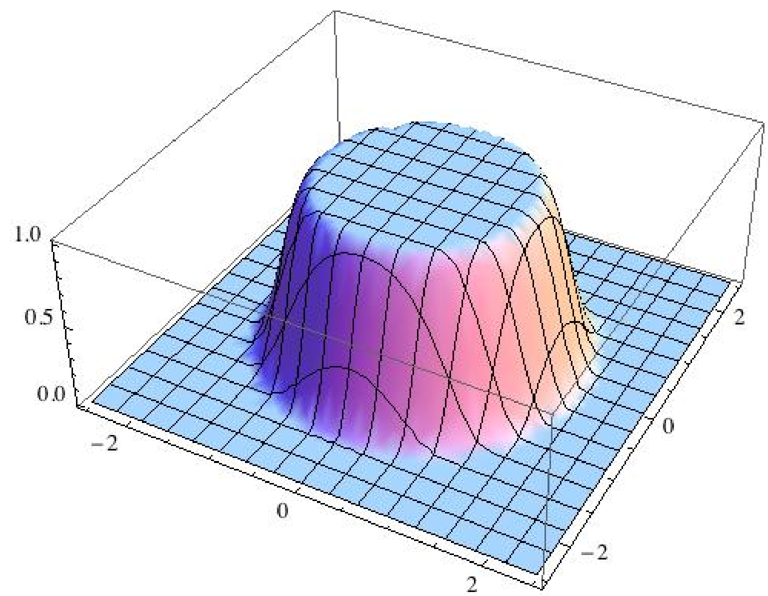

Fig. 13. The function (1.4.7) for $n = 2$, $r = 1$, $R = 2$

Using the above constructions we can easily prove the possibility of extension of locally defined smooth functions onto entire manifold.

Theorem 1.4.4 *Let (U, φ) be a coordinate chart on a smooth manifold M. Then, for an arbitrary smooth function $f \in \mathcal{C}^\infty(U)$ defined on the chart U and an arbitrary point $x_0 \in U$ there exists a smooth function $\hat{f} \in \mathcal{C}^\infty(M)$ such that*

- $\hat{f} = f$ on some neighborhood of the point x_0
- $\hat{f} = 0$ on $M \setminus U$.

Proof: There exists a positive number ϵ such that the open ball

$$B_\epsilon(x_0) = \{x \in U \,|\, |x - x_0| < \epsilon\}$$

of the radius ϵ centered at x_0 belongs to U. The $\mathcal{C}^\infty$-function

$$\hat{f}(x) = \begin{cases} f(x) P_{\frac{\epsilon}{2}, \epsilon}(x - x_0), & x \in B_\epsilon(x_0) \\[2ex] 0, & x \in M \setminus B_\epsilon(x_0) \end{cases}$$

coincides with $f(x)$ on the ball $B_{\frac{\epsilon}{2}}(x_0)$; it is equal to zero on the complement to the ball $B_\epsilon(x_0)$. $\square$

Many properties of smooth functions and smooth maps of manifolds can be established with the help of a gadget called *partition of unity*.

Let M be a smooth manifold with an atlas of charts $(U_\alpha, \varphi_\alpha)_{\alpha\in I}$ that is *locally finite*, i.e., it is such that any point $x \in M$ possesses an open neighborhood intersecting with only finite number of charts. For example, on any compact manifold one can choose an atlas with a finite number of charts.

Definition 1.4.5 *A* partition of unity *on the manifold M with a locally finite atlas is a set of smooth functions $(p_\alpha(x))_{\alpha\in I}$ on M such that*

- $p_\alpha(x) = 0 \quad \text{for} \quad x \in M \setminus U_\alpha$
- $0 \le p_\alpha(x) \le 1 \quad \forall\, \alpha \in I$
- $$\sum_{\alpha\in I} p_\alpha(x) \equiv 1. \tag{1.4.8}$$

Note that the sum in (1.4.8) is finite for any $x \in M$.

Theorem 1.4.6 *Let M be a compact manifold with an atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in I}$. Then there exists a refinement of this atlas and a partition of unity associated with this refinement.*

Proof: For any point $x \in U_\alpha \subset M$ there exists a positive number ϵ such that the open ball $B_\epsilon(x)$ belongs to U_α. In this way one obtains a covering of M with open subsets. The open balls of radius $\rho = \frac{\epsilon}{2}$ still cover M. Due to compactness one can choose a finite subcovering of M by open balls $B_{\rho_k}(z_k)$, $k = 1, 2, \dots, K$. Here we denote $z_1, \dots, z_K$ the centres of the balls. By construction every ball $B_{2\rho_k}(z_k)$ is a subset of a chart $U_\alpha(k)$ for some $\alpha(k)$. So the balls $B_{2\rho_1}(z_1), \dots, B_{2\rho_K}(z_K)$ give a refinement of the original atlas. Define functions

$$\tilde{p}_k(x) = \begin{cases} P_{\rho_k, 2\rho_k}(x - z_k), & x \in B_{2\rho_k}(z_k) \\ \\ 0, & x \in M \setminus B_{2\rho_k}(z_k) \end{cases} \tag{1.4.9}$$

and put

$$p_k(x) = \frac{\tilde{p}_k(x)}{\sum_{k=1}^K \tilde{p}_k(x)}, \quad k = 1, \dots, K.$$

These functions provide us with a partition of unity associated with the atlas $B_{2\rho_1}(z_1), \dots, B_{2\rho_K}(z_K)$. $\square$

Exercise 1.4.7 *Develop a similar construction replacing balls with cubes.*

Remark 1.4.8 *The assumption of compactness of the manifold can be relaxed. Namely, it suffices to assume* paracompactness *of M. By definition the manifold M is paracompact if for any covering of M with open subsets there exists a locally finite refinement.*

Exercise 1.4.9 *Prove existence of a partition of unity for any paracompact manifold.*

Let us now consider the case of an arbitrary paracompact manifold. Without loss of generality one may assume existence of a partition of unity $p_\alpha(x)$ associated with the atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$. Every local coordinate x_α^i can be smoothly extended onto M by using the construction of the Theorem 1.4.4. Define a vector field

$$
\begin{aligned}
& v_\alpha(x) := v_\alpha^i(x) \frac{\partial}{\partial x_\alpha^i}, \quad x \in U_\alpha, \\
& v_\alpha^i(x) := A x_\alpha^i, \quad i = 1, \dots, n \\
& v_\alpha(x) = 0 \quad \text{for} \quad x \in M \setminus U_\alpha.
\end{aligned}
$$

As above one proves that the actions of the vector field and the operator A on smooth vanishing outside U_α coincide. Put

$$
v := \sum_\alpha p_\alpha(x) v_\alpha(x)
$$

This vector field coincides with the operator A everywhere on M. □

1.5 Immersions and submersions

Using the constructions developed in the proof of Theorem 1.4.6 one can prove that any compact manifold can be realized as a multidimensional surface in a Euclidean space of sufficiently large dimension. Before doing this let us recall some elementary constructions from linear algebra. Let $A : V \to W$ be a linear map of finite dimensional vector spaces. There are two natural subspaces: the *kernel* of A

$$
\{x \in V \mid Ax = 0\} =: \operatorname{Ker} A \subset V
$$

and the *image* of A

$$
\operatorname{Im} A := A(V) \subset W.
$$

Dimension of the image is called the *rank* of the linear map

$$
\operatorname{rk} A = \dim \operatorname{Im} A. \tag{1.5.1}
$$

Choosing bases in the spaces V and W one can represent A by a matrix. Then the rank is equal to the number of linearly independent columns of the matrix, or, equivalently, to the number of linearly independent rows. The dimension of the kernel can be computed by the formula

$$
\dim \operatorname{Ker} A = \dim V - \operatorname{rk} A. \tag{1.5.2}
$$

The linear map is called *injective* if $\operatorname{Ker} A = 0$ and *surjective* if $\operatorname{Im} A = W$. A necessary condition for injectivity is the inequality $\dim V \leq \dim W$ while for surjectivity it is necessary to have $\dim V \geq \dim W$.

Exercise 1.5.1 *Define the* cokernel *of the linear map as the quotient*

$$
\operatorname{Coker} A = W / \operatorname{Im} A.
$$

The index *of a linear map is defined as the difference*

$$\operatorname{ind} A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A. \tag{1.5.3}$$

Prove that the index does not depend on A and is given by the formula

$$\operatorname{ind} A = \dim V - \dim W.$$

We will now formulate the following

Definition 1.5.2 *A smooth map $f : M \to N$ of two manifolds is called* immersion *if the differential*

$$f_* : T_x M \to T_{f(x)} N \tag{1.5.4}$$

is an injective linear map at any point $x \in M$. An immersion is called embedding *if $f(x_1) \neq f(x_2)$ for any pair of distinct points x_1, $x_2 \in M$.*

If the manifolds M and N have dimensions m and n respectively, and the smooth map f in local coordinates is represented in the form (1.2.14), then the map f is an immersion *iff* the rank of the Jacobi matrix (1.2.16) of the map is equal to m at every point $x \in M$. In particular for $m = 1$ and $m = 2$ one reproduces the definitions of regularity of a curve or a surface in a Euclidean space. In general one must necessarily have $m \leq n$ for an immersion.

The images of embeddings of smooth manifolds define *submanifolds*[2]. They generalize the curves and surfaces in a parametric representation studied in the first half of the course. As it follows from the Theorem 1.6.10 any compact manifold can be realized as a smooth submanifold in the Euclidean space of a sufficiently large dimension.

Example 1.5.3 *A smooth map $\gamma : \mathbb{R} \to \mathbb{R}^n$ is a vector function*

$$\gamma(t) = \left(x^1(t), \dots, x^n(t)\right).$$

Such a map is an immersion iff *the velocity vector*

$$\dot\gamma(t) = \left(\dot x^1(t), \dots, \dot x^n(t)\right) \neq 0.$$

Example 1.5.4 *Consider a map of a domain D in $\mathbb{R}^2$ to the three-dimensional Euclidean space. It is represented by a vector function of two variables*

$$\mathbf{r} : D \to \mathbb{R}^3, \quad \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)). \tag{1.5.5}$$

Such a map is an immersion iff *the rank of the Jacobi matrix*

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}$$

[2]One has to add more assumptions if the embedded manifold is non compact. Namely, one says that a smooth map $f : M \to N$ is *proper* of the preimage $f^{-1}(K)$ of any compact subset $K \subset N$ is a compact subset in M. By definition the image $f(M)$ of an embedding is called a *submanifold* in N if the map f is proper.

equals two (here and below the subscripts stand for partial derivatives). Equivalently, consider the vectors

$$\mathbf{r}_u, \quad \mathbf{r}_v.$$

The above condition about the rank of the Jacobi matrix means that these two vectors are linearly independent at every point of the surface. Clearly they are tangent to the surface, so they span the tangent plane $T_{(u,v)}M$ at every point (u, v) of the two-dimensional manifold $M = \mathbf{r}(D)$ (here we add an assumption that the map (1.5.5) *is an embedding). Observe that the vector*

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \neq 0 \tag{1.5.6}$$

at every point of the surface. This vector is orthogonal to the surface (i.e., it is orthogonal to the tangent plane $T_{(u,v)}M$ at every point $(u, v) \in M$).

In the particular case of a graph of a smooth function

$$z = f(x, y)$$

the vector function can be written in the form

$$\mathbf{r} = (x, y, f(x, y)) \, .$$

So the basis of tangent vectors reads

$$\mathbf{r}_x = (1, 0, f_x) \, , \quad \mathbf{r}_y = (0, 1, f_y)$$

and the normal vector (1.5.6) *has the form*

$$\mathbf{N} = (-f_x, -f_y, 1) \, .$$

Exercise 1.5.5 *Let $\mathbf{r}(u, v)$ be an embedding of a domain $D \subset R^2$ into the three-dimensional Euclidean space. Denote $M \subset \mathbb{R}^3$ the image of this embedding. Assume that the third component of the normal vector* (1.5.6) *does not vanish at the point (u_0, v_0). Prove that M locally, near the point (u_0, v_0), can be represented as a graph of a smooth function $z = f(x, y)$.*

Submanifolds can also be defined by systems of equations. To be more precise let us first give the following important auxiliary definitions and statements about smooth maps.

Definition 1.5.6 *1) We say that a point $x \in M$ is a* regular point *of a smooth map f if the differential*

$$df(x) : T_xM \to T_{f(x)}N$$

is surjective. In the opposite case the point $x \in M$ is critical *for the smooth map.*

2) A point $y \in N$ is called a regular value *if every point in the preimage $f^{-1}(y)$ is a regular one. In the opposite case the point $y \in N$ is called a* critical value

Remark 1.5.7 *In case a smooth map $f : M \to N$ is regular at every point $x \in M$ they say that f is a* submersion.

Example 1.5.8 *A point x_0 is critical for a smooth function $f : M \to \mathbb{R}$ iff the differential df vanishes at the point. In other words, all partial derivatives vanish at $x = x_0$*

$$\frac{\partial f(x_0)}{\partial x^1} = 0, \quad \dots, \quad \frac{\partial f(x_0)}{\partial x^m} = 0.$$

In particular the points of maximum or minimum are critical points of the function.

More generally, if the manifolds M and N have dimensions m and n respectively, and the smooth map f in local coordinates is represented in the form (1.2.14), then the point $x_0 \in M$ is regular for the map f *iff* the rank of the Jacobi matrix (1.2.16) of the map is equal to n at the point x_0. Recall that this can happen only if $m \geq n$.

Example 1.5.9 *Let TM be the total space of the tangent bundle of a n-dimensional manifold M. The points of TM are pairs (x, v) where $x \in M$ and $v \in T_x M$. This is a smooth manifold of dimension $2n$. The map $TM \to M$ given by projection*

$$(x, v) \to x$$

is a submersion.

Example 1.5.10 *The map $S^n \to \mathbb{R}\mathbf{P}^n$ assigning to a point x of the unit sphere a pair $\pm x$ of opposite points is an immersion and submersion.*

Theorem 1.5.11 *Let $f : M \to N$ be a smooth map having a regular value $y_0 \in f(M)$. Then the preimage*

$$F := f^{-1}(y_0) = \{x \in M \,|\, f(x) = y_0\} \tag{1.5.7}$$

is a smooth submanifold in M of the dimension

$$\dim F = \dim M - \dim N. \tag{1.5.8}$$

They also say that the submanifold F has *codimension* $=\dim N$,

$$\operatorname{codim} F := \dim M - \dim F. \tag{1.5.9}$$

Proof: At every coordinate chart $(U, (x^1, \dots, x^n))$ on M the points of the preimage F have to be determined from a system of m equations with n unknowns

$$\left.\begin{array}{rcl} y^1(x^1, \dots, x^n) & = & y_0^1 \\ \dots\dots\dots & & \dots \\ y^m(x^1, \dots, x^n) & = & y_0^m \end{array}\right\}. \tag{1.5.10}$$

Here $y_0^1, \dots, y_0^m$ are coordinates of the point y_0 in a coordinate chart $(V, (y^1, \dots, y^m))$ on N. We want to apply the implicit function theorem to this system.

At every point $x \in F$ of the preimage at least one of the $m \times m$ minors of the Jacobi matrix

$$\begin{pmatrix} \partial y^1/\partial x^1 & \dots & \partial y^1/\partial x^n \\ \vdots & \dots & \vdots \\ \partial y^m/\partial x^1 & \dots & \partial y^m/\partial x^n \end{pmatrix}$$

does not vanish. For every m-tuple of indices $1 \le j_1 < j_2 < \cdots < j_m \le n$ denote

$$U_{j_1 j_2 \dots j_m} = \{x \in U \cap F \,|\, J_{j_1 j_2 \dots j_m}(x) \neq 0\}$$
$$J_{j_1 j_2 \dots j_m}(x) = \det \begin{pmatrix} \partial y^1/\partial x^{j_1} & \dots & \partial y^1/\partial x^{j_m} \\ \vdots & \dots & \vdots \\ \partial y^m/\partial x^{j_1} & \dots & \partial y^m/\partial x^{j_m} \end{pmatrix}_x . \tag{1.5.11}$$

Every set $U_{j_1 j_2 \dots j_m}$ is an open domain in F; the collection of all these domains covers F. Let us construct a coordinate chart on a neighborhood of a point $x_0 \in U_{j_1 j_2 \dots j_m}$. Represent the set of indices $\{1, 2, \dots, n\}$ as a disjoint union of two subsets

$$\{1, 2, \dots, n\} = \{j_1, j_2, \dots, j_m\} \sqcup \{k_1, k_2, \dots, k_{n-m}\}. \tag{1.5.12}$$

According to the implicit function theorem there exists an open neighborhood of the point

$$x_0 \in U_{j_1 j_2 \dots j_m}(x_0) \subset U_{j_1 j_2 \dots j_m}$$

where the solutions to the system (1.5.10) admit a representation by smooth functions

$$\left.\begin{array}{lcl} x^{j_1} & = & g^{j_1}\left(x^{k_1}, \dots, x^{k_{n-m}}\right) \\ \dots & & \dots\dots\dots \\ x^{j_m} & = & g^{j_m}\left(x^{k_1}, \dots, x^{k_{n-m}}\right) \end{array}\right\} \tag{1.5.13}$$

such that

$$g^{j_s}\left(x_0^{k_1}, \dots, x_0^{k_{n-m}}\right) = x_0^{j_s}, \quad s = 1, \dots, m \tag{1.5.14}$$

(the coordinates of the point x_0). The functions g^{j_1}, ..., g^{j_m} are determined by the system (1.5.10) and the normalization condition (1.5.14) uniquely; their partial derivatives are determined from the linear system

$$\sum_{s=1}^{m} \frac{\partial y^i}{\partial x^{j_s}} \frac{\partial g^{j_s}}{\partial x^{k_t}} + \frac{\partial y^i}{\partial x^{k_t}} = 0, \quad i = 1, \dots, m, \quad t = 1, \dots, n-m. \tag{1.5.15}$$

The determinant of the coefficient matrix of this linear system coincides with $J_{j_1 j_2 \dots j_m}(x)$, $x \in U_{j_1 j_2 \dots j_m}(x_0) \subset U_{j_1 j_2 \dots j_m}$. So this determinant does not vanish. Therefore the variables $\left(x^{k_1}, \dots, x^{k_{n-m}}\right)$ define coordinates on the chart $U_{j_1 j_2 \dots j_m}(x_0)$. We leave as an exercise to verify that the transition functions from the chart $U_{j_1 j_2 \dots j_m}(x_0)$ to another one $U_{j'_1 j'_2 \dots j'_m}(x'_0)$ and back are smooth on the intersection of charts. □

Example 1.5.12 *The level surface of a smooth function $f : M \to \mathbb{R}$ is a submanifold $F := \{x \in M \,|\, f(x) = 0\} \subset M$ of codimension 1* iff

$$\sum_{i=1}^{n} \left|\frac{\partial f(x)}{\partial x^1}\right|^2 + \cdots + \left|\frac{\partial f(x)}{\partial x^n}\right|^2 \neq 0 \quad \forall\, x \in F. \tag{1.5.16}$$

Exercise 1.5.13 *Let* 0 *be a regular value of a smooth function* $f(x, y, z)$ *defined on a domain* $B \subset \mathbb{R}^3$*. Denote*

$$M = \{(x, y, z) \in B \,|\, f(x, y, z) = 0\}$$

the zero level surface of this function. According to Theorem 1.5.11 M *is a two-dimensional submanifold in* $\mathbb{R}^3$ *(i.e., a surface). Prove that the tangent plane to the surface at a point* $(x_0, y_0, z_0) \in M$ *is determined by the linear equation*

$$T_{(x_0,y_0,z_0)}M = \{(X, Y, Z) \in \mathbb{R}^3 \,|\, f_x(x_0, y_0, z_0)X + f_y(x_0, y_0, z_0)Y + f_z(x_0, y_0, z_0)Z = 0\}.$$

That is, the gradient of f *is orthogonal to the level surface* $f = 0$*.*

Exercise 1.5.14 *Prove that the* special linear group

$$\begin{aligned} &SL(n) \subset GL(n) \\ &SL(n) = \{X = (x^i_j)_{1 \le i,\, j \le n} |\, \det X = 1\} \end{aligned} \tag{1.5.17}$$

is a smooth submanifold of dimension $n^2 - 1$*. Prove that the tangent space to this submanifold at the point* $X = \mathbf{1}$ *(the identity matrix) can be identified with the linear space of all* $n \times n$ *matrices of trace zero*

$$T_{\mathbf{1}}SL(n) = \{Y \in Mat(n, \mathbb{R}) \,|\, \operatorname{tr} Y = 0\}. \tag{1.5.18}$$

Prove that $SL(n)$ *is a Lie group in the sense of the Definition 1.1.8.*

Exercise 1.5.15 *Prove that* orthogonal group

$$\begin{aligned} &O(n) \subset GL(n) \\ &O(n) = \{X = (x^i_j)_{1 \le i,\, j \le n} |\, X^{\mathrm{T}}X = 1\} \end{aligned} \tag{1.5.19}$$

is a smooth submanifold of dimension $\frac{n(n-1)}{2}$*. Prove that the tangent space to this submanifold at the point* $X = \mathbf{1}$ *can be identified with the linear space of all antisymmetric matrices*

$$T_{\mathbf{1}}O(n) = \{Y \in Mat(n, \mathbb{R}) \,|\, Y^T + Y = 0\}. \tag{1.5.20}$$

Prove that this is a Lie group in the sense of the Definition 1.1.8. Prove similar statements for the subgroup

$$SO(n) = O(n) \cap SL(n).$$

1.6 Sard theorem. Embeddings of compact manifolds into Euclidean spaces. Transversality.

The following deep result is used quite often in differential topology.

Theorem 1.6.1 (Sard Theorem) *The set* $f(C) \subset N$ *of critical values of a smooth map* $f : M \to N$ *of two manifolds is a subset of measure zero in* N*.*

By definition a subset A of a n-dimensional euclidean space $\mathbb{R}^n$ has measure zero if, for any positive ϵ there exists an at most countable set of n-dimensional cubes covering A of the total volume less than ϵ. The subset B of an n-dimensional manifold N has measure zero if, for any chart $U \subset N$, $\phi : U \to \mathbb{R}^n$ the image $\phi(B \cap U) \subset \mathbb{R}^n$ has measure zero in $\mathbb{R}^n$.

Exercise 1.6.2 *Proof that the definition of a subset of measure zero does not depend on the choice of an atlas on a smooth manifold.*

In the proof of Sard theorem we will use the following statement that can be derived from Fubini theorem.

Proposition 1.6.3 *Let A be a subset in $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ such that, for any $t \in \mathbb{R}$ the intersection A_t of A with the $(n-1)$-dimensional hyperplane $\{t\} \times \mathbb{R}^{n-1}$ has measure zero in $\mathbb{R}^{n-1}$. Then A has measure zero in $\mathbb{R}^n$.*

Proof of the Sard theorem. It suffices to prove the local statement: for a given smooth map $f : U \to \mathbb{R}^n$ of an open subset $U \subset \mathbb{R}^m$ the measure of the subset of critical values $f(C) \subset \mathbb{R}^n$ is zero. Let us use induction in m. For $m = 0$, $n = 0$ there is nothing to prove. So, let us assume that both dimensions m and n are positive. Introduce subsets $C_p \subset U$ as follows

$$C_p = \{x \in U \,|\, \text{all partial derivatives of } f \text{ of order} \leq \ p \text{ vanish at the point } x\}.$$

Clearly

$$C_p \subset C.$$

Moreover, one has a filtration

$$\cdots \subset C_2 \subset C_1 \subset C \tag{1.6.1}$$

The first step will be in proving

Lemma 1.6.4 *The measure of $f(C \setminus C_1) \subset \mathbb{R}^n$ is equal to zero.*

Proof: Observe that, for $n = 1$ the filtration is trivial, $C_1 = C$. So, let us assume $n \geq 2$. Let $x_0 \in C \setminus C_1$. Thus, at least one of the first order partial derivatives of the map

$$f(x) = \left(f^1(x), \ldots, f^n(x)\right), \quad x = \left(x^1, \ldots, x^m\right) \tag{1.6.2}$$

is different from zero at the point x_0 while the rank of the $m \times n$ Jacobi matrix

$$\left(\frac{\partial f^k(x_0)}{\partial x^i}\right)_{1 \leq k \leq n,\ 1 \leq i \leq m}$$

is less than n. Without loss of generality we may assume that

$$\frac{\partial f^1(x_0)}{\partial x^1} \neq 0. \tag{1.6.3}$$

Define a map $h : U \to \mathbb{R}^m$ by the formula

$$h(x) = \left(f^1(x), x^2, \dots, x^m\right).$$

Due to assumption (1.6.3) it is a local diffeomorphism of some neighborhood $V(x_0)$ onto an open neighborhood V_0 of the point $h(x_0)$. Consider the map

$$g = f \circ h^{-1} : V_0 \to \mathbb{R}^n.$$

The set C_0 of critical points of this map coincides with $h\left(V(x_0) \cap C\right)$. Clearly

$$\frac{\partial g^1}{\partial x^1} = 1, \quad \frac{\partial g^1}{\partial x^i} = 0 \quad \text{for} \quad i \geq 2. \tag{1.6.4}$$

Let us fix the restriction g_t of the map g onto the $(m-1)$-dimensional hyperplane

$$x^1 = t$$

for a given constant t close to x_0^1,

$$g_t : \left(\{t\} \times \mathbb{R}^{m-1}\right) \cap V_0 \to \mathbb{R}^n.$$

Due to (1.6.4) the Jacobi matrix of the map g has the form

$$\left(\frac{\partial g^k}{\partial x^j}\right) = \begin{pmatrix} 1 & 0 \\ * & \frac{\partial g_t^k}{\partial x^j} \end{pmatrix}.$$

So, $x = (t, x^2, \dots, x^m)$ is a critical point of g *iff* the point $\{t\} \times (x^2, \dots, x^m)$ is a critical point of g_t. By induction the measure of the set of critical values of g_t has measure zero. Applying Proposition 1.6.3 we conclude that the set of critical values of g has measure zero. The lemma is proved. $\square$

At the next step we deal with the complement $C_p \setminus C_{p+1}$ for $p \geq 1$.

Lemma 1.6.5 *For $p \geq 1$ the measure of the set $f\left(C_p \setminus C_{p+1}\right) \subset \mathbb{R}^n$ is zero.*

Proof: Let $x_0 \in C_p \setminus C_{p+1}$. That is, all partial derivatives of the functions $f^k(x)$ (see eq. (1.6.2)) of order $\leq i$ vanish at this point but, for some indices k, i_1, $\dots$, i_{p+1} the partial derivative

$$\frac{\partial^{p+1} f^k(x_0)}{\partial x^{i_1} \dots \partial x^{i_{p+1}}} \neq 0.$$

Without loss of generality we may assume that $i_1 = 1$. Denote

$$w(x) = \frac{\partial^p f^k(x)}{\partial x^{i_2} \dots \partial x^{i_{p+1}}}.$$

One has

$$w(x_0) = 0, \quad \frac{\partial w(x_0)}{\partial x^1} \neq 0.$$

Like in the proof of Lemma 1.6.4 let us consider the map $h : U \to \mathbb{R}^m$

$$h(x) = \left(w(x), x^2, \dots, x^m\right).$$

It is a diffeomorphism of a neighborhood $V(x_0) \subset U$ of the point $x_0 \in C_p \setminus C_{p+1}$ onto an open domain $V_0 \subset \mathbb{R}^m$. Recall that, at the points in C_p the function $w(x)$ vanishes. So, the image $h\left(C_p \cap V(x_0)\right)$ belongs to the hyperplane $\{0\} \times \mathbb{R}^{m-1}$.

Like in the proof of Lemma 1.6.4 consider the superposition

$$g = f \circ h^{-1} : V_0 \to \mathbb{R}^n.$$

Denote $\tilde{g}$ the restriction of g onto the hyperplane,

$$\tilde{g} : \left(\{0\} \times \mathbb{R}^{m-1}\right) \cap V_0 \to \mathbb{R}^n.$$

Any point of the set $h(C_p \cap V(x_0)$ is critical for $\tilde{g}$. By induction the measure of image of the set of critical points of $\tilde{g}$ belonging to V_0 is zero. Thus, the measure of the set $\tilde{g} \circ h\left(C_p \cap V_0\right) = f\left(C_p \cap V_0\right) \subset \mathbb{R}^n$ is zero. Covering $C_p \setminus C_{p+1}$ with a countable set of such domains V_0 we complete the proof of Lemma. □

The last step in the proof of the Sard theorem is given by

Lemma 1.6.6 *For sufficiently large p the measure of $f(C_p) \in \mathbb{R}^n$ is zero.*

Proof: Let us cover the set C_p with a countable set of cubes of the size δ (so, the volume of every cube equals δ^m). Let I^m be any of such cubes. Let us prove that the measure of $f(C_p \cap I^m)$ is zero. For any point $x \in C_p \cap I^m$ and any vector Δx such that $x + \Delta x \in I_m$ we have

$$f(x + \Delta x) = f(x) + R(x, \Delta x)$$

where the truncation error satisfies the estimate

$$\|R(x, \Delta x)\| \leq \alpha \, \|\Delta x\|^{p+1} \tag{1.6.5}$$

as it readily follows from the Taylor formula. Here α is a constant depending on f and on the cube I^m. Let us divide the cube I^m into r^m smaller cubes of size δ/r, $r \in \mathbb{Z}_{>0}$. Denote I_0^m a cube containing the point x. Every point in I_0^m has the form $x + \Delta x$ where the norm of the vector Δx satisfies inequality

$$\|\Delta x\| \leq \sqrt{m}\,\frac{\delta}{r}.$$

Using the estimate (1.6.5) we conclude that the image $f(I_0^m)$ belongs to a cube of size $\frac{c}{r^{p+1}}$ where

$$c = 2\alpha \left(\sqrt{m}\,\delta\right)^{p+1}.$$

Hence the image $f(C_p \cap I^m)$ belongs to the union of r^m cubes of the total volume less or equal than

$$r^m \left(\frac{c}{r^{p+1}}\right)^n = c^n r^{m-n(p+1)}.$$

If p is such that

$$p+1 > \frac{m}{n}$$

than the total volume tends to zero when $r \to \infty$. This completes the proof of the Lemma and also of the Sard theorem. $\square$

Corollary 1.6.7 *There always exists a regular value $y \in N$ of a smooth map $f : M \to N$.*

A particular case of this corollary is used often:

Corollary 1.6.8 *Let $f : M \to N$ be a smooth map of manifolds of dimensions m and n respectively. If $m < n$ then there exists a point $y \in N$ that does not belong to the image $f(M)$.*

Proof: For $m < n$ the differential $df(x)$ cannot be surjective at any point $x \in M$. So any point in $f(M)$ is a critical value. Therefore the image has measure zero and thus cannot cover the entire N. $\square$

Another corollary from Sard theorem says that

Corollary 1.6.9 *Given a point $y_0 \in N$, then the set of smooth maps $M \to N$ having y_0 as a regular value is dense in the space of all smooth maps $\mathcal{C}^\infty(M, N)$.*

Proof: We have to prove that, for any smooth map $f : M \to N$ there exists a deformed map $\tilde{f}$ such that $y_0 \in N$ is a regular value for $\tilde{f}$. Moreover, the deformed map $\tilde{f}$ can be chosen arbitrarily close to f. Indeed, if y_0 is a critical value of f then, due to Sard theorem, for an arbitrary neighborhood U, $y_0 \in U \subset N$ there exists a point $\tilde{y}_0 \in U$ being a regular value for f. Without loss of generality one may assume that U is diffeomorphic to the standard n-dimensional ball B^n. Denote $\varphi : U \to B^n$ the diffeomorphism. Put $z_0 = \varphi(y_0)$, $\tilde{z}_0 = \varphi(\tilde{y}_0)$. It is easy to construct a diffeomorphism $h : B^n \to B^n$ identical near the boundary ∂B^n moving $\tilde{z}_0$ to z_0 and, moreover, satisfying the inequality

$$\|h(z) - z\| < \epsilon = \|z_0 - \tilde{z}_0\| \quad \text{for any} \quad z \in B^n. \tag{1.6.6}$$

Denote $H = \varphi^{-1} \circ h \circ \varphi$ the corresponding diffeomorphism of U to itself. Extend H to a diffeomorphism $N \to N$ by the identity map outside U. Then for the smooth map

$$\tilde{f} = H \circ f$$

the point y_0 will be regular as the preimage $\tilde{f}^{-1}(y_0)$ coincides with $f^{-1}(\tilde{y}_0)$. The map $\tilde{f}$ is close to f because of (1.6.6). $\square$

Clearly the set of smooth maps from M to N having a given point $y_0 \in N$ as a regular value is open. The Corollary says that this subset is dense in the space of all smooth maps $\mathcal{C}^\infty(M, N)$.

They often represent this idea saying that for a *generic* smooth map $f : M \to N$ a given point $y_0 \in N$ is a regular value.

We will now use Sard theorem in the proof of the following

Theorem 1.6.10 (Whitney) *Given a compact n-dimensional manifold M there exists an immersion*

$$M \to \mathbb{R}^{2n}$$

and an embedding

$$f : M \to \mathbb{R}^{2n+1}.$$

Proof: Let us first construct an embedding of M into a Euclidean space of sufficiently large dimension N. Consider a covering of the manifold M by open balls $B_{\rho_1}(z_1), \ldots B_{\rho_K}(z_K)$ constructed in the proof of Theorem 1.4.6. For every $k = 1, \ldots, K$ we have constructed a smooth function $\tilde{p}_k(x)$ on M such that

$$\begin{aligned}
&\tilde{p}_k(x) \equiv 1 \quad \text{for} \quad x \in B_{\rho_k}(z_k)\\
&\tilde{p}_k(x) \equiv 0 \quad \text{for} \quad x \in M \setminus B_{2\rho_k}(z_k)\\
&0 < \tilde{p}_k(x) < 1 \quad \text{for} \quad x \in B_{2\rho_k}(z_k) \setminus B_{\rho_k}(z_k)
\end{aligned}$$

(see eq. (1.4.9)). Let n be the dimension of the manifold M. Define a smooth map

$$\begin{aligned}
&f_k : M \to \mathbb{R}^{n+1}\\
&f_k(x) = \left(x_k^1 \tilde{p}_k(x), \ldots, x_k^n \tilde{p}_k(x), \tilde{p}_k(x)\right).
\end{aligned}$$

Here $(x_k^1, \ldots, x_k^n)$ are the local coordinates near the point z_k. This map is smooth; it vanishes outside the ball $B_{2\rho_k}(z_k)$. Restricting this map onto the ball $B_{\rho_k}(z_k)$ one obtains

$$f_k(x) = (x_k^1, \ldots, x_k^n, 1). \tag{1.6.7}$$

Hence the map f_k is an embedding when restricted onto the ball $B_{\rho_k}(z_k)$.

Consider now the map

$$\begin{aligned}
&f : M \to \mathbb{R}^{(n+1)\,K}\\
&f(x) = (f_1(x), \ldots, f_K(x)).
\end{aligned} \tag{1.6.8}$$

Due to (1.6.7) the k-th component of this map is an embedding on $B_{\rho_k}(z_k)$. Hence the entire map (1.6.8) is an immersion. Let us prove that this map is also an embedding. Indeed, if a, b are two distinct points in $B_{\rho_k}(z_k)$ then $f_k(a) \neq f_k(b)$. If $a \in B_{\rho_k}(z_k)$ and $b \notin B_{\rho_k}(z_k)$ then the last component of the vector $f_k(a)$ is equal to $\tilde{p}_k(a) = 1$ and the last component of the vector $f_k(b)$ is equal to $\tilde{p}_k(b) < 1$. Therefore $f_k(a) \neq f_k(b)$.

Let us now construct an immersion into $\mathbb{R}^{2n}$. We may assume, due to the first part of the proof, that M is a submanifold in $\mathbb{R}^N$ for some large N. Or goal is to reduce N. Let us apply an orthogonal projection π_ℓ onto the hypeplane $\mathbb{R}^{N-1}$ orthogonal to a line ℓ. Which tangent vectors to M go to zero under the induced map $d\pi_\ell$? They are those tangent vectors $v \in T_x M$ at some $x \in M$ that are parallel to ℓ in $\mathbb{R}^N$.

Call ℓ a *bad direction of the 1st kind* if there exists a pair (x, v), $x \in M$, $0 \neq v \in T_x M$ such that $v \parallel \ell$. Such bad directions can be identified in the following way. Let us introduce the manifold $\mathbf{P}(TM)$ as the set of classes of equivalence

$$\mathbf{P}(TM) = \{(x, v) \in TM,\ v \neq 0 \,|\, v \sim \lambda\, v \quad \text{for some} \quad \lambda \neq 0\} \tag{1.6.9}$$

(the so-called *projectivization* of the tangent bundle. (We leave as an exercise to prove that $\mathbf{P}(TM)$ is a smooth manifold of dimension $2n-1$.) Consider the map

$$\mathbf{P}(TM) \to \mathbb{R}\mathbf{P}^{N-1}, \quad (x, v) \mapsto v. \tag{1.6.10}$$

The bad directions of the first kind are those ℓ in the projective space $\mathbb{R}P^{N-1}$ that belong to the image of this map. According to Corollary 1.6.8, if $2n-1 < N-1$ then the image of the map (1.6.10) does not cover the entire projective space $\mathbb{R}\mathbf{P}^{N-1}$. That means that there exists a direction ℓ such that none of the tangent vectors to M is parallel to it. Choosing such ℓ one obtains the projection π_ℓ with never vanishing differential, i.e., an immersion. In this way, after a finite number of steps one arrives at an immersion of M into $\mathbb{R}^{2n}$.

Let us now clarify when the projection π_ℓ is an embedding. One has to exclude the bad directions of the 1st kind and also of the second kind to be defined as follows. The line ℓ is a *bad direction of the second kind* if there exists a pair of distinct points x, $y \in M$ such that the bisecant $\overline{x\,y}$ is parallel to ℓ. Like above consider the map

$$M \times M \setminus \mathrm{diag} \to \mathbb{R}\mathbf{P}^{N-1}, \quad (x, y) \mapsto \overline{x\,y}. \tag{1.6.11}$$

Bad directions of the second kind are in the image of this map. If $2n < N-1$ then the map (1.6.11) does not cover the entire projective space. Choosing a line ℓ not belonging to the images of the maps (1.6.9), (1.6.11) one obtains an embedding into $\mathbb{R}^{N-1}$. The last time it can be done when $N = 2n+2$. Applying the projection one arrives at an embedding into $\mathbb{R}^{2n+1}$. $\square$

Let us now introduce a generalization of the notion of regularity. Let us consider a smooth map $f : M \to N$ and a submanifold $P \subset N$.

Definition 1.6.11 *We say that the map $f : M \to N$ is* transversally regular *to the submanifold $P \subset N$ at the point $y_0 \in P$ if, for any $x \in M$ such that $f(x) = y_0$ the through map*

$$T_x M \xrightarrow{f_*} T_{y_0 = f(x)} N \to T_{y_0} N / T_{y_0} P$$

is surjective. If the above condition holds true at any point $y_0 \in P$ then the map f is called transversally regular along P.

We will often use the short form t-regular in order to save space.

For the case of P= one point $P = y_0 \in N$ the notion of t-regularity along P of a map $f : M \to N$ coincides with the assumption that y_0 is a regular value.

Example 1.6.12 *Given a smooth function $f(x)$ of one real variable consider the graph map*

$$F : \mathbb{R} \to \mathbb{R}^2, \quad F(x) = (x, f(x)) .$$

Take the x-axis $\{y = 0\}$ as the submanifold $N \subset \mathbb{R}^2$. The intersections $F(\mathbb{R}) \cap N$ correspond to zeroes of the function $f(x)$. Transversal regularity in this case means that all zeroes are simple

$$f(x_0) = 0, \quad f'(x_0) \neq 0.$$

A generalization of Theorem 1.5.11 is straighforward:

Exercise 1.6.13 *Prove the following generalization of the implicit function theorem: given a smooth map $f : M \to N$ transversally regular along $P \subset N$, then the full preimage $f^{-1}(P)$ is a smooth submanifold of M of codimension*

$$\operatorname{codim} f^{-1}(P) = \operatorname{codim} P.$$

Clearly the set of all smooth maps $f : M \to N$ being t-regular along $P \subset N$ is open in $\mathcal{C}^\infty(M, N)$. The following statement says that it is dense in the space of all smooth maps (cf. Corollary 1.6.9).

Theorem 1.6.14 *Given a smooth map $f : M \to N$ and a submanifold $P \subset N$, then there exists a smooth map $g : M \to N$ arbitrarily close to f and t-regular along P.*

Proof: It suffices to prove the following local version of the theorem assuming that $M = U$ and $N = V$ are open domains in Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively and P is a p-dimensional subspace in $\mathbb{R}^n$ of the form

$$P = \{\big(y^1, \ldots, y^p, 0, \ldots, 0\big)\}.$$

The map f is nothing but a n-component vector function of m variables

$$x = \big(x^1, \ldots, x^m\big) \mapsto \big(f^1(x), \ldots, f^p(x), f^{p+1}(x), \ldots, f^n(x)\big)\,.$$

Transversal regularity of f along P means that for the map from $\mathbb{R}^m$ to $\mathbb{R}^{n-p}$

$$x \mapsto \big(f^{p+1}(x), \ldots, f^n(x)\big) \tag{1.6.12}$$

the point 0 is a regular value. According to Corollary 1.6.9 there exists a map

$$x \mapsto \big(g^{p+1}(x), \ldots, g^n(x)\big) \tag{1.6.13}$$

arbitrarily close to (1.6.12) for which 0 is a regular value. Therefore the map $\tilde f$ locally defined by

$$x \mapsto \big(f^1(x), \ldots, f^p(x), g^{p+1}(x), \ldots, g^n(x)\big) \tag{1.6.14}$$

is t-regular along P.

Let us now explain how to extend globally the deformation (1.6.14) of the map (1.6.12). Choose a compact $K \subset V$ construct a smooth function

$$\varphi = \begin{cases} 0 & \text{on} \quad \partial V \\ \\ 1 & \text{on} \quad K \end{cases}, \quad 0 \le \varphi \le 1.$$

Put

$$F = \varphi \tilde f + (1 - \varphi) f = f + \varphi(\tilde f - f).$$

This map coincides with $\tilde f$ inside K and with f outside V. So it is regular everywhere as the difference $\tilde f - f$ is small together with its first derivatives. $\square$

Example 1.6.15 *Let $f : M \to N$ is a smooth map transversally regular along the submanifold $P \subset N$ such that*

$$\dim M + \dim P < \dim N. \tag{1.6.15}$$

Then

$$f(M) \cap P = \emptyset.$$

Indeed, under the assumption (1.6.15) the dimension of T_xM is less than the dimension of the quotient $\dim T_yN/T_yP = \dim N - \dim P$, $y \in P \subset N$. So the through map

$$T_xM \xrightarrow{f_*} T_{f(x)}N \to T_{f(x)}N/T_{f(x)}P$$

cannot be surjective.

From these arguments along with the Theorem 1.6.14 it readily follows

Corollary 1.6.16 *Given a smooth map $f : M \to N$ and a submanifold $P \subset N$ satisfying the dimension condition* (1.6.15)*. Then there exists a smooth map $\tilde{f} : M \to N$ arbitrarily close to f such that*

$$\tilde{f}(M) \cap P = \emptyset.$$

In other words, a *generic* smooth map from M to N is t-regular along a given submanifold $P \subset N$.

Let us consider a particular situation of a pair of submanifolds $M,\ N \subset Q$ in an ambient manifold Q.

Definition 1.6.17 *We say that the submanifolds $M,\ N \subset Q$ are in* general position *if, at any point $x \in M \cap N$ one has*

$$T_xM + T_xN = T_xQ. \tag{1.6.16}$$

The notation $M \pitchfork N$ is often used to state that the submanifolds M and N are in general position.

Observe that the submanifolds are in general position if the embedding map

$$i : M \hookrightarrow Q$$

is t-regular along N or, equivalently, the embedding map

$$j : N \hookrightarrow Q$$

is t-regular along M.

Theorem 1.6.18 *If the submanifolds $M,\ N \subset Q$ are compact then there exist deformed submanifolds $\tilde{M}$ and $\tilde{N}$ arbitrarily close to M and N respectively such that $\tilde{M}$ and $\tilde{N}$ being in general position.*

Proof: If the embedding map

$$i : M \hookrightarrow Q$$

is t-regular along N then the submanifolds are in general position. Otherwise there exists an arbitrarily small deformation $\tilde{i}$ of the embedding map being t-regular along N. If $\tilde{i}$ is sufficiently close to i then $\tilde{i} : M \hookrightarrow Q$ is also an embedding. $\square$

Corollary 1.6.19 *If the submanifolds are compact and their dimensions satisfy the inequality*

$$\dim M + \dim N < \dim Q \tag{1.6.17}$$

then there exist submanifolds $\tilde{M}$ and $\tilde{N}$ arbitrarily close to M and N respectively such that

$$\tilde{M} \cap \tilde{N} = \emptyset.$$

Proof: The submanifolds in general position of dimensions satisfying (1.6.17) do not intersect. $\square$

Exercise 1.6.20 *Given two compact submanifolds in general position $M,\ N \subset Q$. Prove that the intersection $M \cap N$ is a smooth submanifold in M and in N.*

Let us consider two other illustrations of the notion of transversal regularity.

Example 1.6.21 *A given smooth vector field $v(x)$ on a manifold M can be considered as a section of the tangent bundle, i.e., as a map*

$$M \to TM, \quad x \mapsto (x, v(x)). \tag{1.6.18}$$

Denote

$$M_v \subset TM$$

the image of this map. By M_0 denote the zero section. Intersections $M_v \cap M_0$ correspond to the stationary points *of the vector field*

$$v(x_0) = 0.$$

Transversality $M_v \pitchfork M_0$ at a point x_0 means that the stationary point is nondegenerate

$$v(x_0) = 0, \quad \det\left(\frac{\partial v^i(x_0)}{\partial x^j}\right) \neq 0. \tag{1.6.19}$$

Importance of nondegenerate stationary points of a vector field v in the theory of differential equations is due to the following fact: solutions to a system of differential equations

$$\dot{x} = v(x)$$

near a nondegenerate stationary point can be approximated by solutions to a linear system. On a compact manifold M a vector field can have only finite number of stationary points provided all of them are nondegenerate. Number of these stationary points counted with suitably defined multiplicities is a topological invariant of the manifold (see below).

Example 1.6.22 *To a given smooth function f on a manifold M we associate a section of the cotangent bundle defined by the differential df*

$$M \to T^*M, \quad x \mapsto (x, df(x)). \tag{1.6.20}$$

In natural local coordinates $\left(x^1, \dots, x^n, p_1, \dots, p_n\right) \in T^*M$ the map (1.6.20) reads

$$\left(x^1, \dots, x^n\right) \mapsto \left(x^1, \dots, x^n, \frac{\partial f(x)}{\partial x^1}, \dots, \frac{\partial f(x)}{\partial x^n}\right).$$

Denote $L_f \subset T^*M$ the image of the map (1.6.20). Like above denote $L_0 \subset T^*M$ the zero section of the cotangent bundle. The intersection points of L_f with the zero section L_0 correspond to the critical points x_0 of the function f

$$\frac{\partial f(x_0)}{\partial x^1} = 0, \dots, \frac{\partial f(x_0)}{\partial x^n} = 0.$$

Transversality means that x_0 is a Morse critical point

$$\frac{\partial f(x_0)}{\partial x^1} = 0, \dots, \frac{\partial f(x_0)}{\partial x^n} = 0, \quad \det\left(\frac{\partial^2 f(x_0)}{\partial x^i \partial x^j}\right) \neq 0. \tag{1.6.21}$$

Thus, transversality $L_f \pitchfork L_0$ means that f is a Morse function, *i.e., the one having only Morse critical points. On a compact manifold M a Morse function can have only finite number of critical points. Number of these critical points counted with suitably defined multiplicities is a topological invariant of the manifold (see below).*

Remark 1.6.23 *One cannot apply the theorem 1.6.14 in order to prove existence of a Morse function $\tilde f$ arbitrarily close to a given function $f \in \mathcal{C}^\infty(M)$. The problem is with a nontrivial geometrical restriction valid for the image L_f of the map (1.6.20). Namely, for any smooth function $L_f \subset T^*M$ is a* Lagrangian submanifold. *That means that the natural* symplectic 2-form

$$\Omega = \sum_{i=1}^n dp_i \wedge dx^i$$

being restricted onto L_f vanishes identically:

$$\Omega\,|_{L_f} = \sum_{i=1}^n d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i = \sum_{i,\,j=1}^n \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j \wedge dx^i = 0$$

since

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}, \quad dx^j \wedge dx^i = -dx^i \wedge dx^j.$$

Clearly not any n-dimensional submanifold in the total space of the cotangent bundle is Lagrangian: for example, take the following 2-dimensional submanifold in $T^*\mathbb{R}^2 = \mathbb{R}^4 = \{(x, y, p, q)\,|\, x,\, y,\, p,\, q \in \mathbb{R}\}$:

$$L = \{(x, y, p = -y, q = x)\,|\, x,\, y \in \mathbb{R}\}.$$

Restricting the symplectic form $\Omega = dp \wedge dx + dq \wedge dy$ onto L yields

$$\Omega\,|_L = -dy \wedge dx + dx \wedge dy = 2dx \wedge dy \neq 0.$$

In order to prove density of Morse functions in the space $\mathcal{C}^\infty(M)$ one can use the following construction that we will explain for the particular case $M = \mathbb{R}^n$ leaving the general case as an exercise. Let $x_0 \in \mathbb{R}^n$ be a critical point of a function $f \in \mathcal{C}^\infty \mathbb{R}^n$ such that

$$\det\left(\frac{\partial^2 f(x_0)}{\partial x^i \partial x^j}\right) = 0.$$

Consider the function

$$f_a(x) = f(x) - \sum_{i=1}^n a_i x^i$$

depending on n parameters $a_1, \ldots, a_n$. Critical points of f_a are solutions to the system

$$\frac{\partial f(x)}{\partial x^1} = a_1, \ldots, \frac{\partial f(x)}{\partial x^n} = a_n.$$

Such a critical point is a Morse one if a is a regular value of the gradient map $\mathbb{R}^n \to \mathbb{R}^n$

$$x \mapsto \left(\frac{\partial f(x)}{\partial x^1}, \ldots, \frac{\partial f(x_0)}{\partial x^n}\right).$$

Applying Sard theorem choose a regular value a (it can be chose arbitrarily close to $a = 0$). Then the deformed function f_a will be a Morse function.

2 First examples of topological invariants

2.1 Orientation. Topological degree of a smooth map

Definition 2.1.1 *An* orientation *on a smooth manifold M is an atlas $(U_\alpha, (x_\alpha^1, \ldots, x_\alpha^n))_{\alpha \in I}$ such that the Jacobians of all transition maps are positive*

$$\det\left(\frac{\partial x_\alpha}{\partial x_\beta}\right)_P > 0, \quad \forall\, P \in U_\alpha \cap U_\beta. \tag{2.1.1}$$

Two orientations $(U_\alpha, (x_\alpha^1, \ldots, x_\alpha^n))_{\alpha \in I}$ and $(V_\beta, (y_\beta^1, \ldots, y_\beta^n))_{\beta \in J}$ are called equivalent *if they define equivalent smooth structures on M and the Jacobians* $\det\left(\frac{\partial x_\alpha}{\partial y_\beta}\right)$ *are all positive on the intersections $U_\alpha \cap V_\beta$.*

If an orientation on M exists then the manifold is called orientable. *In the opposite case the manifold M is called* non-orientable.

Example 2.1.2 *The manifold $\mathbb{R}^n$ has the orientation corresponding to an ordering of the Euclidean coordinates $x^1, \ldots, x^n$. A permutation of the coordinates $x^{\sigma(1)}, \ldots, x^{\sigma(n)}$, $\sigma \in S_n$ defines an equivalent orientation if σ is an even substitution and an opposite orientation if the substitution σ is odd. More generally, given a frame of n linearly independent vectors $f_1, \ldots, f_n$ in $\mathbb{R}^n$ one can introduce another chart on $\mathbb{R}^n$ considering the coordinates with respect to the new basis. This chart will define the same orientation on $\mathbb{R}^n$* iff *the determinant of the transition matrix*

$$A = (a_{ij})_{1 \le i, j \le n}, \quad f_k = \sum_{i=1}^n a_{ik} e_i$$

is positive

$$\det A > 0.$$

In this case one says that the frames $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ have coherent orientations*.*

More generally, on an oriented manifold M one can choose a class of frames in the tangent space T_xM at any point $x \in M$. Namely, we say that a frame $e_1, \ldots, e_m$ is *positively oriented* if, for any chart $U \ni x$ with local coordinates $x^1, \ldots, x^m$ the orientations of the frames $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^m}$ and $e_1, \ldots, e_m$ are coherent. Clearly such a definition does not depend on the choice of the chart. Conversely, an orientation on the manifold can be defined by choosing an equivalence class of frames in the tangent spaces T_xM at any x depending continuously on the point x.

Example 2.1.3 *Let $M \subset \mathbb{R}^n$ be the level surface $M = \{x \in \mathbb{R}^n \,|\, f(x) = 0\}$ of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\operatorname{grad} f(x) \neq 0 \quad \forall\, x \in M.$$

Choosing an atlas of coordinate charts

$$(U_i, (x_i^1, \ldots, x_i^{n-1})), \qquad \left(\frac{\partial f(x)}{\partial x^i}\right)_{x \in U_i} \neq 0$$

one obtains a structure of a $(n-1)$-dimensional smooth manifold on M, as in the Theorem 1.5.11. Recall that one can choose $(x^1, \ldots, \hat{x^i}, \ldots, x^n)$ as the local coordinates $(x_i^1, \ldots, x_i^{n-1})$. Let us reorder the local coordinates in such a way that the frame of n vectors

$$\frac{\partial}{\partial x_i^1}, \frac{\partial}{\partial x_i^2}, \ldots, \frac{\partial}{\partial x_i^{n-1}}, \operatorname{grad} f(x) \tag{2.1.2}$$

is positively oriented with respect to the standard orientation of $\mathbb{R}^n$ at every point $x \in U_i$. In this way one obtains an orientation on the level surface M.

Let us prove this statement in the simple case of a two-dimensional surface M in $\mathbb{R}^3$ defined by one equation

$$M = \{(x, y, z) \in \mathbb{R}^3 \,|\, f(x, y, z) = 0\}.$$

Recall that $0 \in \mathbb{R}$ is a regular value if the gradient

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)_M \neq 0.$$

The surface M is covered by three charts

$$U_x = \{(x, y, z) \in M \,|\, \frac{\partial f(x, y, z)}{\partial x} \neq 0\}, \quad \text{the coordinates} \quad (y, z)$$
$$U_y = \{(x, y, z) \in M \,|\, \frac{\partial f(x, y, z)}{\partial y} \neq 0\}, \quad \text{the coordinates} \quad (x, z)$$
$$U_z = \{(x, y, z) \in M \,|\, \frac{\partial f(x, y, z)}{\partial z} \neq 0\}, \quad \text{the coordinates} \quad (x, y).$$

According to the above definition the orientation on these charts is defined by the following order of coordinates

$$\begin{aligned} U_x : (y, z) \quad &\text{if} \quad \frac{\partial f(x,y,z)}{\partial x} > 0 \quad \text{and} \quad (z, y) \quad \text{otherwise} \\ U_y : (z, x) \quad &\text{if} \quad \frac{\partial f(x,y,z)}{\partial y} > 0 \quad \text{and} \quad (x, z) \quad \text{otherwise} \\ U_z : (x, y) \quad &\text{if} \quad \frac{\partial f(x,y,z)}{\partial z} > 0 \quad \text{and} \quad (y, x) \quad \text{otherwise.} \end{aligned}$$

Let us compute, for example, the jacobian of the transition functions from U_z to U_y on the intersection of these domains assuming that the partial derivatives are positive

$$f_y := \frac{\partial f}{\partial y} > 0, \quad f_y := \frac{\partial f}{\partial z} > 0.$$

We have

$$\frac{D(z,x)}{D(x,y)} = \det \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ 1 & 0 \end{pmatrix} = -\frac{\partial z}{\partial y} = \frac{f_y}{f_z} > 0.$$

Exercise 2.1.4 *Develop a similar construction of an orientation of submanifolds $M \subset \mathbb{R}^N$ of any codimension k defined as preimages in an oriented manifold N of a point in $\mathbb{R}^k$ with respect to a submersion $f : N \to \mathbb{R}^k$.*

Let $f : M \to N$ be a smooth map of two compact oriented manifolds of the same dimension n. We want to define an important characteristic of f called *topological degree* $\deg f$ that depends only on the homotopy class of f.

Let $y \in N$ be a regular value for this map. All the points $x_1, \ldots, x_K \in M$ of the preimage

$$f^{-1}(y) = \{x_1\} \cup \{x_2\} \cup \ldots \{x_K\} \subset M$$

are regular. Let $(y^1, \ldots, y^n)$ be a positively oriented chart on N near the point y. Denote $(x_i^1, \ldots, x_i^n)$ a positively oriented chart on M near the point x_i. Due to the assumption of regularity the Jacobian

$$\det\left(\frac{\partial y}{\partial x_i}\right) := \det\left(\frac{\partial y^p}{\partial x_i^q}\right)_{1 \le p, q \le n}$$

does not vanish at the point x_i

$$\det\left(\frac{\partial y}{\partial x_i}\right) \neq 0. \tag{2.1.3}$$

Denote

$$\deg_{x_i} f := \operatorname{sign} \det\left(\frac{\partial y}{\partial x_i}\right) \tag{2.1.4}$$

and put

$$\deg f|_y := \sum_{x_i \in f^{-1}(y)} \deg_{x_i} f. \tag{2.1.5}$$

Theorem 2.1.5 *1) The degree of the map $f : M \to N$ of two compact connected oriented manifolds of the same dimension does not depend on the choice of a regular value $y_0 \in N$.*

Such a degree is called simply the degree of the smooth map*.*

2) If two smooth maps f_0, $f_1 : M \to N$ are homotopic then their degrees coincide:

$$\deg f_0 = \deg f_1.$$

Proof: Let y_0 and $y_1 \in N$ be two regular values of f. Choose a smooth curve $\gamma : [0,1] \to N$, γ connecting these points in such a way that the map f is transversally regular along γ. According to the statement of Exercise 1.6.13 the full preimage $f^{-1}(\gamma)$ is a one-dimensional smooth submanifold in M. It consists of a finite number of closed curves diffeomorphic to the circle S^1 and segments (a one-dimensional connected manifold with a boundary consisting of a pair of points) diffeomorphic to a segment on the real line. We have no interest in the closed curves. Let us concentrate on the geometry of one of the segments $\hat{\gamma} : [a, b] \to M$, $f(\hat{\gamma}) \subset \gamma$. Denote $x_0 = \hat{\gamma}(a)$, $x_1 = \hat{\gamma}(b)$. Clearly the images $f(x_0)$ and $f(x_1)$ must be among the endpoints y_0, y_1 of γ. Without loss of generality we may assume that $f(x_0) = y_0$. Let us prove that, if $\deg_{x_1} f = \deg_{x_0} f$ then $f(x_1) = y_1$. In the opposite case $\deg_{x_1} f = -\deg_{x_0} f$ we will prove that $f(x_1) = f(x_0) = y_0$.

We can assume that the points y_0 and y_1 are sufficiently close to each other, that is, they belong to the same coordinate chart. Moreover, one can choose local coordinates $(y^1, \ldots, y^n)$ in such a way that the curve γ has the form

$$\gamma = \left\{ y^1(t) = 0, \ldots, y^{n-1}(t) = 0, y^n(t) = t \right\}.$$

Every component of the preimage of γ is determined by a system of equations

$$y^1(x) = 0, \ldots, y^{n-1}(x) = 0$$

where the map f in the local coordinates is given by n functions of n variables

$$x = \left(x^1, \ldots, x^n\right) \mapsto f(x) = \left(y^1(x), \ldots, y^n(x)\right).$$

The velocity vector of the curve $\hat{\gamma}$ satisfies a system of $(n-1)$ linear homogeneous equations

$$\frac{\partial y^p}{\partial x^q} \frac{dx^q}{ds} = 0, \quad p = 1, \ldots, n-1.$$

So a parameterization $x = x(s)$ on such a component can be chosen in such a way that

$$\frac{dx^i}{ds} = M^i, \quad i = 1, \ldots, n$$

where $M^i = (-1)^{n+i} \times$ the i-th minor of the matrix

$$\begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \ldots & \frac{\partial y^1}{\partial x^n} \\ \ldots & \ldots & \ldots \\ \frac{\partial y^{n-1}}{\partial x^1} & \ldots & \frac{\partial y^{n-1}}{\partial x^n} \end{pmatrix}$$

obtained by deleting of the i-th column. Transversal regularity of f along γ means exactly that this vector never vanishes at the points of $f^{-1}(\gamma)$. The map f of a component to γ can be described by a function $t = t(s)$ defined by

$$t(s) = y^n \left(x^1(s), \dots, x^n(s)\right). \tag{2.1.6}$$

Expanding the determinant of the Jacobi matrix with respect to the last row one easily deduces that

$$\frac{dt}{ds} = \frac{\partial y^n}{\partial x^1} M^1 + \cdots + \frac{\partial y^n}{\partial x^n} M^n = \det \left(\frac{\partial y^p}{\partial x^q} \right)_{1 \le p,\, q \le n}.$$

So, the function $t(s)$ is monotone at the points of $\hat\gamma$ where the Jacobian

$$Jac(s) := \det \left(\frac{\partial y^p}{\partial x^q} \right)_{x = \hat\gamma(s)}$$

does not vanish. If the Jacobian $Jac(s)$ changes sign at some $s = s_0$ then the function $t(s)$ has local maximum or minimum.

Let $\hat\gamma$ be one of segments in the full preimage. Consider the Jacobian $Jac(s)$ as a function on the curve $\hat\gamma = \hat\gamma(s)$, $s \in [a, b]$. By assumption of regularity of the endpoints $Jac(s)$ does not vanish at $x_0 = \hat\gamma(a)$ and $x_1 = \hat\gamma(b)$. If $\deg_{x_0} f = \deg_{x_1} f$ then the number of sign changes of the Jacobian is even. So the function $t(s)$ is monotone increasing or monotone decreasing both near $s = a$ and near $s = b$. This can happen only if $f\left(\hat\gamma(a)\right)$ and $f\left(\hat\gamma(b)\right)$ are two different endpoints of the curve γ. In the opposite case $\deg_{x_0} f = -\deg_{x_1} f$ the number of sign changes of the Jacobian is odd, so the function $t(s)$ is monotone increasing/decreasing near $s = a$ but it is monotone decreasing/increasing near $s = b$. Thus the points $\hat\gamma(a)$ and $\hat\gamma(b)$ go to the same endpoint of γ.

We have proved that connected components of the full preimage$f^{-1}(\gamma)$ can be subdivided into two types

$$f^{-1}(\gamma) = \left(\underset{i \in I}{\cup} \hat\gamma_i^{\text{even}} \right) \bigcup \left(\underset{j \in J}{\cup} \hat\gamma_j^{\text{odd}} \right).$$

For the segments $\hat\gamma_i^{\text{even}}$ of the first type the signs of the Jacobian at the endpoints coincide and, moreover, the images of the two endpoints of the segment go to two different endpoints y_0 or y_1 of γ. For the segments $\hat\gamma_j^{\text{odd}}$ of the second type the signs of the Jacobian at the endpoints are opposite; the images of the two endpoints of the segment go to the same endpoint of γ.

For any segment $\hat\gamma_i^{\text{even}} : [a_i, b_i] \to M$ of the first type, $i \in I$ denote

$$\sigma_i = \deg_{\gamma_i^{\text{even}}(a_i)} f.$$

Observe that

$$\sigma_i = \deg_{\gamma_i^{\text{even}}(b_i)} f. \tag{2.1.7}$$

From above considerations it follows that

$$\deg f|_{y_0} = \sum_{i \in I} \sigma_i = \deg f|_{y_1}.$$

In order to prove invariance with respect to homotopies one has to choose a regular value $y_0 \in N$ for the smooth homotopy map $F : M \times [0, 1] \to N$. In particular y_0 is a regular value for both $f_0 = F|_{M\times\{0\}}$ and $f_1 = F|_{M\times\{1\}}$. The preimage

$$F^{-1}(y_0) = \Big(\underset{i\in I}{\cup}\gamma_i^{\text{even}}\Big)\bigcup\Big(\underset{j\in J}{\cup}\gamma_j^{\text{odd}}\Big)\bigcup\Big(\underset{k\in K}{\cup}\gamma_k^{\circ}\Big).$$

is a collection of smooth closed curves γ_k° or segments γ_i^{even} or γ_j^{odd} with end-points in the boundary of the cylinder $M \times [0,1]$. Denote $\gamma = \gamma(s) = \big(x^1(s), \dots, x^n(s), t(s)\big)$ one of segments in the preimage. The parameter $s \in [a,b]$ for some a, b can be chosen in such a way that

$$\frac{dx^i}{ds} = M_i, \quad i = 1, \dots, n, \quad \frac{dt}{ds} = Jac = \det\left(\frac{\partial y^p(x,t)}{\partial x^q}\right)$$

where

$$M^i = (-1)^{n+i+1} \times \text{ the } i\text{-th minor of the Jacobi matrix } \begin{pmatrix} \frac{\partial y^1(x,t)}{\partial x^1} & \dots & \frac{\partial y^1(x,t)}{\partial x^n} & \frac{\partial y^1(x,t)}{\partial t} \\ \dots & \dots & \dots & \dots \\ \frac{\partial y^n(x,t)}{\partial x^1} & \dots & \frac{\partial y^n(x,t)}{\partial x^n} & \frac{\partial y^n(x,t)}{\partial t} \end{pmatrix}$$

of the map $F(x,t) = \big(y^1(x,t), \dots, y^n(x,t)\big)$ obtained by deleting the i-th column.

We are interested only in the components γ of $F^{-1}(y_0)$ that are segments. Such a component is called even if the number of sign changes of the Jacobian Jac is even; otherwise it is called odd. Like above it is easy to see that an even component starts at one of the pieces $M \times \{0\}$ or $M \times \{1\}$ of the boundary of the cilinder and, moreover, the degrees at the end points $x_0 \in M \times \{0\}$, $x_1 \in M \times \{1\}$ coincide

$$\deg_{x_0} f_0 = \operatorname{sign} Jac(t=0) = \operatorname{sign} Jac(t=1) = \deg_{x_1} f_1.$$

The endpoints of an odd component of $F^{-1}(y_0)$ belong to the same piece of the boundary of the cilinder, i.e., either to $M\times\{0\}$ or to $M\times\{1\}$. The degree of the map f_0 or f_1 respectively at these endpoints are opposite. Therefore

$$\deg f_0|_{y_0} = \deg f_1|_{y_0}.$$

$\square$

Exercise 2.1.6 *Prove that any polynomial of odd degree with real coefficients has a real root.*

Remark 2.1.7 *For a smooth map $f : M \to N$ of compact not necessarily oriented manifolds of the same dimension one can define in a similar way the number* $\deg f$ *modulo 2 by just counting the parity of the number of points in the preimage of a regular value $y \in M$. Like above, it is easy to prove that* $\deg f \mod 2$ *does not depend on the choice of a regular value.*

Exercise 2.1.8 *Consider a smooth map $f : S^1 \to S^1$ of the circle represented as $S^1 = \{x \in \mathbb{R}\}/(x \sim x + 2\pi)$. Derive the following formula for the degree of such a map*

$$\deg f = \frac{1}{2\pi}\int_0^{2\pi} f'(x)\,dx. \tag{2.1.8}$$

Prove that any such map is homotopic to

$$x \mapsto k\,x, \quad k = \deg f.$$

Remark 2.1.9 *More generally, for a smooth map $f : M \to N$ of compact oriented n-dimensional manifolds the following formula holds true*

$$\int_M f^*\Omega = \deg f \int_N \Omega \tag{2.1.9}$$

where Ω is an arbitrary n-form on N. A proof of this formula will be given in Section 3.4 below.

Example 2.1.10 *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial with complex coefficients. It defines a smooth map of the Riemann sphere*

$$S^2 = \mathbb{C} \cup \{\infty\}$$

to itself. By definition $f(\infty) = \infty$. Let us compute the degree of this map.

First, in the real coordinates

$$z = x + iy, \quad w = u + iv$$

the map

$$w = f(z) \tag{2.1.10}$$

satisfies

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = |f'(z)|^2 > 0 \tag{2.1.11}$$

if the derivative $f'(z)$ does not vanish at the point z. (The same is true for any holomorphic map of complex manifolds of the same dimension.) Hence the critical points of the map (2.1.10) are at the roots of the derivative. Because of positivity of the Jacobians (2.1.11) we conclude that the degree of a holomorphic map is equal to the number of points in the preimage of a regular value. For the example of a polynomal

$$f_0(z) = z^n$$

the number of points in the preimage of the value $w = 1$ is equal to n. Indeed, these are the roots $z_0, z_1, \ldots, z_{n-1}$ of the equation

$$z^n = 1, \quad z_k = e^{\frac{2\pi i k}{n}}, \quad k = 0, 1, \ldots, n-1.$$

Any other polynomial is homotopic to $f_0(z)$. The homotopy is given by the formula

$$F(z,t) = z^n + t(a_1 z^{n-1} + \cdots + a_n).$$

Because of invariance of degree of the map with respect to homotopies we derive the Main Theorem of algebra of polynomals: every polynomial with complex coefficients has n complex roots. If the value w_0 is not regular then the equation $f(z) = w_0$ may have multiple roots. In this case one must count the roots with their multiplicities.

2.2 Intersection index

Let P, $Q \subset M$ be two compact oriented submanifolds in an oriented manifold M satisfying the condition

$$\dim P + \dim Q = \dim M \tag{2.2.1}$$

for their dimensions. To any such pair of submanifolds we will assign an integer number $P \circ Q$ called *intersection index* of the submanifolds. Let us first assume that the submanifolds intersect transversally. Then the submanifolds intersect in a finite number of points. The intersection index $P \circ Q$ is the algebraic number of intersection points with signs defined in the following way. Let $y \in P \cap Q$. The transversality together with (2.2.1) implies that

$$T_y M = T_y P \oplus T_y Q.$$

Denote τ_P and τ_Q two frames in the tangent spaces $T_y P$ and $T_y Q$ respectively oriented accordingly to the orientation of these submanifolds. Also choose a positively oriented frame τ_M in $T_y M$. Define

$$\operatorname{sign} y := \operatorname{sign} \det \left(\tau_P \oplus \tau_Q \to \tau_M \right). \tag{2.2.2}$$

Definition 2.2.1 *The number*

$$P \circ Q = \sum_{y \in P \cap Q} \operatorname{sign} y \tag{2.2.3}$$

is called the intersection index *of the submanifolds P and Q.*

Observe that the intersection index depends on the order:

$$Q \circ P = (-1)^{\dim P \cdot \dim Q} P \circ Q. \tag{2.2.4}$$

Indeed, the sign of the permutation

$$(1, \ldots, p, p+1, \ldots, n) \mapsto (p+1, \ldots, n, 1, \ldots, p)$$

is equal to $(-1)^{pq}$, $q = n - p$.

Remark 2.2.2 *For any point $y \in P$ one can introduce a natural orientation on the quotient space $T_y M / T_y P$ in the following way. Let τ_P be a positively oriented frame in $T_y P \subset T_y M$. Complement τ_P with a system of $\dim M - \dim P$ linearly independent vectors $\tau_P^\perp$ in $T_y M$ such that $\left(\tau_P, \tau_P^\perp\right)$ is a positively oriented frame in $T_y M$. Projecting $\tau_P^\perp$ onto the quotient space defines the needed orientation on $T_y M / T_y P$.*

Observe that the sign (2.2.2) assigned to an intersection point $y \in P \cap Q$ of transversally intersecting manifolds of complementary dimensions can be defined in the following equivalent way. Let $i : Q \hookrightarrow M$ be the embedding map. Consider the linear map

$$A : T_y Q \stackrel{i_*}{\to} T_y M \to T_y M / T_y P.$$

Due to t-regularity and the dimension condition it has the rank $\dim M - \dim P = \dim Q$. The sign of the determinant of the matrix of this map computed in the properly oriented bases of vectors in $T_y Q$ and $T_y M / T_y P$ coincides with the sign $\operatorname{sign} y$ defined in (2.2.2).

Let us prove that the intersection index does not depend on deformations of the submanifolds. It suffices to prove independence from deformations of Q. The precise statement is given by

Theorem 2.2.3 *Let $i_0 : Q \hookrightarrow M$, $i_1 : Q \hookrightarrow M$ be two homotopic embeddings of a compact oriented manifold Q into M. Denote $Q_0 = i_0(Q) \subset M$, $Q_1 = i_1(Q) \subset M$ their images. Assume they intersect $P \subset M$ transversally and, moreover, that the dimension condition* (2.2.1) *holds true. Then*

$$P \circ Q_1 = P \circ Q_2. \tag{2.2.5}$$

Proof: Denote $F : Q \times I \to M$ the homotopy between the embeddings $i_0 = F|_{Q\times\{0\}}$ and $i_1 = F|_{Q\times\{1\}}$. Without loss of generality we may assume that the map F is t-regular along the submanifold $P \in M$. Thus the full preimage $F^{-1}(P)$ is a one-dimensional submanifold in Q. Its boundary must belong to the boundary of the cylinder $Q \times I$. Let $\gamma(s) = (x(s), t(s)) \subset F^{-1}(P)$, $s \in [a, b]$, be a connected component in the preimage that is homeomorphic to a segment connecting two points $x_0 = \gamma(a)$ and $x_1 = \gamma(b)$ on the boundary of the cylinder.

Let us begin with considering the case $x_0 \in Q \times \{0\}$, $x_1 \in Q \times \{1\}$. Denote

$$y_0 = F(x_0) = i_0(x_0) \in P \cap Q_0, \quad y_1 = F(x_1) = i_1(x_1) \in P \cap Q_1.$$

At these points the rank of the linear maps

$$A_k : T_{x_k}Q \stackrel{i_{k*}}{\to} T_{y_k}M \to T_{y_k}M/T_{y_k}P, \quad k = 0, 1$$

is equal to $\dim M - \dim P = \dim Q$ due to t-regularity of i_0 and i_1. The sign of the determinant of these maps coincides with the signs at the intersection points $y_0 \in P \cap Q_0$ and $y_1 \in P \cap Q_1$ respectively

$$\text{sign det } A_0 = \text{sign}\, y_0, \quad \text{sign det } A_1 = \text{sign}\, y_1$$

(cf. an alternative definition of these signs given in Remark 2.2.2).

Denote $F_t : Q \to M$ the restriction

$$F_t = F|_{Q\times\{t\}}.$$

For $s \in [a, b]$ consider the linear map

$$A(s) : T_{\gamma(s)}Q \stackrel{F_{t*}}{\to} T_{y(s)}M \to T_{y(s)}M/T_{y(s)}P, \quad t = t(s), \quad y(s) = F(\gamma(s)).$$

We have $A(a) = A_0$, $A(b) = A_1$. Like in the proof of Theorem 2.1.5 we can choose the parameterization of the curve $\gamma(s)$ in such a way that

$$\frac{dt}{ds} = \det A(s).$$

Thus, in the case under consideration $t(a) = 0$, $t(b) = 1$, so the number of changes of the sign must be even (like above without loss of generality we may assume that all zeroes of $\frac{dt}{ds}$ are simple). Hence, in this case, sign det A_0 = sign det A_1. Such a component γ in the preimage $F^{-1}(P)$ will be called *even*.

In a similar way assuming that the end points $\gamma(a)$ and $\gamma(b)$ both belong to $Q \times \{0\}$ or to $Q \times \{1\}$ we prove that the number of changes of sign of $dt/ds = \det A(s)$ is odd. So the signs $\operatorname{sign} y_0$ and $\operatorname{sign} y_1$ at the intersection points $y_0 \in P \cap Q_0$ and $y_1 \in P \cap Q_1$ are opposite. Such a component γ in the preimage $F^{-1}(P)$ will be called *odd*.

From the above considerations it follows that, computing the intersection indices $P \circ Q_0$ and $P \circ Q_1$ it suffices to take into consideration only those points in the intersection $P \cap Q_0$ and $P \cap Q_1$ that are images of endpoints of even components of $F^{-1}(P)$. The algebraic numbers of these intersection points coincide. Hence $P \circ Q_0 = P \circ Q_1$. □

Remark 2.2.4 *If the submanifolds P, $Q \subset M$ are not oriented then the intersection index $P \circ Q$ is defined modulo 2 (cf. Remark 2.1.7 above). The homotopy invariance of $P \circ Q$* mod 2 *can be proved in a similar way.*

The arguments similar to those used in the proof of the Theorem can be applied to the following slightly more general situation.

Exercise 2.2.5 *Let*

$$F : W \to M$$

be a smooth map of an oriented compact manifold of dimension $(q + 1)$ with an oriented boundary $\partial W = Q_1 \cup (-Q_0)$ into an oriented manifold M. Assume that the restrictions $F|_{Q_0}$ and $F|_{Q_1}$ are embeddings of these q-dimensional manifolds and, moreover, their images intersect transversally a compact oriented submanifold $P \subset M$ of the complementary dimension $\dim P = \dim M - q$. *Prove that $P \circ F(Q_0) = P \circ F(Q_1)$.*

We are now in a position to define the intersection index for any pair of compact oriented submanifolds P, $Q \subset M$ of complementary dimensions $\dim P + \dim Q = \dim M$ in an oriented manifold M. According to Theorem 1.6.18 there exist small deformations of P and Q such that the deformed submanifolds $\tilde{P}$ and $\tilde{Q}$ intersect transversally. Define

$$P \circ Q := \tilde{P} \circ \tilde{Q}.$$

The deformed embeddings $\tilde{P}$, $\tilde{Q}$ are homotopic[3] to the original embeddings P, Q. So, due to the Theorem, this definition does not depend on the choice of deformations.

Let us derive few useful corollaries from the Theorem.

Corollary 2.2.6 *Let P, Q be two oriented compact submanifolds of complementary dimensions in $\mathbb{R}^n$. Then $P \circ Q = 0$.*

Proof: Shifting Q along a sufficiently long vector one obtains another submanifold Q' that does not intersect P. The new embedding is homotopic to the old one. Hence $P \circ Q = P \circ Q' = 0$. □

As an application of the above considerations we will prove that a compact submanifold of codimension one in Euclidean space is necessarily orientable.

[3] This will be proved in Section 4.4 below.

Lemma 2.2.7 *Let $M \subset \mathbb{R}^n$ be a compact $(n-1)$-dimensional submanifold. Then the complement $\mathbb{R}^n \setminus M$ is disconnected.*

Proof: Choose an arbitrary point $x_0 \in M$. Denote $\mathbf{n}$ the unit vector orthogonal to $T_{x_0}M$. For sufficiently small $\epsilon > 0$ the points $x_\pm = x_0 \pm \epsilon\,\mathbf{n}$ do not belong to M. If $\mathbb{R}^n \setminus M$ is connected then there exists a curve $\gamma : [-1, 1] \to \mathbb{R}^n \setminus M$ such that $\gamma(\pm 1) = x_\pm$. Adding the segment $[x_-, x_+]$ one obtains a closed curve $\hat{\gamma}$ in $\mathbb{R}^n$ intersecting the submanifold M in one point x_0. Hence $M \circ \hat{\gamma} = \pm 1$ (in the nonoriented case the intersection index modulo 2 is well defined). This contradicts Corollary 2.2.6. $\square$

Corollary 2.2.8 *A connected compact submanifold of codimension 1 in Euclidean space is orientable.*

Proof: At any point $x \in M$ of the submanifold choose a normal vector $\mathbf{n}(x)$ of a sufficiently small length $\epsilon > 0$ "looking" towards one of the two components of the complement (we leave as an easy exercise to the reader to prove that, for a connected submanifold the complement cannot consist of more than two components). By definition a frame of tangent vectors e_1, $\dots$, e_{n-1} at x is positively oriented in T_xM if the frame $e_1, \dots, e_{n-1}, \mathbf{n}(x)$ is positively oriented in $\mathbb{R}^n$. $\square$

2.3 Index of a vector field on a manifold

Let $v(x)$ be a smooth vector field on a compact oriented manifold M. Assume that all stationary points of the vector field are nondegenerate. That means that the graph $M_v = \{(x, v(x)) \,|\, x \in M\} \subset TM$ is t-regular along the zero section $M_0 = \{x, 0\} \subset TM$ (see Example 1.6.21 above). Observe that $\dim M_0 + \dim M_v = \dim TM$. Define

$$\operatorname{ind} v := M_0 \circ M_v. \tag{2.3.1}$$

This definition can be extended to an arbitrary smooth vector field as any vector field can be slightly deformed to obtain another one with all nondegenerate stationary points.

Proposition 2.3.1 *The index* (2.3.1) *does not depend on the choice of a vector field.*

Proof: The submanifolds M_v and M_0 can be considered as two embeddings of M into TM. They are homotopic. Indeed, the homotopy is obtained by rescaling $v \mapsto t\,v$, $t \in [0, 1]$. Therefore for any two vector fields v_1 and v_2 the embeddings M_{v_1} and M_{v_2} are homotopic. Hence $\operatorname{ind} v_1 = \operatorname{ind} v_2$. $\square$

An alternative definition of index of a vector field can be given by the following construction. Let x_0 be a nondegenerate stationary point of a vector field $v(x)$ on an n-dimensional oriented manifold M. We know that, for a sufficiently small $\epsilon > 0$ the vector field does not vanish at the points of the sphere $S_\epsilon^{n-1}(x_0) = \{x \,|\, |x - x_0| = \epsilon\} \subset M$ of radius ϵ with the center at the point x_0. Define a map[4]

$$
\begin{aligned}
g_v : S^{n-1}_\epsilon(x_0) &\to S^{n-1} \subset T_{x_0}M \\
S^{n-1}_\epsilon(x_0) \ni x &\mapsto \frac{v(x)}{|v(x)|}.
\end{aligned}
\tag{2.3.2}
$$

Define the index of the stationary point $\operatorname{ind} x_0$ as the degree of this map

$$
\operatorname{ind} x_0 := \deg g_v. \tag{2.3.3}
$$

Proposition 2.3.2 *The total index index* $\operatorname{ind} v$ *of a vector field* v *with nondegenerate stationary points is equal to the sum of indices of stationary points*

$$
\operatorname{ind} v = \sum_{v(x)=0} \operatorname{ind} x. \tag{2.3.4}
$$

Proof: Without loss of generality we can assume that $x_0 = 0$ in the chosen local coordinates. Denote

$$
A = \left(\frac{\partial v^i(x)}{\partial x^j} \right)_{x=x_0}.
$$

Introduce linear vector field

$$
v_A(x) = A\, x.
$$

Let us prove that the vector fields $v(x)$ and $v_A(x)$ are homotopic in the class of vector fields with isolated stationary point at the origin. Indeed, consider the Taylor expansion of the vector valued function $v(x)$

$$
v(x) = A\, x + \mathcal{O}\left(|x|^2\right).
$$

The homotopy $v_t(x)$ is defined by the following formula

$$
v_t(x) = \frac{v(t\, x)}{t} = A\, x + \mathcal{O}(t), \quad v_1(x) = v(x), \quad v_0(x) = v_A(x).
$$

Due to homotopy invariance of degree it remains to compute the degree of the map (4.11.36) for $v = v_A$. The linear map $x \mapsto A\, x$ is one-to-one on the sphere $|x| = \epsilon$. Moreover, all points of the sphere are regular points. Hence the degree is equal to $\pm =$ sign det A. $\square$

Exercise 2.3.3 *Let $v(x)$ be a vector field on an oriented manifold M with isolated, but not necessarily nondegenerate stationary points. In this case one can define the index of every stationary point as the degree of the spherical map* (4.11.36). *Prove that the sum of indices of all stationary points coincide with the Euler characteristic of M.*

Exercise 2.3.4 *Prove that the index of a nonzero vector field $v(t) = (P(t), Q(t))$ defined on a closed curve $\gamma(t)$, $0 \le t \le 2\pi$ on the plane with an isolated stationary point inside the curve is given by the formula*

$$
\frac{1}{2\pi} \int_0^{2\pi} \frac{P\, \dot{Q} - Q\, \dot{P}}{P^2 + Q^2}\, dt. \tag{2.3.5}
$$

[4]In these formulae the lengths of vectors are computed in local coordinates, i.e., $|x - x_0| = \sqrt{(x^1 - x_0^1)^2 + \cdots + (x^n - x_0^n)^2}$, $|v| = \sqrt{(v^1)^2 + \cdots + (v^n)^2}$.

Exercise 2.3.5 *Prove that the index of a vector field* $\mathbf{v}(s,t)\,(P(s,t),Q(s,t),R(s,t))$ *defined on a closed surface* $\Sigma = \mathbf{r}(s,t)$ *in* $\mathbb{R}^3$ *with an isolated stationary point inside is given by the formula*

$$\frac{1}{4\pi}\int_\Sigma \begin{vmatrix} P & Q & R \\ P_s & Q_s & R_s \\ P_t & Q_t & R_t \end{vmatrix} \frac{ds\,dt}{(P^2+Q^2+R^2)^{3/2}}. \tag{2.3.6}$$

The result of Proposition 2.3.1 justifies the following

Definition 2.3.6 Euler characteristic $\chi(M)$ *of a compact oriented manifold* M *is defined as the index of an arbitrary smooth vector field on* M.

2.4 Morse index

Let f be a Morse function on a compact oriented n-dimensional manifold M (see above Example 1.6.22). Recall that means that the graph of the differential

$$L_f = \{(x, df(x)\} \subset T^*M$$

intersects transversally the zero section $L_0 = \{(x,0)\}$. Define the *index* of the Morse function by

$$\operatorname{ind} f := L_0 \circ L_f. \tag{2.4.1}$$

We will now spell out this definition as the sum of *Morse indices* of critical points of f. Let us first recall the following statement from linear algebra.

Proposition 2.4.1 *For any symmetric matrix* A *there exists a nondegenerate matrix* M *such that*

$$M^T A\, M = \operatorname{diag}(\underbrace{1,1,\dots,1}_{p}, \underbrace{-1,\dots,-1}_{q}, \underbrace{0,\dots,0}_{r}), \quad p+q+r=n \tag{2.4.2}$$

were M^T *is the transposed matrix. The numbers* p *and* q *do not depend on the choice of the reducing matrix* M. *They are called respectively* positive *and* negative inertia indices *of the symmetric matrix* A.

Let x_0 be a Morse critical point of a function f. Consider the matrix $A = (A_{ij})$ of the second derivatives at the critical point

$$A_{ij} = \frac{\partial^2 f(x_0)}{\partial x^i \partial x^j}, \quad i, j = 1, \dots, n. \tag{2.4.3}$$

This is a symmetric nondegenerate matrix.

Definition 2.4.2 *The negative inertia index of the symmetric matrix* (2.4.3) *is called the* index *of the Morse critical point. It will be denoted* $\operatorname{ind} x_0$.

Lemma 2.4.3 *Index of a Morse critical point does not depend on the choice of local coordinates,* *Proof:* In another coordinate system $(y^1, \dots, y^n)$, $y^k = y^k(x^1, \dots, x^n)$ the second derivatives of the function can be computed by the following rule

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial y^k \partial y^l}\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j} + \frac{\partial f}{\partial y^k}\frac{\partial^2 y^k}{\partial x^i \partial x^j}.$$

At the critical point the second term vanishes. So, the matrices A_y and A_x of second derivatives of f at the critical point in the coordinates y^k and x^i respectively are related by the equation

$$A_x = J^T A_y\, J, \quad J = \left(\frac{\partial y^k(x_0)}{\partial x^i}\right).$$

The inertia indices do not change under such transformations. □

Remark 2.4.4 *The above considerations show that the second differential of the function f at a critical point x_0*

$$d^2 f(x_0) = \frac{\partial^2 f(x_0)}{\partial x^i \partial x^j} dx^i dx^j \tag{2.4.4}$$

is a well defined quadratic form on the tangent space $T_{x_0}M$. For a Morse critical point the negative inertia index is the maximal dimension of a subspace $V \subset T_{x_0}M$ such that the restriction

$$d^2 f(x_0)|_V$$

is a negative definite quadratic form.

Proposition 2.4.5 *The index of a Morse function f on a compact oriented manifold M is equal to*

$$\operatorname{ind} f = \sum_{df(x)=0} (-1)^{\operatorname{ind} x}. \tag{2.4.5}$$

Proof: At a Morse critical point x_0 the signs of $\det\left(\frac{\partial^2 f(x_0)}{\partial x^i \partial x^j}\right)$ and $(-1)^{\operatorname{ind} x_0}$ coincide. So the statement of the Proposition follows from the definition of the intersection index

$$L_0 \circ L_f = \sum_{df(x)=0} \operatorname{sign}\det\left(\frac{\partial^2 f(x)}{\partial x^i \partial x^j}\right).$$

□

2.5 Lefschetz number. Brouwer theorem

3 Tensors on a manifold. Integration of differential forms. Cohomology

3.1 Tensors on manifolds

Definition 3.1.1 *A* tensor of type (p,q) *at a point P of a manifold M is described by a table of n^{p+q} real numbers*

$$A^{i_1 \dots i_p}_{j_1 \dots j_q}, \quad i_1, \dots, i_p,\, j_1, \dots, j_q = 1, \dots, n$$

called components *of the tensor in a given coordinate chart* $(U, (x^1, \dots, x^n))$ *containing the point* P. *In another coordinate chart* $(U', (x^{1'}, \dots, x^{n'}))$ *containing* P *the tensor is described by another table of components*

$$A^{i'_1 \dots i'_p}_{j'_1 \dots j'_q}$$

related with the old one by the following transformation law

$$A^{i'_1 \dots i'_p}_{j'_1 \dots j'_q} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} A^{i_1 \dots i_p}_{j_1 \dots j_q}. \tag{3.1.1}$$

In this equation all partial derivatives have to be evaluated at the point P.

Example 0. A tensor of type $(0, 0)$ is described by a real number depending on the point P but independent from the choice of a coordinate system. Thus, it is just a smooth function on M.

Example 1. For $(p, q) = (1, 0)$ the transformation law (3.1.1) specializes to

$$A^{i'} = \frac{\partial x^{i'}}{\partial x^i}\Big|_P A^i.$$

So the $(1, 0)$ tensors are just vectors at the point P.

Example 2. Tensors of type $(0, 1)$ can be identified with covectors at the point P since the transformation law (3.1.1) specializes to

$$A_{j'} = \frac{\partial x^j}{\partial x^{j'}}\Big|_P A_j.$$

Given two tensors $A = (A^{i_1 \dots i_p}_{j_1 \dots j_q})$ and $B = (B^{i_1 \dots i_p}_{j_1 \dots j_q})$ of the same type (p, q) at the point P, their linear combination $\alpha\, A + \beta\, B$

$$\alpha A^{i_1 \dots i_p}_{j_1 \dots j_q} + \beta B^{i_1 \dots i_p}_{j_1 \dots j_q} \tag{3.1.2}$$

is again a tensor of the same type. One obtains a linear space $\mathcal{T}^p_q(P)$ of tensors of a given type (p, q) at a given point $P \in M$. Clearly the dimension of this space is equal to n^{p+q} where $n = \dim M$.

The operators of *permutation* of two upper or two lower indices act on this linear space if $p \geq 2$ or $q \geq 2$. Choosing a pair i_k, i_l of two upper indices, $1 \leq k < l \leq p$ define the permutation operator Π^{kl} acting as follows

$$\left(\Pi^{kl} A\right)^{i_1 \dots i_k \dots i_l \dots i_p}_{j_1 \dots j_q} = A^{i_1 \dots i_l \dots i_k \dots i_p}_{j_1 \dots j_q}. \tag{3.1.3}$$

In a similar way one can define the operator Π_{kl} of permutation of two lower indices $1 \leq k < l \leq q$.

Lemma 3.1.2 *The operators* Π^{kl} *and* Π_{kl} *are well defined linear operators acting on the space* $\mathcal{T}^p_q(P)$.

There are also important tensor operations that change the type of tensors. The first one is an operation of *tensor product* of two tensors $A = (A^{i_1\dots i_p}_{j_1\dots j_q})$ and $B = (B^{i_1\dots i_r}_{j_1\dots j_s})$ of the types (p,q) and (r,s) respectively. It produces a tensor $A\otimes B$ of type $(p+r, q+s)$ with the components

$$(A\otimes B)^{i_1\dots i_p k_1\dots k_r}_{j_1\dots j_q l_1\dots l_s} = A^{i_1\dots i_p}_{j_1\dots j_q} B^{k_1\dots k_r}_{l_1\dots l_s}. \tag{3.1.4}$$

This operation is linear with respect to every factor:

$$(\alpha A + \beta B)\otimes C = \alpha\, A\otimes C + \beta\, B\otimes C, \quad A\otimes(\alpha B + \beta C) = \alpha\, A\otimes B + \beta\, A\otimes C. \tag{3.1.5}$$

Remark 3.1.3 *Let us remind the operation of tensor product of linear spaces known from linear algebra. By definition the tensor product of spaces X and Y of linear spaces of dimensions n and m respectively is a linear space $X\otimes Y$ of dimension nm. The vectors of this space are finite sums*

$$\sum_i \lambda_i\, x_i\otimes y_i, \quad x_i\in X,\ y_i\in Y, \quad \lambda_i\in\mathbb{R} \tag{3.1.6}$$

considered modulo the following equivalence relations

$$(\alpha x_1 + \beta x_2)\otimes y \sim \alpha\, x_1\otimes y + \beta\, x_2\otimes y, \quad x\otimes(\alpha y_1 + \beta y_2) \sim \alpha\, x\otimes y_1 + \beta\, x\otimes y_2 \tag{3.1.7}$$
$$\alpha,\ \beta\in\mathbb{R}.$$

If $e_1, \dots, e_n$ is a basis in X and $f_1, \dots, f_m$ is a basis in Y then the vectors $e_i\otimes f_j$ make a basis in $X\otimes Y$. The decomposition of the tensor product of vectors $x = x^i e_i\in X$, $y = y^j f_j \in Y$ with respect to this basis reads

$$x\otimes y = x^i y^j\, e_i\otimes f_j. \tag{3.1.8}$$

A generic vector in $X\otimes Y$ can be written as

$$z = z^{ij} e_i\otimes f_j \tag{3.1.9}$$

where the entries of the $n\times m$ matrix z^{ij} can be considered as the coordinates of this vector.

The space $\mathcal{T}^p_q(P)$ of tensors of type (p,q) can be identified with the tensor product of p copies of the tangent space T_PM and q copies of cotangent space T^*_PM. A choice of local coordinates $(x^1, \dots, x^n)$ on M provides one with a basis

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$$

in the tangent space and a basis

$$dx^1, \dots, dx^n$$

in the cotangent space. In this way one obtains a basis in the tensor product $\mathcal{T}^p_q(P)$

$$\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_q}. \tag{3.1.10}$$

A decomposition of a tensor $A = (A^{i_1\dots i_p}_{j_1\dots j_q})$ with respect to this basis reads

$$A = A^{i_1\dots i_p}_{j_1\dots j_q}\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_q}. \tag{3.1.11}$$

Another important operation is *contraction* of tensors. It depends on the choice of one upper and one lower index. Applying this operation to a tensor of type (p, q), $p \geq 1$ and $q \geq 1$ one obtains a tensor of type $(p-1, q-1)$. If the chosen upper index is i_k and chosen lower index is j_l then the contraction C_l^k with respect to these two indices applied to a tensor $A_{j_1 \dots j_q}^{i_1 \dots i_p}$ produces a tensor with the components

$$\left(\mathrm{C}_l^k A\right)_{j_1 \dots j_{q-1}}^{i_1 \dots i_{p-1}} = A_{j_1 \dots j_{l-1}\, s\, j_l \dots j_{q-1}}^{i_1 \dots i_{k-1}\, s\, i_k \dots i_{p-1}}. \tag{3.1.12}$$

Lemma 3.1.4 *For any* $1 \leq k \leq p$, $1 \leq l \leq q$ *the contraction* C_l^k *is a well defined linear operator*

$$\mathrm{C}_l^k : \mathcal{T}_q^p(P) \to \mathcal{T}_{q-1}^{p-1}(P). \tag{3.1.13}$$

Example 1. A combination of tensor product of a vector $v = (v^i)$ and a $(1,1)$-tensor $A = (a_j^i)$

$$(A, v) \mapsto a_j^i v^k$$

with contraction with respect to the indices k and j produces a vector

$$Av = (a_j^i v^j).$$

One obtains a realization of $(1,1)$-tensors as linear operators on the tangent space

$$A : T_P M \to T_P M, \quad v \mapsto Av. \tag{3.1.14}$$

The same $(1,1)$-tensor A can also be identified with linear operation on the cotangent space

$$\omega \mapsto (a_j^i \omega_i) =: A^* \omega. \tag{3.1.15}$$

This is the adjoint linear operator to (3.1.14). Both realizations can be easily obtained from the natural isomorphisms of the space of $(1,1)$-tensors $\mathcal{T}_1^1(P) \simeq T_P \otimes T_P^*$ with

$$T_P \otimes T_P^* \simeq Hom(T_P, T_P) \simeq Hom(T_P^*, T_P^*). \tag{3.1.16}$$

Example 2. A $(0,2)$-tensor $B = (b_{ij})$ can be realized as a bilinear form on the tangent space by means of the operations of tensor product and double contraction

$$(B, v, w) \mapsto b_{ij} v^k w^l \mapsto b_{ij} v^i w^j =: b(x, y). \tag{3.1.17}$$

Cf. also the isomorphism

$$\mathcal{T}_2^0(P) \simeq T_P^* \otimes T_P^* \simeq Hom(T_P \otimes T_P, \mathbb{R}). \tag{3.1.18}$$

The same tensors can also be realized by the operator of *lowering the indices*

$$B : T_P \to T_P^*, \quad v^i \mapsto b_{ij} v^j \tag{3.1.19}$$

that can also be understood in view of the natural isomorphism

$$\mathcal{T}_2^0(P) \simeq T_P^* \otimes T_P^* \simeq Hom(T_P, T_P^*). \tag{3.1.20}$$

In particular the tensor b_{ij} is nondegenerate if the operator (3.1.19) is an isomorphism.

Example 3. In a similar way any tensor $A = (a_{i_1\dots i_k})$ of type $(0,k)$ can be realized as a k-linear form on the tangent space

$$\begin{aligned}
&A(x_1,\dots,x_n) = a_{i_1\dots i_k} x_1^{i_1}\dots x_k^{i_k} \\
&A(\alpha x_1+\beta y_1, x_2,\dots,x_k) = \alpha A(x_1,x_2,\dots,x_k) + \beta A(y_1,x_2,\dots,x_k) \\
&\dots \qquad \dots \qquad \dots \qquad \dots \\
&A(x_1,\dots,x_{k-1},\alpha x_k+\beta y_k) = \alpha A(x_1,\dots,x_{k-1},x_k) + \beta A(x_1,\dots,x_{k-1},y_k).
\end{aligned} \tag{3.1.21}$$

The k-linear form is symmetric/antisymmetric

$$A(x_1,\dots,x_j,\dots,x_i,\dots,x_k) = \pm A(x_1,\dots,x_i,\dots,x_j,\dots,x_k) \quad \forall\ 1\le i<j\le k \tag{3.1.22}$$

iff the tensor $a_{i_1\dots i_k}$ is invariant/antiinvariant with respect to the operators Π_{ij} of permutation of indices. Recall that, according to the Lemma 3.1.2 the property of symmetry/antisymmetry of a tensor does not depend on the choice of local coordinates.

Remark 3.1.5 *The permutation group S_k of k symbols acts on the space of $(0,k)$-tensors by permuting the indices. A permutation*

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & k \\ \sigma(1) & \sigma(2) & \dots & \sigma(k)\end{pmatrix} \in S_k \tag{3.1.23}$$

acts as follows

$$a_{i_1 i_2\dots i_k} \mapsto a_{i_{\sigma(1)} i_{\sigma(2)}\dots i_{\sigma(k)}}. \tag{3.1.24}$$

Symmetric tensors remain invariant with respect to this action; antisymmetric tensors transform according to the following rule

$$a_{i_{\sigma(1)} i_{\sigma(2)}\dots i_{\sigma(k)}} = \operatorname{sign}\sigma\, a_{i_1 i_2\dots i_k} \tag{3.1.25}$$

where $\operatorname{sign}\sigma = \pm 1$ *is the sign of the permutation. To any $(0,k)$-tensor one can apply the operator of* symmetrization Sym *and* alternation Alt *producing symmetric/antisymmetric tensors respectively:*

$$\operatorname{Sym} a_{i_1\dots i_k} = \frac{1}{k!}\sum_{\sigma\in S_k} a_{i_{\sigma(1)} i_{\sigma(2)}\dots i_{\sigma(k)}} \tag{3.1.26}$$

$$\operatorname{Alt} a_{i_1\dots i_k} = \frac{1}{k!}\sum_{\sigma\in S_k} \operatorname{sign}\sigma\, a_{i_{\sigma(1)} i_{\sigma(2)}\dots i_{\sigma(k)}}. \tag{3.1.27}$$

Similarly to vector fields and differential forms one can consider *tensor fields*. A tensor field of type (p,q) is described by a collection of functions $a^{i_1\dots i_p}_{j_1\dots j_q}(x)$ smoothly depending on the point x. The functions depend on the choice of local coordinates; a change of local coordinates $(x^1,\dots,x^n)\mapsto(x^{1'},\dots,x^{n'})$ yields a transformation of these functions

$$a^{i_1\dots i_p}_{j_1\dots j_q}(x) \mapsto a^{i'_1\dots i'_p}_{j'_1\dots j'_q}(x')$$

according to the tensor law (3.1.1). The decompostion (3.1.11) of a tensor with respect to the natural basis associated with the coordinate chart does not change the form under changes of local coordinates

$$A^{i_1\dots i_p}_{j_1\dots j_q}(x)\frac{\partial}{\partial x^{i_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\cdots\otimes dx^{j_q} = A^{i'_1\dots i'_p}_{j'_1\dots j'_q}(x')\frac{\partial}{\partial x^{i'_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{i'_p}}\otimes dx^{j'_1}\otimes\cdots\otimes dx^{j'_q}. \tag{3.1.28}$$

3.2 Vector bundles

Informally speaking a *vector bundle* of rank k over a manifold B is a family of k-dimensional vector spaces called *fibers* parameterized by points of B called the *base* of the vector bundle. Let us proceed to the precise definition.

Definition 3.2.1 *A rank k vector bundle over an n-dimensional manifold B consists of*

1) a $(n+k)$-dimensional manifold E called the total space *of the vector bundle;*

2) a smooth submersion

$$\pi : E \to B \tag{3.2.1}$$

called projection*;*

3) a collection of open domains $(U_\alpha)_{\alpha\in I}$ covering the base B and diffeomorphisms

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k \tag{3.2.2}$$

called local trivialization *of E over U_α such that the diagrams*

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \overset{\Phi_\alpha}{\to} & U_\alpha \times \mathbb{R}^k \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U_\alpha & \overset{\mathrm{id}}{\to} & U_\alpha \end{array} \tag{3.2.3}$$

are all commutative. Observe that the full preimage of any point $P \in B$ is isomorphic to $\mathbb{R}^k$

$$F_P := \pi^{-1}(P) \simeq \mathbb{R}^k.$$

It is called the fiber *over the point P.*

4) On the intersection $\pi^{-1}(U_\alpha \cap U_\beta)$ one has transition functions of the form

$$\begin{aligned} &\Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k \\ &(P;\xi_\alpha) \mapsto (P;\xi_\beta = T_{\beta\alpha}(P)\,\xi_\alpha) \end{aligned} \tag{3.2.4}$$

where $P \in U_\alpha \cap U_\beta$, $\xi_\alpha \in \mathbb{R}^k$,

$$T_{\beta\alpha} : U_\alpha \cap U_\beta \to GL(k,\mathbb{R}) \tag{3.2.5}$$

is a smooth map.

Remark 3.2.2 *The collection of smooth matrix valued functions $T_{\alpha\beta}$ defined on intersections of domains $U_\alpha \cap U_\beta$, α, $\beta \in I$ clearly satisfies the following properties*

$$T_{\alpha\alpha} = \mathrm{id}_{\mathbb{R}^k}$$

$$T_{\beta\alpha} = T_{\alpha\beta}^{-1}$$

$$T_{\alpha\beta} T_{\beta\gamma} T_{\gamma\alpha} = \mathrm{id}_{\mathbb{R}^k} \quad \text{on triple intersection of} \quad U_\alpha,\ U_\beta,\ \text{and}\ U_\gamma. \tag{3.2.6}$$

Such functions completely determine the structure of the fiber bundle.

Definition 3.2.3 *A* morphism *of vector bundles $\pi_1 : E_1 \to B_1$ to $\pi_2 : E_2 \to B_2$ is a pair of smooth maps $f : B_1 \to B_2$ and $F : E_1 \to E_2$ such that the diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

is commutative.

3.3 Integration of differential forms. Cohomology

In this section we will consider the particular case of antisymmetric tensors and tensor fields of type $(0, k)$. The space of antisymmetric $(0, k)$-tensors at the point $P \in M$ will be denoted $\Lambda^k T_P^* M \subset T_P^* \otimes \cdots \otimes T_P^*$ (k copies of the cotangent space $T_P^* M$).

Lemma 3.3.1 *Denote n the dimension of the manifold M. Then the dimension of the space $\Lambda^k = \Lambda^k T_P^* M$ of $(0, k)$-antisymmetric tensors is equal to*

$$\dim \Lambda^k = \begin{cases} \left(\begin{array}{c} n \\ k \end{array} \right), & k \leq n \\ \\ 0, & k > n \end{cases} \tag{3.3.1}$$

Proof: Due to antisymmetry the component $\omega_{i_1 i_2 \ldots i_k}$ of the tensor $\omega \in \Lambda^k$ having a pair of equal indices is equal to zero. If $k > n$ then there is at least one pair of equal indices among $i_1,\ i_2,\ \ldots,\ i_k$. Hence $\omega = 0$. For $k \leq n$ the independent coordinates in the space Λ^k are

$$\omega_{i_1 i_2 \ldots i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n. \tag{3.3.2}$$

The number of these components is equal to the binomial coefficient

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}.$$

$\square$

Combining the operation of tensor products with the alternation one obtains the operation of *exterior product* (or *wedge product*)

$$\begin{aligned} &\Lambda^k \times \Lambda^l \to \Lambda^{k+l} \\ &(\alpha, \beta) \mapsto \alpha \wedge \beta := \mathrm{Alt}(\alpha \otimes \beta). \end{aligned} \tag{3.3.3}$$

An explicit expression for the components of $\alpha \wedge \beta$ is given by the following formula

$$(\alpha \wedge \beta)_{j_1 \dots j_{k+l}} = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} \operatorname{sign} \sigma \, \alpha_{j_{\sigma_1} \dots j_{\sigma_k}} \beta_{j_{\sigma_{k+1}} \dots j_{\sigma_{k+l}}}. \tag{3.3.4}$$

Taking the wedge products of basic 1-forms

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \operatorname{sign} \sigma \, dx^{i_{\sigma(1)}} \otimes \dots \otimes dx^{i_{\sigma(k)}} \tag{3.3.5}$$

one obtains a basis in Λ^k. A decomposition of an antisymmetric tensor $\omega = (\omega_{i_1 \dots i_k})$ with respect to this basis is written as a *differential k-form*

$$\omega = \sum_{i_1 < i_2 < \dots < i_k} \omega_{i_1 i_2 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}. \tag{3.3.6}$$

Such a form was defined so far at one point of the manifold. Considering the corresponding tensor fields one obtains differential forms $\omega(x)$ defined on the manifold M or on some part of it.

The exterior product is a bilinear associative operation satisfying the following *graded commutativity property*

$$\beta \wedge \alpha = (-1)^{k\, l} \alpha \wedge \beta, \quad \alpha \in \Lambda^k, \quad \beta \in \Lambda^l. \tag{3.3.7}$$

The total space of antisymmetric tensors at a given point

$$\Lambda T_P^* M = \Lambda^0 \oplus \Lambda^1 \oplus \dots \oplus \Lambda^n \tag{3.3.8}$$

acquires a structure of *exterior algebra*. The dimension of this space is equal to 2^n. Using the representation of antisymmetric tensors by differential forms (3.3.6) one can reformulate the definition of the wedge product in the following simple way:

$$\begin{aligned} &dx^i \wedge dx^i = 0 \\ &dx^j \wedge dx^i = -dx^i \wedge dx^j, \quad i \neq j \end{aligned}$$

along with the bilinearity and associativity.

Example 3.3.2 *The wedge product of two one-forms $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3$ and $\beta = \beta_1 dx^1 + \beta_2 dx^2 + \beta_3 dx^3$ on a three-dimensional manifold is equal to*

$$\begin{aligned} &\alpha \wedge \beta = (\alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3) \wedge (\beta_1 dx^1 + \beta_2 dx^2 + \beta_3 dx^3) \\ &= \alpha_1 \beta_2 dx^1 \wedge dx^2 + \alpha_2 \beta_1 dx^2 \wedge dx^1 + \alpha_1 \beta_3 dx^1 \wedge dx^3 + \alpha_3 \beta_1 dx^3 \wedge dx^1 + \alpha_2 \beta_3 dx^2 \wedge dx^3 + \alpha_3 \beta_2 dx^3 \wedge dx^2 \\ &= (\alpha_2 \beta_3 - \alpha_3 \beta_2) dx^2 \wedge dx^3 + (\alpha_3 \beta_1 - \alpha_1 \beta_3) dx^3 \wedge dx^1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) dx^1 \wedge dx^2. \end{aligned}$$

Exercise 3.3.3 *Prove that the product of three one-forms* $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3$, $\beta = \beta_1 dx^1 + \beta_2 dx^2 + \beta_3 dx^3$ *and* $\gamma = \gamma_1 dx^1 + \gamma_2 dx^2 + \gamma_3 dx^3$ *in the three-dimensional space is given by*

$$\alpha \wedge \beta \wedge \gamma = \det \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} dx^1 \wedge dx^2 \wedge dx^3. \tag{3.3.9}$$

Let us concentrate our attention on the particular case of n-forms on a n-dimensional smooth manifold M. In a coordinate chart $(U, (x^1, \dots, x^n))$ such a form is defined by just one smooth function $\omega_{12\dots n}(x)$

$$\omega = \omega_{12\dots n}(x) dx^1 \wedge dx^2 \wedge \dots \wedge dx^n. \tag{3.3.10}$$

This function depends, however, on the choice of the chart. In this particular case the general tensor law (3.1.1) takes a particularly simple form.

Lemma 3.3.4 *Under a change of the local coordinates* $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ *the coefficient of a* n*-form transforms as follows*

$$\omega_{1'2'\dots n'}(x') = \det\left(\frac{\partial x}{\partial x'}\right) \omega_{12\dots n}(x). \tag{3.3.11}$$

Proof: The tensor law (3.1.1) in the particular case of $(0, n)$-tensors takes the following form

$$\omega_{1'2'\dots n'} = \frac{\partial x^{i_1}}{\partial x^{1'}} \frac{\partial x^{i_2}}{\partial x^{2'}} \dots \frac{\partial x^{i_n}}{\partial x^{n'}} \omega_{i_1 i_2 \dots i_n}.$$

In the sum over repeated indices in the right hand side of this equation only the terms with all pairwise distinct indices will survive. In this case

$$\omega_{i_1 i_2 \dots i_n} = \operatorname{sign} \sigma \, \omega_{12\dots n}, \quad \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix} \in S_n.$$

Since

$$\sum_{\sigma \in S_n} \operatorname{sign} \sigma \, \frac{\partial x^{i_1}}{\partial x^{1'}} \frac{\partial x^{i_2}}{\partial x^{2'}} \dots \frac{\partial x^{i_n}}{\partial x^{n'}} = \det\left(\frac{\partial x}{\partial x'}\right)$$

the formula (3.3.11) readily follows. □

We will use the transformation formula (3.3.11) for introducing the important operation of integral of differential forms.

Let $D \subset M$ be an open domain in a n-dimensional manifold such that $\bar{D}$ is compact. We want to integrate a differential n-form over this subset. This can be done under an additional assumption for the manifold that we are going to explain now.

Let us consider first the particular case $\bar{D} \subset U$. Here $(U, (x^1, \dots, x^n))$ is a coordinate chart. Consider the multiple integral

$$I_U(\omega, D) = \int \dots \int_{x(D)} \omega_{1\dots n}(x) \, dx^1 \dots dx^n. \tag{3.3.12}$$

Lemma 3.3.5 *If $(U', (x^{1'}, \dots, x^{n'}))$ is another chart containing $\bar D$ then the integrals $I_U(\omega, D)$ and $I_{U'}(\omega, D)$ are equal up to a sign*

$$I_{U'}(\omega, D) = \pm I_U(\omega, D), \quad \pm = \text{sign}\,\det\left(\frac{\partial x}{\partial x'}\right). \tag{3.3.13}$$

Proof: Doing a change of integration variables in the multiple integral (3.3.12) one obtains

$$\int \cdots \int_{x(D)} \omega_{1\dots n}(x)\, dx^1 \dots dx^n = \int \cdots \int_{x'(D)} \omega_{1\dots n}(x(x')) \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| dx^{1'} \dots dx^{n'}$$
$$= \text{sign}\,\det\left(\frac{\partial x}{\partial x'}\right) \int \cdots \int_{x'(D)} \omega_{1\dots n}(x(x'))\det\left(\frac{\partial x}{\partial x'}\right) dx^{1'} \dots dx^{n'}$$
$$= \text{sign}\,\det\left(\frac{\partial x}{\partial x'}\right) \int \cdots \int_{x'(D)} \omega_{1'\dots n'}(x')\, dx^{1'} \dots dx^{n'}.$$

where we have used the transformation law (3.3.11) in the derivation of the last equality.
□

We are now ready to define integration of differential n-forms over domains in n-dimensional oriented manifolds. Let us first consider the case of a domain contained inside $\bar D \subset U$. In this case define

$$\int_D \omega := \int \cdots \int_{x(D)} \omega_{1\dots n}(x)\, dx^1 \dots dx^n. \tag{3.3.14}$$

According to the Lemma 3.3.5 such an integral does not change if choosing another chart covering D from the same atlas. In the general case use a partition of unity $p_\alpha(x)$ associated with the atlas and put

$$\int_D \omega = \sum_\alpha \int_{D\cap U_\alpha} p_\alpha(x)\,\omega. \tag{3.3.15}$$

The last step in the justification of such a definition is in

Lemma 3.3.6 *The integral* (3.3.15) *does not depend from the choice of a partition of unity.*

Proof: Let $q_\beta(x)$ be another partition of unity associated with an atlas V_β. Then the products

$$r_{\alpha\beta}(x) = p_\alpha(x) q_\beta(x)$$

is a partition of unity associated with the atlas $U_\alpha \cap V_\beta$. The integral defined by this partition is equal to

$$I = \sum_{\alpha,\,\beta} \int_{U_\alpha \cap V_\beta} p_\alpha(x) q_\beta(x)\,\omega.$$

Performing first summation in β one obtains

$$I = \sum_\alpha \int_{U_\alpha} p_\alpha(x)\,\omega \sum_\beta q_\beta(x) = \sum_\alpha \int_{U_\alpha} p_\alpha(x)\,\omega.$$

Changing the order of summation results in

$$I = \sum_\beta \int_{V_\beta} q_\beta(x)\,\omega \sum_\alpha p_\alpha(x) = \sum_\beta \int_{V_\beta} q_\beta(x)\,\omega.$$

So the two definitions of the integral yield the same result. $\square$

Remark 3.3.7 *An important particular case is the integral of a n-form over an oriented n-dimensional manifold M. Such an integral always exists if the manifold is compact. Observe that a change of orientation on M to the opposite one changes the sign of the integral.*

Let us now explain the operation of pullback of differential forms. Let

$$f : N \to M$$

be a smooth map. Given a k-form

$$\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1\dots i_k}(y) dy^{i_1} \wedge \cdots \wedge dy^{i_k}$$

on M one obtains a k-form $f^*\omega$ on N:

$$f^*\omega = \sum_{i_1<\cdots<i_k} \omega_{i_1\dots i_k}(y(x)) dy^{i_1}(x) \wedge \cdots \wedge dy^{i_k}(x)$$

where the map f in local coordinates is given by

$$x = (x^1, \dots, x^n) \mapsto (y^1(x), \dots, y^m(x)).$$

In the rhs there is the exterior product of k one-forms

$$dy^i(x) = \frac{\partial y^i}{\partial x^j} dx^j.$$

If the manifold N has dimension k and is oriented then one can consider the integral of the pullback

$$\int_N f^*\omega. \tag{3.3.16}$$

In particular one can integrate a k-form ω over a k-dimensional oriented submanifold

$$i : N \hookrightarrow M, \quad \dim N = k, \quad \int_N \omega := \int_N i^*\omega. \tag{3.3.17}$$

Example 1. The integral of a 1-form in $\mathbb{R}^3$

$$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$

over a smooth curve

$$\gamma : \{(x = x(t), y = y(t), z = z(t)) \,|\, a \le t \le b\}$$

is equal to

$$\int_\gamma \omega = \int_a^b \left[P(x(t), y(t), z(t))\dot{x}(t) + Q(x(t), y(t), z(t))\dot{y}(t) + R(x(t), y(t), z(t))\dot{z}(t)\right] dt$$
$$= \int_\gamma \langle \mathbf{X}, \mathbf{v}\rangle \, ds$$

where the vector field $\mathbf{X}$ is defined by

$$\mathbf{X} = (P, Q, R)$$

and $\mathbf{v}$ is the unit tangent vector to the oriented curve.

Example 2. The integral of a 2-form in $\mathbb{R}^3$

$$\omega = P(x, y, z)\, dy \wedge dz + Q(x, y, z)\, dz \wedge dx + R(x, y, z)\, dx \wedge dy$$

over a domain $D \subset \mathbb{R}^2_{(u,v)}$ on a parametrized two-dimensional surface

$$\mathbf{r} = \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

is equal to

$$\iint_D \langle \mathbf{X}, \mathbf{n}\rangle \, dA \tag{3.3.18}$$

where the vector field $\mathbf{X}$ is composed from the coefficients[5] of the 2-form

$$\mathbf{X} = (P, Q, R),$$

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

is the unit normal vector to the surface and

$$dA = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$$

is the area element on the surface (see Exercise **??** above).

Remark 3.3.8 *One can also integrate 0-forms, i.e., just functions* $f \in \Omega^0(M) = \mathcal{C}^\infty(M)$ *over zero-dimensional submanifold. By definition an oriented zero-dimensional submanifold is just a collection of points* P_1, P_2, ... *in* M *with signs* $\pm$ *assigned. The integral is defined as the algebraic sum of values*

$$\pm f(P_1) \pm f(P_2) \pm \dots.$$

Let us now proceed with the differential calculus of differential forms. We will now define an operator calle *exterior differential* (or also simply *differential*)

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \tag{3.3.19}$$

[5]Here and below the *Hodge duality* has been used for a correspondence between vector fields and two-forms in $\mathbb{R}^3$. This duality will be explained below.

for any $k \geq 0$. For $k = 0$ the operator (3.3.19) coincides with the differential of a function

$$df = \frac{\partial f(x)}{\partial x^i} dx^i \in \Omega^1(M). \tag{3.3.20}$$

For $k = n$ the differential is trivial since $\Omega^{n+1}(M) = 0$. For any $0 < k < n$ the differential of a k-form

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \dots i_k}(x) dy^{i_1} \wedge \cdots \wedge dx^{i_k}$$

can be defined in two equivalent ways. First, as an antisymmetric $(k+1)$-tensor

$$(d\omega)_{j_1 \dots j_{k+1}} = \sum_{m=1}^{k+1} (-1)^{m+1} \frac{\partial \omega_{j_1 \dots \hat{j}_m \dots j_{k+1}}(x)}{\partial x^{j_m}} \tag{3.3.21}$$

The hat over the m-th index means that this index j_m has to be omitted. An alternative form is

$$d\omega = \sum_{j_1 < \cdots < j_k} \frac{\partial \omega_{j_1 \dots j_k}(x)}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}. \tag{3.3.22}$$

Theorem 3.3.9 *The exterior differential is a well defined linear operator* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ *satisfying*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta, \quad \alpha \in \Omega^p(M) \tag{3.3.23}$$

and

$$d^2 = 0 \quad \text{where} \quad d^2 : \Omega^k(M) \to \Omega^{k+1}(M) \to \Omega^{k+2}(M). \tag{3.3.24}$$

Example 3.3.10 *For a 1-form* $\omega = P\,dx + Q\,dy + R\,dz$ *in a three-dimensional Euclidean space one has*

$$d\omega = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy. \tag{3.3.25}$$

This operation is related to the curl of a vector field $\mathbf{X} = (P, Q, R)$

$$\operatorname{curl} \mathbf{X} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix}. \tag{3.3.26}$$

Example 3.3.11 *For a two-form* $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$ *in* $\mathbb{R}^3$ *one has*

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz. \tag{3.3.27}$$

This operation is related to the divergence of the vector field $\mathbf{X} = (P, Q, R)$

$$\operatorname{div} \mathbf{X} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{3.3.28}$$

Thus, the exterior differential generalizes the operations of gradient, curl and divergence known from multivariable calculus.

For differential of functions one has the following important property usually referred to as *invariance of the differential*

$$
\begin{aligned}
df(y(x)) &= \left(\frac{\partial f}{\partial y^j} dy^j\right)_{y=y(x)} = \frac{\partial f}{\partial y^j} dy^j(x) \\
&= \frac{\partial f}{\partial x^i} dx^i.
\end{aligned}
$$

In the second term of the first line it is understood that

$$
dy^j(x) = \frac{\partial y^j}{\partial x^i} dx^i
$$

is the differential of the functions $y^j = y^j(x^1, \dots, x^n)$. A generalization of this property is formulated in the following

Exercise 3.3.12 *Prove that the exterior differential commutes with the operation of pullbacks of differential forms: if*

$$
f : M \to N
$$

is a smooth map and ω a differential form on N then

$$
f^* d\omega = d\, f^* \omega. \tag{3.3.29}
$$

In order to formulate the main result of calculus of differential forms we need to define manifolds with a boundary.

Definition 3.3.13 *A subset $M \subset \hat{M}$ in an n-dimensional manifold $\hat{M}$ is called a* manifold with boundary *if, for any coordinate chart $(U, (x^1, \dots, x^n))$ on $\hat{M}$ the intersection $M \cap U$ has one of the following two types:*

$$
\begin{aligned}
&\text{or} \quad M \cap U = U \\
&\text{or} \quad M \cap U = \{x \in U \mid f_U(x) \leq 0\}, \quad df_U(x) \neq 0 \quad \forall\, x \in U \quad \text{such that} \quad f_U(x) = 0.
\end{aligned}
$$

Here $f_U : U \to \mathbb{R}$ is a smooth function. It is required that on the intersection of charts U, V of the second type the domains $\{f_U(x) \leq 0\}$ and $\{f_V(x) \leq 0\}$ coincide.

The subset $\partial M \subset M$ defined by

$$
\partial M \cap U = \{x \in U \mid f_U(x) = 0\}
$$

is called the boundary *of M. This is a smooth submanifold in $\hat{M}$ of dimension $(n-1)$ (see Theorem 1.5.11 and Example 1.5.12).*

Smooth functions on M are defined as restrictions onto M of smooth functions on some neighborhood of M in $\hat{M}$.

Example 3.3.14 *The cylinder*

$$M \times [0,1] \subset M \times \mathbb{R} \tag{3.3.30}$$

is a $(n+1)$-dimensional manifold with a boundary consisting of two copies of the manifold M:

$$\partial\, (M \times [0,1]) = (M \times 0) \cup (M \times 1).$$

The function $f = f(x,t)$, $x \in M$, $t \in \mathbb{R}$ specifying the first piece of the boundary is $f(x,t) = -t$. For the second piece one can take $f(x,t) = t - 1$.

Remark 3.3.15 *A compact manifolds without boundary are often called* closed *manifolds.*

Definition 3.3.16 *Let $(M, \partial M)$ be a manifold with boundary in a n-dimensional oriented manifold $\hat{M}$. The* induced orientation *on the boundary is defined as follows (cf. Exercise 2.1.3 above). If at some point $x_0 \in \partial M \cap U$ the i-th derivative of the function $f_U(x)$ does not vanish then the $(n-1)$ variables $(x^1, \dots \hat{x^i}, \dots x^n)$ can be used as local coordinates on this part of the boundary specified in the chart $(U, (x^1, \dots, x^n))$ by equation $f_U(x) = 0$. By definition the order of these coordinates coincides with the orientation on ∂M if*

$$(-1)^{n-i} \frac{\partial f_U(x_0)}{\partial x^i} < 0.$$

in the opposite case one has to change the orientation to the opposite one.

Example 1. The boundary of the cylinder $M \times [0,1]$ for an oriented manifold M consists of one copy $M \times \{1\}$ of M taken with the same orientation and another copy $M \times \{0\}$ of the manifold M taken with the opposite orientation:

$$\partial\, (M \times [0,1]) = (M \times 1) \cup (-M \times 0). \tag{3.3.31}$$

In the particular case $M = [0,1]$ the boundary of M consists of two points

$$\partial[0,1] = \{1\} \cup (-\{0\}). \tag{3.3.32}$$

Example 2. In the integral calculus of differential forms we will also consider manifolds with piecewise smooth boundaries. They become manifolds with boundary after deleting some subsets of codimension ≥ 2. Such subsets do not contribute into integrals. For example, the boundary of the n-dimensional unit cube

$$I^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \,|\, 0 \leq x^i \leq 1, \quad i = 1, \dots, n\} \tag{3.3.33}$$

obtained by taking Cartesian product of n copies of the unit interval consists of $2n$ unit cubes

$$\partial I^n = \bigcup_{i=1}^{n} (-1)^{n-i} I^{n-1}_{i,1} \cup (-1)^{n-i+1} I^{n-1}_{i,0} \tag{3.3.34}$$

where

$$\begin{aligned} I^{n-1}_{i,1} &= \{(x^1, \dots, x^{i-1}, x^i = 1, x^{i+1}, \dots, x^n)\} \subset I^n \\ I^{n-1}_{i,0} &= \{(x^1, \dots, x^{i-1}, x^i = 0, x^{i+1}, \dots, x^n)\} \subset I^n. \end{aligned} \tag{3.3.35}$$

For $n = 1$ one again obtains (3.3.32).

We are now ready to formulate the main result of this section

Theorem 3.3.17 (Stokes' formula) *For any compact smooth n-dimensional oriented manifold M with a piecewise smooth boundary ∂M and any differential $(n-1)$-form ω on M the following formula holds true*

$$\int_{\partial M} \omega = \int_M d\omega. \tag{3.3.36}$$

Before proving the theorem let us consider the case $n = 1$, $M = [0, 1]$, $\omega = f(x)$ a smooth function. Then the left hand side of (3.3.36) is equal to

$$\int_{\partial M} \omega = f(1) - f(0)$$

since $\partial[0, 1] = \{1\} \cup (-\{0\})$ (see Remark 3.3.8). In the right hand side, applying the Fundamental Theorem of calculus, we have

$$\int_0^1 f'(t)\, dt = f(1) - f(0).$$

This proves the Stokes' formula for this particular case.

Proof: Let us begin with the proof of the Stokes' formula for the case $M = I^n$ (n-dimensional cube).

In order to avoid complicated notations we will perform calculations only for the case $n = 2$. The one-form ω reads

$$\omega = \omega_1(x, y)dx + \omega_2(x, y)dy.$$

The oriented boundary of the square I^2 is the union of four segments

$$\partial I^2 = \{(1, y)\} \cup (-\{(x, 1)\}) \cup (-\{(0, y)\}) \cup \{(x, 0)\}, \quad 0 \le x,\, y \le 1.$$

(see (3.3.34)). So

$$\int_{\partial I^2} \omega = \int_0^1 \omega_2(1, y)\, dy - \int_0^1 \omega_1(x, 1)\, dx - \int_0^1 \omega_2(0, y)\, dy + \int_0^1 \omega_1(x, 0)\, dx.$$

In the right hand side of the Stokes' formula we have a double integral

$$\int_{I^2} d\omega = \iint_{0 \le x,\, y \le 1} \left(\frac{\partial \omega_2}{\partial x} - \frac{\partial \omega_1}{\partial y} \right) dx\, dy.$$

We will apply the Fubini theorem in two different ways to the two parts of the double integral

$$\iint_{0 \le x,\, y \le 1} \left(\frac{\partial \omega_2}{\partial x} - \frac{\partial \omega_1}{\partial y} \right) dx\, dy = \int_0^1 dy \int_0^1 \frac{\partial \omega_2(x, y)}{\partial x}\, dx - \int_0^1 dx \int_0^1 \frac{\partial \omega_1(x, y)}{\partial y}\, dy.$$

Both double integrals can be easily reduced to single integrals

$$\int_0^1 dy \int_0^1 \frac{\partial \omega_2(x, y)}{\partial x}\, dx = \int_0^1 [\omega_2(1, y) - \omega_2(0, y)]\, dy$$
$$\int_0^1 dx \int_0^1 \frac{\partial \omega_1(x, y)}{\partial y}\, dy = \int_0^1 [\omega_1(x, 1) - \omega_1(x, 0)]\, dx.$$

Taking the difference of these two integrals one arrives at the proof of the Stokes' formula for the square.

In the general case one can construct an oriented atlas U_α on the compact manifold $(M, \partial M)$ consisting of open cubes $(0,1)^n$ or half-cubes $(0,1)^{n-1} \times (0,1]$. Let $p_\alpha(x)$ be a partition of unity associated with such an atlas. Then

$$\int_M d\omega = \sum_\alpha \int_{U_\alpha} p_\alpha(x)\, d\omega = \sum_\alpha \int_{U_\alpha} d(p_\alpha(x)\omega) - \sum_\alpha \int_{U_\alpha} dp_\alpha(x) \wedge \omega.$$

The last integral vanishes since

$$\sum_\alpha dp_\alpha(x) = d \sum_\alpha p_\alpha(x) = 1$$

since

$$\sum_\alpha p_\alpha(x) \equiv 1.$$

Now we reduce the general proof to the local case

$$\int_{\partial U_\alpha} p_\alpha(x)\omega = \int_{U_\alpha} d(p_\alpha(x)\omega).$$

The differential form $p_\alpha(x)\omega$ vanishes on the boundary of the cubes of the first type. For the half-cubes the last equality follows from the already proven Stokes' theorem for cube. □

Definition 3.3.18 *A differential k-form ω is called* closed *if $d\omega = 0$. It is called* exact *if $\omega = d\alpha$ for some $(k-1)$-form α.*

From the Stokes' formula one derives an important corollary

Corollary 3.3.19 *1) The integral of a closed differential form over the boundary ∂M of a manifold M is equal to zero:*

$$\int_{\partial M} \omega = 0 \quad \text{if} \quad d\omega = 0. \tag{3.3.37}$$

2) The integral of an exact differential form over a closed manifold is equal to zero:

$$\int_M d\alpha = 0 \quad \text{if} \quad \partial M = \emptyset. \tag{3.3.38}$$

Since the square of the exterior differential is equal to zero (see the Theorem 3.3.9 above), any exact differential form is always closed. The converse statement is false. Let us construct a counterexample. Let $M = \mathbb{R}^2 \setminus \{0\}$,

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

One has

$$d\omega = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2} = 0.$$

To prove that this form is not exact let us consider the integral of this 1-form on the unit circle S^1

$$\left.\begin{array}{rcl} x & = & \cos t \\ y & = & \sin t \end{array}\right\}$$

$0 \leq t \leq 2\pi$. The pullback of the 1-form onto the circle is equal to

$$\frac{\cos t\, d\sin t - \sin t\, d\cos t}{\sin^2 t + \cos^2 t} = dt.$$

So

$$\int_{S^1} \omega = \int_0^{2\pi} dt = 2\pi \neq 0.$$

This contradicts the assumption of exactness of ω.

Definition 3.3.20 *Two closed differential forms ω_1, ω_2 on a manifold M are called* equivalent *if their difference is an exact form*

$$\omega_2 - \omega_1 = d\alpha.$$

The linear space of classes of equivalence of closed k-forms on M is called the k-th De Rham cohomology space *of M*

$$\begin{array}{l} H^k(M) := \operatorname{Ker} d_k / \operatorname{Im} d_{k-1}, \quad 0 \leq k \leq n = \dim M \\ d_k = d : \Omega^k(M) \to \Omega^{k+1}(M). \end{array} \tag{3.3.39}$$

It is understood that $\operatorname{Im} d_{k-1}$ is equal to zero for $k = 0$; for $k = n = \dim M$ the kernel $\operatorname{Ker} d_n$ coincides with the entire space $\Omega^n(M)$ of differential n-forms on M.

Example 3.3.21 *The cohomology of a point are equal to*

$$H^k(\mathrm{pt}) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k > 0. \end{cases} \tag{3.3.40}$$

Example 3.3.22 *Let M be a closed connected manifold. Then*

$$H^0(M) = \mathbb{R}. \tag{3.3.41}$$

Indeed, a zero-form on M is just a function $f \in \mathbb{C}^\infty(M)$. Such a zero-form is closed, $df = 0$ iff *the function $f(x)$ is locally constant. Since the manifold is connect the function must identically constant everywhere. This gives an isomorphism*

$$H^0(M) = \operatorname{Ker} d_0 \to \mathbb{R}.$$

In a more general case $H^0(M) = \mathbb{R}^N$ where N is the number of connected components of a closed manifold M.

Example 3.3.23 *Let us compute the cohomologies of the circle S^1. Since S^1 is connect the zero-dimensional cohomology is isomorphic to $\mathbb{R}$ (see the previous example). For $k \geq 2$ the cohomology $H^k(S^1)$ is trivial. It remains to compute $H^1(S^1)$. Let ϕ be the 2π-periodic coordinate on the circle. Every 1-form on S^1 can be written in the form*

$$\omega = f(\phi)d\phi, \quad f(\phi + 2\pi) = f(\phi).$$

Consider a linear map

$$\begin{aligned} &h : \Omega^1(S^1) \to \mathbb{R} \\ &\omega \mapsto h(\omega) = \frac{1}{2\pi} \int_{S^1} \omega = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi. \end{aligned} \tag{3.3.42}$$

Let us prove that the 1-form ω is exact iff $h(\omega) = 0$. *Indeed, if $\omega = dg$, $g = g(\phi)$ a 2π-periodic function, then*

$$h(\omega) = \frac{1}{2\pi} \int_0^{2\pi} g'(\phi)\, d\phi = \frac{1}{2\pi}[g(2\pi) - g(0)] = 0.$$

Conversely, let

$$f(\phi) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi)$$

be the Fourier expansion of the 2π-periodic function $f(\phi)$. We have

$$h(\omega) = \frac{a_0}{2}.$$

So, if $h(\omega) = 0$ then one can find a periodic primitive for the function $f(\phi)$

$$f(\phi) = g'(\phi), \quad g(\phi) = \sum_{n=1}^{\infty} \frac{1}{n}(a_n \sin n\phi - b_n \cos n\phi).$$

Thus $\omega = dg$.

3.4 Homotopy invariance of cohomologies. Degree of a smooth map and integrals of differential forms

We begin with the following simple statement.

Exercise 3.4.1 *Let $f : M \to N$ be a smooth map. Prove that the pullback*

$$f^* : \Omega^k(N) \to \Omega^k(M)$$

induces a homomorphism of cohomologies that will be denoted by the same symbol

$$f^* : H^k(N) \to H^k(M) \tag{3.4.1}$$

for all $k = 0, 1, \ldots$

Hint: use the result of Exercise 3.3.12.

A very important property of the induced homomorphism (3.4.1) is its *invariance with respect to homotopies*. Roughly speaking a homotopy is a deformation of a map $f : M \to N$, i.e., a family of maps depending on a parameter. More precisely,

Definition 3.4.2 *Two maps*

$$f_0, f_1 : M \to N$$

are said to be homotopic *if there exists a map of a cylinder* $M \times [0,1] = \{(x,t) \,|\, x \in M, \quad 0 \leq t \leq 1\}$

$$\begin{aligned} F : M \times [0,1] &\to N \\ (x,t) &\mapsto F(x,t) \end{aligned} \tag{3.4.2}$$

such that

$$F(x,0) = f_0(x), \quad F(x,1) = f_1(x). \tag{3.4.3}$$

The map F itself is called a homotopy *between f_0 and f_1.*

We will use notation

$$f_0 \sim f_1$$

for two homotopic maps.

Remark 3.4.3 *Recall that, by default all maps are assumed to be smooth. However, sometimes it is more convenient to work with piecewise smooth homotopies. It is not difficult to prove that they can be approximated by smooth homotopies.*

Example 3.4.4 *Let $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ be the identity map and $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ the constant map $f_0(x) \equiv 0$. Let us prove that these two maps are homotopic. The needed homotopy $F(x,t)$ can be constructed as follows*

$$F(x,t) = x\,t. \tag{3.4.4}$$

Theorem 3.4.5 *Let $f_0 \sim f_1 : M \to N$ be two homotopic maps. Then the induced homomorphisms of cohomologies coincide*

$$f_0^* = f_1^* : H^k(N) \to H^k(M) \tag{3.4.5}$$

for all $k = 0, 1, \ldots$

Proof: Let us first define a linear map

$$D : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M) \tag{3.4.6}$$

for any $k \geq 0$ (for $k = 0$ we put $D = 0$) and an arbitrary manifold M. Let n be the dimension of the manifold M; choosing a chart $(U, (x^1, \ldots, x^n))$ one obtains local coordinates

$(x^1, \dots, x^n, t)$ on $M \times [0,1]$. In these charts any k-form $\omega \in \Omega^k(M \times [0,1])$ naturally splits into two parts $\omega = \alpha + \beta$,

$$\omega = \sum_{1 \le i_1 < \dots < i_k \le n} \alpha_{i_1 \dots i_k}(x,t)\, dx^{i_1} \wedge \dots dx^{i_k} + \sum_{1 \le i_1 < \dots < i_{k-1} \le n} \beta_{i_1 \dots i_{k-1}}(x,t)\, dx^{i_1} \wedge \dots dx^{i_{k-1}} \wedge dt.$$

Put

$$D\omega := \sum_{1 \le i_1 < \dots < i_{k-1} \le n} \left(\int_0^1 \beta_{i_1 \dots i_{k-1}}(x,t)\, dt \right) dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}} \tag{3.4.7}$$

Lemma 3.4.6 *For any k-form $\omega \in \Omega^k(M \times [0,1])$ the following identity holds true*

$$d\, D\, \omega - D\, d\, \omega = (-1)^{k+1} \left(\omega|_{t=1} - \omega|_{t=0} \right). \tag{3.4.8}$$

Proof: Let us do the calculations for the particular case $k = 1$; the proof in the general case is similar. For

$$\omega = \alpha_i(x,t) dx^i + \beta(x,t)\, dt$$

we have

$$D\, \omega = \int_0^1 \beta(x,t)\, dt.$$

So

$$d\, D\, \omega = \left(\int_0^1 \frac{\partial \beta(x,t)}{\partial x^i}\, dt \right) dx^i.$$

Computation of the differential $d\omega$ yields

$$d\omega = \sum_{i<j} \left(\frac{\partial \alpha_j(x,t)}{\partial x^i} - \frac{\partial \alpha_i(x,t)}{\partial x^j} \right) dx^i \wedge dx^j + \left(\frac{\partial \beta(x,t)}{\partial x^i} - \frac{\partial \alpha_i(x,t)}{\partial t} \right) dx^i \wedge dt.$$

Applying to this 2-form the operator D one obtains

$$\begin{aligned} D\, d\, \omega &= \left[\int_0^1 \left(\frac{\partial \beta(x,t)}{\partial x^i} - \frac{\partial \alpha_i(x,t)}{\partial t} \right) dt \right] dx^i \\ &= \left(\int_0^1 \frac{\partial \beta(x,t)}{\partial x^i}\, dt \right) dx^i - \left[\alpha_i(x,1) dx^i - \alpha_i(x,0) dx^i \right]. \end{aligned}$$

Thus

$$d\, D\, \omega - D\, d\, \omega = \left[\alpha_i(x,1) dx^i - \alpha_i(x,0) dx^i \right] = \omega|_{t=1} - \omega|_{t=0}$$

since

$$dt|_{t=1} = dt|_{t=0} = 0.$$

$\square$

We are now in a position for completing the proof of the Theorem. Let ω be a closed k-form on N. We have to prove that the difference

$$f_1^* \omega - f_0^* \omega$$

is an exact form. Indeed, put

$$\rho := (-1)^{k+1} D F^* \omega.$$

According to the Lemma applied to the k-form $F^*\omega$ on $M \times [0,1]$ one has an identity

$$f_1^*\omega - f_0^*\omega = (-1)^{k+1} \left[d\, D\, F^*\omega - D\, d\, F^*\omega\right]$$

Since the differential commutes with pullbacks one has

$$d\, F^*\omega = F^* d\omega = 0$$

due to closedness of the form ω. Hence

$$f_1^*\omega - f_0^*\omega = d\rho.$$

□

Definition 3.4.7 *Two manifolds M, N are called* homotopicaly equivalent *if there exist two maps*

$$f : M \to N, \quad g : N \to M$$

such that the superpositions $f \circ g$ and $g \circ f$ are homotopic to the identity maps

$$g \circ f \sim \mathrm{id}_M : M \to M$$
$$f \circ g \sim \mathrm{id}_N : N \to N.$$

We will use notation

$$M \sim N$$

for homotopicaly equivalent manifolds.

Two diffeomorphic manifolds are homotopicaly equivalent, but not vice versa, as it follows from

Example 3.4.8 *The Euclidean space $\mathbb{R}^n$ is homotopically equivalent to a point. Indeed, the map*

$$f : \mathrm{pt} \to \mathbb{R}^n$$

is an embedding, while

$$g : \mathbb{R}^n \to \mathrm{pt}$$

is the constant map $g(x) \equiv \mathrm{pt}$.

The homotopy between the superposition $g \circ f$ and $\mathrm{id}_{\mathrm{pt}}$ is trivial; in order to construct a homotopy between $f_0 := f \circ g : \mathbb{R}^n \to \mathrm{pt} \in \mathbb{R}^n$ and $f_1 := \mathrm{id}_{\mathbb{R}^n}$ one can use the map

$$F(x,t) = x\, t.$$

(cf. (3.4.4) above).

In a similar way one can prove that any *star shaped* domain $D \subset \mathbb{R}^n$ in the Euclidean space is homotopicaly equivalent to a point. By definition a domain is called star shaped if there exists a point $P_0 \in D$ that can be connected to any other point $P \in D$ by a segment of a straight line belonging to D.

From the Theorem 3.4.5 it immediately follows

Theorem 3.4.9 *Homotopicaly equivalent manifolds have isomorphic cohomologies*

$$H^k(M) \simeq H^k(N) \quad \forall\, k \geq 0 \quad \text{if} \quad M \sim N. \tag{3.4.9}$$

Proof: Let

$$f : M \to N \quad \text{and} \quad g : N \to M$$

be maps establishing the homotopy equivalence. Since

$$g \circ f \sim \mathrm{id}_M$$

we have, due to the Theorem 3.4.5

$$f^* g^* = (g \circ f)^* : H^k(M) \to H^k(M)$$

is the identity isomorphism for any $k \geq 0$. In a similar way, from

$$f \circ g \sim \mathrm{id}_N$$

it follows

$$g^* f^* = (f \circ g)^* = \mathrm{id} : H^k(N) \to H^k(N).$$

So the two induced homorphisms f^* and g^* are mutually inverse. Hence they are isomorphisms. $\square$

Corollary 3.4.10 (Lemma Poincaré) *Every closed k form on Euclidean space is exact, if $k > 0$.*

Let $f : M \to N$ be a smooth map of compact oriented manifolds of the same dimension n. Let Ω be any n-form on the manifold N. For example, one can take the volume form (4.1.17) assuming a Riemannian metric has been chosen on N. Then the following formula holds true

Theorem 3.4.11

$$\int_M f^*\Omega = \deg f \cdot \int_N \Omega. \tag{3.4.10}$$

Proof: For the particular case of a diffeomorphism the degree is equal to ± 1. In this case the formula (3.4.10) readily follows from the theorem about changes of variables in a multiple integral. In the general case let $y \in N$ be a regular value of the map. Denote $x_1, \ldots, x_K$ the points in the preimage $f^{-1}(K)$. There exists a ball $B_y \subset N$ centered at y such that

$$f^{-1}(B_y) = B_{x_1} \cup \cdots \cup B_{x_K} \subset M$$

and the map f restricted onto every ball B_{x_i} is a diffeomorphism $f : B_{x_i} \to B_y$. Applying the formula of changing variables in a multiple integral one obtains

$$\int_{B_{x_i}} f^*\Omega = \deg_i \cdot \int_{B_y} \Omega, \quad i = 1, \ldots, K.$$

Here $\deg_i$ is the sign (2.1.4) of the Jacobian $\det\left(\frac{\partial y}{\partial x_i}\right)$ at the point x_i. Hence

$$\int_{f^{-1}(B_y)} f^*\Omega = \deg f \cdot \int_{B_y} \Omega$$

for any regular value $y \in N$. The set of critical values has measure zero, they do not contribute to the integral over N. Moreover, the pullback $f^*\Omega$ vanishes at the critical points in M where, by definition, the Jacobian of the map f is equal to zero. Therefore the of critical points does not contribute to the integral over M. This completes the proof of the formula. □

4 Riemannian Manifolds

4.1 Riemannian metrics

Definition 4.1.1 *A* Riemannian metric *on a manifold is a symmetric bilinear form*

$$\begin{aligned} &v,\ w \in T_xM \mapsto \langle v,\ w\rangle \in \mathbb{R} &(4.1.1)\\ &\langle a_1v_1 + a_2v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle, \quad \langle v, a_1w_1 + a_2w_2\rangle = a_1\langle v, w_1\rangle + a_2\langle v, w_2\rangle \\ &\langle w, v\rangle = \langle v, w\rangle \end{aligned}$$

defined at every point $x \in M$ smoothly depending on x such that

$$\langle v, v\rangle > 0 \quad \text{for any} \quad v \neq 0. \tag{4.1.2}$$

Recall that the property (4.1.2) is usually referred to as *positive definiteness* of the symmetric bilinear form.

Denote n the dimension of the manifold M. In a chart one can associate the *Gram matrix* to the bilinear form

$$g_{ij}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle. \tag{4.1.3}$$

It is a symmetric matrix of smooth functions. The condition of positive definiteness is equivalent to positivity of the *principal minors* of the matrix

$$g_{11} > 0, \quad \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} > 0, \quad \dots, \quad \det\left(g_{ij}\right)_{1\le i,\, j\le n} > 0 \tag{4.1.4}$$

(the *Sylvester criterion*). Such a matrix determines the bilinear form according to the following formula

$$\langle v, w\rangle = g_{ij}(x)v^iw^j \quad \text{where} \quad v = v^i(x)\frac{\partial}{\partial x^i}, \quad w = w^j(x)\frac{\partial}{\partial x^j} \in T_xM. \tag{4.1.5}$$

The result does not depend on the choice of local coordinates if, on the intersection of coordinate charts $\left(U, (x^1, \dots, x^n)\right)$ and $\left(V, (y^1, \dots, y^n)\right)$ the corresponding Gram matrices

$$g_{ij}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle \quad \text{and} \quad g_{kl}(y) = \left\langle \frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l}\right\rangle, \quad y = y(x)$$

are related by the transformation law

$$g_{ij}(x) = g_{kl}(y)\frac{\partial y^k}{\partial x^i}\frac{\partial y^l}{\partial x^j}. \tag{4.1.6}$$

This is nothing that the transformation law of a $(0,2)$-tensor. So, one can reformulate the above definition in the following way.

Definition 4.1.2 *A Riemannian metric on a manifold is a symmetric* $(0,2)$*-tensor* $g_{ij}(x)$ *satisfying the Sylvester positivity conditions* (4.1.4)*. A manifold equipped with a Riemannian metric is called* Riemannian manifold*.*

Length of a curve

$$\gamma : \quad \{x^i = x^i(t) \,|\, a \leq t \leq b\}$$

on a Riemannian manifold can be defined in a way familiar from geometry in Euclidean space or on a surface

$$s[\gamma] := \int_a^b \sqrt{g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)}\, dt. \tag{4.1.7}$$

Lemma 4.1.3 *The length does not change under monotone reparametrizations of the curve.*

Proof: Let $t = t(t')$ be a monotone change of parameter, $t(a') = a$, $t(b') = b$. To be more specific let us consider the monotone decreasing case

$$\frac{dt}{dt'} < 0, \quad a' > b'.$$

Then

$$\int_a^b \sqrt{g_{ij}(x(t))\frac{dx^i(t)}{dt}\frac{dx^j(t)}{dt}}\, dt = \int_{a'}^{b'} \sqrt{g_{ij}(x(t(t')))\frac{dx^i(t(t'))}{dt'}\frac{dx^j(t(t'))}{dt'}\left(\frac{dt'}{dt}\right)^2}\,\frac{dt}{dt'}dt'$$

$$= \int_{a'}^{b'} \sqrt{g_{ij}(x(t(t')))\frac{dx^i(t(t'))}{dt'}\frac{dx^j(t(t'))}{dt'}}\,\left|\frac{dt'}{dt}\right|\frac{dt}{dt'}dt' = -\int_{a'}^{b'} \sqrt{g_{ij}(x(t(t')))\frac{dx^i(t(t'))}{dt'}\frac{dx^j(t(t'))}{dt'}}\, dt'$$

$$= \int_{b'}^{a'} \sqrt{g_{ij}(x(t(t')))\frac{dx^i(t(t'))}{dt'}\frac{dx^j(t(t'))}{dt'}}\, dt'.$$

□

It is convenient to introduce the *square length element*

$$ds^2 = g_{ij}(x)dx^i dx^j. \tag{4.1.8}$$

The formula for the length of the curve can be written as follows

$$s[\gamma] = \int_\gamma ds \tag{4.1.9}$$

where the restriction of the square length element onto the curve is defined by the usual procedure

$$ds^2|_\gamma = g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)\, dt^2.$$

Example 4.1.4 *Euclidean metric in $\mathbb{R}^n$. In the Cartesian coordinates x^1, ..., x^n the metric reads*

$$ds^2 = \left(dx^1\right)^2 + \cdots + \left(dx^n\right)^2 .$$

The length of the curve is given by the well known formula

$$s[\gamma] = \int_a^b \sqrt{\left(\dot x^1\right)^2 + \cdots + \left(\dot x^n\right)^2}\, dt.$$

One can choose another coordinate system and rewrite the metric. E.g., let us consider polar coordinates r, ϕ on the Euclidean case with Cartesian coordinates x, y, so that

$$x = r\,\cos\phi$$
$$y = r\,\sin\phi.$$

We have

$$dx = \cos\phi\, dr - r\,\sin\phi\, d\phi$$
$$dy = \sin\phi\, dr + r\,\cos\phi\, d\phi$$

so

$$ds^2 = dx^2 + dy^2 = (\cos\phi\, dr - r\,\sin\phi\, d\phi)^2 + (\sin\phi\, dr + r\,\cos\phi\, d\phi)^2 = dr^2 + r^2 d\phi^2.$$

Thus, the length of the curve γ: $r = r(\phi)$, $\alpha \le \phi \le \beta$ equals

$$s[\gamma] = \int_\alpha^\beta \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2(\phi)}\, d\phi.$$

Relaxing the positive definiteness condition to simply non-degeneratness one arrives at the definition of a *pseudo-Riemannian metric.*

Definition 4.1.5 *A pseudo-Riemannian metric on a manifold is a symmetric $(0,2)$-tensor $g_{ij}(x)$ satisfying the condition*

$$\det\left(g_{ij}\right)_{1\le i,\, j\le n} \neq 0. \qquad (4.1.10)$$

A manifold equipped with a pseudo-Riemannian metric is called pseudo-Riemannian manifold*.*

Example 4.1.6 *Given a pair of positive integers p, q satisfying $p+q=n$, the n-dimensional pseudo-Euclidean space $\mathbb{R}^{p,q} = \{(x^1, \dots, x^n)\,|\, x^i \in \mathbb{R}, \quad i = 1, \dots, n\}$ is a pseudo-Riemannian manifold with the metric*

$$ds^2 = \left(dx^1\right)^2 + \cdots + \left(dx^p\right)^2 - \left(dx^{p+1}\right)^2 - \cdots - \left(dx^n\right)^2 . \qquad (4.1.11)$$

One can also define *volumes* of compact domains with a piecewise smooth boundary in a Riemannian manifold $D \subset M$. By definition

$$Vol(D) := \int_{x(D)} dV \qquad (4.1.12)$$
$$dV = \sqrt{g(x)}\, dx^1 \dots dx^n, \quad g(x) := \det(g_{ij}(x))$$

if the domain D is covered by one coordinate chart. Here we denote $x(D)$ the coordinate representation of the domain.

A change of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ yields

$$g_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} g_{ij}$$

or, in the matrix form

$$G' = A^T G\, A \tag{4.1.13}$$

where G and G' are the matrices of the metric with respect to the two coordinate systems

$$G = (g_{ij})\,, \quad G' = \left(g_{i'j'}\right)$$

and

$$A = \left(\frac{\partial x^i}{\partial x^{i'}}\right)$$

is the Jacobi matrix. Thus

$$\det G' = (\det A)^2 \det G$$

that is,

$$g' = \left[\det\left(\frac{\partial x}{\partial x'}\right)\right]^2 g, \quad g' = \det G', \quad g = \det G. \tag{4.1.14}$$

Therefore the definition (4.1.12) of the volume does not depend on the choice of coordinates:

$$\int_{x'(D)} \sqrt{g'} dx^{1'} \dots dx^{n'} = \int_{x'(D)} \sqrt{g} \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| dx^{1'} \dots dx^{n'} = \int_{x(D)} \sqrt{g}\, dx^1 \dots dx^n$$

due to the formula of changing variables in the multiple integral.

One can pass from local to the global definition of the volume by using a partition of unity $p_\alpha(x)$ associated with a given atlas $\left(U_\alpha, (x^1_\alpha, \dots, x^n_\alpha)\right)_{\alpha \in I}$

$$Vol(D) = \sum_{\alpha \in I} \int_{x_\alpha(D \cap U_\alpha)} p_\alpha(x) \sqrt{g_\alpha}\, dx^1_\alpha \dots dx^n_\alpha \tag{4.1.15}$$

where g_α is the determinant of the metric tensor in the coordinates $(x^1_\alpha, \dots, x^n_\alpha)$. In particular if the manifold itself is compact then the volume

$$Vol(M) = \int_M dV > 0 \tag{4.1.16}$$

is defined.

Let M be an oriented Riemannian manifold of dimension n. Then one can construct a *volume form* $\Omega \in \Omega^n(M)$

$$\Omega = \sqrt{g(x)}\, dx^1 \wedge \dots \wedge dx^n. \tag{4.1.17}$$

This formula is invariant with respect to orientation preserving changes of coordinates. Hence the n-form Ω is defined globally on an oriented manifold.

For the dimension reason any n-form on an n-dimensional manifold M is closed.

Theorem 4.1.7 *For a compact orientable manifold M the n-form Ω is not exact.*

Proof: The integral

$$\int_M \Omega = Vol(M) > 0.$$

So assumption of exactness of Ω contradicts to the second part of the Corollary 3.3.19. $\square$

Corollary 4.1.8 *For a compact orientable n-dimensional manifold M the n-th cohomology is non-trivial*

$$H^n(M) \neq 0.$$

Actually, one can prove a stronger result that $H^n(M) = \mathbb{R}$ for a closed connected orientable n-dimensional manifold, but the proof requires some techniques not explained in this course.

Let us now prove an existence theorem.

Theorem 4.1.9 *A Riemannian metric exists on any paracompact manifold M.*

Proof: Let $(U_\alpha, (x_\alpha^1, \dots, x_\alpha^n))_{\alpha \in I}$ be an atlas on M equipped with a partition of unity $p_\alpha(x)$. Let us define a metric in every chart by

$$g_{ij}^\alpha = \delta_{ij}.$$

Denote $\langle\ ,\ \rangle_\alpha$ the inner product of tangent vectors at the points of the chart. Define the inner product of two vectors v, $w \in T_x M$ at any point $x \in M$ by

$$\langle v, w\rangle = \sum_{\alpha \in I} p_\alpha(x) \langle v, w\rangle_\alpha. \tag{4.1.18}$$

Such an inner product is bilinear; it depends smoothly from the point $x \in M$. It is positive definite since positive definite inner products on $\mathbb{R}^n$ form a convex cone in the space of symmetric matrices:

Lemma 4.1.10 *Given two positive definite symmetric $n \times n$ matrices g_{ij} and h_{ij}, their arbitrary linear combination*

$$\lambda\, g_{ij} + \mu\, h_{ij}, \quad \lambda \geq 0,\ \mu \geq 0, \quad \lambda^2 + \mu^2 > 0$$

with nonnegative coefficients is again a positive definite matrix.

Proof: For an arbitrary vector $v \in \mathbb{R}^n$ the linear combination

$$\begin{aligned} &\lambda\, g(v, v) + \mu\, h(v, v) \\ &g(v, v) := g_{ij} v^i v^j, \quad h(v, v) = h_{ij} v^i v^j \end{aligned}$$

with nonnegative coefficients vanishes *iff* $\lambda\, g(v, v) = \mu\, h(v, v) = 0$. Hence $v = 0$, $\square$

Let $M \subset \mathbb{R}^N$ be a submanifold in the Euclidean space. One can define the *induced metric* on M in the following way. Let v, $w \in T_x M$ be two tangent vectors. Considering them as vectors in $\mathbb{R}^N$ we compute their inner product $\langle v, w\rangle_{\mathbb{R}^N}$ and put

$$\langle v, w\rangle_M := \langle v, w\rangle_{\mathbb{R}^N}. \tag{4.1.19}$$

Example 4.1.11 *The induced metric on a smooth curve*

$$x(t) = \left(x^1(t), \dots, x^n(t)\right)$$

in a Riemannian manifold has the form

$$ds^2 = g(t)dt^2 \quad \text{where} \quad g(t) = |\dot{x}(t)|^2 = g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t).$$

Observe that $g(t) > 0$ for any t since $\dot{x}(t) \neq 0$.

Example 4.1.12 *For a two-dimensional surface in $\mathbb{R}^3$*

$$\mathbf{r} = \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad \mathbf{N} := \mathbf{r}_u \times \mathbf{r}_v \neq 0 \tag{4.1.20}$$

the induced metric is often written as follows

$$\begin{aligned} &ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2 \qquad (4.1.21)\\ &E = E(u, v) = \langle \mathbf{r}_u, \mathbf{r}_u \rangle, \quad F = F(u, v) = \langle \mathbf{r}_u, \mathbf{r}_v \rangle, \quad G = G(u, v) = \langle \mathbf{r}_v, \mathbf{r}_v \rangle. \end{aligned}$$

It is also called the first fundamental form *of the surface. In this case the volume of a domain defined by the formula* (4.1.15) *coincides with the area of this domain.*

Exercise 4.1.13 *Prove that the area of a domain D on the surface* (4.1.20) *is given by the following integral*

$$Area(D) = \iint_D |\mathbf{N}|\, du\, dv \tag{4.1.22}$$

where $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v$ is the vector normal to the surface.

Exercise 4.1.14 *Consider the particular case of the sphere of radius R. Representing it in the spherical coordinates*

$$\begin{aligned} x &= R\, \cos\phi\, \sin\theta \\ y &= R\, \sin\phi\, \sin\theta \\ z &= R\, \cos\theta \end{aligned}$$

one obtains the induced metric in the form

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{4.1.23}$$

The Gram matrix of the metric becomes degenerate at the poles $\theta = 0$ or $\theta = \pi$ since the coordinate system becomes singular at these points

$$\mathbf{r}_\theta \times \mathbf{r}_\phi = 0 \quad \text{at the poles.}$$

Exercise 4.1.15 *Compute the induced metric on the surface represented as a graph of a smooth function $z = f(x, y)$. Prove that the area of a domain D on such a surface is given by the familiar formula*

$$Area(D) = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

In certain cases the construction of induced metrics is applicable also to submanifolds in pseudo-Riemannian manifolds.

Example 4.1.16 *In the three-dimensional pseudo-Euclidean space* $\mathbb{R}^{1,2}$ *with the coordinates* (x, u, z) *and the metric*

$$ds^2 = -dx^2 - dy^2 + dz^2 \tag{4.1.24}$$

consider the surface defined by equation

$$z^2 - x^2 - y^2 = R^2. \tag{4.1.25}$$

It is often called pseudosphere *due to the similarity with the equation of sphere in Euclidean space*

$$\langle \mathbf{r}, \mathbf{r} \rangle = R^2, \quad \mathbf{r} = (x, y, z) \in \mathbb{R}^{1,2}.$$

Clearly this surface is a hyperboloid of two sheets (see Fig. 2). One can introduce global

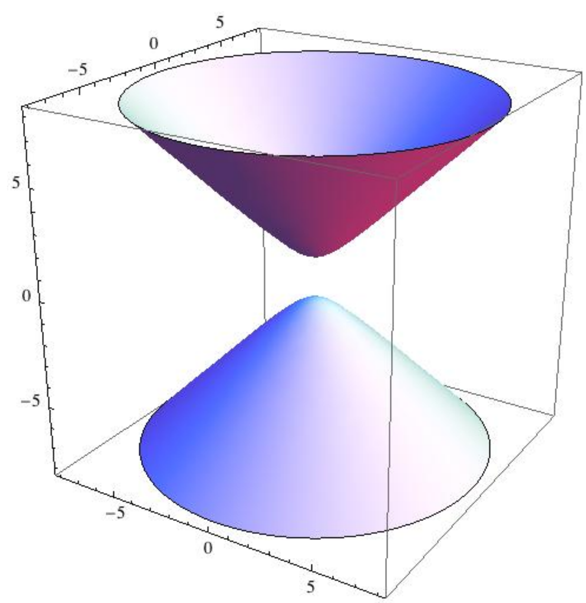

Figure 2: Pseudosphere

pseudospherical coordinates *on the upper sheet*

$$\begin{aligned} x &= R\,\cos\phi\,\sinh\theta \\ y &= R\,\sin\phi\,\sinh\theta \\ z &= R\,\cosh\theta. \end{aligned}$$

In these coordinates the induced metric becomes equal to

$$ds^2 = -R^2\left(d\theta^2 + \sinh^2\theta\, d\phi^2\right). \tag{4.1.26}$$

Such a metric is negative definite; changing the overall sign one arrives at a Riemannian metric on the two-dimensional surface. We will call it the metric of pseudosphere. As it will become clear below, this metric is of a fundamental importance for the hyperbolic geometry*.*

More generally one can construct in a similar way the induced metric on any submanifold in a Riemannian manifold. Namely, given an embedding $f : M \to N$ to a Riemannian manifold N equipped with a metric $\langle\ ,\ \rangle_N$, define a metric on the manifold M by

$$\langle v, w\rangle_M := \langle f_* v, f_* w\rangle_N. \tag{4.1.27}$$

Exercise 4.1.17 *Given a smooth curve*

$$\gamma : \quad t \mapsto \left(x^1(t), \dots, x^m(t)\right), \quad a \le t \le b$$

on the manifold M consider its image $f(\gamma)$ on N with respect to an embedding $f : M \to N$. Prove that the length of the curve γ with respect to the induced metric $\langle\ ,\ \rangle_M$ coincides with the length of the image $f(\gamma)$ with respect to the metric $\langle\ ,\ \rangle_N$.

4.2 Tensors on a Riemannian manifold

There are some important additions to tensor algebra on Riemannian manifolds. The metric at the point $x \in M$ defines an isomorphism

$$\begin{aligned} &g : T_x M \to T_x^* M \\ &v^i \mapsto g_{ij}(x) v^j. \end{aligned} \tag{4.2.1}$$

The explicit formula justifies the name *lowering of indices* for this isomorphism. The inverse isomorphism is often called *raising of indices*

$$\begin{aligned} &g^{-1} : T_x^* M \to T_x M \\ &\omega_i \mapsto g^{ij}(x) \omega_j. \end{aligned} \tag{4.2.2}$$

Here $g^{ij}(x)$ are entries of the inverse matrix

$$\left(g^{ij}(x)\right) = \left(g_{ij}(x)\right)^{-1}. \tag{4.2.3}$$

Exercise 4.2.1 *Prove that the inverse matrix* (4.2.3) *is a* $(2,0)$*-tensor on the manifold M. Prove that the inner product on the cotangent spaces defined by*

$$\langle \alpha, \beta\rangle^* = g^{ij}(x) \alpha_i \beta_j, \quad \alpha,\, \beta \in T_x^* M \tag{4.2.4}$$

is positive definite.

In a similar way one can construct isomorphisms between the spaces of tensors

$$\mathcal{T}_l^k \simeq \mathcal{T}_{k+l} \simeq \mathcal{T}^{k+l} \tag{4.2.5}$$

at any point of the Riemannian manifold.

Finally, since the operations of lowering and raising indices do not require positivity but only nondegenerateness of the Gram matrix $(g_{ij}(x))$. So, they can be defined also on a pseudo-Riemannian manifold.

There is also a useful operator

$$* : \Omega^k(M) \to \Omega^{n-k}(M) \tag{4.2.6}$$

on an oriented Riemannian n-dimensional manifold called *Hodge duality.* The construction uses the volume form (4.1.17). Let us first observe that the components of the antisymmetric tensor Ω can be written as follows

$$\Omega_{i_1 \dots i_n} = \sqrt{g}\, \epsilon_{i_1 \dots i_n} \tag{4.2.7}$$

where

$$\epsilon_{i_1 i_2 \dots i_n} = \begin{cases} \operatorname{sign} \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}, & \text{all indices } i_1, i_2, \dots, i_n \text{ are distinct} \\ \\ 0, & \text{otherwise} \end{cases} \tag{4.2.8}$$

To apply this operator to a k-form $\omega = \omega_{i_1 \dots i_k}$ one first has to raise all indices

$$\omega_{i_1 \dots i_k} \mapsto \omega^{i_1 \dots i_k} = g^{i_1 s_1} \dots g^{i_k s_k} \omega_{s_1 \dots s_k}.$$

At the second step apply contraction with the tensor (4.2.7)

$$(*\,\omega)_{j_1 \dots j_l} = \frac{1}{k!} \sqrt{g}\, \epsilon_{i_1 \dots i_k j_1 \dots j_l} \omega^{i_1 \dots i_k}, \quad k + l = n. \tag{4.2.9}$$

Exercise 4.2.2 *Prove that the square of the Hodge operator is equal to* $\pm\mathrm{id}$

$$*^2 = (-1)^{k(n-k)} \mathrm{id} : \Omega^k \to \Omega^k. \tag{4.2.10}$$

Essentially the Hodge duality appears already in the calculations with differential forms in the three-dimensional Euclidean space where

$$\begin{aligned} &*\,dx = dy \wedge dz, \quad *\,dy = dz \wedge dx, \quad *\,dz = dx \wedge dy \\ &*\,1 = dx \wedge dy \wedge dz. \\ &*^2 = \mathrm{id}. \end{aligned}$$

We finish this section with an application of Riemannian geometry to the theory of algebraic equations. Consider an algebraic equation with real coefficients

$$P(x) := x^n + a_1 x^{n-1} + \dots + a_n = 0, \quad a_1, \dots, a_n \in \mathbb{R}.$$

Under what conditions all roots of this equation are real? To this end let us consider the space $\mathcal{P}_n$ of all degree n monic polynomials with all real roots. It can be considered as the n-dimensional Euclidean space using the roots as the Cartesian coordinates

$$P(x) = \prod_{i=1}^{n} (x - x_i), \quad x_1, \dots, x_n \in \mathbb{R}.$$

Denote $\mathcal{A}_n$ another n-dimensional space with coefficients $a_1, \ldots, a_n$ as the coordinates (in these considerations all coordinates will be labelled by lower indices for the sake of simplicity of explicit expressions). Viète's formulae

$$a_i = (-1)^i \sigma_i(x_1, \ldots, x_n), \quad i = 1, \ldots n$$

where $\sigma_1 = \sigma_1(x_1, \ldots, x_n)$, ..., $\sigma_n = \sigma_n(x_1, \ldots, x_n)$ are elementary symmetric functions

$$\sigma_1 = x_1 + \cdots + x_n, \quad \sigma_2 = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n, \ \ldots, \ \sigma_n = x_1 \ldots x_n \tag{4.2.11}$$

define a smooth map $\mathcal{P}_n \xrightarrow{\sigma} \mathcal{A}_n$.

Exercise 4.2.3 *Prove the following formula for the Jacobian of the Viète map*

$$\det\left(\frac{\partial a_i}{\partial x_j}\right) = (-1)^{\frac{n(n+1)}{2}} \prod_{i<j} (x_i - x_j). \tag{4.2.12}$$

So, the Viète map is a local diffeomorphism on a sufficiently small neighborhood of any polynomial with pairwise distinct roots.

Our goal is to describe the image of this map. To this end let us introduce Euclidean metric on $\mathcal{P}_n$ and take the inverse matrix defining an inner product on the cotangent bundle to $\mathcal{P}_n$

$$\langle dx_i, dx_j \rangle = \delta_{ij}.$$

The Viète map σ induces a metric on the cotangent spaces to $\mathcal{A}_n$. Namely, by definition

$$g^{ij}(a) := \langle da_i, da_j \rangle = \sum_{k=1}^{n} \frac{\partial a_i}{\partial x_k} \frac{\partial a_j}{\partial x_k}, \quad i, j = 1, \ldots, n. \tag{4.2.13}$$

Observe that $g^{ij}(a)$ is a *symmetric polynomial* in the roots $x_1, \ldots, x_n$. Thus, according to the main theorem of the theory of symmetric polynomials it can be represented as a polynomial in $a_1, \ldots, a_n$. For example,

$$\langle da_1, da_1 \rangle = n.$$

Computing other elements of the matrix $\left(g^{ij}\right)$ for $n = 2$ one obtains (recall that in the concrete examples we write all indices of coordinates as lower)

$$\left(g^{ij}\right) = \begin{pmatrix} 2 & a_1 \\ a_1 & a_1^2 - 2a_2 \end{pmatrix}$$

and for $n = 3$

$$\left(g^{ij}\right) = \begin{pmatrix} 3 & 2a_1 & a_2 \\ 2a_1 & 2(a_1^2 - a_2) & a_1 a_2 - 3a_3 \\ a_2 & a_1 a_2 - 3a_3 & a_2^2 - 2a_1 a_3 \end{pmatrix}. \tag{4.2.14}$$

Theorem 4.2.4 *The polynomial $P(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ with real coefficients has all roots real and pairwise distinct* iff *the Gram matrix* (4.2.13) *is positive definite.*

Applying the Sylvester criterion (4.1.4) one obtains, for $n = 2$, the well known inequality

$$a_1^2 - 4a_2 > 0.$$

For $n = 3$ the Sylvester criterion yields two inequalities

$$a_1^2 - 3a_2 > 0, \quad a_1^2 a_2^2 - 4a_2^3 - 4a_1^3 a_3 + 18 a_1 a_2 a_3 - 27 a_3^2 > 0$$

that ensure that a cubic polynomial $x^3 + a_1 x^2 + a_2 x + a_3$ has all real and distinct roots.

There is another trick from the theory of symmetric functions that helps to derive expressions for the metric. To this end we choose another system of coordinates on the space of polynomials. Instead of elementary symmetric functions we will use *Newton polynomials*

$$s_k(x_1, \dots, x_n) = x_1^k + \dots + x_n^k, \quad k = 1, 2, \dots.$$

They can be easily expressed via the coefficients of the polynomial due to the following simple identity

$$\log\left(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_n}{x^n}\right) = \sum_{i=1}^n \log\left(1 - \frac{x_i}{x}\right) = -\sum_{k=1}^\infty \frac{s_k}{k}\frac{1}{x^k}. \tag{4.2.15}$$

For the example $n = 3$ one has

$$\begin{aligned} &s_1 = -a_1, \quad s_2 = -2a_2 + a_1^2, \quad s_3 = -3a_3 + 3a_1 a_2 - a_1^3 \\ &s_4 = a_1^4 - 4a_1^2 a_2 + 2a_2^2 + 4a_1 a_3 \end{aligned}$$

etc. In this way one obtains a triangular change of coordinates

$$(a_1, \dots, a_n) \mapsto (s_1, \dots, s_n), \quad s_k = -k\, a_k + \tilde{s}_k(a_1, \dots, a_{k-1}), \quad k = 1, \dots, n.$$

The inverse transform can easily be derived from the identity

$$e^{-\sum_{k=1}^n s_k \frac{x^k}{k}} = 1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_n}{x^n} + \mathcal{O}\left(\frac{1}{x^{n+1}}\right)$$

(cf. (4.2.15)). For the example $n = 3$ one has

$$a_1 = -s_1, \quad a_2 = \frac{1}{2}\left(s_1^2 - s_2\right), \quad a_3 = \frac{1}{6}\left(-s_1^3 + 3s_1 s_2 - 2s_3\right).$$

Let us now compute the metric (4.2.13) in the coordinates $s_1, \dots, s_n$. We have

$$\langle ds_k, ds_l \rangle = \sum_{i=1}^n \frac{\partial s_k}{\partial x_i}\frac{\partial s_l}{\partial x_i} = k\, l\, s_{k+l-2} \tag{4.2.16}$$

where we put

$$s_0 = n.$$

Since positive definiteness of the metric does not depend on the choice of coordinates we arrive at the following

Corollary 4.2.5 (Sturm theorem). *All the roots of the polynomial $x^n + a_1 x^{n-1} + \cdots + a_n$ are pairwise distinct and real* iff *the symmetric matrix*

$$\left(g^{kl}\right) = \begin{pmatrix} n & 2s_1 & 3s_2 & \dots & n\,s_{n-1} \\ 2s_1 & 4s_2 & 6s_3 & \dots & 2n\,s_n \\ 3s_2 & 6s_3 & 9s_4 & \dots & 3n\,s_{n+1} \\ . & & & & . \\ . & & & & . \\ n\,s_{n-1} & . & . & \dots & n^2 s_{2n-2} \end{pmatrix} \tag{4.2.17}$$

is positive definite. In this matrix the polynomials $s_k = s_k(a_1, \dots, a_n)$, $k = 1, \dots, 2n-2$ are defined from the eq. (4.2.15).

The above differential-geometric derivation of the Sturm theorem is due to Sylvester. For $n = 3$ one obtains the matrix (4.2.17) reads

$$\begin{pmatrix} 3 & -2a_1 & 3(a_1^2 - 2a_2) \\ -2a_1 & 4(a_1^2 - 2a_2) & 6(-a_1^3 + 3a_1 a_2 - 3a_3) \\ 3(a_1^2 - 2a_2) & 6(-a_1^3 + 3a_1 a_2 - 3a_3) & 9(a_1^4 - 4a_1^2 a_2 + 2a_2^2 + 4a_1 a_3) \end{pmatrix}. \tag{4.2.18}$$

Needless to say that the conditions of positive definiteness of the matrices (4.2.14) and (4.2.18) coincide.

Exercise 4.2.6 *Prove that all the roots of the polynomial $x^n + a_1 x^{n-1} + \cdots + a_n$ are pairwise distinct and exactly m of them are real* iff *the matrix* (4.2.17) *defines a pseudo-Riemannian metric of signature $(m+k, k)$ (k negative squares), where the number k is defined by the equation $m + 2k = n$.*

4.3 Riemannian manifolds as metric spaces

On a connected Riemannian manifold one can define *distance between two points*

$$\rho(x, y) = \inf\{\text{lengths of piecewise smooth curves connecting } x \text{ and } y\} \tag{4.3.1}$$

Before proving that the distance function defines on M a structure of a metric space we prove the following

Lemma 4.3.1 *For a coordinate chart $U \subset M$ consider a function*

$$\lambda(x, v) = \sqrt{g_{ij}(x) v^i v^j}, \quad x \in U, \quad v \in \mathbb{R}^n. \tag{4.3.2}$$

Denote

$$\|v\| = \sqrt{(v^1)^2 + \cdots + (v^n)^2}$$

the Euclidean norm of the vector v. Let $B^n \subset U$ be an n-dimensional ball. Then there exists a positive constant k such that the following inequalities hold true

$$\frac{1}{k}\,\|v\| \le \lambda(x, v) \le k\,\|v\| \quad \text{for any} \quad (x, v) \in \bar{B}^n \times \mathbb{R}^n. \tag{4.3.3}$$

Proof: Denote S^{n-1} the unit sphere in the Euclidean space, $S^{n-1} = \{v \in \mathbb{R}^n \mid \|v\| = 1\}$. The restriction of the function $\lambda(x, v)$ onto the compact $\bar{B}^n \times S^{n-1}$ attains its minimum $\lambda_{\min}$ and maximum $\lambda_{\max}$ at some points (x_1, v_1) and (x_2, v_2) in $\bar{B}^n \times S^{n-1}$ respectively. Clearly $\lambda_{\min}$ is a positive number. The choice

$$k = \max\left(\lambda_{\min}^{-1}, \lambda_{\max}\right)$$

implies inequalities

$$\frac{1}{k} \leq \lambda(x, v) \leq k \quad \text{for any} \quad x \in \bar{B}^n, \quad \|v\| = 1.$$

Normalizing an arbitrary vector $v = \|v\| \cdot \tilde{v}$, $\|\tilde{v}\| = 1$ one completes the proof of Lemma. □

Theorem 4.3.2 *The distance function* (4.3.1) *satisfies the following properties*

$$\begin{aligned} &\rho(y, x) = \rho(x, y) \\ &\rho(x, z) \leq \rho(x, y) + \rho(y, z) \\ &\rho(x, y) \geq 0, \quad \rho(x, y) = 0 \quad \textit{iff} \quad y = x. \end{aligned} \tag{4.3.4}$$

Proof: The first two properties of the distance function are obvious from the definition and from the invariance of the length under a reparameterization $t \mapsto -t$. Let us prove that $\rho(x_0, y_0) > 0$ for any $x_0 \neq y_0$. Let U be a coordinate chart containing x_0. Choose a sufficiently small positive number r such that the ball

$$\bar{B}_r^n(x_0) := \{x \in U \mid \|x - x_0\| \leq r\}$$

entirely belongs to U. Without loss of generality we may assume that $y_0 \notin B_r^n(x_0)$. According to Lemma 4.3.1 one has inequality

$$\sqrt{g_{ij}(x) v^i v^j} \geq \frac{1}{k} \|v\| \quad \forall\, x \in \bar{B}_r^n(x_0), \quad \forall\, v \in \mathbb{R}^n$$

for a suitable positive constant k. Now, let γ be any piecewise smooth curve connecting $x_0 = \gamma(0)$ and y_0. We will derive the lower bound for its length

$$s[\gamma] \geq \frac{r}{k}.$$

It suffices to prove such an inequality for the part of the curve belonging to the ball

$$\gamma' = \gamma \cap \bar{B}_r^n(x_0).$$

Reducing, if necessary, the radius r we can assume that γ' is a connected piecewise smooth curve having it end point $x_1 = \gamma(t_1)$ on the boundary of the ball,

$$\|x_1 - x_0\| = r.$$

We have

$$s[\gamma'] = \int_0^{t_1} \sqrt{g_{ij}(x(t)) \dot{x}^i \dot{x}^j}\, dt \geq \frac{1}{k} \int_0^{t_1} \sqrt{\left(\dot{x}^1\right)^2 + \cdots + \left(\dot{x}^n\right)^2}\, dt.$$

The integral in the rhs is the Euclidean length of the curve γ'. As it is well known from Euclidean geometry its length is greater or equal than the Euclidean distance between the points $\gamma(0)$ and $\gamma(t_1)$. The latter is equal to r. So $s[\gamma'] \geq r/k$. $\square$

Let us consider some particular metrics. We have derived above three important examples of two-dimensional Riemannian metrics:

$$\begin{array}{ll} ds^2 = d\theta^2 + \theta^2 d\phi^2 & \text{Euclidean plane} \\ ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 & \text{sphere of radius} \quad 1 \\ ds^2 = d\theta^2 + \sinh^2\theta\, d\phi^2 & \text{pseudosphere of radius} \quad 1 \end{array}$$

(for the Euclidean plane we have redenoted the radial coordinate $r \mapsto \theta$). Observe that for small θ the above three metrics are approximately equal due to the wellknown formulae

$$\sin\theta \simeq \theta, \quad \sinh\theta \simeq \theta.$$

In order to better understand the difference between these three metrics let us consider the *disks* of a given radius on these three Riemannian manifolds. By definition the disk D_r of radius r with the centre at a point x is defined as follows

$$D_r = \{y \,|\, \rho(x, y) \leq r\}.$$

Let us choose the point $\theta = 0$ as the centre of the disk. On the Euclidean plane the disk is given by

$$D_r = \{(\theta, \phi) \,|\, 0 \leq \theta \leq r, \quad 0 \leq \phi \leq 2\pi\}. \tag{4.3.5}$$

Let us prove that the disk on the unit sphere has the same form.

Lemma 4.3.3 *The distance from the north pole $\theta = 0$ of the unit sphere to the point (θ_0, ϕ_0) for any $\theta_0 \leq \pi$, $0 \leq \phi_0 < 2\pi$ is equal to θ_0.*

Proof: We have to prove that the length of any piecewise smooth curve from the north pole to (θ_0, ϕ_0) is greater or equal than θ_0. Without loss of generality we may assume that the curve is written in the form

$$\phi = \phi(\theta), \quad 0 \leq \theta \leq \theta_0.$$

The length of such a curve is equal to

$$\int_0^{\theta_0} \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}\, d\theta \geq \int_0^{\theta_0} d\theta = \theta_0.$$

$\square$

A similar argument works also for the pseudosphere. So, for all three examples of the metrics the disks have are defined by the same inequality (4.3.5). Let us now compare the areas of these disks. For Euclidean space we have, of course,

$$Area^{\text{Eucl}}\,(D_r) = \pi\, r^2.$$

For the unit sphere

$$Area^{\rm sph}\left(D_r\right) = \int_0^r d\theta \int_0^{2\pi} \sin\theta\, d\phi = 2\pi(1-\cos r)$$

while for the pseudosphere

$$Area^{\rm pseudosph}\left(D_r\right) = \int_0^r d\theta \int_0^{2\pi} \sinh\theta\, d\phi = 2\pi(\cosh r - 1).$$

Using the wellknown asymptotic formulae

$$\cos r = 1 - \frac{r^2}{2} + \frac{r^4}{24} + \mathcal{O}\left(r^6\right), \quad \cosh r = 1 + \frac{r^2}{2} + \frac{r^4}{24} + \mathcal{O}\left(r^6\right)$$

we derive that, for sufficiently small radius

$$Area^{\rm sph}\left(D_r\right) = \pi\, r^2 - \frac{\pi\, r^4}{12} + \mathcal{O}\left(r^6\right), \quad Area^{\rm pseudosph}\left(D_r\right) = \pi\, r^2 + \frac{\pi\, r^4}{12} + \mathcal{O}\left(r^6\right). \tag{4.3.6}$$

The coefficient of r^4 can be used to measure the deviation from the Euclidean formula. This is the starting point for definition of *curvature* of a Riemannian metric that will be introduced below.

In order to better understand the properties of this distance one has to study the curves minimizing the length functional

$$s[\gamma] = \int_\gamma |\dot x|\, dt = \int_\gamma \sqrt{g_{ij}(x)\dot x^i \dot x^j}\, dt. \tag{4.3.7}$$

Since the length functional (4.9.4) is invariant with respect to monotone changes of the parameter it suffices to minimize the functional on the subspace of curves parameterized by the arc length.

Lemma 4.3.4 *The stationary points of the functional* (4.9.4) *on the subspace of curves with the arc length parameterization are determined by the following system of differential equations*

$$\ddot x^k + \Gamma^k_{ij}(x)\dot x^i \dot x^j = 0, \quad k = 1, \dots, n \tag{4.3.8}$$

where

$$\Gamma^k_{ij} = \frac12 g^{ks}\left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s}\right). \tag{4.3.9}$$

Proof: One has to derive the Euler–Lagrange equations

$$\frac{\partial \ell}{\partial x^m} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot x^m} = 0, \quad m = 1, \dots, n \tag{4.3.10}$$

for the Lagrangian

$$\ell(x, \dot x) = \sqrt{g_{ij}(x)\dot x^i \dot x^j}. \tag{4.3.11}$$

We have

$$\frac{\partial \ell}{\partial x^m} - \frac{d}{dt}\frac{\partial \ell}{\partial \dot x^m} = \frac{1}{\sqrt{g_{ij}(x)\dot x^i \dot x^j}}\left[\frac12 \frac{\partial g_{ij}}{\partial x^m}\dot x^i \dot x^j - \frac{d}{dt}\left(g_{mi}\dot x^i\right)\right] - g_{mi}\dot x^i \frac{d}{dt}\frac{1}{\sqrt{g_{ij}(x)\dot x^i \dot x^j}}.$$

The last term in the rhs vanishes since

$$g_{ij}(x(t))\dot{x}^i\dot{x}^j \equiv 1$$

on the curves parameterized by arc length. The expression in the square brackets yields

$$\frac{1}{2}\frac{\partial g_{ij}}{\partial x^m}\dot{x}^i\dot{x}^j - \frac{d}{dt}\left(g_{mi}\dot{x}^i\right) = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^m}\dot{x}^i\dot{x}^j - \frac{\partial g_{mi}}{\partial x^j}\dot{x}^i\dot{x}^j - g_{mi}\ddot{x}^i.$$

Since $\dot{x}^i\dot{x}^j$ is symmetric in i, j the second double sum can be written as follows

$$\frac{\partial g_{mi}}{\partial x^j}\dot{x}^i\dot{x}^j = \frac{1}{2}\left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i}\right)\dot{x}^i\dot{x}^j.$$

So, the m-th equation for the critical points of the functional after changing the sign becomes

$$g_{mi}\ddot{x}^i + \frac{1}{2}\left(\frac{\partial g_{mi}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right)\dot{x}^i\dot{x}^j = 0, \quad m = 1, \dots, n.$$

Multiplying the last equation by the entries g^{km} of the inverse matrix and taking the sum in m one completes the proof of the Lemma. $\square$

Definition 4.3.5 *The solutions $x = x(t)$ of the Euler–Lagrange equations* (4.3.8) *are called* geodesics*.*

Example 4.3.6 *In the Euclidean space*

$$g_{ij} = \delta_{ij},$$

so

$$\Gamma_{ij}^k \equiv 0.$$

The geodesics are straight lines

$$x(t) = v_0 t + x_0.$$

Applying to the system (4.3.8) the existence and uniqueness theorem of solutions to ODEs one arrives at

Theorem 4.3.7 *Given a point $x_0 \in M$ on a Riemannian manifold M and a tangent vector $v_0 \in T_{x_0}M$, there exists a positive number $\epsilon > 0$ and a geodesic $\gamma(t)$ defined for $|t| < \epsilon$ such that $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v_0$.*

Exercise 4.3.8 *Prove that the parameter on any geodesic is proportional to the arc length, i. e., given a solution $\left(x^1(t), \dots, x^n(t)\right)$ to the system* (4.3.8)*, prove that*

$$|\dot{x}|^2 = g_{ij}\left(x(t)\right)\dot{x}^i\dot{x}^j = \text{const}.$$

Alternatively one can consider the *action functional*

$$S[\gamma] = \int_\gamma \frac{1}{2}|\dot{x}|^2 dt = \int_\gamma \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j\, dt \tag{4.3.12}$$

that can be considered as an analogue of the kinetic energy for the free motion on the manifold of a point particle of mass 1. Deriving the Euler–Lagrange equations for the new Lagrangian

$$L(x,\dot{x}) = \frac{1}{2} g_{ij}(x)\dot{x}^i\dot{x}^j \tag{4.3.13}$$

one obtains the same equations of motion (4.3.8).

Exercise 4.3.9 *Derive the statement of Exercise 4.3.8 using the property of conservation of energy on solutions to Euler–Lagrange equations with a Lagrangian $L = L(x, \dot{x})$ that does not depend explicitly on time*

$$\frac{\partial L(x,\dot{x})}{\partial x^m} - \frac{d}{dt}\frac{\partial L(x,\dot{x})}{\partial \dot{x}^m} = 0, \quad m = 1,\dots,n \quad \Rightarrow \quad \frac{d}{dt}E(x,\dot{x}) = 0$$

where

$$E(x,\dot{x}) = \dot{x}^i \frac{\partial L(x,\dot{x})}{\partial \dot{x}^i} - L(x,\dot{x}). \tag{4.3.14}$$

Example 4.3.10 *The geodesics on the sphere*

$$|x|^2 = R^2$$

in the Euclidean space are arcs of the so-called great circles *obtained as sections of the sphere by hyperplanes passing through the origin. From this example it is clear that, in general the geodesics minimize the length only locally.*

4.4 Approximation theorems

Any smooth manifold is a topological manifold. The definition of a topological manifold is similar to that of a smooth manifold with just one modification: the transition functions between charts are homeomorphisms. Moreover, any smooth map $f : M \to N$ of smooth manifolds is continuous. In particular, any diffeomorphism between smooth manifolds is also a homeomorphism (but not vice versa!). One of the most striking examples (S.Donaldson) is an infinite set of smooth manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^4$. There is still an open problem of classification, up to a diffeomorphism, of smooth manifolds homeomorphic to the 4-dimensional sphere.

Approximation theorems give a justification of working in the theory of smooth manifolds with only smooth maps, smooth homotopies etc. For simplicity we will formulate few simple statements of this kind for approximation of smooth functions and smooth maps of compact connected manifolds. It will also be convenient to assume that the manifolds carry a Riemannian structure. Indeed, denote $\mathcal{C}(M, N)$ the space of continuous maps between compact connected Riemannian manifolds M and N. One can equip this infinite dimensional space with a structure of a metric space defining the distance between two functions $f,\ g \in \mathcal{C}(M,N)$ by

$$\rho(f,g) := \max_{x\in M} \rho\left(f(x), g(x)\right). \tag{4.4.1}$$

In the right hand side of this formula we use the distance function on the Riemannian manifold N defined in the previous section. It is easy to prove that the distance function (4.4.1) satisfies all axioms of a metric space.

Our first statement says that $\mathcal{C}^\infty(M,N) \subset \mathcal{C}(M,N)$ is a dense subset. More precisely, the following statement holds true.

Theorem 4.4.1 *Let M, N be compact connected Riemannian manifolds. Then for any $f \in \mathcal{C}(M,N)$ and an arbitrary $\epsilon > 0$ there exists $g \in \mathcal{C}^\infty(M,N)$ such that*

$$\rho(f,g) < \epsilon.$$

Proof: is based on the following

Lemma 4.4.2 *Let $U \subset \mathbb{R}^n$ be an arbitrary bounded open domain and f a continuous function $f : U \to \mathbb{R}$. Then for an arbitrary $\epsilon > 0$ and an arbitrary open V such that $\bar{V} \subset U$ there exists a function $g : U \to \mathbb{R}$ smooth on V such that*

$$g|_{U\setminus V} = f|_{U\setminus V}$$

and

$$\max_{x\in\bar{V}} |f(x) - g(x)| \leq \epsilon.$$

Moreover, g is smooth also at all the points of smoothness of f.

Recall that two continuous maps $f_0,\ f_1 : M \to N$ are homotopic if there exists a continuous map

$$F : M \times [0,1] \to N$$

such that

$$F|_{M\times\{0\}} = f_0, \quad F|_{M\times\{1\}} = f_1.$$

Theorem 4.4.3 *Let M, N be compact connected Riemannian manifolds. Then there exists $\epsilon > 0$ such that, any two continuous maps $f_0,\ f_1 : M \to N$ satisfying*

$$\rho(f,g) < \epsilon$$

are homotopic.

From these two theorems the following statement readily follows.

Corollary 4.4.4 *Let $f : M \to N$ be a continuous map of compact connected Riemannian manifolds. Then there exists a smooth map $g : M \to N$ homotopic to f.*

Finally, the following statement says that, working with smooth maps it suffices to deal with smooth homotopies only.

Theorem 4.4.5 *Given two homotopic smooth maps $f_0 \sim f_1 : M \to N$ of compact connected Riemannian manifolds, then there exists a smooth homotopy $F : M \times [0,1] \to N$ between them.*

4.5 Isometries of Riemannian manifolds

Let $(M, \langle\ ,\ \rangle_M)$, $(N, \langle\ ,\ \rangle_N)$ be two Riemannian manifolds.

Definition 4.5.1 *A diffeomorphism $f : M \to N$ is called* isometry *if*

$$\langle f_* v, f_* w\rangle_N = \langle v, w\rangle_M \quad \forall\, v,\, w \in T_x M, \quad \forall\, x \in M. \tag{4.5.1}$$

A local version of this definition gives *local isometries*. In this case f is a local diffeomorphism.

Exercise 4.5.2 *Let M, N be two Riemannian manifolds. Denote ρ_M, ρ_N the corresponding distance functions. Let $f : M \to N$ be an isometry. Prove that, for any two points x, $y \in M$*

$$\rho_M(x, y) = \rho_N\left(f(x), f(y)\right).$$

One of the main problems of Riemannian geometry is to classify Riemannian manifolds up to an isometry.

Example 4.5.3 *For $n = 1$ a Riemannian metric has the form*

$$ds^2 = g(t)dt^2, \quad g(t) > 0.$$

Introducing the arc length parameter

$$s(t) = \int_0^t \sqrt{g(t)}\, dt$$

reduces the metric to the Euclidean form

$$ds^2 = g(t)dt^2 = (ds(t))^2.$$

So, any one-dimensional Riemannian manifold is locally isometric to the one-dimensional Euclidean space.

For $n > 1$ in general there is no local isometry between a Riemannian manifold (M, ds^2) and Euclidean space. Below we define the obstacle for existence of such a local isometry. It will be defined in terms of the *curvature* of the Riemannian manifold.

Another important geometrical object come from the study of isometries of a Riemannian manifold (M, ds^2) to itself.

Exercise 4.5.4 *Prove that the set of all isometries of a Riemannian manifold to itself is a group.*

In this way we obtain *the group of isometries* of the Riemannian manifolds.

Example 4.5.5 *For $M = \mathbb{R}^n$ with the Euclidean metric $ds^2 = \left(dx^1\right)^2 + \dots + (dx^n)^2$ any map $M \to M$ of the form*

$$x \mapsto Ax + b, \quad A \in O(n), \quad b \in \mathbb{R}^n \tag{4.5.2}$$

is an isometry. Clearly such maps form a subgroup in the group of all isometries. Later we will show that this group coincides with the group of all isometries $\mathbb{R}^n$ to itself.

Example 4.5.6 *For the standard sphere $\{x^2 + y^2 + z^2 = R^2\} = S^2 \subset \mathbb{R}^3$ equipped with the induced metric the transformations*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A \in O(3)$$

are isometries. We obtain a three-dimensional group of isometries of the sphere S^2. Also in this case, as it will be shown below, the full group of isometries coincides with $O(3)$.

Similar considerations can be applied to the standard sphere $S^{n-1} \subset \mathbb{R}^n$. The orthogonal group $O(n)$ acts by isometries on the sphere of an arbitrary radius.

Let us now show that the pseudosphere also possesses a three-dimensional group of isometries.

Let us first observe that the definition (4.5.1) of isometries makes sense also for pseudo-Riemannian manifolds.

Exercise 4.5.7 *1) Prove that the linear map*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad A \in Mat(3, \mathbb{R})$$

is an isometry of the three-dimensional Minkowski space $\mathbb{R}^{2,1}$ with the metric $ds^2 = -dx^2 - dy^2 + dz^2$ iff the matrix $A = (a_{ij})$ satisfies

$$A^T G\, A = G, \quad G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.5.3}$$

2) Prove that matrices satisfying (4.5.3) form a three-dimensional Lie group.

3) Prove that the a_{33} entry of such a matrix A never vanishes.

Denote $O(2,1)$ the group of transformations (4.5.3). It is often called *Lorentz group*. Observe that transformations of this group leave invariant the pseudosphere $x^2 + y^2 - z^2 = -R^2$ of any radius R. By $O_+(2,1) \subset O(2,1)$ denote the subgroup consisting of transformations satisfying $a_{33} > 0$. Such a subgroup acts on the upper sheet $z > 0$ of the pseudosphere.

Exercise 4.5.8 *Prove that the transformation of the form*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad A \in O_+(2,1)$$

defines an isometry of the pseudosphere as a Riemannian manifold with the metric

$$ds^2 = R^2(d\theta^2 + \sinh^2\theta\, d\phi^2). \tag{4.5.4}$$

Also in this case it will be shown that the group of isometries of the Riemannian manifold (4.5.4) coincides with $O_+(2,1)$.

4.6 Affine connections

In this section M is an arbitrary smooth manifold of dimension n. Let us first recall the definition of the derivative $\partial_X f$ of a smooth function $f \in \mathcal{C}^\infty(M)$ along a vector field $X = X^i(x)\frac{\partial}{\partial x^i}$

$$\partial_X f := X^i(x)\frac{\partial f(x)}{\partial x^i}. \tag{4.6.1}$$

Consider the integral curves $x(t)$ of the vector field X, i.e., solutions to the dynamical system

$$\dot{x}^i = X^i(x), \quad i = 1, \dots, n.$$

Then the derivative (4.6.1) measures how fast the function f changes along the integral curves

$$\frac{d}{dt}f(x(t)) = \partial_X f(x)|_{x=x(t)}. \tag{4.6.2}$$

In particular, the functions satisfying $\partial_X f = 0$ take constant values along the integral curves. They are called *first integrals* of the dynamical system.

We want now to define the derivative of one vector field along another one. This requires an introduction of an additional structure on the manifold.

Definition 4.6.1 *An* affine connection *on M is an operation that assigns to any pair of smooth vector fields X, $Y \in Vect(M)$ a new smooth vector field $\nabla_X Y \in Vect(M)$ called* covariant derivative *of Y along X. The operation must depend linearly on X and Y, i.e.,*

$$\nabla_{a_1 X_1 + a_2 X_2} Y = a_1 \nabla_{X_1} Y + a_2 \nabla_{X_2} Y, \quad \nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2 \quad \forall\, a_1,\, a_2 \in \mathbb{R}$$

and also satisfy the following properties

$$\nabla_{fX} Y = f\,\nabla_X Y \tag{4.6.3}$$

$$\nabla_X (f\,Y) = f\,\nabla_X Y + \partial_X f \cdot Y \tag{4.6.4}$$

for any $f \in \mathcal{C}^\infty(M)$.

In a system of local coordinates $\left(x^1, \ldots, x^n\right)$ define a collection of smooth functions $\Gamma_{ij}^k(x)$ called *Christoffel coefficients* of the affine connection by taking covariant derivatives of one basic field along another one

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}|_{x\in M} = \Gamma_{ij}^k(x)\frac{\partial}{\partial x^k}. \tag{4.6.5}$$

These coefficients locally completely determine the affine connection due to

Lemma 4.6.2 *For arbitrary smooth vector fields*

$$X = X^i \frac{\partial}{\partial x^i}, \quad Y = Y^j \frac{\partial}{\partial x^j}$$

the following formula takes place

$$\nabla_X Y = X^i \left(\frac{\partial Y^k}{\partial x^i} + \Gamma_{ij}^k Y^j\right)\frac{\partial}{\partial x^k}. \tag{4.6.6}$$

Proof: Because of (4.6.3)

$$\nabla_X = X^i \nabla_{\frac{\partial}{\partial x^i}}.$$

Because of (4.6.4)

$$\nabla_{\frac{\partial}{\partial x^i}} Y = \nabla_{\frac{\partial}{\partial x^i}}\left(Y^j \frac{\partial}{\partial x^j}\right) = Y^j \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} + \frac{\partial Y^j}{\partial x^i}\cdot\frac{\partial}{\partial x^j} = Y^j\Gamma_{ij}^k\frac{\partial}{\partial x^k} + \frac{\partial Y^k}{\partial x^i}\cdot\frac{\partial}{\partial x^k}$$

(in the last term we have changed the notation for the summation index from j to k). From the above two equations one easily derives formula (4.6.6). □

Notation: we will denote $\nabla_i X^j$ the coordinates of the vector field $\nabla_{\partial/\partial x^i} X$ with respect to the basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$

$$\nabla_{\frac{\partial}{\partial x^i}} X =: \nabla_i X^j \frac{\partial}{\partial x^j}. \tag{4.6.7}$$

Explicitly

$$\nabla_i X^j = \frac{\partial X^j}{\partial x^i} + \Gamma_{ik}^j X^k. \tag{4.6.8}$$

In these notations the formula (4.6.6) reads

$$\nabla_X Y = X^i \nabla_i Y^j \frac{\partial}{\partial x^j}. \tag{4.6.9}$$

Lemma 4.6.3 *For any vector field Y the functions $\nabla_i Y^j$ are components of a $(1,1)$-tensor.*

Proof: The corresponding linear operator on tangent spaces reads

$$X \mapsto \nabla_X Y.$$

□

Corollary 4.6.4 *Let $y = (y^1, \dots, y^n)$ be another system of coordinates on a neighborhood of a given point $x = (x^1, \dots, x^n)$, $y = y(x)$. Denote $\Gamma_{ij}^k(x)$ the Christoffel coefficients of an affine connection in the coordinates x and $\Gamma_{pq}^r(y)$ the Christoffel coefficients of the same affine connection in the coordinates y. Then the following transformation law holds true*

$$\Gamma_{pq}^r(y) = \frac{\partial y^r}{\partial x^k} \frac{\partial x^i}{\partial y^p} \frac{\partial x^j}{\partial y^q} \Gamma_{ij}^k(x) + \frac{\partial y^r}{\partial x^k} \frac{\partial^2 x^k}{\partial y^p \partial y^q}. \tag{4.6.10}$$

Proof: We have

$$\frac{\partial}{\partial y^p} = \frac{\partial x^i}{\partial y^p} \frac{\partial}{\partial x^i}, \quad \frac{\partial}{\partial y^q} = \frac{\partial x^j}{\partial y^q} \frac{\partial}{\partial x^j}.$$

So

$$\Gamma_{pq}^r(y) \frac{\partial}{\partial y^r} = \nabla_{\frac{\partial}{\partial y^p}} \frac{\partial}{\partial y^q} = \frac{\partial x^i}{\partial y^p} \nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial x^j}{\partial y^q} \frac{\partial}{\partial x^j} \right) = \frac{\partial x^i}{\partial y^p} \frac{\partial x^j}{\partial y^q} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial y^p} \frac{\partial}{\partial x^i} \left(\frac{\partial x^j}{\partial y^q} \right) \frac{\partial}{\partial x^j}.$$

In the last term we can use the chain rule

$$\frac{\partial x^i}{\partial y^p} \frac{\partial}{\partial x^i} \left(\frac{\partial x^j}{\partial y^q} \right) = \frac{\partial^2 x^j}{\partial y^p \partial y^q}.$$

We arrive at the equation

$$\Gamma_{pq}^r(y) \frac{\partial}{\partial y^r} = \frac{\partial x^i}{\partial y^p} \frac{\partial x^j}{\partial y^q} \Gamma_{ij}^k(x) \frac{\partial}{\partial x^k} + \frac{\partial^2 x^j}{\partial y^p \partial y^q} \frac{\partial}{\partial x^j}.$$

The last step is to change the notation $j \mapsto k$ for the summation index in the last term and to use

$$\frac{\partial}{\partial y^r} = \frac{\partial x^k}{\partial y^r} \frac{\partial}{\partial x^k}.$$

$\square$

Remark 4.6.5 *An alternative definition of affine connection can be formulated as follows: an affine connection is a collection of smooth functions $\Gamma_{ij}^k(x)$ assigned to any chart $(x^1, \dots, x^n)$ such that, on the intersection of charts the transformation law* (4.6.10) *holds true.*

Example 4.6.6 *$M = \mathbb{R}^n$. The* Euclidean connection *is defined by trivial Christoffel coefficients in the Euclidean coordinates*

$$\Gamma_{ij}^k = 0, \quad i,\, j,\, k = 1, \dots, n.$$

Observe that the Christoffel coefficients of the Euclidean connection vanish also in any system of affine coordinates on $\mathbb{R}^n$. In a system of curvilinear coordinates *$y = y(x)$ the Christoffel coefficients in general do not vanish:*

$$\Gamma_{pq}^r(y) = \frac{\partial y^r}{\partial x^k} \frac{\partial x^k}{\partial y^p \partial y^q}.$$

Exercise 4.6.7 *1) Given an affine connection Γ^k_{ij}, prove that*

$$T^k_{ij} := \Gamma^k_{ij} - \Gamma^k_{ji} \tag{4.6.11}$$

is a $(1,2)$-tensor. It is called the torsion tensor *of the connection.*

2) Prove that the value of the torsion tensor on any pair of smooth vector fields X, Y is given by the formula

$$T(X,Y) := X^i Y^j T^k_{ij} \frac{\partial}{\partial x^k} = \nabla_X Y - \nabla_Y X - [X,Y]. \tag{4.6.12}$$

Here

$$[X,Y] = \left(X^s \frac{\partial Y^k}{\partial x^s} - Y^s \frac{\partial X^k}{\partial x^s} \right) \frac{\partial}{\partial x^k}$$

is the commutator of vector fields.

3) Prove that the torsion of the Euclidean connection identically vanishes.

Definition 4.6.8 *An affine connection with vanishing torsion is called* symmetric.

For a symmetric connection the Christoffel coefficients Γ^k_{ij} are symmetric in i and j. Such a symmetry does not depend on the choice of a coordinate system.

Given an affine connection on a manifold M and a vector field X one can define a differential operator

$$\begin{aligned} &\nabla_X : \mathcal{T}^p_q(M) \to \mathcal{T}^p_q(M) \\ &T^{i_1\dots i_p}_{j_1\dots j_q} \mapsto \nabla_X T^{i_1\dots i_p}_{j_1\dots j_q} = X^k \nabla_k T^{i_1\dots i_p}_{j_1\dots j_q} \end{aligned} \tag{4.6.13}$$

on tensors of the type (p,q), i.e., on sections of the vector bundle

$$\underbrace{TM \otimes \cdots \otimes TM}_{p \text{ times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{q \text{ times}}.$$

Here $T \in \mathcal{T}^p_q(M)$ is defined by its components

$$T = T^{i_1\dots i_p}_{j_1\dots j_q}(x) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}$$

in a given coordinate system. The general formula for the operators (4.6.13) can be derived from the following requirements:

- for $\quad p = q = 0 \quad \nabla_m f = \dfrac{\partial f}{\partial x^m}$
- for $\quad p = 1,\ q = 0 \quad \nabla_i X^j \quad$ is given by the formula(4.6.8)
- Leibnitz rule $\quad \nabla(T \otimes S) = \nabla T \otimes S + T \otimes \nabla S.$

One has to also take into account that, due to linearity the covariant derivative commutes with contractions.

Theorem 4.6.9 *The above conditions uniquely determine the operation* (4.6.13) *on a manifold equipped with an affine connection. Namely, components of the covariant derivative of a* (p, q)*-tensor* $T^{i_1\dots i_p}_{j_1\dots j_q}$ *is a* $(p, q+1)$*-tensor* $\nabla_k T^{i_1\dots i_p}_{j_1\dots j_q}$ *given by the formula*

$$\nabla_k T^{i_1\dots i_p}_{j_1\dots j_q} = \frac{\partial T^{i_1\dots i_p}_{j_1\dots j_q}}{\partial x^k} + \Gamma^{i_1}_{ks} T^{s\, i_2\dots i_p}_{j_1\dots j_q} + \dots + \Gamma^{i_p}_{ks} T^{i_1\dots i_{p-1}\, s}_{j_1\dots j_q} - \Gamma^s_{k\, j_1} T^{i_1\dots i_p}_{s\, j_2\dots j_q} - \dots - \Gamma^s_{k\, j_q} T^{i_1\dots i_p}_{j_1\dots j_{q-1}\, s}. \tag{4.6.14}$$

Proof: Let us first derive that, for a $(0, 1)$-tensor $\omega = (\omega_i)$ (a 1-form) the covariant derivative is a $(0, 2)$-tensor given by

$$\nabla_i \omega_j = \frac{\partial \omega_j}{\partial x^i} - \Gamma^k_{ij} \omega_k. \tag{4.6.15}$$

Indeed, for any vector field $X = X^i \frac{\partial}{\partial x^i}$ the contraction $\omega_i X^i$ is a smooth function on M. Therefore

$$\nabla_k \left(\omega_i X^i\right) = \frac{\partial}{\partial x^k} \left(\omega_i X^i\right) = \frac{\partial \omega_i}{\partial x^k} X^i + \omega_i \frac{\partial X^i}{\partial x^k}.$$

On another side, using the Leibnitz rule for the covariant derivative we obtain

$$\nabla_k \left(\omega_i X^i\right) = \nabla_k \omega_i\, X^i + \omega_i\, \nabla_k X^i = \nabla_k \omega_i\, X^i + \omega_i \left(\frac{\partial X^i}{\partial x^k} + \Gamma^i_{ks} X^s\right).$$

A comparison of the two expressions yields

$$\nabla_k \omega_i\, X^i = \left(\frac{\partial \omega_i}{\partial x^k} - \Gamma^s_{ki} \omega_s\right) X^i$$

(we have interchanged the notations for the summation indices $i \leftrightarrow s$ in one of the terms of the formula). Since X is an arbitrary vector field, the formula (4.6.15) is proved.

Let us now proceed to tensors of higher rank. The idea of the proof will be explained for tensors of the type $(1, 1)$. Let us first consider the case where the tensor T^i_j is equal to the tensor product of a $(1, 0)$-tensor X and a $(0, 1)$-tensor ω,

$$T^i_j = X^i \omega_j. \tag{4.6.16}$$

Then, using Leibnitz rule one obtains

$$\begin{aligned} \nabla_k T^i_j &= \nabla_k X^i\, \omega_j + X^i\, \nabla_k \omega_j = \left(\frac{\partial X^i}{\partial x^k} + \Gamma^i_{ks} X^s\right) \omega_j + X^i \left(\frac{\partial \omega_j}{\partial x^k} - \Gamma^s_{kj} \omega_s\right) \\ &= \frac{\partial}{\partial x^k} \left(X^i \omega_j\right) + \Gamma^i_{ks} X^s \omega_j - \Gamma^s_{kj} X^i \omega_s = \frac{\partial T^i_j}{\partial x^k} + \Gamma^i_{ks} T^s_j - \Gamma^s_{kj} T^i_s. \end{aligned}$$

So, for factorizable $(1, 1)$-tensors (4.6.16) the formula (4.6.14) is proved. Since any $(1, 1)$-tensor can be represented as a linear combination of factorizable tensors, the formula is proved also for an arbitrary $(1, 1)$-tensor. The derivation of the formula (4.6.14) in the general case is completely similar and it will be omitted. □

The particular formula for covariant derivatives of a $(0, 2)$-tensor g_{ij} will be often used in sequel

$$\nabla_m g_{ij} = \frac{\partial g_{ij}}{\partial x^m} - \Gamma^k_{mi} g_{mj} - \Gamma^k_{mj} g_{ik}. \tag{4.6.17}$$

Exercise 4.6.10 *Consider a* $(1,1)$*-tensor* T *with components* $T^i_j = \delta^i_j$. *Prove that*

$$\nabla T = 0$$

for an arbitrary affine connection.

4.7 Parallel transport. Curvature of an affine connection

Let $\gamma(t) = \left(x^1(t), \dots, x^n(t)\right)$ be a smooth curve on a manifold equipped with an affine connection.

Definition 4.7.1 *The velocity of a vector field* $X \in Vect(M)$ *along the curve* γ *is the vector field* $\nabla_{\dot\gamma} X$.

Here

$$\dot\gamma = \left(\dot x^1(t), \dots, \dot x^n(t)\right)$$

is the velocity vector of the curve. Explicitly,

$$\nabla_{\dot\gamma} X = \dot x^i \nabla_i X^k \frac{\partial}{\partial x^k}$$

So, the components of the vector field $\nabla_{\dot\gamma} X$ at the points of the curve are equal to

$$\dot x^i \left(\frac{\partial X^k\left(x(t)\right)}{\partial x^i} + \Gamma^k_{ij} X^j\left(x(t)\right)\right) = \frac{d X^k\left(x(t)\right)}{dt} + \dot x^i \Gamma^k_{ij} X^j\left(x(t)\right).$$

Observe that the rhs depends only on the values of the vector field *at the points of the curve* γ. Such an observations motivates the following

Definition 4.7.2 *A vector field* $X = X(t)$ *on the curve* γ *is called* parallel along γ *if*

$$\nabla_{\dot\gamma} X = 0.$$

From the above calculation we derive a system of n linear ODEs for a parallel vector field $X(t)$

$$\frac{d\,X^k}{dt} + \dot x^i(t)\Gamma^k_{ij}(x(t))X^j = 0, \quad k = 1, \dots, n. \tag{4.7.1}$$

Theorem 4.7.3 *Given a smooth curve* $\gamma : [0,1] \to M$, $\gamma(0) = x_0$, $\gamma_1 = x_1$, *and a vector* $X_0 \in T_{x_0}M$, *then there exists a unique parallel along* γ *vector field* $X(t)$ *on the curve,* $0 \le t \le 1$.

Proof: Solving the system of *linear* differential equations (4.7.1) with the initial data $X(0) = X_0$ we obtain a unique solution $X(t)$ defined on the entire domain of smoothness of coefficients of the linear system. □

Definition 4.7.4 *For a given vector* $X_0 \in T_{x_0}M$ *and a given smooth curve* $\gamma : [0,1] \to M$ *starting at* x_0 *and ending at* x_1 *the value* $X_1 := X(1) \in T_{x_1}M$ *of the parallel along* γ *vector field* $X(t)$ *is called* the parallel transport of X_0 along γ.

Exercise 4.7.5 *Prove that parallel transport of vectors along a curve γ from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$ defines a linear map of tangent spaces*

$$T_{x_0}M \to T_{x_1}M.$$

Moreover, prove that this linear map is an isomorphism.

Example 4.7.6 *In Euclidean space $\mathbb{R}^n$ with the Euclidean connection the vector field $X(t) = \big(X^1(t), \dots, X^n(t)\big)$ is parallel along γ* iff $X^k(t) \equiv X_0^k$, $k = 1, \dots, n$.

So, the result of parallel transport in Euclidean space does not depend on the choice of the curve. We will use this observation in order to develop the theory of *curvature* of an affine connection. Intuitively, curvature measures the dependence of the parallel transport from the curve. An analytic approach to the curvature is based on the following statement.

Theorem-Definition 4.7.7 *Let M be a manifold equipped with an affine connection. Then there exists a $(1,3)$-tensor R^k_{ijl} called* curvature tensor *of the connection such that, for an arbitrary vector field $X = X^i \frac{\partial}{\partial x^i}$ the following formula holds true*

$$\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = -R^k_{ijl} X^l - T^s_{ij} \nabla_s X^k \tag{4.7.2}$$

Here $T^s_{ij} = \Gamma^s_{ij} - \Gamma^s_{ji}$ are components of the torsion tensor.

Proof: We know that $\nabla_j X^k$ is a $(1,1)$-tensor. Applying the formula (4.6.14) we obtain

$$\begin{aligned}
\nabla_i \left(\nabla_j X^k\right) &= \frac{\partial}{\partial x^i}\left(\nabla_j X^k\right) + \Gamma^k_{is}\nabla_j X^s - \Gamma^s_{ij}\nabla_s X^k \\
&= \frac{\partial}{\partial x^i}\left(\frac{\partial X^k}{\partial x^j} + \Gamma^k_{jl}X^l\right) + \Gamma^k_{is}\left(\frac{\partial X^s}{\partial x^j} + \Gamma^s_{jl}X^l\right) - \Gamma^s_{ij}\nabla_s X^k \\
&= \frac{\partial^2 X^k}{\partial x^i \partial x^j} + \frac{\partial \Gamma^k_{jl}}{\partial x^i} X^l + \Gamma^k_{jl}\frac{\partial X^l}{\partial x^i} + \Gamma^k_{is}\frac{\partial X^s}{\partial x^j} + \Gamma^k_{is}\Gamma^s_{jl}X^l - \Gamma^s_{ij}\nabla_s X^k.
\end{aligned}$$

Similarly,

$$\nabla_j\left(\nabla_i X^k\right) = \frac{\partial^2 X^k}{\partial x^j \partial x^i} + \frac{\partial \Gamma^k_{il}}{\partial x^j} X^l + \Gamma^k_{il}\frac{\partial X^l}{\partial x^j} + \Gamma^k_{js}\frac{\partial X^s}{\partial x^i} + \Gamma^k_{js}\Gamma^s_{il}X^l - \Gamma^s_{ji}\nabla_s X^k.$$

Subtracting we obtain

$$\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k = -R^k_{ijl} X^l - T^s_{ij} \nabla_s X^k$$

where

$$R^k_{ijl} = \frac{\partial \Gamma^k_{il}}{\partial x^j} - \frac{\partial \Gamma^k_{jl}}{\partial x^i} + \Gamma^k_{js}\Gamma^s_{il} - \Gamma^k_{is}\Gamma^s_{jl}. \tag{4.7.3}$$

Since the lhs $\nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k$ is a $(1,2)$-tensor and also $T^s_{ij}\nabla_s X^k$ is a $(1,2)$-tensor we derive that $R^k_{ijl}X^l$ is a $(1,2)$-tensor for any vector field X. It depends linearly on X. Since the space of linear maps from T_xM to $T_xM \otimes T^*_xM \otimes T^*_xM$ (i.e., to the space of $(1,2)$-tensors at the point $x \in M$) is isomorphic to the space $T_xM \otimes T^*_xM \otimes T^*_xM \otimes T^*_xM$, we conclude that R^k_{ijl} is a $(1,3)$-tensor. $\square$

The formula (4.7.3) derived in the proof of the Theorem gives an explicit expression for the curvature tensor in terms of Christoffel coefficients and their derivatives.

Exercise 4.7.8 *For a 1-form $\omega = \omega_k dx^k$ prove the following formula*

$$\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = R^l_{ijk}\omega_l - T^s_{ij}\nabla_s \omega_k. \tag{4.7.4}$$

For a Euclidean connection all Christoffel coefficients vanish in the affine coordinates. We will now prove the converse statement.

Definition 4.7.9 *An affine connection on a manifold M is called* locally Euclidean *if, for any point $x_0 \in M$ there exists a neighborhood U_{x_0} and a system of coordinates $\left(y^1, \dots, y^n\right)$ on it such that in these coordinates all Christoffel coefficients vanish.*

Theorem 4.7.10 *An affine connection on M is locally Euclidean* iff *its torsion and curvature tensors identically vanish.*

Proof: It has already been explained that the conditions $T^k_{ij} = 0$ and $R^k_{ijl} = 0$ are necessary for existence of a system of locally Euclidean coordinates. Let us prove that these conditions are also sufficient. Denote $\Gamma^k_{ij}(x)$ the Christoffel coefficients of the connection in a coordinate system $\left(x^1, \dots, x^n\right)$. We are looking for new coordinates $y^s = y^s(x)$ such that, in new coordinates all Christoffel coefficients $\Gamma^r_{pq}(y)$ vanish. Using the transformation law (4.6.10)

$$\Gamma^k_{ij}(x) = \frac{\partial x^k}{\partial y^r}\frac{\partial y^p}{\partial x^i}\frac{\partial y^q}{\partial x^j}\Gamma^r_{pq}(y) + \frac{\partial x^k}{\partial y^r}\frac{\partial^2 y^r}{\partial x^i \partial x^j}$$

we arrive at equations for the functions $y(x)$:

$$\Gamma^k_{ij}(x) = \frac{\partial x^k}{\partial y^r}\frac{\partial^2 y^r}{\partial x^i \partial x^j}.$$

Multiplying by the inverse Jacobi matrix $\frac{\partial y^s}{\partial x^k}$ rewrite the system of equations in the form

$$\frac{\partial^2 y^s}{\partial x^i \partial x^j} = \Gamma^k_{ij}(x)\frac{\partial y^s}{\partial x^k}. \tag{4.7.5}$$

Our goal is to find n independent functions $y^s(x)$, $s = 1, \dots, n$ satisfying equations (4.7.5) for all i, $j = 1, \dots, n$.

Let us fix a particular value of the index s. We will rewrite a system (4.7.5) as a first order system. To this end denote

$$f = y^s, \quad F_j = \frac{\partial y^s}{\partial x^j}, \quad j = 1, \dots, n.$$

The system (4.7.5) for a given s takes the form

$$\begin{aligned} \frac{\partial f}{\partial x^i} &= F_i \\ \frac{\partial F_j}{\partial x^i} &= \Gamma^k_{ij} F_k. \end{aligned} \tag{4.7.6}$$

We are looking for a solution to this system satisfying the following initial data at a given point $x_0 = \left(x_0^1, \ldots, x_0^n\right) \in M$

$$f(x_0) = 0, \quad F_j(x_0) = \delta_j^s, \quad j = 1, \ldots, n.$$

The problem is that the system (4.7.6) is *overdetermined*, i.e., there are n equations for one vector-valued function

$$X = \begin{pmatrix} f \\ F_1 \\ . \\ . \\ . \\ F_n \end{pmatrix} \tag{4.7.7}$$

Our main goal is to establish a criterion for existence of a common solution for equations (4.7.6) with arbitrary initial data.

Let us rewrite the system (4.7.6) in matrix notations

$$\frac{\partial X}{\partial x^i} = M_i X, \quad i = 1, \ldots, n \tag{4.7.8}$$

where the $(n+1) \times (n+1)$ matrices M_i have the form

$$M_i = \begin{pmatrix} 0 & 0 & \ldots & 1 & \ldots & 0 \\ 0 & \Gamma_{i\,1}^1 & \ldots & \ldots & \ldots & \Gamma_{i\,1}^n \\ 0 & \Gamma_{i\,2}^1 & \ldots & \ldots & \ldots & \Gamma_{i\,2}^n \\ . & \ldots & \ldots & \ldots & \ldots & . \\ . & \ldots & \ldots & \ldots & \ldots & . \\ . & \ldots & \ldots & \ldots & \ldots & . \\ 0 & \Gamma_{i\,n}^1 & \ldots & \ldots & \ldots & \Gamma_{i\,n}^n \end{pmatrix} \tag{4.7.9}$$

(the only non-zero element in the first row is at the $(i+1)$-th position).

Lemma 4.7.11 *Let the coefficient matrices of the system* (4.7.8) *be smooth functions on a cube*

$$I_{x_0}^n = \{|x^i - x_0^i| < a, \quad i = 1, \ldots, n\}$$

for some positive a. A common solution $X = X(x)$, $x \in I_{x_0}^n$ to the equations of the system with an arbitrary initial data

$$X(x_0) = X_0$$

exists iff *the matrices M_i satisfy the following system of equations*

$$\frac{\partial M_i}{\partial x^j} - \frac{\partial M_j}{\partial x^i} + [M_i, M_j] = 0, \quad i,\, j = 1, \ldots, n. \tag{4.7.10}$$

Here

$$[M_i, M_j] = M_i M_j - M_j M_i$$

is matrix commutator.

Needless to say that the solution, if exists, is unique.

Proof: The common solution $X(x)$, if exists, must satisfy the identities

$$\frac{\partial^2 X}{\partial x^j \partial x^i} = \frac{\partial^2 X}{\partial x^i \partial x^j} \tag{4.7.11}$$

for any pair of indices i, j. From (4.7.8) it follows

$$\frac{\partial^2 X}{\partial x^j \partial x^i} = \frac{\partial}{\partial x^j}\left(M_i X\right) = \frac{\partial M_i}{\partial x^j} X + M_i \frac{\partial X}{\partial x^j} = \left(\frac{\partial M_i}{\partial x^j} + M_i M_j\right) X.$$

Therefore the required symmetry (4.7.11) implies

$$\left(\frac{\partial M_i}{\partial x^j} - \frac{\partial M_j}{\partial x^i} + [M_i, M_j]\right) X = 0.$$

This equation must hold true for an arbitrary vector X, as the initial data $X(x_0)$ is an arbitrary vector. Thus we have proved necessity of the conditions (4.7.10) for existence of a common solution to the system (4.7.8).

In order to prove sufficiency of the conditions (4.7.10) let us first consider the case of a system of two overdetermined linear equations

$$\frac{\partial X}{\partial u} = M_u X, \quad \frac{\partial X}{\partial v} = M_v X \tag{4.7.12}$$

for one vector-valued function $X = X(u, v)$. The coefficient matrices $M_u = M_u(u, v)$, $M_v = M_v(u, v)$ are smooth functions of variables u, v satisfying

$$\frac{\partial M_u}{\partial v} - \frac{\partial M_v}{\partial u} + [M_u, M_v] = 0.$$

We want to construct a solution satisfying given initial data

$$X(u_0, v_0) = X_0.$$

Let us first construct a solution $Y = Y(u)$ to the following Cauchy problem

$$\frac{\partial Y}{\partial u} = M_u(u, v_0) Y, \quad Y(u_0) = X_0.$$

Such a solution exists and is unique. Next, consider another Cauchy problem depending on u as on a parameter

$$\frac{\partial X}{\partial v} = M_v(u, v) X, \quad X(u, v_0) = Y(u).$$

Such a solution $X = X(u, v)$ also exists and is unique for all u. Moreover it depends smoothly on the parameter u. Let us prove that X also satisfies the first equation of the system (4.7.12). Denote

$$\tilde{X} = \frac{\partial X}{\partial u} - M_u X.$$

Let us prove that $\tilde{X}$ satisfies the second equation of (4.7.12). Indeed,

$$\begin{aligned}
\frac{\partial \tilde{X}}{\partial v} - M_v \tilde{X} &= \frac{\partial^2 X}{\partial u \partial v} - \frac{\partial M_u}{\partial v} X - M_u \frac{\partial X}{\partial v} - M_v \left(\frac{\partial X}{\partial u} - M_u X \right) \\
&= \frac{\partial^2 X}{\partial u \partial v} - \left(\frac{\partial M_v}{\partial u} - M_u M_v + M_v M_u \right) X - M_u M_v X - M_v \frac{\partial X}{\partial u} + M_v M_u X \\
&= \frac{\partial^2 X}{\partial u \partial v} - \frac{\partial M_v}{\partial u} X - M_v \frac{\partial X}{\partial u} = \frac{\partial}{\partial u} \left(\frac{\partial X}{\partial v} - M_v X \right) = 0.
\end{aligned}$$

For $v = v_0$ one has

$$\tilde{X}(u, v_0) = \frac{\partial}{\partial u} X(u, v_0) - M_u(u, v_0) X(u, v_0) = \frac{\partial}{\partial u} Y(u) - M_u(u, v_0) Y(u) = 0.$$

Hence $\tilde{X} \equiv 0$. The Lemma is proved for $n = 2$. For $n > 2$ the proof can be easily completed by induction. □

In order to finish the proof of the Theorem it remains to check the compatibility conditions of the equations (4.7.10) for the matrices (4.7.9)

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}, \quad \frac{\partial^2 F_k}{\partial x^i \partial x^j} = \frac{\partial^2 F_k}{\partial x^j \partial x^i}.$$

It is easy to see that these conditions are equivalent to vanishing of the torsion and the curvature of the affine connection; we leave this calculation as an exercise for the reader.
□

Remark 4.7.12 *The linear differential equations* (4.7.6) *can be rewritten in the form*

$$\mathcal{M}_i X = 0, \quad i = 1, \dots, n \tag{4.7.13}$$

where

$$\mathcal{M}_i = \frac{\partial}{\partial x^i} - M_i \tag{4.7.14}$$

is a linear differential operator with matrix coefficients. The compatibility conditions (4.7.10) *is nothing but the commutativity of the operators*

$$[\mathcal{M}_i, \mathcal{M}_j] = 0 \quad \Leftrightarrow \quad \frac{\partial M_i}{\partial x^j} - \frac{\partial M_j}{\partial x^i} + [M_i, M_j] = 0 \tag{4.7.15}$$

as it follows from the following statement.

Exercise 4.7.13 *Given a smooth function $f = f(x)$ of one variable x, prove that the commutator of the operator $\frac{d}{dx}$ and of the operator of multiplication by the function f is the operator of multiplication by the derivative f':*

$$\left[\frac{d}{dx}, f \right] = f'. \tag{4.7.16}$$

We recommend to the reader to prove again the Lemma 4.7.11 using the operator notations.

We will now outline an alternative approach to the definition of the curvature tensor.

Exercise 4.7.14 *Given two smooth vector fields X, Y on M, consider a linear operator*

$$\begin{aligned} &R(X,Y) : Vect(M) \to Vect(M) \\ &R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \end{aligned} \tag{4.7.17}$$

1) Prove that for an arbitrary vector field Z and arbitrary smooth function f one has

$$R(f\,X,Y)Z = R(X, f\,Y)Z = R(X,Y)(f\,Z) = f\,R(X,Y)Z. \tag{4.7.18}$$

2) Derive from this property that, for any point $x \in M$ the value of the vector field $R(X,Y)Z$ at x depends only on the values of the vector fields X, Y, Z at the same point.

3) For a vectors $X = X^i \frac{\partial}{\partial x^i}$, $Y = Y^j \frac{\partial}{\partial x^j}$, $Z = Z^k \frac{\partial}{\partial x^k}$ prove that

$$R(X,Y)Z = -R^l_{ijk} X^i Y^j Z^k \frac{\partial}{\partial x^l}. \tag{4.7.19}$$

Let us now explain in what sense the curvature tensor "measures" the dependence of the parallel transport from the curve.

First, we extend the notion of parallel transport to piecewise smooth curves. It is easy: if $0 < t_1 < t_2 < \cdots < t_m < 1$ are the points of discontinuity of the derivative $\dot\gamma$ of the curve $\gamma : [0,1] \to M$ from $x_0 = \gamma(0)$ to $x_1 = \gamma(1)$ then we first perform a parallel transport of the initial vector $X_0 \in T_{x_0}M$ from x_0 to $\gamma(t_1)$, then the resulting vector from $\gamma(t_1)$ to $\gamma(t_2)$ etc.

Given a point $x_0 \in M$, choose a pair of indices $i < j$ and consider a small rectangle

$$x_0^i \le x^i \le x_0^i + \Delta_1, \quad x_0^j \le x^j \le x_0 + \Delta_2, \quad x^k = x_0^k \quad \text{for} \quad k \ne i,\, j \tag{4.7.20}$$

where

$$\Delta_1 \sim \epsilon, \quad \Delta_2 \sim \epsilon$$

for a sufficiently small positive ϵ. Denote x_1, x_2, y_0 the remaining vertices of the rectangle

$$\begin{array}{lcl} x_1 &=& (x_0^1, \quad \dots, \quad x_0^i + \Delta_1, \quad \dots, \quad x_0^j, \quad \dots, \quad x_0^n) \\ x_2 &=& (x_0^1, \quad \dots, \quad x_0^i, \quad \dots, \quad x_0^j + \Delta_2, \quad \dots, \quad x_0^n) \\ y_0 &=& (x_0^1, \quad \dots, \quad x_0^i + \Delta_1, \quad \dots, \quad x_0^j + \Delta_2, \quad \dots, \quad x_0^n) \end{array} \tag{4.7.21}$$

Let γ_1 be the part of the boundary of the rectangle from x_0 to y_0 passing via x_1. In a similar way γ_2 is another part of the boundary going from x_0 to y_0 via x_2. Let us compare the results $Y_{0,1}$ and $Y_{0,2}$ of parallel transport of an arbitrary vector $X_0 \in T_{x_0}M$ to the vertex y_0 of the rectangle along its boundary in two possible ways. The result of such comparison is given by

Theorem 4.7.15 *The difference between the results $Y_{0,1}$ and $Y_{0,2}$ of two parallel transports of a vector $X_0 \in T_{x_0}M$ from x_0 to y_0 along the curves γ_1 and γ_2 respectively admits the following expansion at $\epsilon \to 0$*

$$Y^k_{0,2} - Y^k_{0,1} = R^k_{ijs}(x_0) X^s_0 \Delta_1 \Delta_2 + \mathcal{O}\left(\epsilon^3\right). \tag{4.7.22}$$

Proof: Let us first compute the result of parallel transport along γ_1 from x_0 to x_1 (i.e., along the x^i-axis). To this end we have to find the solution $X(t)$ to the Cauchy problem

$$\frac{\partial X^k}{\partial x^i} + \Gamma^k_{is} X^s = 0, \quad X(0) = X_0. \tag{4.7.23}$$

Expanding the solution in Taylor series for small $t = x^i - x^i_0$

$$X(t) = X_0 + \frac{\partial X(0)}{\partial x^i}\, t + \frac{1}{2}\frac{\partial^2 X(0)}{\partial {x^i}^2}\, t^2 + \mathcal{O}\left(t^3\right)$$

and using equations (4.7.23) we obtain

$$X_1^k := X^k\left(t = \Delta_1\right) = X_0^k - \Gamma^k_{is}(x_0) X_0^s\, \Delta_1 + \frac{1}{2}\left(-\frac{\partial \Gamma^k_{is}(x_0)}{\partial x^i} + \Gamma^k_{ir}(x_0)\Gamma^r_{is}(x_0)\right) X_0^s \Delta_1^2 + \mathcal{O}\left(\Delta_1^3\right).$$

In a similar way, the parallel transport $Y_{0,1}$ of the vector X_1 from x_1 to y_0 along the x^j-axis reads

$$\begin{aligned}
Y^k_{0,1} &= X_1^k - \Gamma^k_{js}(x_1) X_1^s\, \Delta_2 + \frac{1}{2}\left(-\frac{\partial \Gamma^k_{js}(x_1)}{\partial x^i} + \Gamma^k_{jr}(x_1)\Gamma^r_{js}(x_1)\right) X_1^s \Delta_2^2 + \mathcal{O}\left(\Delta_2^3\right)\\
&= X_0^k - \Gamma^k_{is}(x_0) X_0^s\, \Delta_1 - \left(\Gamma^k_{js}(x_0) + \frac{\partial \Gamma^k_{js}(x_0)}{\partial x^i}\Delta_1\right)\left(X_0^s - \Gamma^s_{il}(x_0) X_0^l\, \Delta_1\right)\, \Delta_2\\
&+ \frac{1}{2}\left(-\frac{\partial \Gamma^k_{is}(x_0)}{\partial x^i} + \Gamma^k_{ir}(x_0)\Gamma^r_{is}(x_0)\right) X_0^s \Delta_1^2 + \frac{1}{2}\left(-\frac{\partial \Gamma^k_{js}(x_0)}{\partial x^i} + \Gamma^k_{jr}(x_0)\Gamma^r_{js}(x_0)\right) X_0^s \Delta_2^2 + \mathcal{O}\left(\epsilon^3\right)\\
&= X_0^k - \Gamma^k_{is}(x_0) X_0^s\, \Delta_1 - \Gamma^k_{js}(x_0) X_0^s\, \Delta_2 + \left(-\frac{\partial \Gamma^k_{js}(x_0)}{\partial x^i} + \Gamma^k_{jl}(x_0)\Gamma^l_{is}(x_0)\right) X_0^s \Delta_1 \Delta_2\\
&+ \frac{1}{2}\left(-\frac{\partial \Gamma^k_{is}(x_0)}{\partial x^i} + \Gamma^k_{ir}(x_0)\Gamma^r_{is}(x_0)\right) X_0^s \Delta_1^2 + \frac{1}{2}\left(-\frac{\partial \Gamma^k_{js}(x_0)}{\partial x^i} + \Gamma^k_{jr}(x_0)\Gamma^r_{js}(x_0)\right) X_0^s \Delta_2^2 + \mathcal{O}\left(\epsilon^3\right).
\end{aligned}$$

A similar computation yields the followng expression for the vector $Y_{0,2}$ obtained from X_0 via parallel transport along γ_2 from x_0 to y_0

$$\begin{aligned}
Y^k_{0,2} &= X_0^k - \Gamma^k_{js}(x_0) X_0^s\, \Delta_1 - \Gamma^k_{is}(x_0) X_0^s\, \Delta_2 + \left(-\frac{\partial \Gamma^k_{js}(x_0)}{\partial x^j} + \Gamma^k_{il}(x_0)\Gamma^l_{js}(x_0)\right) X_0^s \Delta_1 \Delta_2\\
&+ \frac{1}{2}\left(-\frac{\partial \Gamma^k_{js}(x_0)}{\partial x^j} + \Gamma^k_{jr}(x_0)\Gamma^r_{js}(x_0)\right) X_0^s \Delta_1^2 + \frac{1}{2}\left(-\frac{\partial \Gamma^k_{is}(x_0)}{\partial x^j} + \Gamma^k_{ir}(x_0)\Gamma^r_{is}(x_0)\right) X_0^s \Delta_2^2 + \mathcal{O}\left(\epsilon^3\right).
\end{aligned}$$

Subtracting one obtains the needed formula. □

4.8 The Levi-Civita connection and curvature of Riemannian manifolds

In this section we will construct a particular affine connection on a Riemannian manifold.

Theorem-Definition 4.8.1 *An affine connection on a Riemannian manifold* $\left(M, ds^2\right)$ *is called* compatible with the metric *if it satisfies one of the following equivalent properties.*

1) The covariant derivatives of the metric tensor vanish

$$\nabla_k g_{ij} = 0, \quad i,\, j,\, k = 1, \dots, n. \tag{4.8.1}$$

2) For arbitrary vector fields $X,\, Y,\, Z \in Vect(M)$

$$\nabla_Z \langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle . \tag{4.8.2}$$

3) Parallel transport along any curve $\gamma : [0,1] \to M$ *is an orthogonal transformation* $T_{x_0}M \to T_{x_1}M$, $x_0 = \gamma(0)$, $x_1 = \gamma(1)$, *that is, for any pair of vectors* $X_0,\, Y_0 \in T_{x_0}M$ *their parallel transports* $X_1,\, Y_1 \in T_{x_1}M$ *satisfy*

$$\langle X_1, Y_1\rangle = \langle X_0, Y_0\rangle . \tag{4.8.3}$$

Observe that the covariant derivative in the lhs of eq. (4.8.2) coincides with the partial one

$$\nabla_Z \langle X, Y\rangle = \partial_Z \langle X, Y\rangle .$$

Proof: 1) $\Rightarrow$ 2). It suffices to check validity of (4.8.2) for $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$. In this case eq. (4.8.2) becomes

$$\nabla_{\frac{\partial}{\partial x^k}} \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\rangle \equiv \frac{\partial}{\partial x^k} g_{ij} = \left\langle \Gamma^s_{ki}\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^j}\right\rangle + \left\langle \frac{\partial}{\partial x^i}, \Gamma^s_{kj}\frac{\partial}{\partial x^s}\right\rangle \equiv \Gamma^s_{ki} g_{sj} + \Gamma^s_{kj} g_{is}.$$

On another side, using formula (4.6.17) for the covariant derivatives of a $(0,2)$-tensor we find that

$$\nabla_k g_{ij} = 0 \quad \Leftrightarrow \quad \frac{\partial g_{ij}}{\partial x^k} = \Gamma^s_{ki} g_{sj} + \Gamma^s_{kj} g_{is}. \tag{4.8.4}$$

This proves (4.8.2).

2) $\Rightarrow$ 3). Let the vector fields $X(t)$, $Y(t)$ be parallel alone the curve γ, i.e., $\nabla_{\dot\gamma} X(t) = 0$, $\nabla_{\dot\gamma} Y(t) = 0$. Using (4.8.2) we obtain

$$\frac{d}{dt}\langle X(t), Y(t)\rangle \equiv \nabla_{\dot\gamma}\langle X(t), Y(t)\rangle = \langle \nabla_{\dot\gamma} X(t), Y(t)\rangle + \langle X(t), \nabla_{\dot\gamma} Y(t)\rangle = 0.$$

Hence

$$\langle X(0), Y(0)\rangle = \langle X(1), Y(1)\rangle ,$$

q.e.d.

3) $\Rightarrow$ 1). Choosing, as above, vector fields $X(t)$, $Y(t)$ parallel along γ one must have

$$0 = \frac{d}{dt}\left(g_{ij}(x(t))X^i(t)Y^j(t)\right) = \dot x^k \frac{\partial g_{ij}}{\partial x^k} X^i Y^j + g_{ij}\dot X^i Y^j + g_{ij} X^i \dot Y^j.$$

Using equations of parallel transport

$$\dot X^i + \dot x^k \Gamma^i_{ks} X^s = 0, \quad \dot Y^j + \dot x^k \Gamma^j_{ks} Y^s = 0$$

recast the above expression into the form

$$0=\frac{d}{dt}\left(g_{ij}(x(t))X^i(t)Y^j(t)\right)=\dot{x}^k\left(\frac{\partial g_{ij}}{\partial x^k}-\Gamma^s_{ki}g_{sj}-\Gamma^s_{kj}g_{is}\right)X^iY^j=0.$$

Since X and Y are arbitrary vectors and also the velocity vector $\dot{\gamma}$ can be arbitrary we conclude that the expression in the parenthesis is equal to zero. □

Definition 4.8.2 *An affine connection on Riemannian manifold is called* Levi-Civita connection *if it is symmetric and compatible with the metric.*

Theorem 4.8.3 *On an arbitrary Riemannian manifold there exists a unique Levi-Civita connection.*

Proof: Let us derive expressions for the Christoffel coefficients of the Levi-Civita connection using eqs. (4.8.4) along with the symmetry condition $\Gamma^k_{ji}=\Gamma^k_{ij}$. Adding to (4.8.4) equations obtained by cyclic permutations of indices i, j, k arrive at a system

$$\begin{aligned}\frac{\partial g_{ij}}{\partial x^k}&=\Gamma^s_{ki}g_{sj}+\Gamma^s_{kj}g_{is}\\ \frac{\partial g_{ki}}{\partial x^j}&=\Gamma^s_{jk}g_{si}+\Gamma^s_{ji}g_{ks}\\ \frac{\partial g_{jk}}{\partial x^i}&=\Gamma^s_{ij}g_{sk}+\Gamma^s_{ik}g_{js}.\end{aligned}$$

Adding the second and the third equations and subtracting the first one we obtain

$$2\Gamma^s_{ij}g_{sk}=\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}.$$

Multiplying by the inverse matrix g^{kl} and taking summation over k we come at the needed formula

$$\Gamma^k_{ij}=\frac{1}{2}g^{ks}\left(\frac{\partial g_{sj}}{\partial x^i}+\frac{\partial g_{is}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^s}\right). \tag{4.8.5}$$

In this way we have proved uniqueness of the Levi-Civita connection. In order to prove existence it remains to check that the Christoffel coefficients defined for any coordinate system by the formulae (4.8.5) satisfy the transformation law (4.6.10). We leave this computation as an exercise for the reader. □

The formulae (4.8.5) for the Levi-Civita connection are called *Christoffel formulae.*

Example 4.8.4 *The Levi-Civita connection on the Euclidean space equipped with the Euclidean metric coincides with the Euclidean connection.*

Exercise 4.8.5 *Prove the following identities for contractions of Christoffel coefficients of the Levi-Civita connection*

$$\Gamma^k_{ik}=\frac{\partial}{\partial x^i}\log\sqrt{g} \tag{4.8.6}$$

$$g^{ij}\Gamma^k_{ij}=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^s}\left(\sqrt{g}\,g^{sk}\right). \tag{4.8.7}$$

In these formulae

$$g=\det\left(g_{ij}\right).$$

Exercise 4.8.6 *Prove that the covariant divergence $\nabla_i X^i$ of a vector field X^i with respect to the Levi-Civita connection can be written in the following form*

$$\nabla_i X^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, X^i \right). \tag{4.8.8}$$

Exercise 4.8.7 *For a vector field X^i consider the 1-form*

$$\omega = X_i dx^i, \quad X_i = g_{ij} X^j.$$

Prove that the Hodge-dual $(n-1)$-form $*\omega$ (see definition in (4.2.9)) reads

$$*\omega = \sum_{i=1}^{n} X^i \sqrt{g}\, dx^1 \wedge \dots \hat{dx^i} \dots \wedge dx^n. \tag{4.8.9}$$

Here a hat over dx^i means that this factor is omitted from the product.

Exercise 4.8.8 *Let X^i be a vector field on a compact n-dimensional oriented Riemannian manifold M with a boundary ∂M equipped with the induced orientation. Prove that*

$$\int_{\partial M} \sum_{i=1}^{n} X^i \sqrt{g}\, dx^1 \wedge \dots \hat{dx^i} \dots \wedge dx^n = \int_M \nabla_i X^i \sqrt{g}\, dx^1 \wedge \dots \wedge dx^n. \tag{4.8.10}$$

Remark 4.8.9 *On a Riemannian oriented manifold one can consider the* Laplace–Beltrami operator $\Delta : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ *as follows*

$$\Delta f = * \, d * df \tag{4.8.11}$$

From the above exercises it follows that

$$\Delta f = \nabla_i \left(g^{ik} \frac{\partial f}{\partial x^k} \right) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, g^{ik} \frac{\partial f}{\partial x^k} \right). \tag{4.8.12}$$

Definition 4.8.10 *The curvature tensor R^l_{ijk} of the Levi-Civita connection of a Riemannian manifold is called the* Riemann curvature *of the Riemannian manifold.*

Theorem 4.8.11 *A Riemannian manifold is locally isometric to Euclidean space* iff *it has zero curvature*

$$R^l_{ijk} = 0, \quad i,\, j,\, k,\, l = 1, \dots, n. \tag{4.8.13}$$

Proof: For Euclidean space $\Gamma^k_{ij} = 0 \Rightarrow R^l_{ijk} = 0$. Conversely, due to Theorem 4.7.10 on a manifold with a symmetric connection with vanishing curvature there exists a system of local coordinates $\left(y^1, \dots, y^n\right)$ such that $\Gamma^r_{pq}(y) = 0$. In these coordinates the Gram matrix g_{pq} is constant. Doing if necessary a linear transformation of the coordinates y one reduces the Gram matrix to the standard form $g_{pq} = \delta_{pq}$. $\square$

Remark 4.8.12 *The definition of Levi-Civita connection as well as the proofs of Theorems 4.8.3 and 4.8.11 remain valid also for pseudo-Riemannian manifolds.*

One may ask how many independent equations (4.8.13) one has to analyze in order to check if a given Riemannian manifold is locally isometric to Euclidean space. The naive answer n^4 does not work as there are some universal relations between various components of the Riemann curvature tensor. For example, due to the definition the tensor is antisymmetric in i, j

$$R^l_{jik} = -R^l_{ijk}. \tag{4.8.14}$$

In order to analyze other *symmetries* of the Riemann curvature tensor it is convenient to lower the index l, i.e., to work with the $(0,4)$-tensor with components

$$R_{ijkl} = g_{ls} R^s_{ijk}. \tag{4.8.15}$$

Theorem 4.8.13 *The Riemann curvature tensor of a Riemannian manifold satisfies the following constraints*

$$R_{jikl} = -R_{ijkl} \tag{4.8.16}$$
$$R_{ijkl} + R_{kijl} + R_{jkil} = 0 \tag{4.8.17}$$
$$R_{ijlk} = -R_{ijkl} \tag{4.8.18}$$
$$R_{klij} = R_{ijkl}. \tag{4.8.19}$$

Proof: The first equation readily follows from (4.8.14). The identity (4.8.17) follows from the symmetry of the connection

$$\nabla_i \nabla_j \frac{\partial}{\partial x^k} - \nabla_j \nabla_i \frac{\partial}{\partial x^k} + \nabla_k \nabla_i \frac{\partial}{\partial x^j} - \nabla_i \nabla_k \frac{\partial}{\partial x^j} + \nabla_j \nabla_k \frac{\partial}{\partial x^i} - \nabla_k \nabla_j \frac{\partial}{\partial x^i}$$

$$= \nabla_i \left(\nabla_j \frac{\partial}{\partial x^k} - \nabla_k \frac{\partial}{\partial x^j} \right) + \nabla_j \left(\nabla_k \frac{\partial}{\partial x^i} - \nabla_i \frac{\partial}{\partial x^k} \right) + \nabla_k \left(\nabla_i \frac{\partial}{\partial x^j} - \nabla_j \frac{\partial}{\partial x^i} \right) = 0.$$

Proof of antisymmetry (4.8.18) follows from compatibility of the connection with the metric. Namely, for arbitrary vector fields Y, Z using (4.8.2) one obtains

$$\begin{aligned} \frac{\partial^2}{\partial x^i \partial x^j} \langle Y, Z \rangle &= \frac{\partial}{\partial x^i} \left(\langle \nabla_j Y, Z \rangle + \langle Y, \nabla_j Z \rangle \right) \\ &= \langle \nabla_i \nabla_j Y, Z \rangle + \langle \nabla_j Y, \nabla_i Z \rangle + \langle \nabla_i Y, \nabla_j Z \rangle + \langle Y, \nabla_i \nabla_j Z \rangle . \end{aligned}$$

The same expression can be computed in a different way

$$\begin{aligned} \frac{\partial^2}{\partial x^j \partial x^i} \langle Y, Z \rangle &= \frac{\partial}{\partial x^j} \left(\langle \nabla_i Y, Z \rangle + \langle Y, \nabla_i Z \rangle \right) \\ &= \langle \nabla_j \nabla_i Y, Z \rangle + \langle \nabla_i Y, \nabla_j Z \rangle + \langle \nabla_j Y, \nabla_i Z \rangle + \langle Y, \nabla_j \nabla_i Z \rangle . \end{aligned}$$

Subtracting one obtains

$$\langle [\nabla_i, \nabla_j] Y, Z \rangle + \langle Y, [\nabla_i, \nabla_j] Z \rangle = 0.$$

This proves (4.8.17).

The last identity (4.8.19) follows from the previous three. To see this let us consider eq. (4.8.17) together with three other eqs. obtained by cyclic permutations of indices i, j, k, l

$$\begin{aligned} R_{ijkl} + R_{kijl} + R_{jkil} &= 0 \\ R_{lijk} + R_{jlik} + R_{ijlk} &= 0 \\ R_{klij} + R_{iklj} + R_{likj} &= 0 \\ R_{jkli} + R_{ljki} + R_{klji} &= 0. \end{aligned}$$

Adding the first and fourth equations and subtracting the second and third one obtains, with the help of (4.8.16) and (4.8.18)

$$2R_{ijkl} - 2R_{klij} = 0.$$

□

For $n = 1$ all components of the curvature tensor vanish, as it should be. Indeed, any one-dimensional Riemann manifold is locally isometric to Euclidean space. For $n = 2$ there is only one non-zero component R_{1212}. All other components are or equal to $\pm R_{1212}$ or vanish.

Exercise 4.8.14 *Prove that the dimension of the space of tables R_{ijkl}, $i, j, k, l = 1, \ldots, n$, satisfying constraints* (4.8.16)–(4.8.19) *is equal to* $\frac{n^2(n^2-1)}{12}$.

Hint: prove that one can use the following components as independent coordinates in the space of tensors R_{ijkl} satisfying (4.8.16)–(4.8.19)

$$R_{ijkl} \quad \text{for} \quad i < j \leq k < l \quad \text{or} \quad k \leq i < j \leq l.$$

Other two quantities are often considered in the study of curvature of Riemannian manifolds. The first one is *Ricci tensor* obtained from the Riemann curvature tensor by a contraction

$$R_{ij} = g^{kl} R_{ikjl}. \tag{4.8.20}$$

It is a symmetric tensor of rank 2

$$R_{ji} = R_{ij}. \tag{4.8.21}$$

The contraction of Ricci tensor

$$R = g^{ij} R_{ij} \tag{4.8.22}$$

is called *scalar curvature*. Its value at a given point does not depend on the choice of a system of coordinates, i.e., the scalar curvature is just a smooth function on the manifold.

Exercise 4.8.15 *For $n = 2$ prove that*

$$R_{ijkl} = \frac{1}{2} R \det \begin{pmatrix} g_{ik} & g_{il} \\ g_{jk} & g_{jl} \end{pmatrix}.$$

and

$$R_{ij} = R\, g_{ij}.$$

Derive that any two-dimensional Riemannian manifold with vanishing scalar curvature is locally isometric to Euclidean plane.

For $n = 3$ the Riemann curvature tensor has 6 independent components R_{1212}, R_{1213}, R_{1223} R_{1313}, R_{1323}, R_{2323}. The Ricci tensor has the same number of independent components R_{11}, R_{12}, R_{13}, R_{22}, R_{23}, R_{33}. Such a coincidence suggests that the Riemann curvature of a three-dimensional manifold is completely determined by the Ricci curvature. Indeed, this is the case, as it follows from the following

Exercise 4.8.16 *For $n = 3$ prove the following formula*

$$R_{ijkl} = \det \begin{pmatrix} R_{ik} & R_{il} \\ g_{jk} & g_{jl} \end{pmatrix} - \det \begin{pmatrix} R_{jk} & R_{jl} \\ g_{ik} & g_{il} \end{pmatrix} - \frac{1}{2} R \det \begin{pmatrix} g_{ik} & g_{il} \\ g_{jk} & g_{jl} \end{pmatrix}.$$

Definition 4.8.17 *A Riemannian manifold is called* Ricci flat *if $R_{ij} = 0$, $i, j = 1, \ldots, n$.*

From the results of the last two exercises it follows that any Ricci flat Riemannian manifold of dimension ≤ 3 is locally isometric to Euclidean space. For $n \geq 4$ this is not true any more. For example, for $n = 4$ the dimension of the space of Ricci tensors is equal to 10 while the space of Riemann curvature tensors has dimension 20.

Importance of Ricci flat manifolds is mainly due to the Einstein's general relativity. According to this theory the space-time in absence of matter is a Ricci flat pseudo-Riemannian manifold of signature $(1, 3)$ (needless to say that the above definitions of curvature makes sense also for pseudo-Riemannian manifolds; also all its properties remain valid).

Exercise 4.8.18 *Let the metric on a two-dimensional Riemannian manifold have a diagonal form*

$$ds^2 = h_1^2 \left(dx^1\right)^2 + h_2^2 \left(dx^2\right)^2, \quad h_1 = h_i \left(x^1, x^2\right). \tag{4.8.23}$$

Prove that the scalar curvature of this metric is given by the following formula

$$R = -\frac{2}{h_1 h_2} \left[\frac{\partial}{\partial x^1} \left(\frac{h_{2,1}}{h_1} \right) + \frac{\partial}{\partial x^2} \left(\frac{h_{1,2}}{h_2} \right) \right] \tag{4.8.24}$$

where

$$h_{1,2} := \frac{\partial h_1}{\partial x^2}, \quad h_{2,1} := \frac{\partial h_2}{\partial x^1}.$$

4.9 Geodesics on a Riemannian manifold

Geodesics on Riemannian manifolds are analogues of straight lines. Let us give a precise definition.

Let $\gamma = \left(x^1(t), \ldots, x^n(t)\right)$ be a smooth curve on a Riemannian manifold M. As above the symbol ∇ will denote the Levi-Civita connection for M.

Definition 4.9.1 *The curve γ is called* geodesic *if its velocity vector $\dot\gamma$ is parallel along γ*

$$\nabla_{\dot\gamma} \dot\gamma = 0.$$

Using eq. (4.7.1) one arrives at a system of the second order ODEs for geodesics

$$\ddot{x}^k + \Gamma^k_{ij}(x) \dot{x}^i \dot{x}^j = 0, \quad k = 1, \ldots, n. \tag{4.9.1}$$

We have already derived this system in the analysis of the Euler–Lagrange equations for the length functional

$$s[\gamma] = \int_\gamma \sqrt{g_{ij}(x) \dot{x}^i \dot{x}^j}\, dt. \tag{4.9.2}$$

More precisely, it was shown that the critical points of the length functional on the subset of curves parameterized by the arc length are solutions to the differential equations (4.9.1). An alternative variational formulation of the theory of geodesics is given by the *action functional*

$$S[\gamma] = \int_\gamma \frac{1}{2} g_{ij}(x)\dot x^i \dot x^j \, dt. \tag{4.9.3}$$

Lemma 4.9.2 *The Euler–Lagrange equations*

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial \dot x^k} = 0, \quad k = 1, \dots, n$$

for the Lagrangian

$$L = L(x, \dot x) = \frac{1}{2} g_{ij}(x)\dot x^i \dot x^j \tag{4.9.4}$$

are equivalent to eqs. (4.9.1).

Proof: is similar to the proof of Lemma 4.3.4. □

We emphasize that, for the Lagrangian (4.9.4) one does not need to assume that the parameter on the curve is equal to the arc length. Moreover, the following statement holds true

Lemma 4.9.3 *For a geodesic* $|\dot\gamma(t)| = \text{const}$.

In other words, the parameter t along the geodesic is proportional to the arc length.

Proof: We have

$$\frac{d}{dt}\langle\dot\gamma, \dot\gamma\rangle \equiv \nabla_{\dot\gamma}\langle\dot\gamma, \dot\gamma\rangle = 2\langle\nabla_{\dot\gamma}\dot\gamma, \dot\gamma\rangle = 0.$$

□

Applying to the system of ODEs (4.9.1) the Cauchy theorem along with the standard results about smooth dependence of solutions on the initial data one obtains

Theorem 4.9.4 *Given a point* $x_0 \in M$ *of a Riemannian manifold and a vector* $v_0 \in T_{x_0}M$ *there exists a neighborhood* $W = W(x_0, v_0) \subset TM$ *and a positive number* ϵ *such that for any* $(x_1, v_1) \in W$ *there exists a unique geodesic* $\gamma : (-\epsilon, \epsilon) \to M$ *such that*

$$\gamma(0) = x_1, \quad \dot\gamma(0) = v_1. \tag{4.9.5}$$

The map

$$W \times (-\epsilon, \epsilon) \to M, \quad (x_1, v_1, t) \mapsto \gamma(t) \tag{4.9.6}$$

is smooth.

Example 4.9.5 *In a Euclidean space the equations of geodesics, written in Euclidean coordinates, become*

$$\ddot x^k = 0, \quad k = 1, \dots, n.$$

Thus geodesics are straight lines

$$x^k(t) = a^k t + b^k.$$

The following simple statement is helpful in the study of geodesics.

Theorem 4.9.6 *Given an isometry $f : M \to N$ of Riemannian manifolds and a geodesic γ on M. Then $f(\gamma)$ is a geodesic on N.*

Example 4.9.7 *Let us prove that geodesics on the standard sphere $S^2 \subset \mathbb{R}^3$ are big circles. Indeed, let a geodesic γ for $t = 0$ passes through $x_0 = \gamma(0)$ and has a (nonzero) initial vector $v_0 = \dot\gamma(0) \in T_{x_0}S^2$. Consider the two-dimensional plane P passing through the origin, the point x_0 and parallel to the vector v_0. The reflection with respect to P is an isometry of the sphere. Denote γ_r the image of γ with respect to the reflection. Due to the Theorem it is again a geodesic. Since γ_r satisfies the same initial data*

$$\gamma_r(0) = x_0, \quad \dot\gamma_r(0) = v_0,$$

it coincides with γ. Therefore $\gamma = S^2 \cap P$.

Similar arguments allow to find geodesics on the pseudosphere

$$x^2 + y^2 - z^2 = -R^2$$

in the Minkowski space $\mathbb{R}^{1,2}$. They are sections of the pseudosphere by planes passing through the origin $x = y = z = 0$.

Euler–Lagrange equations (4.3.10), (4.3.11) give a necessary condition for finding curves minimizing the value of the length functional (4.9.2). Our next goal is to prove that, locally, geodesics *minimize* the length. We will first introduce an analogue of polar coordinates on a sufficiently small neighborhood of any Riemannian manifold. To this end let us modify the statement of the theorem of existence and uniqueness 4.9.4.

Theorem 4.9.8 *Given a point $x_0 \in M$ of a Riemannian manifold, there exists a neighborhood $U = U(x_0)$ and a positive number ϵ such that, for an arbitrary point $x_1 \in U$ and an arbitrary vector $v_1 \in T_{x_1}M$ of the length $|v_1| < \epsilon$ there exists a unique geodesics $\gamma : (-2, 2) \to M$ satisfying initial conditions*

$$\gamma(0) = x_1, \quad \dot\gamma(0) = v_1.$$

Proof: We will use invariance of equations of geodesics with respect to rescalings of independent variable

$$\gamma(t) \mapsto \gamma(ct)$$

So, choosing the neighborhood $W = W(x_0, 0)$ of the Theorem 4.9.4 in the form

$$(x_1, v_1) \in W \quad \Leftrightarrow \quad x_1 \in U, \quad |v_1| < \delta$$

for some $U \subset M$ and some $\delta > 0$ we obtain a geodesic $\gamma : (-2\epsilon_1, 2\epsilon_1) \to M$ with initial conditions

$$\gamma(0) = x_1, \quad \dot\gamma(0) = v_1 \quad \text{for} \quad (x_1, v_1) \in W$$

for some $\epsilon_1 > 0$. The geodesic $\gamma_c(t) := \gamma(ct)$ will be defined on the interval $\left(-\frac{2\epsilon_1}{c}, \frac{2\epsilon_1}{c}\right)$ and satisfy the initial conditions

$$\gamma_c(0) = x_1, \quad \dot\gamma_c(0) = c\, v_1.$$

Choosing c in such a way that

$$0 < c < \epsilon_1$$

we obtain a geodesic γ_c defined on the interval $(-2, 2)$ (or bigger). The length of the initial vector $c\, v_1$ will be less than $\epsilon_1 \delta =: \epsilon$. □

Corollary 4.9.9 *Let $U \subset M$ and $\epsilon > 0$ be such that for any $x \in U$ and any $v \in T_x M$ satisfying $|v| < \epsilon$ there exists a geodesic $\gamma : (-2, 2) \to M$ with the initial condition $\gamma(0) = x$, $\dot\gamma(0) = v$. Then the map*

$$\exp : \{(x, v) \in TM \,|\, x \in U, \quad |v| < \epsilon\} \to M, \quad (x, v) \mapsto \exp_x(v) := \gamma(1) \tag{4.9.7}$$

is well defined and smooth.

Definition 4.9.10 *The map* (4.9.7) *is called* exponential map.

Observe that the geodesics itself can be represented in the form

$$\begin{aligned} &\gamma(t) = \exp_x(t\, v), \quad x \in U, \quad |v| < \epsilon \\ &\gamma(0) = x, \quad \dot\gamma(0) = v. \end{aligned} \tag{4.9.8}$$

Its length between the points $\gamma(0)$ and $\gamma(1)$ is equal to $|v| < \epsilon$.

Fix a point $x_0 \in M$ and consider the exponential map

$$\exp_{x_0} : T_{x_0} M \to M$$

defined for a sufficiently small neighborhood of $0 \in T_{x_0} M$. From the equation of geodesics one obtains the following Taylor expansion

$$x^k(t) = x^k(0) + t\, \dot x^k(0) + \frac{t^2}{2} \ddot x^k(0) + \mathcal{O}\left(t^3\right) = x_0^k + t\, v^k - \frac{t^2}{2} \Gamma^k_{ij}(x_0) v^i v^j + \mathcal{O}\left(t^3\right) \tag{4.9.9}$$

of the geodesic $\gamma(t) = \left(x^1(t), \dots, x^n(t)\right)$ with the initial data

$$\gamma(0) = x_0, \quad \dot\gamma(0) = v.$$

Therefore the exponential map for small $|v|$ can be represented by the following series expansion

$$x^k(v) := \left[\exp_{x_0}(v)\right]^k = x_0^k + v^k - \frac{1}{2} \Gamma^k_{ij}(x_0) v^i v^j + \mathcal{O}\left(|v|^3\right). \tag{4.9.10}$$

The following statement readily follows:

Proposition 4.9.11 *The map $\exp_{x_0} : T_{x_0} M \to M$ is a local diffeomorphism.*

Proof: From (4.9.10) the following expression for the Jacobi matrix $\partial x^k / \partial v^i$ readily follows

$$\frac{\partial x^k}{\partial v^i} = \delta^k_i - \Gamma^k_{ij}(x_0) v^j + \mathcal{O}\left(|v|^2\right). \tag{4.9.11}$$

At the origin $v = 0$ one obtains the identity matrix

$$\frac{\partial x^k}{\partial v^i}|_{v=0} = \delta^k_i.$$

□

Due to the previous statement one can use the components $\left(v^1, \ldots, v^n\right)$ of the tangent vector $v \in T_{x_0}M$ with sufficiently small $|v|$ as coordinates on a neighborhood of the point x_0. Denote

$$h_{ij}(v) := \left\langle \frac{\partial}{\partial v^i}, \frac{\partial}{\partial v^j} \right\rangle \tag{4.9.12}$$

the Gram matrix of the Riemannian metric in the coordinates $\left(v^1, \ldots, v^n\right)$. The following statement is often used in various calculations in Riemannian geometry.

Proposition 4.9.12 *All first derivatives of the Riemannian metric in the coordinates $\left(v^1, \ldots, v^n\right)$ vanish at the origin $v = 0$.*

Proof: From (4.9.11) obtain

$$\begin{aligned} h_{ij}(v) &= \left\langle \frac{\partial}{\partial x^i} - \Gamma^k_{is}(x_0)v^s \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} - \Gamma^l_{jt}(x_0)v^t \frac{\partial}{\partial x^l} \right\rangle + \mathcal{O}\left(|v|^2\right) \\ &= g_{ij}\left(x(v)\right) - \left[\Gamma^k_{is}(x_0) g_{kj}\left(x(v)\right) + \Gamma^k_{js}(x_0) g_{ki}\left(x(v)\right)\right] v^s + \mathcal{O}\left(|v|^2\right) \\ &= g_{ij}(x_0) + \frac{\partial g_{ij}(x_0)}{\partial x^s} v^s - \left[\Gamma^k_{is}(x_0) g_{kj}\left(x_0\right) + \Gamma^k_{js}(x_0) g_{ki}\left(x_0\right)\right] v^s + \mathcal{O}\left(|v|^2\right). \end{aligned}$$

So

$$\frac{\partial h_{ij}(v)}{\partial v^s}|_{v=0} = \frac{\partial g_{ij}(x_0)}{\partial x^s} v - \Gamma^k_{is}(x_0) g_{kj}\left(x_0\right) + \Gamma^k_{js}(x_0) g_{ki}\left(x_0\right) = \nabla_s g_{ij}(x_0) = 0.$$

□

Corollary 4.9.13 *For any point x_0 on a Riemannian manifold there exists a system of local coordinates such that all Christoffel coefficients of the Levi-Civita connection vanish at the point x_0.*

We will now extend the previous constructions allowing also the initial point of the exponential map to vary. Let U and ϵ be same as before. Denote $W \subset TM$ the subset of the form

$$W = \{(x, v) \in TM \,|\, x \in U,\ |v| < \epsilon\}.$$

One can define a map

$$\begin{aligned} &\mathrm{Exp} : W \to M \times M \\ &\mathrm{Exp}(x, v) = \left(x, \exp_x(v)\right). \end{aligned} \tag{4.9.13}$$

Observe that on the zero section $U \times \{0\} \subset W$ one has $\mathrm{Exp}(x, 0) = (x, x)$.

Exercise 4.9.14 *Prove that the differential* Exp_* *of the map* (4.9.13) *at the points of zero section has the matrix*

$$\mathrm{Exp}_*|_{(x,0)} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} \end{pmatrix}. \tag{4.9.14}$$

Here $\mathbf{1}$ *denotes* $n \times n$ *identity matrix.*

Corollary 4.9.15 *The map* Exp *establishes a diffeomorphism of a domain*

$$W' = \{(x, v) \in TM \,|\, x \in U',\ |v| < \epsilon'\}$$

for some $U' \subset U$, $0 < \epsilon' < \epsilon$ *to a domain* $\mathcal{W}' \subset M \times M$ *containing points of the form* (x, x), $x \in U'$.

In sequel we will omit primes of U' and ϵ'.

Corollary 4.9.16 *1) For any* x, $y \in U$ *there exists a unique geodesic* $\gamma : [0, 1] \to U$ *connecting* $x = \gamma(0)$ *with* $y = \gamma(1)$ *of length* $s[\gamma] < \epsilon$.

2) The geodesic γ *depends smoothly on the endpoints* x, y.

3) For any $x \in U$ *the map* $\exp_x$ *is a diffeomorphism of the open* ϵ*-ball in* T_xM *onto an open domain* $U_x \subset U$.

Let us use the exponential map to construct a system of "polar coordinates" on a neighborhood of a given point x_0 in a Riemannian manifold M. Consider the unit sphere

$$S^{n-1}_{x_0} = \{v \in T_{x_0}M \,|\, |v| = 1\}$$

in the tangent space at the point x_0. Let the positive number ϵ be such as above. For every pair (r, v), $0 \le r \le \epsilon$, $v \in S^{n-1}_{x_0}$ consider the point $\exp_{x_0}(r\,v) \in M$. The pair (r, v) can be considered as coordinates of this point. Recall that the curve $\gamma : [0, \epsilon] \to M$ defined by $\gamma(r) = \exp_{x_0}(r\,v)$ is a geodesic. We call it *radial geodesic.*

We will now prove the following statement, due to Gauss.

Lemma 4.9.17 *For a given* $0 < r \le \epsilon$ *the surface*

$$S_r = \{x = \exp_{x_0}(r\,v) \,|\, v \in S^{n-1}_{x_0}\}. \tag{4.9.15}$$

is orthogonal to radial geodesics.

Proof: Choose a curve $v = v(t)$ on the sphere $S^{n-1}_{x_0}$. Consider the surface

$$x(r, t) = \exp_{x_0}(r\,v(t)). \tag{4.9.16}$$

We have to prove that

$$\left\langle \frac{\partial x}{\partial r}, \frac{\partial x}{\partial t} \right\rangle = 0.$$

To this end introduce operators

$$\nabla_r := \nabla_{\frac{\partial x}{\partial r}}, \quad \nabla_t := \nabla_{\frac{\partial x}{\partial t}}.$$

Using the equation of geodesics

$$\nabla_r \frac{\partial x}{\partial r} = 0$$

obtain

$$\frac{\partial}{\partial r} \left\langle \frac{\partial x}{\partial r}, \frac{\partial x}{\partial t} \right\rangle = \left\langle \nabla_r \frac{\partial x}{\partial r}, \frac{\partial x}{\partial t} \right\rangle + \left\langle \frac{\partial x}{\partial r}, \nabla_r \frac{\partial x}{\partial t} \right\rangle = \left\langle \frac{\partial x}{\partial r}, \nabla_r \frac{\partial x}{\partial t} \right\rangle.$$

Using symmetry of the connection

$$\nabla_r \frac{\partial x}{\partial t} = \nabla_t \frac{\partial x}{\partial r}$$

rewrite the last expression in the form

$$\left\langle \frac{\partial x}{\partial r}, \nabla_r \frac{\partial x}{\partial t} \right\rangle = \left\langle \frac{\partial x}{\partial r}, \nabla_t \frac{\partial x}{\partial r} \right\rangle = \frac{1}{2} \frac{\partial}{\partial t} \left\langle \frac{\partial x}{\partial r}, \frac{\partial x}{\partial r} \right\rangle = 0$$

since

$$\left\langle \frac{\partial x}{\partial r}, \frac{\partial x}{\partial r} \right\rangle = \mathrm{const}$$

on a geodesic. □

We are now ready to prove the main result of this section.

Theorem 4.9.18 *Let $U \subset M$ and $\epsilon > 0$ be the same as in Corollary 4.9.16. For a given pair of points x, $y \in U$ denote $\gamma : [0,1] \to M$ the geodesic of length less than ϵ connecting these two points. Then any other piecewise smooth curve connecting x and y has the length greater or equal than the length of γ.*

Proof is based on the following

Lemma 4.9.19 *Represent a sufficiently small piecewise smooth curve $\gamma : [a,b] \to M$ not passing through x_0 in the polar coordinates*

$$\gamma(t) = \exp_{x_0}(r(t)\, v(t)), \quad 0 < r(t) < \epsilon, \quad |v(t)| = 1. \tag{4.9.17}$$

Then

$$s[\gamma] \geq |r(b) - r(a)|;$$

the equality takes place only for a monotone function $r(t)$ and constant $v(t) \equiv v_0$ for some $v_0 \in S^{n-1}_{x_0}$

In other words the shortest curve connecting two spheres centered at x_0 is a radial geodesic, up to a reparameterization.

Proof: Consider a two-dimensional surface

$$x(r,t) = \exp_{x_0}(r\, v(t)).$$

We have

$$\dot\gamma = \frac{\partial x}{\partial r}\dot r + \frac{\partial x}{\partial t}.$$

We know that

$$\left|\frac{\partial x}{\partial r}\right| = 1, \quad \frac{\partial x}{\partial r} \perp \frac{\partial x}{\partial t}.$$

So

$$|\dot\gamma|^2 = |\dot r|^2 + \left|\frac{\partial x}{\partial t}\right|^2 \geq \dot r^2.$$

The equality takes place *iff* $\frac{\partial x}{\partial t} = 0$. Therefore

$$s[\gamma] = \int_a^b |\dot\gamma|\,dt \geq \int_a^b |\dot r|\,dt \geq |r(b) - r(a)|.$$

□

End of the proof of the Theorem. Consider any path ω from x to $x' = \exp_x(rv)$, $0 < r < \epsilon$, $|v| = 1$. For any $\delta > 0$ the path ω must contain a segment connecting the sphere of radius δ with the sphere of radius r. Length of this segment is greater or equal than $r - \delta$, In the limit $\delta \to 0$ we conclude that $s[\omega] \geq r$, and equality takes place only when ω is a geodesic. □

Corollary 4.9.20 *Let $\gamma : [0, l] \to M$ be a curve from $\gamma(0)$ to $\gamma(l)$ parameterized by arc length. Assume that γ is shorter than any other curve between $\gamma(0)$ and $\gamma(l)$. Then γ is a geodesic.*

Recall that we defined the distance function on a connected Riemannian manifold by

$$\rho(x, y) = \inf_\gamma s[\gamma], \quad \gamma : [0, 1] \to M, \quad \gamma(0) = x, \quad \gamma(1) = y.$$

Corollary 4.9.21 *For any compact subset $K \subset M$ there exists $\delta > 0$ such that, for arbitrary x, $y \in K$ satisfying $\rho(x, y) < \delta$ there exists a unique geodesic between x and y of length less than δ. It is minimal and depends smoothly on the end points.*

Let us now describe global properties of geodesics.

Definition 4.9.22 *A Riemannian manifold M is called* geodesically complete *if any geodesic $\gamma(t)$ can be extended for all values of $t \in \mathbb{R}$.*

The following important result is due to Hopf and Rinow.

Theorem 4.9.23 *Arbitrary points x, y of a geodesically complete Riemannian manifold M can be connected by a geodesic of length $\rho(x, y)$.*

Proof: Denote $R = \rho(x, y)$. Let $U_x \subset M$ be a neighborhood of x described in Corollary 4.9.16. Choose $\epsilon > 0$ from the same Corollary. For a positive $\delta < \epsilon$ denote a sphere $S_x(\delta)$ of radius δ centered at x. Due to compactness there exists a point on the sphere

$$z_0 = \exp_x(\delta\, v), \quad |v| = 1$$

minimizing the distance to y. Our goal is to prove that, continuing the geodesic $\exp_x(t\, v)$ we will hit the point y, i.e.,

$$\exp_x(R\, v) = y.$$

We will begin with proving that, going along the geodesic $\gamma(t) = \exp_x(t\, v)$ we are approaching y. Namely, we will prove that

$$\rho(\gamma(t), y) = R - t \quad \text{for} \quad t \in [\delta, R]. \tag{4.9.18}$$

Let us first check validity of (4.9.18) for $t = \delta$. We have

$$R = \rho(x, y) = \min_{z \in S_x(\delta)} [\rho(x, z) + \rho(z, y)] = \delta + \rho(z_0, y).$$

Hence $\rho(z_0, y) = R - \delta$. This proves (4.9.18) for $t = \delta$ as $z_0 = \gamma(\delta)$.

Let $t_0 \in [\delta, R]$ be the supremum of values of the parameter t for which eq. (4.9.18) holds true. Due to continuity this equation is valid also for $t = t_0$. It remains to prove that $t_0 = R$. In the opposite case $t_0 < R$ consider a small sphere $S_{x'}(\delta')$ of radius $\delta' > 0$ centered at $x' := \gamma(t_0)$. Choose a point $z_0' \in S_{x'}(\delta')$ closest to y. Then

$$\rho(x', y) = \min_{z \in S_{x'}(\delta')} \left[\rho(x', z) + \rho(z, y)\right] = \delta' + \rho(z_0', y).$$

Hence

$$\rho(z_0', y) = (R - t_0) - \delta'. \tag{4.9.19}$$

The claim is that $z_0' = \gamma(t_0 + \delta')$. Indeed, from the triangle inequality along with eq. (4.9.19) we obtain

$$\rho(x, z_0') \geq \rho(x, y) - \rho(z_0', y) = t_0 + \delta'. \tag{4.9.20}$$

It remains to observe that a curve from x to z_0' of exactly same length can be obtained by going along γ from x to $x' = \gamma(t_0)$ and then along the minimal geodesic from x' to z_0'. Due to (4.9.20) such a piecewise smooth path is minimal. Hence it must be a geodesic clearly coinciding with γ.

We proved that $\gamma(t_0 + \delta') = z_0'$. So, eq. (4.9.19) takes the form

$$\rho\left(\gamma(t_0 + \delta'), y\right) = R - (t_0 + \delta').$$

That is, eq. (4.9.18) remains valid for $t = t_0 + \delta' > t_0$. Such a contradiction proves that $t_0 = R$. $\square$

Corollary 4.9.24 *The closure of any bounded subset in a geodesically complete Riemannian manifold M is compact.*

Proof: Let the distance between points of a subset $X \subset M$ be bounded from above by d. Then, for any point $x \in M$ the exponential map $\exp_x$ maps a ball of radius d in $T_x M$ into a compact subset of M. Due to Hopf–Rinow theorem such a subset contains X. Therefore the closure $\bar{X}$ is compact. □

The following immediate consequence of Corollary 4.9.24 is also often called Hopf–Rinow theorem.

Corollary 4.9.25 *A geodesically complete Riemannian manifold is complete as a metric space.*

Recall that complete metric spaces are those for which any fundamental sequence converges.

One can also prove the converse statement to the Corollary.

At the end of this long section we will apply the theory of geodesics to the study of the group of isometries of a Riemannian manifold. The group of isometries of a Riemannian manifold is a Lie group. We will not prove this statement. However we will describe the Lie algebra of this group. It corresponds to the *infinitesimal* isometries, i.e., to vector fields X such that

$$Lie_X g_{ij} = 0. \tag{4.9.21}$$

Here the Lie derivatives of the metric tensor is defined by the formula

$$Lie_X g_{ij} = X^k \frac{\partial g_{ij}}{\partial x^k} + \frac{\partial X^k}{\partial x^i} g_{kj} + g_{ik} \frac{\partial X^k}{\partial x^j}. \tag{4.9.22}$$

Vector fields X satisfying (4.19.16) are called *Killing vector fields.*

Exercise 4.9.26 *Prove that the equations* (4.19.16) *for the Killing vector fields can be recast into the form*

$$\nabla_i X_j + \nabla_j X_i = 0, \quad i,\, j = 1, \ldots, n. \tag{4.9.23}$$

Here

$$X_i = g_{is} X^s.$$

One can derive an upper estimate for the dimension of the group of isometries studying the space of solutions to the system (4.19.18). We explain another approach based on the theory of geodesics.

Proposition 4.9.27 *The dimension of the group of isometries of a n-dimensional geodesically complete connected Riemannian manifold M is less or equal than $\frac{n(n+1)}{2}$.*

Proof: Denote G the group of isometries of M. The stabilizer

$$G_{x_0} = \{g \in G \,|\, g(x_0) = x_0\} \tag{4.9.24}$$

of a given point $x_0 \in M$ is a subgroup in G. For $g \in G_{x_0}$ denote $g_* : T_{x_0}M \to T_{x_0}M$ the induced map. It is an orthogonal transformation of the tangent space. In this way we obtain a homomorphism

$$G_{x_0} \to O\left(T_{x_0}M, \langle\ ,\ \rangle\right). \tag{4.9.25}$$

Let us prove that (4.9.25) is injective. Indeed, let $g \in G_{x_0}$ satisfy $g_* = \mathrm{id}$. Connect an arbitrary point $x \in M$ with x_0 by a geodesic $\gamma : [0,1] \to M$. The geodesic $g(\gamma)$ passes through the same point $x_0 = \gamma(0)$ and has the same initial vector $g_*\dot\gamma(0) = \dot\gamma(0)$. Hence $g(\gamma) = \gamma$. In particular $g(x) = g\left(\gamma(1)\right) = \gamma(1) = x$. As

$$\dim O\left(T_{x_0}M, \langle\ ,\ \rangle\right) = \frac{n(n-1)}{2}$$

we conclude that

$$\dim G_{x_0} \le \frac{n(n-1)}{2}.$$

The coset space G/G_{x_0} can be identified with the orbit

$$Gx_0 = \{g(x_0)\,|\,g \in G\} \subset M$$

of the point x_0. Therefore

$$\dim G/G_{x_0} \le \dim M = n.$$

So

$$\dim G = \dim G_{x_0} + \dim G/G_{x_0} \le n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

□

4.10 Gaussian connection on surfaces. Curvature of curves and surfaces

Let $M \subset \mathbb{R}^N$ be a n-dimensional submanifold in Euclidean space. The Euclidean metric on $\mathbb{R}^N$ induces a Riemannian metric ds^2 on M. In this section we give an explicit realization of the Levi-Civita connection on $\left(M, ds^2\right)$.

Denote $\mathbf{r}(u) = \left(x^1(u), \dots, x^N(u)\right) \in \mathbb{R}^N$ the embedding map, $u = \left(u^1, \dots, u^n\right)$ are local coordinates on a chart on M. Recall that the vectors

$$\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial u^i}, \quad i = 1, \dots, n$$

span the tangent space $T_uM \subset \mathbb{R}^N$. The Gram matrix of the induced metric reads

$$g_{ij}(u) = \langle \mathbf{r}_i, \mathbf{r}_j\rangle.$$

Given a point $u \in M$ and a vector $X \in \mathbb{R}^N$, there exists a unique decomposition of the form

$$X = \mathrm{pr}_u X + X^\perp \tag{4.10.1}$$

where the vector $X^\perp$ is orthogonal to T_uM. The first part $\mathrm{pr}_u X$ is called *orthogonal projection* of X onto T_uM.

We are now ready to define the main construction of this section. Let X, Y be two vector fields in $\mathbb{R}^N$ tangent to M. Define another vector field tangent to M by

$$\nabla_X Y|_u = \mathrm{pr}_u\left(\partial_X Y\right) \quad \text{for any} \quad u \in M. \tag{4.10.2}$$

Lemma 4.10.1 *The vector field $\nabla_X Y$ depends only on the restrictions $X|_M$, $Y|_M$ of the vector fields on $M \subset \mathbb{R}^N$.*

Proof: For any smooth function $f \in \mathcal{C}^\infty\left(\mathbb{R}^N\right)$ one has

$$\frac{\partial}{\partial u^i}\left(f|_M\right) = \sum_{a=1}^N \frac{\partial f}{\partial x^a}\frac{\partial x^a}{\partial u^i} = \left(\partial_{\mathbf{r}_i} f\right)_M .$$

In particular the restriction of $\partial_{\mathbf{r}_i} f$ onto M vanishes if $f|_M = 0$. Clearly, for a vector field $X = X^a \frac{\partial}{\partial x^a}$ vanishing on M the restriction of $\partial_X f$ onto M vanishes. Therefore, for a vector field X tangent to M one has

$$\left(\partial_X f\right)_M = \sum_{i=1}^n X^i(u)\frac{\partial}{\partial u^i}\left(f|_M\right) \quad \text{where} \quad X|_M = \sum_{i=1}^n X^i(u)\mathbf{r}_i$$

So, the above definition yields the following expression for the covariant derivative

$$\nabla_X Y = \mathrm{pr}_u \sum_{i,j=1}^n X^i(u)\frac{\partial}{\partial u^i}\left(Y^j(u)\mathbf{r}_j\right). \tag{4.10.3}$$

where, like above

$$Y|_M = \sum_{j=1}^n Y^j(u)\mathbf{r}_j.$$

Clearly this expression depends only on the restrictions $X(u)$ and $Y(u)$ of the vector fields on the submanifold. □

Definition 4.10.2 *The connection* (4.10.2) *defined on a submanifold M in a Euclidean space $\mathbb{R}^N$ is called* Gaussian connection *on the submanifold.*

Theorem 4.10.3 *The Levi-Civita connection for the induced metric on the submanifold M in a Euclidean space coincides with the Gaussian connection.*

Proof: From the formula (4.10.3) it follows that

$$\nabla_{\frac{\partial}{\partial u^i}}\frac{\partial}{\partial u^j} = \mathrm{pr}_u \mathbf{r}_{ij}.$$

Symmetry of the second derivatives $\mathbf{r}_{ji} = \mathbf{r}_{ij}$ implies symmetry of the Gaussian connection. Next, we have

$$\partial_Z X = \nabla_Z X + \left(\partial_Z X\right)^\perp, \quad \partial_Z Y = \nabla_Z Y + \left(\partial_Z Y\right)^\perp .$$

Here the vector fields X, Y, Z are assumed to be tangent to M. Using

$$\partial_Z\langle X, Y\rangle = \langle \partial_Z X, Y\rangle + \langle X, \partial_Z Y\rangle$$

and orthogonality $\left\langle \left(\partial_Z X\right)^\perp, Y\right\rangle = 0$ and $\left\langle X, \left(\partial_Z Y\right)^\perp\right\rangle = 0$ we arrive at

$$\partial_Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle .$$

Hence the Gaussian connection is compatible with the induced metric. □

Let us consider a particular case of hypersurfaces $M \subset \mathbb{R}^{n+1}$. In this case the orthogonal complement to the tangent space $T_u M$ at a given point $u \in M$ is one-dimensional. Locally one can choose an orthogonal unit vector $\mathbf{n} = \mathbf{n}(u)$ smoothly depending on the point of the submanifold.

$$\mathbf{n}(u) \perp T_u M, \quad \langle \mathbf{n}, \mathbf{n} \rangle = 1.$$

Define a bilinear form on $Vect(M)$ by the formula

$$b(X, Y) = \langle \partial_X Y, \mathbf{n} \rangle \,. \tag{4.10.4}$$

It is understood, like above, that X, $Y \in Vect\left(\mathbb{R}^{n+1}\right)$ tangent to M.

Lemma 4.10.4 *The formula* (4.10.4) *is a well defined symmetric bilinear form on $T_u M$ at any point $u \in M$.*

Proof: In order to check that the value of the bilinear form $b(X, Y)$ at a given point $u \in M$ depends only on the values of the vector fields at this point it suffices to prove that

$$b(f\, X, Y) = b(X, f\, Y) = f\, b(X, Y)$$

for any function f. The only non-obvious part is the last equality:

$$b(X, f\, Y) = \langle \partial_X (f\, Y), \mathbf{n} \rangle = \langle \partial_X f\, Y, \mathbf{n} \rangle + \langle f \partial_X Y, \mathbf{n} \rangle \,.$$

The first term in the rhs vanishes since $Y \perp \mathbf{n}$.

For proving symmetry of the bilinear form we use that the vector field

$$\partial_X Y - \partial_Y X = [X, Y]$$

is tangent to M for any pair of vector fields tangent to M (see Exercise 1.3.16). □

Definition 4.10.5 *The bilinear form* (4.10.4) *is called the* second fundamental form *of the hypersurface M.*

The definition of Gaussian connection for hypersurfaces can be rewritten in the form

$$\partial_X Y = \nabla_X Y + b(X, Y)\, \mathbf{n}. \tag{4.10.5}$$

Equivalently, in local coordinates

$$\mathbf{r}_{ij} = \Gamma_{ij}^k \mathbf{r}_k + b_{ij} \mathbf{n} \tag{4.10.6}$$

where Γ_{ij}^k are Christoffel coefficients of the Gaussian connection and

$$b_{ij} = b\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \langle \mathbf{r}_{ij}, \mathbf{n} \rangle \tag{4.10.7}$$

is the matrix of the second fundamental form in the standard basis $\mathbf{r}_1, \ldots, \mathbf{r}_n$ in $T_u M$.

We will now prove the main result of this section.

Theorem 4.10.6 *The Riemann curvature tensor of the Gaussian connection on a hypersurface can be written in terms of the second fundamental form in the following way*

$$R_{ijkl} = \det \begin{pmatrix} b_{ik} & b_{il} \\ b_{jk} & b_{jl} \end{pmatrix}. \tag{4.10.8}$$

Moreover, the second fundamental form satisfies the following equations

$$\frac{\partial b_{ij}}{\partial u^k} - \frac{\partial b_{ik}}{\partial u^j} = \Gamma^s_{ik} b_{sj} - \Gamma^s_{ij} b_{sk}, \quad i,\, j,\, k = 1, \ldots, n. \tag{4.10.9}$$

The formula (4.10.8) is due to Gauss (for $n = 2$). It plays a crucial role in the proof of Gauss *Teorema Egregium* (see below). The equation (4.10.9) was rediscovered many times; for $n = 2$ it is due to Peterson, Mainardi and Codazzi[6].

Proof: We first prove the following

Lemma 4.10.7 *The following equations hold true*

$$\frac{\partial \mathbf{n}}{\partial u^i} = -b_i^j \mathbf{r}_j \quad \text{where} \quad b_i^j = g^{js} b_{is}. \tag{4.10.10}$$

Proof: Differentiating the identity $\langle \mathbf{n}, \mathbf{n} \rangle = 1$ obtain

$$0 = \frac{\partial}{\partial u^i} \langle \mathbf{n}, \mathbf{n} \rangle = 2 \left\langle \frac{\partial \mathbf{n}}{\partial u^i}, \mathbf{n} \right\rangle.$$

That is, the vector $\frac{\partial \mathbf{n}}{\partial u^i}$ is orthogonal to $\mathbf{n}$. Therefore

$$\frac{\partial \mathbf{n}}{\partial u^i} = a_i^j \mathbf{r}_j$$

for some matrix $a_i^j = a_i^j(u)$. Using

$$g_{kj} a_i^j = \left\langle \mathbf{r}_k, \frac{\partial \mathbf{n}}{\partial u^i} \right\rangle = \frac{\partial}{\partial u^i} \langle \mathbf{r}_k, \mathbf{n} \rangle - \langle \mathbf{r}_{ik}, \mathbf{n} \rangle = -b_{ik}$$

we complete the proof of Lemma. □

Putting together eqs. (4.10.2) and (4.10.10) we obtain an overdetermined system of equations (sometimes called *Weingarten formulae*)

$$\frac{\partial \mathbf{r}_j}{\partial u^i} = \Gamma^s_{ij} \mathbf{r}_s + b_{ij} \mathbf{n} \tag{4.10.11}$$

$$\frac{\partial \mathbf{n}}{\partial u^i} = -b_i^s \mathbf{r}_s. \tag{4.10.12}$$

In order to complete the proof of the Theorem one has to analyze the compatibility conditions of this system. Differentiating (4.10.11) in u^k we obtain

$$\frac{\partial^3 \mathbf{r}}{\partial u^i \partial u^j \partial u^k} = \frac{\partial \Gamma^s_{ij}}{\partial u^k} \mathbf{r}_s + \Gamma^s_{ij} \left(\Gamma^t_{sk} \mathbf{r}_t + b_{sk} \mathbf{n} \right) + \frac{\partial b_{ij}}{\partial u^k} \mathbf{n} - b_{ij} b_k^s \mathbf{r}_s$$

The rhs must be symmetric in i, k. Collecting the coefficients of $\mathbf{r}_1, \ldots, \mathbf{r}_n$ one obtains (4.10.8) while the symmetry in i, k of the coefficient of $\mathbf{n}$ yields (4.10.9). □

[6] In the chronological order.

Exercise 4.10.8 *Prove that compatibility of eqs.* (4.10.12) *follows from* (4.10.9)*.*

The following *Bonnet theorem* says that the system of eqs. (4.10.8), (4.10.9) suffices for local reconstruction of a hypersurface.

Theorem 4.10.9 *Let the symmetric matrix valued function $b_{ij}(u)$ and a symmetric positive definite matrix valued function $g_{ij}(u)$ satisfy the Gauss and Peterson–Mainardi–Codazzi equations* (4.10.8), (4.10.9) *where the functions $\Gamma_{ij}^k = \Gamma_{ij}^k(u)$ are defined by the Christoffel formulae* (4.8.5)*. Then there exists a sufficiently small piece of a hypersurface in the $(n+1)$-dimensional Euclidean space with the given first and second fundamental forms g_{ij} and b_{ij}. Moreover, such an embedding is determined uniquely up to an isometry of the ambient space $\mathbb{R}^{n+1}$.*

Proof: Existence of an embedding follows from Lemma 4.7.11 due to the compatibility (4.10.8) and (4.10.9). Let us prove uniqueness. Let $\tilde{\mathbf{r}}(u)$ be another embedding. Choose a point $u_0 = \left(u_0^1, \dots, u_0^n\right)$. Consider two bases $\mathbf{r}_1(u_0)$, $\dots$, $\mathbf{r}_n(u_0)$, $\mathbf{n}(u_0)$ and $\tilde{\mathbf{r}}_1(u_0)$, $\dots$, $\tilde{\mathbf{r}}_n(u_0)$, $\tilde{\mathbf{n}}(u_0)$ in $\mathbb{R}^{n+1}$. The Gram matrices of the Euclidean inner product for these two bases coincide

$$\begin{aligned}
&\langle \mathbf{r}_i(u_0), \mathbf{r}_j(u_0)\rangle = \langle \tilde{\mathbf{r}}_i(u_0), \tilde{\mathbf{r}}_j(u_0)\rangle = g_{ij}(u_0)\\
&\langle \mathbf{r}_i(u_0), \mathbf{n}(u_0)\rangle = \langle \tilde{\mathbf{r}}_i(u_0), \tilde{\mathbf{n}}(u_0)\rangle = 0,\\
&\langle \mathbf{n}(u_0), \mathbf{n}(u_0)\rangle = \langle \tilde{\mathbf{n}}(u_0), \tilde{\mathbf{n}}(u_0)\rangle = 1.
\end{aligned}$$

Therefore there exists a non-degenerate matrix $A \in Mat(n+1, \mathbb{R})$ satisfying orthogonal transformation $A \in O(n+1)$ transforming one basis to another one

$$(\tilde{\mathbf{r}}_1(u_0), \dots, \tilde{\mathbf{r}}_n(u_0), \tilde{\mathbf{n}}(u_0)) = (\mathbf{r}_1(u_0), \dots, \mathbf{r}_n(u_0), \mathbf{n}(u_0))\ A.$$

Applying uniqueness theorem for solutions of the system (4.10.11), (4.10.12) we obtain identity

$$(\tilde{\mathbf{r}}_1(u), \dots, \tilde{\mathbf{r}}_n(u), \tilde{\mathbf{n}}(u)) = (\mathbf{r}_1(u), \dots, \mathbf{r}_n(u), \mathbf{n}(u))\ A.$$

□

For $n = 2$ the formula (4.10.8) together with the result of Exercise 4.8.15 imply the following expresson for the scalar curvature of a two-dimensional surface in $\mathbb{R}^3$

$$R = 2\,\frac{\det B}{\det G}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}. \tag{4.10.13}$$

In the next section we will explain the geometric meaning of this formula in terms of *Gaussian curvature* of a surface.

4.11 Curvature of surfaces in $\mathbb{R}^3$

The curvature of surfaces can be characterized by the curvature of certain curves on the surface. Let us introduce the tools useful for computing these curvatures.

Let

$$\mathbf{r} = \mathbf{r}(u, v) \tag{4.11.1}$$

be a regular smooth two-dimensional surface in the three-dimensional Euclidean space. Define the unit *normal* vector at the point $\mathbf{r}(u, v)$

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}. \tag{4.11.2}$$

The vector $\mathbf{n}$ is orthogonal to $\mathbf{r}_u$ and $\mathbf{r}_\mathbf{v}$ and, hence, it is orthogonal to the tangent plane $T_{(u,v)}$ to the surface.

Introduce functions

$$\begin{aligned} b_{11}(u, v) &= \langle \mathbf{r}_{uu}, \mathbf{n} \rangle \\ b_{12}(u, v) &= \langle \mathbf{r}_{uv}, \mathbf{n} \rangle \\ b_{22}(u, v) &= \langle \mathbf{r}_{vv}, \mathbf{n} \rangle. \end{aligned} \tag{4.11.3}$$

Definition 4.11.1 *The quadratic form*

$$b_{11}(u, v)du^2 + 2b_{12}(u, v)du\, dv + b_{22}(u, v)dv^2 \tag{4.11.4}$$

is called the second fundamental form *of the surface* (4.11.1).

A geometric meaning of the value of the second fundamental form on tangent vectors to the surface becomes clear from the following statement.

Lemma 4.11.2 *Let $(u(t), v(t))$ be a smooth curve on the surface. The normal component of the acceleration vector $\ddot{\mathbf{r}}$ at a point $(u = u(t), v = v(t))$ is equal to the value of the second fundamental form on the velocity vector $(\dot{u}, \dot{v})$ at this point*

$$\langle \ddot{\mathbf{r}}, \mathbf{n} \rangle = b_{11}(u, v)\dot{u}^2 + 2b_{12}(u, v)\dot{u}\,\dot{v} + b_{22}(u, v)\dot{v}^2. \tag{4.11.5}$$

Proof: In the expression

$$\ddot{\mathbf{r}} = \mathbf{r}_{uu}\dot{u}^2 + 2\mathbf{r}_{uv}\dot{u}\dot{v} + \mathbf{r}_{vv}\dot{v}^2 + \mathbf{r}_u\ddot{u} + \mathbf{r}_v\ddot{v}$$

the last two terms are orthogonal to $\mathbf{n}$. Hence

$$\langle \ddot{\mathbf{r}}, \mathbf{n} \rangle = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle \dot{u}^2 + 2\langle \mathbf{r}_{uv}, \mathbf{n} \rangle \dot{u}\dot{v} + \langle \mathbf{r}_{vv}, \mathbf{n} \rangle \dot{v}^2.$$

□

Denote ν the principal normal to the curve

$$\mathbf{r}(u(t), v(t))$$

on the surface.

Theorem 4.11.3 *The curvature of a smooth curve on the surface* (4.11.1) *multiplied by the cosine of the angle between the principal normal to the curve and the normal to the surface is equal to the ratio of values of the second and first fundamental forms on the velocity vector of the curve*

$$k\,\langle \nu, \mathbf{n}\rangle = \frac{b_{11}(u,v)\dot{u}^2 + 2b_{12}(u,v)\dot{u}\dot{v} + b_{22}(u,v)\dot{v}^2}{g_{11}(u,v)\dot{u}^2 + 2g_{12}(u,v)\dot{u}\dot{v} + g_{22}(u,v)\dot{v}^2}. \tag{4.11.6}$$

Proof: Recall that the principal normal to the curve is the normalized vecor of acceleration

$$\frac{d^2\mathbf{r}}{ds^2} = k\,\nu, \quad k > 0, \quad |\nu| = 1.$$

Applying the Lemma one obtains

$$k\,\langle \nu, \mathbf{n}\rangle = b_{11}(u,v)\left(\frac{du}{ds}\right)^2 + 2b_{12}(u,v)\frac{du}{ds}\frac{dv}{ds} + b_{22}(u,v)\left(\frac{dv}{ds}\right)^2.$$

This proves the formula (4.11.6) for the curves parametrized by length since, in that case, the denominator in (4.11.6) is equal to 1. Since both sides of (4.11.6) do not depend on the parametrization of the curve, the formula holds trues also for an arbitrary parametrization. □

Let us consider the curve obtained by intersecting the surface by the plane passing through the normal $\mathbf{n}$. It is called the *normal section*. It is a plane curve; its principal normal vector ν is collinear with $\mathbf{n}$. Denote τ a unit tangent vector to the surface belonging to the normal plane. It coincides with the velocity vector of the normal section passing through $\mathbf{n}$ and τ. We obtain

Corollary 4.11.4 *The absolute value of the second fundamental form on a unit tangent vector τ to the surface is equal to the curvature of the normal section passing through τ and* $\mathbf{n}$.

Let us slightly modify the definition of the curvature for the case of a plane section of an oriented surface: it will coincide with the old one if the direction of the principal normal to the curve coincides, $\nu = \mathbf{n}$, with the direction of the normal to the surface; in the opposite case, $\nu = -\mathbf{n}$, the new curvature will be equal to the negative old one. With such a definition the result of the Corollary for the curvature $k = k(\tau)$ of a plane section passing through the unit tangent vector $\tau = (\tau^1, \tau^2)$ at a point (u, v) can be represented in the following form:

$$k(\tau) = b_{11}(u,v)(\tau^1)^2 + 2b_{12}(u,v)\tau^1\tau^2 + b_{22}(u,v)(\tau^2)^2 \tag{4.11.7}$$

$$g_{11}(u,v)(\tau^1)^2 + 2g_{12}(u,v)\tau^1\tau^2 + g_{22}(u,v)(\tau^2)^2 = 1. \tag{4.11.8}$$

Example. On the sphere of radius R all normal sections are circles of the same radius R. The curvature of these circles is equal to $1/R$. With the choice of the orientation on the surface by ordering the spherical coordinates $u = \phi$, $v = \theta$ the curvature of any normal section is equal to $-1/R$. Hence the second fundamental form of the sphere in the spherical coordinates reads

$$-R(d\theta^2 + \cos^2\theta\, d\phi^2).$$

In order to get more clear idea about dependence of the curvature of a normal section on the direction τ at a given point of the surface let us study the minima and maxima of the function $k(\tau)$. This problem is tantamount to finding the maxima/minima of the function (4.11.7) of two variables τ^1, τ^2 constrained by the equation (4.11.8). In order to simplify notations let us redenote

$$x := \tau^1, \quad y := \tau^2.$$

We will also omit writing explicitly the dependence of the coefficients of the first and second fundamental forms on u and v.

We arrive at the following constraint maximum/minimum problem:

$$b_{11}x^2 + 2b_{12}xy + b_{22}y^2 \to \max/\min \tag{4.11.9}$$

$$g_{11}x^2 + 2g_{12}xy + g_{22}y^2 = 1. \tag{4.11.10}$$

To resolve this problem let us consider the following auxiliary function

$$F = b_{11}x^2 + 2b_{12}xy + b_{22}y^2 - \lambda\left(g_{11}x^2 + 2g_{12}xy + g_{22}y^2 - 1\right). \tag{4.11.11}$$

One has to find the stationary points of $F = F(x, y, \lambda)$ from the system

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial \lambda} = 0.$$

The last equation is nothing but the constraint (4.11.10). The first two, after division by 2 yield a linear homogeneous system

$$b_{11}x + b_{12}y = \lambda(g_{11}x + g_{12}y)$$
$$b_{12}x + b_{22}y = \lambda(g_{12}x + g_{22}y)$$

or, in the matrix form,

$$B\,X = \lambda\, G\, X \tag{4.11.12}$$

where

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.11.13}$$

Recall that the matrices B and G are both symmetric and, moreover, the matrix G is positive definite.

We arrive at the theory of invariants of pairs of quadratic forms with the Gram matrices B and G. Let us briefly explain the main points of this theory in a linear space of an arbitrary dimension n

$$b(x, y) = b_{ij}x^i y^j, \quad g(x, y) = g_{ij}x^i y^j.$$

Definition 4.11.5 *A nonzero vector X satisfying the linear homogeneous system* (4.11.12) *is called* an eigenvector *of a pair of quadratic forms with the* eigenvalue λ.

Lemma 4.11.6 *The eigenvalues of a pair of quadratic forms satisfy the* characteristic equation

$$\det(B - \lambda G) = 0. \tag{4.11.14}$$

Proof: The linear homogeneous system (4.11.12) has a nonzero solution *iff* its determinant vanishes. □

Lemma 4.11.7 *The eigenvalues of a pair of quadratic forms do not depend on the choice of the basis in the space.*

Proof: Changing the basis transforms the Gram matrices of the quadratic forms to

$$B' = T^{\mathrm{T}} B\, T, \quad G' = T^{\mathrm{T}} G\, T.$$

So the new characteristic equation

$$\det(B' - \lambda\, G') = \det\left[T^{\mathrm{T}}(B - \lambda\, G)\, T\right] = (\det T)^2 \det(B - \lambda\, G) \tag{4.11.15}$$

is proportional to the old one. □

In order to complete the theory of normal forms of a pair of bilinear forms we will use the connection between self-adjoint operators and symmetric bilinear forms in a Euclidean space. Recall that a linear operator

$$A : X \to X \tag{4.11.16}$$

on a Euclidean space $(X, \langle\ ,\ \rangle)$ is called *self-adjoint* if it satisfies

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall\, x,\, y. \tag{4.11.17}$$

The bilinear form

$$b(x, y) := \langle x, Ay\rangle \tag{4.11.18}$$

is symmetric *iff* the operator A is self-adjoint. Given the matrix (a^i_j) of the operator in a basis $e_1, \ldots, e_n$,

$$Ae_j = a^i_j e_i, \quad j = 1, \ldots, n \tag{4.11.19}$$

and the Gram matrix of the inner product in the same basis

$$\langle e_i, e_j\rangle = g_{ij}, \quad i,\, j = 1, \ldots, n \tag{4.11.20}$$

one can can compute the Gram matrix of the bilinear form $b(x, y)$:

$$b_{ij} \equiv b(e_i, e_j) = g_{ik} a^k_j, \quad i,\, j = 1, \ldots \tag{4.11.21}$$

or, in the matrix form

$$B = G\, A. \tag{4.11.22}$$

Inverting one reconstructs the operator A by

$$A = G^{-1} B \tag{4.11.23}$$

or, in the index notations

$$a^i_j = g^{ik} b_{kj}, \quad i,\, j = 1, \ldots, n \tag{4.11.24}$$

where g^{ij} are the entries of the matrix inverse to $G = (g_{ij})$

$$G = (g_{ij}), \quad G^{-1} = \left(g^{ij}\right). \tag{4.11.25}$$

They say that the matrix of the bilinear form $b(x, y)$ is obtained from the matrix of the operator A by *lowering the index* (see (4.11.21)) while the inverse procedure (4.11.24) of reconstructing the operator from the bilinear form is called *raising of indices*.

Let us return to the eigenvalues and eigenvectors of a pair of quadratic forms. They coincide with the eigenvalues and eigenvectors of the self-adjoint linear operator A. At this point it is crucial that the quadratic form g defining the inner product in the space is positive definite. Under this assumption the following theorem is fundamental in the theory of self-adjoint operators.

Theorem 4.11.8 *Let*

$$A : X \to X$$

be a self-adjoint operator in a n-dimensional Euclidean space $(X, \langle\ ,\ \rangle)$. Then there exists an orthonormal basis $e_1, \ldots, e_n$ in X consisting of eigenvectors of the operator A

$$\begin{aligned} &Ae_i = \lambda_i e_i, \quad i = 1, \ldots, n \\ &\langle e_i, e_j \rangle = \delta_{ij}. \end{aligned}$$

Applying this theorem to the self-adjoint operator (4.11.23) we arrive at the following

Corollary 4.11.9 *Let B and G be two symmetric $n \times n$ matrices, and the matrix G is positive definite. Then*

1) the characteristic equation (4.11.14) has n real roots $\lambda_1, \ldots, \lambda_n$.

2) There exists a system of coordinates in the linear space such that the quadratic forms b and g take the following diagonal form

$$\begin{aligned} b(x, x) &= \lambda_1 (x^1)^2 + \cdots + \lambda_n (x^n)^2 \\ g(x, x) &= (x^1)^2 + \cdots + (x^n)^2. \end{aligned} \tag{4.11.26}$$

Exercise 4.11.10 *Prove that*

$$\lambda_1 \lambda_2 \ldots \lambda_n = \frac{\det B}{\det G}. \tag{4.11.27}$$

Let us come back to the curvature of normal sections of a surface in $\mathbb{R}^3$. We have proved that the characteristic equation (4.11.14) has two real roots λ_1 and λ_2 depending on the point of the surface and, moreover, at a given point there exist two tangent vectors e_1, e_2 such that

$$b(e_1, e_1) = \lambda_1, \quad b(e_2, e_2) = \lambda_2, \quad b(e_1, e_2) = b(e_2, e_1) = 0 \tag{4.11.28}$$

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_1, e_2) = g(e_2, e_1) = 0. \tag{4.11.29}$$

The second line means that the tangent vectors e_1 and e_2 are orthogonal as vectors in the three-dimensional Euclidean space and, moreover, they have unit length.

Definition 4.11.11 *The linear operator* $A = G^{-1}B$ *is called the* shape operator *of the surface. The numbers* $k_1 = \lambda_1$ *and* $k_2 = \lambda_2$ *are called the* principal curvatures *of the surface at a given point. The directions of the vectors* e_1 *and* e_2 *are called the* principal directions *at the same point.*

We will now see that the principal curvatures give the maximal and minimal values of curvatures normal sections we were looking after.

Theorem 4.11.12 (Euler formula) *Let* φ *be the angle between a unit tangent vector* τ *and* e_1*. Then the curvature* k *of the normal section of the surface passing through* τ *and the normal* $\mathbf{n}$ *is equal to*

$$k = k_1^2 \cos^2 \varphi + k_2 \sin^2 \varphi. \tag{4.11.30}$$

Proof: In the basis e_1, e_2 the first and the second fundamental forms become equal to

$$\begin{aligned} g(\tau, \tau) &= \quad (\tau^1)^2 \;\;+\;\; (\tau^2)^2 \\ b(\tau, \tau) &= k_1(\tau^1)^2 + k_2(\tau^2)^2 \end{aligned}$$

$$\tau = \tau^1 e_1 + \tau^2 e_2.$$

In this basis the vector τ reads

$$\tau = \cos\varphi\, e_1 + \sin\varphi\, e_2.$$

For the curvature of normal section passing through τ one obtains

$$k = \frac{k_1(\tau^1)^2 + k_2(\tau^2)^2}{(\tau^1)^2 + (\tau^2)^2} = k_1 \cos^2 \varphi + k_2 \sinh^2 \varphi.$$

□

Corollary 4.11.13 *Let the principal curvatures at a given point of the surface satisfy*

$$k_2 \leq k_1.$$

Then the curvature k *of an arbitrary normal section passing through the same point satisfies*

$$k_2 \leq k \leq k_1.$$

Definition 4.11.14 *The product of principal curvatures*

$$K = k_1 k_2 \tag{4.11.31}$$

is called the Gaussian curvature *of the surface at a given point. The mean value*

$$H = \frac{k_1 + k_2}{2} \tag{4.11.32}$$

is called the mean curvature *at the point.*

From the result of Exercise 4.11.10 it follows that the Gaussian curvature is equal to the ration of the determinants of the second and the first fundamental forms

$$K = \frac{\det B}{\det G}. \tag{4.11.33}$$

Comparing with the result of Exercise 4.8.15 along with the formula (4.10.13) one arrives to the following

Theorem 4.11.15 *The component R_{1212} of the Riemann curvature tensor and the scalar curvature R of the induced metric on a two-dimensional surface in $\mathbb{R}^3$ are given by the following formulae*

$$R_{1212} = K \det(g_{ij}), \quad R = 2K. \tag{4.11.34}$$

Example 1. For the sphere of radius R with the standard orientation the Gaussian curvature is equal to $K = 1/R^2$ and the mean curvature is $H = -1/R$.

Example 2. Let the surface in the Euclidean space be represented as a graph of a function

$$z = f(x, y).$$

The tangent vectors have the already familiar form

$$\mathbf{r}_x = (1, 0, f_x), \quad \mathbf{r}_y = (0, 1, f_y).$$

Computing their cross-product we obtain the unit normal vector

$$\mathbf{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}. \tag{4.11.35}$$

So the coefficients of the second fundamental form are equal to

$$\begin{aligned} b_{11} &= \langle \mathbf{r}_{xx}, \mathbf{n} \rangle = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}} \\ b_{12} &= \langle \mathbf{r}_{xy}, \mathbf{n} \rangle = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}} \\ b_{22} &= \langle \mathbf{r}_{yy}, \mathbf{n} \rangle = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}. \end{aligned}$$

Computing the determinant

$$\det B = b_{11} b_{22} - b_{12}^2$$

and dividing by the determinant of the first fundamental form (**??**)

$$\det G = 1 + f_x^2 + f_y^2$$

(see (**??**)) one obtains

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}. \tag{4.11.36}$$

One observes that, at a stationary point

$$f_x = f_y = 0$$

the Gaussian curvature is positive near a point of a maximum/minimum where the graph is convex but it is negative near a saddle point where the Hessian $f_{xx}f_{yy} - f_{xy}^2$ is negative.

Let us now compute the mean curvature of the graph surface. Inverting the matrix G

$$G^{-1} = \frac{1}{\det G}\begin{pmatrix} 1+f_y^2 & -f_x f_y \\ -f_x f_y & 1+f_x^2 \end{pmatrix}$$

and computing the trace of the matrix of the shape operator $G^{-1}B$ one obtains

$$\begin{aligned}
H &= \frac{1}{2}\operatorname{tr} G^{-1}B = \frac{1}{(\det G)^{3/2}}\left[(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy}\right] \\
&= \frac{1}{2}\left[\frac{f_{xx}+f_{yy}}{(\det G)^{1/2}} - \frac{f_x(f_x f_{xx} + f_y f_{xy})}{(\det G)^{3/2}} - \frac{f_y(f_x f_{xy} + f_y f_{yy}}{(\det G)^{3/2}}\right] \\
&= \frac{1}{2}\left[\frac{f_{xx}+f_{yy}}{(\det G)^{1/2}} - \frac{f_x(\det G)_x}{2(\det G)^{3/2}} - \frac{f_y(\det G)_y}{2(\det G)^{3/2}}\right] \\
&= \frac{1}{2}\left[\frac{\partial}{\partial x}\frac{f_x}{\sqrt{1+f_x^2+f_y^2}} + \frac{\partial}{\partial y}\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right] = \frac{1}{2}\operatorname{div}\frac{\operatorname{grad} f}{\sqrt{1+f_x^2+f_y^2}}.
\end{aligned}$$

Let us now consider the important particular case of *minimal surfaces* having zero mean curvature. Clearly, the Gaussian curvature of such a surface must be negative since the principal curvatures k_1 and k_2 have opposite signs.

Assuming that the minimal surface is represented as a graph of function $z = f(x,y)$ one obtains the following PDE for the function $f = f(x,y)$

$$\operatorname{div}\frac{\operatorname{grad} f}{\sqrt{1+f_x^2+f_y^2}} = 0 \tag{4.11.37}$$

or, equivalently

$$(1+f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1+f_x^2)f_{yy} = 0. \tag{4.11.38}$$

This equation describes the shape of soap films that, in the absence of external forces tend to minimize their area. Indeed, let us consider the area of a small piece of the surface

$$A[f] = \iint_\Omega \sqrt{1+f_x^2+f_y^2}\,dx\,dy \tag{4.11.39}$$

as a *functional* of the shape function f. Here Ω is a sufficiently small domain on the (x,y)-plane. A necessary condition to minimize the value of the functional is that, under an arbitrary small variation of the function f,

$$f(x,y) \mapsto f(x,y) + \delta f(x,y)$$

the variation of the functional must satisfy

$$A[f+\delta f] - A[f] = \mathcal{O}(\|\delta f\|^2). \tag{4.11.40}$$

Here the function $\delta f(x,y)$ must vanish together with its derivatives on the boundary of the domain Ω; the definition of the norm $\|\delta f\|$ will be clear from subsequent calculations. In other words, the equation (4.11.40) says that f is a "stationary point" of the "function" $A[f]$ on the infinite-dimensional space of functions $f = f(x,y)$.

Let us prove that stationary point condition (4.11.40) reduces to equation (4.11.39). The left hand side of this condition can be written in the following way

$$A[f+\delta f] - A[f] = \iint_\Omega \left[\sqrt{1+(f_x+\delta f_x)^2+(f_y+\delta f_y)^2} - \sqrt{1+f_x^2+f_y^2}\right] dx\, dy.$$

The part of the increment linear in δf can be obtained by expanding the above expression in Taylor series

$$A[f+\delta f] - A[f] = \iint_\Omega \left[\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\,\delta f_x + \frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\,\delta f_y\right] dx\, dy + \mathcal{O}(\|\delta f\|^2).$$

Thus the stationarity condition (4.11.40) can be recast into the form

$$\iint_\Omega \left[\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\,\delta f_x + \frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\,\delta f_y\right] dx\, dy = 0 \tag{4.11.41}$$

for an arbitrary function $\delta f(x,y)$ vanishing on the boundary of the domain Ω. Applying in two different ways the Fubini theorem

$$\iint_\Omega \quad dx\, dy = \int dx \int \quad dy = \int dy \int \quad dx$$

to the two parts of the double integral and integrating by parts one reduces the equation (4.11.41) to

$$\begin{aligned} &\iint_\Omega \left[\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\,\delta f_x + \frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\,\delta f_y\right] dx\, dy \\ &= -\iint_\Omega \left[\frac{\partial}{\partial x}\frac{f_x}{\sqrt{1+f_x^2+f_y^2}} + \frac{\partial}{\partial y}\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right] \delta f\, dx\, dy = 0. \end{aligned} \tag{4.11.42}$$

Since $\delta f(x,y)$ is an arbitrary function one obtains the equation for the stationary points of the area functional $A[f]$ written in the form

$$H = 0$$

where H is the mean curvature of the surface.

4.12 Gauss–Bonnet theorem

Let us begin with recalling the following theorem from spherical geometry. Consider a triangle Δ formed by three geodesics (we already know that they are the big circles on the sphere) on the sphere of radius R. Denote α, β, γ the angles of this triangle. Then

$$\text{Area of spherical triangle } \Delta = (\alpha + \beta + \gamma - \pi)R^2. \tag{4.12.1}$$

To prove this formula one can consider first the triangle with one vertex at the north pole and other two vertices on the equator. Denote γ the angle at the north pole; two other angles are equal to $\frac{\pi}{2}$. Clearly

$$\text{Area of the triangle} = \frac{\gamma}{2\pi} \times \text{Area of hemisphere} = \gamma\, R^2$$

that coincides, in this particular case with (4.12.1). The general case can be proven by cutting and glueing operations.

As the sphere has constant Gaussian curvature $K = 1/R^2$, the above statement can be rewritten in the following form

$$\int_\Delta K\, dA = \alpha + \beta + \gamma - \pi. \tag{4.12.2}$$

The right hand side can be interpreted in the following way. Choose a nonzero vector $\mathbf{v}$ at one of the vertices of the triangle. As the sides of the triangle are geodesics, the parallel transport of the vector along any of these curves preserves the angle between the vector and the geodesic. Denote $\mathbf{v}'$ the result of parallel transport of the vector $\mathbf{v}$ along the contour of the triangle. It is easy to see that the angle between $\mathbf{v}'$ and $\mathbf{v}$ is exactly equal to $\alpha+\beta+\gamma-\pi$. Recall that this angle does not depend on the choice of the initial vector $\mathbf{v}$.

A far-reaching generalization of the theorem of spherical geometry was obtained by Gauss. Namely,

Gauss Theorem. *Consider a triangle Δ on a two-dimensional surface in the Euclidean space $\mathbb{R}^3$ formed by three geodesics. Denote α, β, γ the angles of the triangle. Then*

$$\int_\Delta K\, dA = \alpha + \beta + \gamma - \pi$$

where K is the Gaussian curvature of the surface.

More general version of the Gauss' theorem was obtained by P.O.Bonnet, so it is usually called *Gauss–Bonnet theorem.* To formulate this theorem we need to introduce the notion of geodesic curvature of smooth curves on a two-dimensional oriented Riemannian manifold M.

Let $\gamma(t)$ be such a curve parameterized by arc length, i.e., $|\dot\gamma(t)| \equiv 1$. Then the geodesic curvature k_g is defined as the length of the *covariant acceleration* vector $\nabla_{\dot\gamma}\dot\gamma$

$$k_g = \pm\, |\nabla_{\dot\gamma}\dot\gamma|$$

where the sign is defined as follows.

Denote $\boldsymbol{\tau} = \dot{\gamma}(t)$ the unit tangent vector to the curve at the point $\gamma(t)$ and $\boldsymbol{\nu} \in T_{\gamma(t)}M$ the unit vector orthogonal to $\boldsymbol{\tau}$ such that the orientation of the frame $(\boldsymbol{\tau}, \boldsymbol{\nu})$ coincides with the orientation of M. The covariant derivative $\nabla_{\dot{\gamma}}\dot{\gamma}$ is orthogonal to the curve. Indeed, differentiating the identity $\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle \equiv 1$ we obtain

$$0 = \frac{d}{dt}\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle = 2\langle \nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\rangle.$$

Thus the acceleration vector $\nabla_{\dot{\gamma}}\dot{\gamma}$ is proportional to $\boldsymbol{\nu}$,

$$\nabla_{\dot{\gamma}}\dot{\gamma} = k_g\,\boldsymbol{\nu}. \tag{4.12.3}$$

If γ is a geodesic then its geodesic curvature equals zero.

Theorem 4.12.1 *Let $U \subset M$ be a compact domain in a two-dimensional oriented Riemannian manifold M with a closed piecewise smooth positively oriented boundary ∂U with n vertices $\mathbf{r}_1, \ldots \mathbf{r}_n$ connected by smooth curves*

$$\gamma_i \quad \text{from} \quad \mathbf{r}_i \text{ to } \mathbf{r}_{i+1}, \quad i = 1, \ldots, n$$

(it is understood that $\mathbf{r}_{n+1} = \mathbf{r}_1$). Denote θ_i the internal angle between γ_i and γ_{i+1} at the vertex $\mathbf{r}_i$, $i = 1, \ldots, n$. Let $K := \frac{R}{2}$ be the Gaussian curvature of the manifold. Then

$$\int_U K\,dA + \oint_{\partial U} k_g ds = 2\pi\,\chi(U) - \sum_{i=1}^{n}(\pi - \theta_i) \tag{4.12.4}$$

where $\chi(U)$ is the Euler characteristic of U. In particular, for a closed oriented two-dimensional Riemannian manifold M one has

$$\int_M K\,dA = 2\pi\,\chi(M). \tag{4.12.5}$$

Proof: Let us first prove the Theorem for a small domain U that can be covered by one system of "geodesic polar coordinates" r, ϕ, so that the metric has the form

$$ds^2 = dr^2 + g(r,\phi)d\phi^2$$

for some smooth function $g = g(r,\phi)$. In particular the domain U is homeomorphic to a disk. The Gram matrix (g_{ij}) and its inverse $\left(g^{ij}\right)$ have the form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix}.$$

It is easy to compute the Christoffel coefficients of the Levi-Civita connection for the metric. The only non-zero coefficients are

$$\Gamma^1_{22} = -\frac{1}{2}\frac{\partial g}{\partial r}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2g}\frac{\partial g}{\partial r}, \quad \Gamma^2_{22} = \frac{1}{2g}\frac{\partial g}{\partial \phi}. \tag{4.12.6}$$

The Gaussian curvature of the metric can be computed with the help of the formula (4.8.23):

$$K = -\frac{1}{\sqrt{g}}\frac{\partial^2\sqrt{g}}{\partial r^2}. \tag{4.12.7}$$

Let $(r(s), \phi(s))$ be a smooth curve in U parameterized by arc length, so that

$$\dot r^2(s) + g\left(r(s), \phi(s)\right)\dot\phi^2(s) \equiv 1.$$

The unit normal vector $\boldsymbol{\nu} = (\nu_1, \nu_2)$ must satisfy the orthogonality

$$\dot r\,\nu_1 + g\,\dot\phi\,\nu_2 = 0$$

so $\boldsymbol{\nu} \sim \left(-g\,\dot\phi, \dot r\right)$. Normalizing we obtain

$$\boldsymbol{\nu} = \frac{\left(-g\,\dot\phi, \dot r\right)}{\sqrt{g}}.$$

Let us now compute the geodesic curvature of the curve. We will use the formula for the k-th component of the acceleration vector

$$(\nabla_{\dot\gamma}\dot\gamma)^k = \ddot x^k + \Gamma^k_{ij}\dot x^i \dot x^j$$

along with the above explicit expressions for the Christoffel coefficients to derive that

$$(\nabla_{\dot\gamma}\dot\gamma)^1 = \ddot r - \frac{1}{2}\frac{\partial g}{\partial r}\,\dot\phi^2, \quad (\nabla_{\dot\gamma}\dot\gamma)^2 = \ddot\phi + \frac{1}{g}\frac{\partial g}{\partial r}\,\dot r\,\dot\phi + \frac{1}{2g}\frac{\partial g}{\partial \phi}\,\dot\phi^2.$$

So

$$k_g = \sqrt{g}\left[\dot r\,\ddot\phi - \dot\phi\,\ddot r + \frac{1}{2}\left(g_r\dot\phi^3 + 2\,\frac{g_r}{g}\dot r^2\dot\phi + \frac{g_\phi}{g}\dot r\,\dot\phi^2\right)\right]. \tag{4.12.8}$$

Consider now the angle θ between the tangent vector to the curve γ and a given fixed direction. For example, cosine of the angle between $\dot\gamma$ and the unit vector $(1,0)$ along the r-axis is equal to

$$\cos\theta = \frac{\langle(\dot r, \dot\phi), (1,0)\rangle}{\sqrt{\dot r^2 + g\,\dot\phi^2}} = \dot r$$

(here we use that $|\dot\gamma|^2 = \dot r^2 + g\,\dot\phi^2 = 1$). Therefore

$$\tan\theta = \sqrt{g}\,\frac{\dot\phi}{\dot r}.$$

Hence

$$d\theta = d\arctan\sqrt{g}\,\frac{\dot\phi}{\dot r} = \sqrt{g}\left[\dot r\,\ddot\phi - \dot\phi\,\ddot r + \frac{1}{2}\left(\frac{g_r}{g}\,\dot\phi\,\dot r^2 + \frac{g_\phi}{g}\,\dot r\,\dot\phi^2\right)\right]ds.$$

Thus

$$k_g ds - d\theta = \frac{g_r}{2\sqrt{g}}\dot\phi\left[\dot r^2 + g\,\dot\phi^2\right]ds = \left(\sqrt{g}\right)_r d\phi.$$

Consider now the integral of the geodesic curvature over the boundary of the domain U. Due to the above formula we obtain

$$\oint_{\partial U} k_g\,ds = \oint_{\partial U}\left(\sqrt{g}\right)_r d\phi + \oint_{\partial U} d\theta. \tag{4.12.9}$$

The second integral is the total rotation angle of the tangent vector to the contour ∂U. If the boundary of the disk U is a smooth curve then the total rotation angle is equal to 2π,

$$\oint_{\partial U} d\theta = 2\pi.$$

For a piecewise smooth boundary contour the tangent vector makes jumps at the corners $\mathbf{r}_i$. In this case one has to add the jumps, namely

Lemma 4.12.2 *If the domain U is simply connected then*

$$\sum_{i=1}^{n} \int_{\gamma_i} d\theta + \sum_{i=1}^{n} (\pi - \theta_i) = 2\pi. \tag{4.12.10}$$

Observe that $\pi - \theta_i$ is the *external* angle of the boundary curve at the i-th vertex.

The first integral in (4.12.9) will be transformed using the Stokes formula:

$$\oint_{\partial U} (\sqrt{g})_r \, d\phi = \int_U d\left((\sqrt{g})_r \, d\phi\right) = \int_U \frac{1}{\sqrt{g}} (\sqrt{g})_{rr} \sqrt{g}\, dr \wedge d\phi = -\int_U K\, dA$$

where we use the expression (4.12.7) for the Gaussian curvature and the standard formula $dA = \sqrt{g}\, dr \wedge d\phi$ for the area element. Together with (4.12.10) this proves the Gauss–Bonnet formula (4.12.4) for sufficiently small domains U as in this case $\chi(U) = 1$.

Let us now proceed to the general case. The simplest way is to use a triangulation of the domain U that is, a representation

$$U = \bigcup_{i=1}^{F} \Delta_i$$

where every subset Δ_i is a sufficiently small triangle that can be covered by one system of geodesic polar coordinates. It has three distinct vertices connected by smooth curves called *edges* of the triangle. It is required that, for any pair of intersecting triangles $\Delta_i \cap \Delta_j \neq \emptyset$ the intersection must be a common edge of Δ_i and Δ_j or a common vertex of these triangles. Denote

$$\begin{aligned} V &= \#\,\{\text{vertices of the triangulation}\} \\ E &= \#\,\{\text{edges of the triangulation}\} \\ F &= \#\,\{\text{triangles}\}. \end{aligned}$$

The Euler charcteristic of the domain U is equal to

$$\chi(U) = V - E + F. \tag{4.12.11}$$

Observe that for such a small triangle Δ with internal angles α, β, γ the already proven local version of the Gauss–Bonnet theorem reads

$$\int_\Delta K\, dA + \oint_{\partial\Delta} k_g ds = \alpha + \beta + \gamma - \pi \tag{4.12.12}$$

for the positively oriented boundary $\partial\Delta$. Denote α_i, β_i, γ_i the angles of the triangle Δ_i. Taking sum of eqs. (4.12.12) over all triangles yields

$$\int_U K\,dA + \oint_{\partial U} k_g ds = \sum_{i=1}^{F} (\alpha_i + \beta_i + \gamma_i - \pi) \tag{4.12.13}$$

as the integrals of $k_g ds$ over the edges in the internal part of U will be cancelled. We will first consider the case $U = M$, $\partial U = \emptyset$ (see eq. (4.12.5)) of integrating over the closed compact oriented manifold M. In this case the sum in the rhs can be rewritten as follows

$$\sum_{i=1}^{F} (\alpha_i + \beta_i + \gamma_i - \pi) = 2\pi V - \pi\, F.$$

Denote v_i the valency of the i-th vertex of the triangulation, i.e., the number of edges passing through this vertex. We have the following two equations: first,

$$\sum_{i=1}^{F} v_i = 2E$$

since every edge is counted twice; second,

$$\sum_{i=1}^{F} v_i = 3F$$

since every triangle is counted three times. Thus we have the following identity for a triangulation of a closed two-dimensional manifold

$$3F = 2E. \tag{4.12.14}$$

With the help of this identity we obtain

$$\pi(2V - F) = \pi(2V + 2F - 3F) = \pi(2V + 2F - 2E) = 2\pi\chi(M).$$

This proves the version (4.12.5) of the Gauss–Bonnet theorem for the case $U = M$.

Let us consider the general case. Write

$$V = V_{int} + V_{ext}, \quad E = E_{int} + E_{ext}$$

where V_{int}, E_{int} are the numbers of vertices and edges belonging to the internal part of U and V_{ext}, E_{ext} are the numbers of vertices and edges on the boundary ∂U. It is easy to see that the identity (4.12.14) modifies to

$$3F = 2E_{int} + E_{ext}. \tag{4.12.15}$$

Another obvious identity says that

$$V_{ext} = E_{ext}. \tag{4.12.16}$$

We also split the set of vertices on the boundary in two parts

$$V_{ext} = \tilde{V}_{ext} + n$$

where $\tilde{V}_{ext}$ is the number of vertices on the *smooth* part of the boundary ∂U.

The sum in the rhs of (4.12.13) can be rewritten as follows

$$\int_U K\, dA + \oint_{\partial U} k_g ds = 2\pi V_{int} + \pi \tilde{V}_{ext} + \sum_{j=1}^n \theta_j - \pi F$$

$$= 2\pi V_{int} + \pi \tilde{V}_{ext} - \sum_{j=1}^n (\pi - \theta_j) + \pi n - \pi F = 2\pi V_{int} + \pi V_{ext} - \sum_{j=1}^n (\pi - \theta_j) - \pi F$$

where $\theta_1, \ldots, \theta_n$ are the internal angles at the points $\mathbf{r}_1, \ldots, \mathbf{r}_n$ on the boundary. Using eqs. (4.12.15), (4.12.16) we obtain

$$\pi\left(2V_{int} + V_{ext} - F\right) = \pi\left(2V_{int} + V_{ext} + 2F - 3F\right) = \pi\left(2V_{int} + V_{ext} + 2F - 2E_{int} - E_{ext}\right)$$
$$= \pi\left(2V_{int} + 2V_{ext} + 2F - 2E_{int} - 2E_{ext}\right) = 2\pi(V - E + F) = 2\pi\chi(U).$$

The Gauss–Bonnet theorem is proved. □

From topology of surfaces it is known that every oriented connected compact two-dimensional manifold[7] is homeomorphic to the sphere with g handles. The number g is called the genus of the surface. The Euler characteristic of a surface M_g of genus g is equal to $\chi(M_g) = 2 - 2g$. Combining this statement with the Gauss–Bonnet theorem we obtain

Corollary 4.12.3 *If the Gaussian curvature is positive at every point of a closed oriented surface then the surface is homeomorphic to the sphere.*

Exercise 4.12.4 *Let γ_1, γ_2 be two closed geodesics on a closed oriented surface of positive Gaussian curvature. Then they necessarily intersect.*

4.13 Conformal structures on two-dimensional Riemannian manifolds and Laplace–Beltrami equation

Let $\left(M, ds^2\right)$ be a two-dimensional Riemannian manifold with the metric[8]

$$ds^2 = E dx^2 + 2F dx\, dy + G dy^2.$$

We will prove existence of local coordinates such that the metric takes the form

$$ds^2 = g(u, v)\left(du^2 + dv^2\right). \tag{4.13.1}$$

Theorem 4.13.1 *Assume that the coefficients E, F, G are real analytic functions of the coordinates x, y. Then there exist local coordinates $u = u(x, y)$, $v = v(x, y)$ such that*

$$g(u, v)\left(du^2 + dv^2\right) = E(x, y) dx^2 + 2F(x, y) dx\, dy + G(x, y) dy^2 \tag{4.13.2}$$

[7] Here we will use the short name *closed oriented surfaces* for such manifolds.

[8] In this section we will use the "old-fashion" notations for the coefficients of the metric tensor

$$g_{11} = E, \quad g_{12} = F, \quad g_{22} = G$$

where E, F, G are some function of the coordinates x, y.

for some function $g(u,v)$ also satisfying the condition of positivity of the Jacobian

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} > 0. \tag{4.13.3}$$

Clearly the last condition means that the change of local coordinates

$$(x,y) \mapsto (u,v)$$

preserves the orientation.

Proof: Denote

$$A = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

the Jacobi matrix of the coordinate change we are looking for. The transformation law of components of the metric tensor

$$A^T \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} A = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{4.13.4}$$

imply

$$(g \det A)^2 = EG - F^2.$$

Due to the orientation preserving assumption this gives

$$g \det A = \sqrt{EG - F^2}. \tag{4.13.5}$$

Multiplying eq. (4.13.4) on the right by the inverse matrix we obtain

$$\begin{aligned} A^T = \frac{1}{g} \begin{pmatrix} E & F \\ F & G \end{pmatrix} A^{-1} &= \frac{1}{g \det A} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix} \\ &= \frac{1}{\sqrt{EG - F^2}} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} v_y & -u_y \\ -v_x & u_x \end{pmatrix}. \end{aligned}$$

Therefore we arrive at two systems of PDEs

$$\begin{aligned} u_x &= \frac{-F\, v_x + E\, v_y}{\sqrt{EG - F^2}} \\ u_y &= \frac{-G\, v_x + F\, v_y}{\sqrt{EG - F^2}} \end{aligned} \tag{4.13.6}$$

and

$$\begin{aligned} v_x &= \frac{F\, u_x - E\, u_y}{\sqrt{EG - F^2}} \\ v_y &= \frac{G\, u_x - F\, u_y}{\sqrt{EG - F^2}}. \end{aligned} \tag{4.13.7}$$

Clearly the second system is equivalent to the first one (just interchanging $u \leftrightarrow v$).

We will now reduce the system (4.13.6) to one second order PDE for the function $v = v(x,y)$. To this end we use equality of mixed derivatives

$$(u_x)_y - (u_y)_x = 0.$$

Substituting (4.13.6) we obtain

$$-\frac{\partial}{\partial x}\frac{-G\,v_x + F\,v_y}{\sqrt{EG - F^2}} + \frac{\partial}{\partial y}\frac{-F\,v_x + E\,v_y}{\sqrt{EG - F^2}} = 0. \tag{4.13.8}$$

It is easy to see that the lhs of the above equation coincides, up to a common factor $\sqrt{EG - F^2}$ with the Laplace–Beltrami equation (4.8.12). Applying the Cauchy–Kovalevskaya theorem from the theory of PDEs with analytic coefficients[9] we establish local existence of a solution $v = v(x,y)$ near a given point (x_0, y_0) satisfying the condition

$$v_x^2(x_0, y_0) + v_y^2(x_0, y_0) \neq 0.$$

It remains to find, for a given $v = v(x,y)$, the function $u = u(x,y)$. To this end we recall that the Laplace–Beltrami equation can be represented in the form

$$* \, d * dv = 0$$

(cf. eq. (4.8.11) above). Thus the 1-form $*dv$ is closed. Hence it locally exists a smooth function u such that

$$du = *dv.$$

It is easy to see that the last equation coincides with the system (4.13.6). The last step is to check positivity (4.13.3) of the Jacobian. Indeed, from (4.13.6) it follows that

$$\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \frac{1}{EG - F^2}\det\begin{pmatrix} -Fv_x + Ev_y & -Gv_x + Fv_y \\ v_x & v_y \end{pmatrix} =$$

$$= \frac{1}{EG - F^2}\left[Gv_x^2 - 2Fv_xv_y + Ev_y^2\right] = g^{ij}\frac{\partial v}{\partial x^i}\frac{\partial v}{\partial x^j} > 0. \tag{4.13.9}$$

Finally, the diagonal entry g of the metric in the coordinates u, v can be easily computed from the above calculations. □

Definition 4.13.2 *The coordinates (u,v) constructed in the Theorem are called* conformal *or also* isothermic *coordinates on the two-dimensional Riemannian manifold.*

It will be convenient to use complex coordinates $w = u + iv$, $\bar{w} = u - iv$ to recast the metric into the form

$$ds^2 = g\,(w, \bar{w})\,dw\,d\bar{w}, \tag{4.13.10}$$

Lemma 4.13.3 *Holomorphic changes of coordinates*

$$(w, \bar{w}) \mapsto \left(w' = f(w), \bar{w}' = \overline{f(w)}\right), \quad \frac{\partial f}{\partial \bar{w}} = 0 \tag{4.13.11}$$

with non-vanishing complex derivative

$$\frac{df}{dw} \neq 0$$

preserve the form (4.13.10) *of the metric.*

[9] Instead of the assumption of analyticity one can apply to (4.13.8) the theory of elliptic PDEs.

Recall that the holomorphic changes of coordinates also preserve orientation.

Proof: We have

$$dw' = \frac{df}{dw}dw, \quad d\bar{w}' = \overline{\frac{df}{dw}}d\bar{w}.$$

So

$$ds^2 = g'\, dw'\, d\bar{w}' = g' \left|\frac{df}{dw}\right|^2 dw\, d\bar{w}.$$

Thus

$$g'\left(w', \bar{w}'\right) = \left|\frac{dw}{dw'}\right|^2 g\left(w, \bar{w}\right). \tag{4.13.12}$$

□

The converse statement is given in the following

Exercise 4.13.4 *Let u', v' be another system of conformal coordinates for the same metric ds^2 satisfying the orientation preserving assumption. Prove that the coordinate change*

$$(u, v) \mapsto (u', v')$$

is given by a holomorphic function

$$w' = u' + iv' = f(w), \quad w = u + iv, \quad \frac{\partial f}{\partial \bar{w}} = 0 \tag{4.13.13}$$

with non-vanishing complex derivative $\frac{dw'}{dw} \neq 0$.

Remark 4.13.5 *The system* (4.13.6) *of two equations for conformal coordinates can be rewritten in complex form*

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z} \tag{4.13.14}$$

for their complex combination $w = u + iv$ as a function of complex variables $z = x + iy$ and $\bar{z} = x - iy$ where

$$\mu = \frac{E - G + 2iF}{E + G + \sqrt{EG - F^2}}. \tag{4.13.15}$$

The equation (4.13.14) *is called* Beltrami equation. *It appears in the theory of quasiconformal maps, in Teichmüller theory and also in other branches of mathematics.*

Exercise 4.13.6 *Prove that the orientation preserving assumption* (4.13.3) *is equivalent to the inequality*

$$|\mu| < 1.$$

4.14 Geometry of sphere and pseudosphere in conformal coordinates

Example 4.14.1 *Let us construct conformal coordinates on the standard sphere* S^2

$$x^2 + y^2 + z^2 = R^2.$$

To this end we use the stereographic projection $s : S^2 \to \mathbb{R}^2$ *from the south pole* $(0, 0, -R)$ *to the* (x, y)*-plane*

$$(x, y, z) \mapsto \left(u = \frac{R\,x}{R+z}, v = \frac{R\,y}{R+z} \right)$$

where (u, v) *are the* (x, y)*-coordinates of the point* $s(x, y, z)$*. Using*

$$u^2 + v^2 = R^2 \frac{x^2 + y^2}{(R+z)^2} = R^2 \frac{R^2 - z^2}{(R+z)^2} = R^2 \frac{R-z}{R+z}$$

we find

$$z = R\,\frac{1 - \frac{u^2+v^2}{R^2}}{1 + \frac{u^2+v^2}{R^2}}$$

so

$$x = \frac{2u}{1 + \frac{u^2+v^2}{R^2}}, \quad y = \frac{2v}{1 + \frac{u^2+v^2}{R^2}}.$$

From here it readily follows that the metric on the sphere takes the following form in the coordinates (u, v)

$$ds^2 = dx^2 + dy^2 + dz^2 = 4\,\frac{du^2 + dv^2}{\left(1 + \frac{u^2+v^2}{R^2}\right)^2}.$$

Thus u*,* v *are conformal coordinates on the sphere. After rescaling*

$$u \to R\,u, \quad v \to R\,v$$

the metric becomes

$$ds^2 = 4R^2 \frac{du^2 + dv^2}{(1 + u^2 + v^2)^2} = 4R^2 \frac{dw\, d\bar{w}}{(1 + |w|^2)^2}. \tag{4.14.1}$$

The complex variable $w = u + iv \in \mathbb{C}$ *takes arbitrary value. When the point* (x, y, z) *on the sphere approaches the south pole then* $w \to \infty$*. It is easy to see that, using the stereographic projection from the north pole one obtains the same expression. Warning: in this case we choose the new conformal coordinates* u'*,* v' *as the* $(x, -y)$*-coordinates of the stereographic projection!*

Example 4.14.2 *We consider now the pseudosphere*

$$x^2 + y^2 - z^2 = -R^2, \quad z > 0$$

in the three-dimensional Minkowski space $\mathbb{R}^{1,2}$*. The stereographic projection from the "south pole"* $(0, 0, -R)$ *works also in this case. After simple calculations we obtain*

$$x = \frac{2u}{1 - \frac{u^2+v^2}{R^2}}, \quad y = \frac{2v}{1 - \frac{u^2+v^2}{R^2}}, \quad z = R\,\frac{1 + \frac{u^2+v^2}{R^2}}{1 - \frac{u^2+v^2}{R^2}},$$

so
$$dx^2 + dy^2 - dz^2 = 4\,\frac{du^2 + dv^2}{\left(1 - \frac{u^2+v^2}{R^2}\right)^2}.$$

After rescaling $u \to R\,u$, $v \to R\,v$ finally obtain the metric on the pseudosphere in complex coordinates

$$ds^2 = 4R^2 \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2} = 4R^2 \frac{dw\, d\bar w}{(1 - |w|^2)^2}. \tag{4.14.2}$$

The above formula defines a Riemannian metric on the unit disk $D = \{w \in \mathbb{C} \mid |w| < 1\}$. From complex analysis it is well known that there exists a biholomorphic map from the unit disk to the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}.$$

For example, one can use the following fractional-linear transformation $\mathbb{H} \to D$

$$w = \frac{z - i}{z + i}. \tag{4.14.3}$$

After susbtitution obtain the following Riemannian metric of constant curvature $K = -1/R^2$ on the upper half-plane $z = x + iy$, $y > 0$

$$dz^2 = -4R^2 \frac{dz\, d\bar z}{(z - \bar z)^2} = 4R^2\, \frac{dx^2 + dy^2}{y^2}. \tag{4.14.4}$$

We will now consider the orientation preserving isometries of the sphere (4.14.1) and pseudosphere (4.14.2) or (4.14.4). According to Lemma 4.13.3 and Exercise 4.13.4 such isometries must be given by holomorphic changes of complex coordinates. We will restrict ourselves to fractional-linear changes of complex coordinates

$$z \mapsto w = \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \tag{4.14.5}$$

for the reasons that will be explained later.

Proposition 4.14.3 *The fractional-linear trasformation* (4.14.5) *is an isometry of the sphere with the metric*

$$ds^2 = 4R^2 \frac{dz\, d\bar z}{(1 + |z|^2)^2}$$

iff *the matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$.

Recall that a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to the subgroup $SU(2) \subset SL(2, \mathbb{C})$ if it satisfies

$$A^\dagger A = \mathbf{1}, \quad \det A = 1 \tag{4.14.6}$$

where $A^\dagger = \bar A^T$.

Proof: We have
$$dw = \frac{dz}{(cz+d)^2}, \quad d\bar{w} = \frac{d\bar{z}}{(\bar{c}\bar{z}+\bar{d})^2}$$
so
$$dw\, d\bar{w} = \frac{dz\, d\bar{z}}{|cz+d|^4}.$$
Next,
$$|w|^2 = w\,\bar{w} = \frac{az+b}{cz+d} \cdot \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$$
so
$$1+|w|^2 = \frac{(a\,\bar{a}+c\,\bar{c})z\,\bar{z}+(a\,\bar{b}+c\,\bar{d})\,z+(\bar{a}\,b+\bar{c}\,d)\,\bar{z}+b\,\bar{b}+d\,\bar{d}}{|cz+d|^4}.$$
Finally
$$4R^2\frac{dw\,d\bar{w}}{(1+|w|^2)^2} = 4R^2\frac{dz\,d\bar{z}}{\left[(a\,\bar{a}+c\,\bar{c})z\,\bar{z}+(a\,\bar{b}+c\,\bar{d})\,z+(\bar{a}\,b+\bar{c}\,d)\,\bar{z}+b\,\bar{b}+d\,\bar{d}\right]^2}$$
Thus the isometry condition
$$4R^2\frac{dw\,d\bar{w}}{(1+|w|^2)^2} = 4R^2\frac{dz\,d\bar{z}}{(1+|z|^2)^2}$$
is equivalent to the following system of equations
$$a\,\bar{a}+c\,\bar{c}=1, \quad b\,\bar{b}+d\,\bar{d}=1, \quad a\,\bar{b}+c\,\bar{d}=0 \tag{4.14.7}$$
along with the unimodularity condition
$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1. \tag{4.14.8}$$
This is the system of defining equations for the subgroup $SU(2)$. □

Remark 4.14.4 *One may ask whether the above calculation produces the* full *group of orientation preserving isometries of the sphere? The answer is 'yes'. Indeed, the dimension of the group of isometries of a two-dimensional Riemannian manifold cannot be bigger than 3. The dimension of the group $SU(2)$ is equal to 3. One can see this in the following way. Denote $a_{\rm Re}$, $a_{\rm Im}$ and $c_{\rm Re}$, $c_{\rm Im}$ the real and imaginary parts of a and c. The first of eqs.* (4.14.7) *then takes the form*
$$a_{\rm Re}^2 + a_{\rm Im}^2 + c_{\rm Re}^2 + c_{\rm Im}^2 = 1.$$
This equations defines the standard three-dimensional *sphere S^3 in the four-dimensional space. It is easy to see that other equations* (4.14.7), (4.14.8) *yield the following expressions of the entries b and d in terms of a and c*
$$b = -\bar{c}, \quad d = a.$$
Therefore the group $SU(2)$ as a manifold coincides with S^3.

Remark 4.14.5 *Observe that the group of orientation preserving isometries of the sphere is isomorphic to the quotient of $SU(2)$ over the subgroup $\{\pm\mathbf{1}\}$ of two elements. In this way we arrive at the following important isomorphism*

$$SO(3) \simeq SU(2)/\{\pm\mathbf{1}\}. \tag{4.14.9}$$

Similar arguments yield the following descrption of the group of orientation-preserving isometries of the pseudosphere realized by the metric (4.14.2) on the unit disk. Denote $SU(1,1)$ the group of 2×2 matrices A satisfying the conditions

$$A^\dagger G\, A = G, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \det A = 1. \tag{4.14.10}$$

Proposition 4.14.6 *The group of orientation-preserving isometries of the pseudosphere* (4.14.2) *is isomorphic to $SU(1,1)/\{\pm\mathbf{1}\}$.*

Remark 4.14.7 *From complex analysis it is known that the group of complex automorphisms of the unit disk is isomorphic to $SU(1,1)/\{\pm\mathbf{1}\}$.*

Another important realization of the group of isometries of the pseudosphere is associated with the metric (4.14.4) on the upper half-plane $\mathbb{H}$. Recall that the group of complex automorphisms of the upper half-plane is isomorphic to

$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\mathbf{1}\}.$$

Remarkably, this group coincides with the group of isometries of the pseudosphere:

Proposition 4.14.8 *The group of orientation-preserving isometries of the pseudosphere* (4.14.4) *is isomorphic to $PSL(2,\mathbb{R})$.*

Needless to say that the two realizations of the group of orientation-preserving isometries of the pseudosphere produce isomorphic groups

$$SU(1,1)/\{\pm\mathbf{1}\} \simeq PSL(2,\mathbb{R}).$$

They are also isomorphic to the group $SO_+(1,2)$ of Lorentz transformations preserving the direction of time.

Let us now consider the geodesics of the sphere and pseudosphere represented in the conformal coordinates. For the sphere we already know that the geodesics are the big circles obtained by intersecting the sphere with a plane passing through the origin. Consider first the geodesics passing through the poles of the sphere. After the stereographic projection one obtains straight lines on the plane passing through the origin. Applying an isometry

$$z \mapsto w = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$$

to a straight line one obtains a circle or a straight line on the complex plane. Clearly these are all the geodesics of the sphere.

Let us now consider the case of pseudosphere (4.14.2). Also in this case applying the stereographic projection to the geodesics obtained by intersection of the pseudosphere $x^2 + y^2 - z^2 + R^2 = 0$ in $\mathbb{R}^{1,2}$ with planes passing through the pole $(0, 0, R)$ one obtains diameters of the unit circle. Applying an isometry

$$z \mapsto w = \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$$

to such a diameter one obtains an arc of a circle or another diameter. As the diameters are orthogonal to the boundary of the unit disk, their images will also be orthogonal to the boundary of the disk. Clearly in this way one obtains all geodesics of the pseudosphere drawn on the unit disk.

Finally, consider the pseudosphere modelled on the upper half plane (4.14.4). Applying the fractional-linear transformation from the unit disk to the upper half-plane to the diameters or arcs orthogonal to the boundary one obtains vertical half-lines starting from a point on the real axis or half-circles with centres on the real line. This construction provides the full list of geodesics on the upper half-plane (4.14.4). In such a realization it is very easy to see that, given a geodesic γ (say, a vertical half-line) and a point $P \notin \gamma$, then there exists an infinite family of geodesics γ' passing through P but not intersecting γ.

4.15 Surfaces of constant curvature. Liouville equation

Our nearest goal is the classification of two-dimensional Riemannian manifolds with constant scalar curvature

$$R = 2K = \text{const}.$$

It will be shown that, under certain analytic assumption any such Riemannian manifold is locally isometric to the sphere, pseudosphere or Euclidean plane, depending on the sign of the curvature.

Remark 4.15.1 *Gaussian curvature K makes sense only for surfaces in $\mathbb{R}^3$. Otherwise it can be defined as half of the scalar curvature R. In this section we will use K instead of R for notational reasons.*

Theorem 4.15.2 *Any two-dimensional analytic Riemannian manifold of constant curvature $K = \text{const}$ is locally isometric to*

sphere, if $K > 0$;

Euclidean plane, if $K = 0$;

pseudosphere, if $K < 0$.

Proof: Let us first spell out the condition of constancy of the curvature of the metric written in conformal coordinates

$$ds^2 = e^{2\varphi}\left(du^2 + dv^2\right), \quad \varphi = \varphi(u, v) \tag{4.15.1}$$

(we have just redenoted $g = e^{2\varphi}$).

Lemma 4.15.3 *The curvature* $K = \frac{R}{2}$ *of the metric* (4.15.1) *is equal to*

$$K = -e^{-2\varphi}\Delta\varphi \tag{4.15.2}$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

is the Laplace operator.

Proof: It readily follows from the formula (4.8.24). $\square$

Thus, for a metric of constant curvature K the function $\varphi = \varphi(u, v)$ in (4.15.1) must satisfy the following *Liouville equation*

$$\Delta\varphi = -K\, e^{2\varphi}. \tag{4.15.3}$$

It is convenient to introduce complex variables

$$z = u + iv, \quad \bar{z} = u - iv, \quad \text{so that} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \partial\bar{z} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

to recast the Liouville equation into the complex form

$$\frac{\partial^2\varphi}{\partial z\partial\bar{z}} = -\frac{K}{4}e^{2\varphi}. \tag{4.15.4}$$

Our goal is to describe the general solution to the Liouville equation for an arbitrary constant K.

Lemma 4.15.4 *Let* $\varphi = \varphi\,(z, \bar{z})$ *be a solution to the Liouville equation. Then the function*

$$\omega = \frac{\partial^2\varphi}{\partial z^2} - \left(\frac{\partial\varphi}{\partial z}\right)^2 \tag{4.15.5}$$

is holomorphic,

$$\frac{\partial\omega}{\partial\bar{z}} = 0. \tag{4.15.6}$$

Proof: We have

$$\frac{\partial\omega}{\partial\bar{z}} = \frac{\partial^3\varphi}{\partial z^2\partial\bar{z}} - 2\frac{\partial\varphi}{\partial z}\frac{\partial^2\varphi}{\partial z\partial\bar{z}} = \frac{\partial}{\partial z}\left(-\frac{K}{4}e^{2\varphi}\right) - 2\left(-\frac{K}{4}e^{2\varphi}\right)\frac{\partial\varphi}{\partial z} = 0.$$

$\square$

Let us investigate dependence of the holomorphic function ω on the choice of conformal coordinates.

Lemma 4.15.5 *Under a holomorphic changes of coordinates*

$$z \mapsto z' = f(z), \quad \frac{\partial f}{\partial \bar{z}} = 0$$

the function $\omega(z)$ transforms according to the following rule

$$\omega'(z') = \frac{1}{f_z^2}\left[\omega(z) - \frac{1}{2} S_z f\right] \tag{4.15.7}$$

where

$$S_z f = \frac{f_{zzz}}{f_z} - \frac{3}{2}\left(\frac{f_{zz}}{f_z}\right)^2 \tag{4.15.8}$$

is the Schwarzian derivative *of the holomorphic function $f(z)$.*

Proof: Differentiating the transformation rule

$$\varphi' = \varphi - \frac{1}{2}\left(\log f_z + \overline{\log f_z}\right)$$

(see eq. (4.13.12) above) in z' obtain

$$\frac{\partial \varphi'}{\partial z'} = \frac{1}{f_z}\frac{\partial}{\partial z}\left[\varphi - \frac{1}{2}\left(\log f_z + \overline{\log f_z}\right)\right] = \frac{1}{f_z}\left[\varphi_z - \frac{1}{2}\frac{f_{zz}}{f_z}\right].$$

Differentiating once more yields

$$\frac{\partial^2 \varphi'}{\partial z'^2} = \frac{1}{f_z^2}\left[\varphi_{zz} - \frac{1}{2}\frac{f_{zzz}}{f_z} + \frac{1}{2}\frac{f_{zz}^2}{f_z^2}\right] - \frac{f_{zz}}{f_z^3}\left[\varphi_z - \frac{1}{2}\frac{f_{zz}}{f_z}\right].$$

Hence

$$\frac{\partial^2 \varphi'}{\partial z'^2} - \left(\frac{\partial \varphi'}{\partial z'}\right)^2 = \frac{1}{f_z^2}\left[\frac{\partial^2 \varphi}{\partial z^2} - \left(\frac{\partial \varphi}{\partial z}\right)^2 - \frac{1}{2}\left(\frac{f_{zzz}}{f_z} - \frac{3}{2}\left(\frac{f_{zz}}{f_z}\right)^2\right)\right].$$

$\square$

We want to choose a particular system of conformal coordinates in order to kill the holomorphic function ω. To this end we have to solve the *Schwarz equation*

$$S_z f = 2\omega(z) \tag{4.15.9}$$

for the unknown holomorphic change of coordinates.

Lemma 4.15.6 *Let $\psi_1(z)$, $\psi_2(z)$ be two linearly independent solutions to the (complex) Sturm–Liouville equation*

$$\frac{\partial^2 \psi}{\partial z^2} + \omega(z)\psi = 0 \tag{4.15.10}$$

such that $\psi_1(z_0) \neq 0$. Then the function

$$f(z) = \frac{\psi_2(z)}{\psi_1(z)} \tag{4.15.11}$$

satisfies the Schwarz equation (4.15.9). *Conversely, any solution to* (4.15.9) *holomorphic on some neighbourhood of z_0 can be represented in such a form.*

Proof: In this proof we will use short notations

$$\psi' = \frac{d\psi}{dz}, \psi'' = \frac{d^2\psi}{z^2}$$

etc. Differentiating the ratio (4.15.11) obtain

$$f_z = \frac{\psi_2'\psi_1 - \psi_2\psi_1'}{\psi_1^2} = \frac{C}{\psi_1^2}$$

where we use the well-known fact saying that the Wronskian $\psi_2'\psi_1 - \psi_2\psi_1'$ of two linearly independent solutions to the Sturm–Liouville equation (4.15.10) is a nonzero constant C. Furthermore,

$$f_{zz} = -\frac{2C\psi_1'}{\psi_1^3}$$

$$f_{zzz} = -\frac{2C\psi_1''}{\psi_1^3} + \frac{6C{\psi_1'}^2}{\psi_1^4} = \frac{2C\omega}{\psi_1^2} + \frac{6C{\psi_1'}^2}{\psi_1^4}.$$

Thus

$$\frac{f_{zzz}}{f_z} = 2\omega + 6\left(\frac{\psi_1'}{\psi_1}\right)^2, \quad \frac{f_{zz}}{f_z} = -2\frac{\psi_1'}{\psi_1}.$$

This implies eq. (4.15.9) for the function (4.15.11).

The converse statement can be proved just by counting the number of arbitrary constants for solutions to the equations (4.15.9), (4.15.10). Indeed, every solution ψ_1, ψ_2 to the Sturm–Liouville equation depends on two arbitrary constants. However their ratio ψ_2/ψ_1 depends on three arbitrary constants as one common factor in the numerator/denominator disappears. On the other side the general solution to the third order equation (4.15.9) depends on three arbitrary constants. This completes the proof of the Lemma. □

Example 4.15.7 *Consider the homogeneous Schwarz equation*

$$S_z f = 0.$$

According to the Lemma its solutions have the form

$$f(z) = \frac{az+b}{cz+d}, \quad a,\, b,\, c,\, d \in \mathbb{C}, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

Indeed, in this case the Sturm–Liouville equation (4.15.10) *reduces to* $d^2\psi/dz^2 = 0$. *Its solutions have the form* $\psi_1 = cz + d$, $\psi_2 = az + b$.

The last step in proving the main Theorem of this section is the following

Lemma 4.15.8 *For any analytic two-dimensional metric of constant curvature there exists a system of conformal coordinates such that*

$$ds^2 = \frac{dz\, d\bar z}{\left[a\, z\, \bar z + b\, \bar z + \bar b\, z + c\right]^2} \tag{4.15.12}$$

for some constants

$$a,\, c \in \mathbb{R}, \quad b \in \mathbb{C}, \quad a^2 + c^2 + |b|^2 \neq 0.$$

Proof: From the above arguments it follows existence of a system of conformal coordinates

$$ds^2 = e^{2\varphi} dz\, d\bar z$$

such that the function $\varphi = \varphi(z, \bar z)$ satisfies equation

$$\varphi_{zz} - \varphi_z^2 = 0.$$

Observe that

$$\frac{\partial^2}{\partial z^2} e^{-\varphi} = \left[\varphi_{zz} - \varphi_z^2\right] e^{-\varphi} = 0.$$

Hence the function $e^{-\varphi}$ depends at most linearly on z. Since this function is real-valued it also satisfies

$$\frac{\partial^2}{\partial \bar z^2} e^{-\varphi} = \overline{\frac{\partial^2}{\partial z^2} e^{-\varphi}} = 0.$$

So, it dependence on $\bar z$ is also at most linear. Hence

$$e^{-\varphi} = a\, z\, \bar z + b\, \bar z + \bar b\, z + c.$$

□

End of the proof of the Theorem. Computing the curvature of the metric (4.15.12) with the help of the formula

$$K = -4 e^{-2\varphi} \frac{\partial^2 \varphi}{\partial z \partial \bar z}$$

obtain

$$K = 4\left(a\, c - |b|^2\right). \tag{4.15.13}$$

Let us first consider the case of zero curvature, $K = 0$. If $a = 0$ then $b = 0$, so $c \neq 0$. The metric takes the Euclidean form

$$ds^2 = \frac{dz\, d\bar z}{c^2}.$$

In the case $a \neq 0$ one has $c = \frac{|b|^2}{a}$, so we can factorise

$$a\, z\, \bar z + b\, \bar z + \bar b\, z + c = a\left(z + \frac{b}{a}\right)\left(\bar z + \frac{\bar b}{a}\right).$$

The substitution

$$w = \frac{1}{z + \frac{b}{a}}$$

reduces the metric to the Euclidean form

$$ds^2 = \frac{dw\, d\bar w}{a^2}.$$

Consider now the case of positive curvature $K > 0$. From (4.15.13) it follows $a \neq 0$. Write

$$a\, z\, \bar z + b\, \bar z + \bar b\, z + c = a\left(z + \frac{b}{a}\right)\left(\bar z + \frac{\bar b}{a}\right) + \frac{K}{4a}.$$

So, after the substitution

$$w = \frac{\sqrt{K}}{2|a|}\left(z + \frac{b}{a}\right)$$

one obtains the metric of sphere (4.14.1) of the radius $R = K^{-1/2}$,

$$ds^2 = \frac{4}{K}\frac{dw\, d\bar{w}}{(1 + |w|^2)^2}.$$

In the case of negative curvature, let us first assume that $a \neq 0$. Then a similar substitution

$$w = \frac{\sqrt{-K}}{2|a|}\left(z + \frac{b}{a}\right)$$

yields the metric of pseudosphere (4.14.2) with $R = (-K)^{-1/2}$,

$$ds^2 = -\frac{4}{K}\frac{dw\, d\bar{w}}{(1 - |w|^2)^2}.$$

In the case $a = 0$ we have $|b| = \frac{1}{2}\sqrt{-K}$, so $b = \rho e^{i\alpha}$ for some $\alpha \in \mathbb{R}$ where we denote $\rho = \frac{1}{2}\sqrt{-K}$. After the substitution

$$z = \rho e^{i\alpha}\left[i\, w - \frac{c}{2\rho^2}\right]$$

obtain

$$\bar{b}z + b\bar{z} + c = i\,\rho^2(w - \bar{w})$$

that gives the metric of pseudosphere in the form (4.14.4)

$$ds^2 = -\frac{1}{\rho^2}\frac{dw\, d\bar{w}}{(w - \bar{w})^2} = \frac{4}{K}\frac{dw\, d\bar{w}}{(w - \bar{w})^2}.$$

The Theorem is proved. □

4.16 Differential geometry versus topology: Gauss–Bonnet formula and Gauss map

One of the main problem in the theory of smooth manifolds is the problem of classification. An approach to this problem is based on constructing invariants of smooth manifolds, i.e., numerical characteristics taking the same values for diffeomorphic manifolds. In particular, one may look for a construction of such topological invariants in terms of differential geometric structures on the manifolds. One of the simplest example of such invariant is given by the following version of the Gauss–Bonnet theorem.

Theorem 4.16.1 (Gauss–Bonnet) *Let M be a compact connected oriented two-dimensional Riemannian manifold. Denote dA the area element on the manifold and $R = R(x)$ the scalar curvature of the manifold at the point $x \in M$. Then*

1) the quantity

$$\chi(M) := \frac{1}{4\pi}\int_M R\, dA \tag{4.16.1}$$

does not depend on the Riemannian metric on M;

2) If M admits an embedding[10] *in $\mathbb{R}^3$ then $\chi(M)$ is an integer.*

Proof: Our first step will be to prove that the quantity (4.16.1) does not depend on the metric. Consider a more general setting of a n-dimensional compact connected oriented Riemannian manifold. Denote

$$dV = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n, \quad g = \det(g_{ij})$$

the volume form on M and R, as above, the scalar curvature of the Riemannian metric. For a given manifold M the integral

$$S[g_{ij}] := \int_M R\sqrt{g}\, d^n x \tag{4.16.2}$$

can be considered as a functional of the metric on the manifold. (Here and below we use short notation $d^n x = dx^1 \wedge \cdots \wedge dx^n$.) Our goal is to investigate the dependence of this functional on the metric. To this end let us compute the principal linear part of the increment

$$\delta S = S[g_{ij} + \delta g_{ij}] - S[g_{ij}] \text{ (modulo } |\delta g_{ij}|^2) \tag{4.16.3}$$

called *first variation* of the functional S. Instead of the variation δg_{ij} it is convenient to use the variation of the inverse matrix δg^{ij} related to δg_{ij} by the obvious formula

$$\delta g^{ij} = -g^{ik}\delta g_{kl}\, g^{lj} \tag{4.16.4}$$

(cf. the formula

$$\frac{d}{dx}A^{-1} = -A^{-1}\frac{dA}{dx}A^{-1}$$

for the derivative of the inverse of a matrix-valued function $A = A(x)$).

Lemma 4.16.2 (Hilbert) *The first variation of the functional* (4.16.2) *is given by the following formula*

$$\delta S = \int_M \left(R_{ij} - \frac{1}{2}R\, g_{ij}\right)\delta g^{ij}\sqrt{g}\, d^n x. \tag{4.16.5}$$

Here R_{ij} is the Ricci tensor of the metric.

Proof: Recall that the scalar curvature is the contraction

$$R = g^{ij}R_{ij}$$

of the Ricci tensor

$$R_{ij} = R^k_{ikj} = \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{kj}}{\partial x^i} + \Gamma^k_{ks}\Gamma^s_{ij} - \Gamma^k_{is}\Gamma^s_{kj}$$

[10] Actually, any smooth compact oriented two-dimensional manifold does admit an embedding into $\mathbb{R}^3$. Moreover, any compact two-dimensional submanifold in $\mathbb{R}^3$ is orientable. The proof of these statements goes beyond the scope of this course.

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{ks} \left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s} \right).$$

The variation δS can be written as a sum of three terms

$$\delta \int_M R_{ij} g^{ij} \sqrt{g}\, d^n x = \int_M \left[g^{ij} \delta R_{ij} + R_{ij} \delta g^{ij} + R \frac{\delta g}{2g} \right] \sqrt{g}\, d^n x. \tag{4.16.6}$$

Let us begin with the third term in this expression. Denote $G = (g_{ij})$, so $g = \det G$. Therefore

$$\begin{aligned}
&\delta g = \det[G + \delta G] - \det G = \det \left[G \left(\mathbf{1} + G^{-1} \delta G \right) \right] - \det G \\
&= \det G \cdot \det \left[\mathbf{1} + G^{-1} \delta G \right] - \det G \\
&= \det G \cdot \operatorname{tr} \left(G^{-1} \delta G \right) + \mathcal{O} \left(|\delta G|^2 \right) = - \det G \cdot \operatorname{tr} \left(\delta G^{-1}\, G \right) + \mathcal{O} \left(|\delta G|^2 \right) = -g \cdot g_{ij} \delta g^{ij}.
\end{aligned}$$

In this caluclation we have used the following formula for determinant of a square matrix close to identity

$$\det(\mathbf{1} + A) = 1 + \operatorname{tr} A + \mathcal{O} \left(|A|^2 \right)$$

and also the formula for the derivatives of the inverse function

$$G^{-1} \cdot \delta G = -\delta G^{-1} \cdot G.$$

We see that the contributions of the second and third terms in eq. (4.16.6) give exactly the right hand side of the needed formula (4.16.5). So, it remains to prove that the first term

$$\begin{aligned}
&\int_M g^{ij} \delta R_{ij} \sqrt{g}\, d^n x \qquad\qquad (4.16.7)\\
&= \int_M g^{ij} \left(\frac{\partial}{\partial x^k} \delta \Gamma_{ij}^k - \frac{\partial}{\partial x^i} \delta \Gamma_{kj}^k + \delta \Gamma_{ks}^k \Gamma_{ij}^s + \Gamma_{ks}^k \delta \Gamma_{ij}^s - \delta \Gamma_{is}^k \Gamma_{kj}^s - \Gamma_{is}^k \delta \Gamma_{kj}^s \right) \sqrt{g}\, d^n x
\end{aligned}$$

vanishes.

The strategy is to apply to the above expression the Stokes formula written in the form (4.8.10). The crucial observation is that, the variation $\delta \Gamma_{ij}^k$ of the Christoffel coefficients is a tensor of type $(1, 2)$. Indeed, from (4.6.10) it readily follows that the difference $\tilde{\Gamma}_{ij}^k - \Gamma_{ij}^k$ of two affine connections under a change of coordinates transforms according to the tensor law (the terms with the second derivatives cancel).

For covariant derivatives of the tensor $\delta \Gamma_{ij}^k$ we have the standard formula

$$\nabla_k \delta \Gamma_{ij}^k = \frac{\partial}{\partial x^k} \delta \Gamma_{ij}^k + \Gamma_{ks}^k \delta \Gamma_{ij}^s - \Gamma_{is}^k \delta \Gamma_{kj}^s - \Gamma_{kj}^s \delta \Gamma_{is}^k.$$

Moreover, the contraction $\delta \Gamma_{kj}^k$ is a tensor of type $(0, 1)$. The covariant derivative of this tensor reads

$$\nabla_i \delta \Gamma_{kj}^k = \frac{\partial}{\partial x^i} \delta \Gamma_{kj}^k - \Gamma_{ij}^s \delta \Gamma_{ks}^k.$$

So, the expression (4.16.7) can be recast into the form

$$\int_M g^{ij} \delta R_{ij} \sqrt{g}\, d^n x = \int_M g^{ij} \left(\nabla_k \delta \Gamma_{ij}^k - \nabla_i \delta \Gamma_{kj}^k \right) \sqrt{g}\, d^n x.$$

The last step is to observe that

$$g^{ij}\nabla_k\delta\Gamma^k_{ij} = \nabla_k\left(g^{ij}\delta\Gamma^k_{ij}\right), \quad g^{ij}\nabla_i\delta\Gamma^k_{kj} = \nabla_i\left(g^{ij}\delta\Gamma^k_{kj}\right)$$

since

$$\nabla g^{ij} = 0$$

for the Levi-Civita connection. Introduce two vector fields

$$X^k = g^{ij}\delta\Gamma^k_{ij}, \quad Y^i = g^{ij}\delta\Gamma^k_{kj}.$$

Then the expression (4.16.7) takes the following form

$$\int_M g^{ij}\delta R_{ij}\sqrt{g}\,d^nx = \int_M\left(\nabla_k X^k - \nabla_i Y^i\right)\sqrt{g}\,dx = \int_{\partial M}\sum_{i=1}^n (X^i - Y^i)\sqrt{g}\,dx^1\wedge\dots\hat{dx^i}\dots\wedge dx^n = 0$$

since the boundary of the manifold M is empty. $\square$

Remark 4.16.3 *The first variation formula* (4.16.5) *remains true also for noncompact manifolds provided the variation of the metric vanishes together with its two derivatives outside a compact domain.*

Remark 4.16.4 *The formula* (4.16.5) *makes sense also for pseudo-Riemannian manifolds; it suffices just to replace* $\sqrt{g}$ *with* $\sqrt{|g|}$. *In particular an analogue of the least action principle for the Einstein equations in vacuum*

$$R_{ij} - \frac{1}{2}R\,g_{ij} = 0, \quad i,\, j = 1, \dots, 4 \tag{4.16.8}$$

readily follows from the Hilbert formula applied to a pseudo-Riemannian manifold of signature $(1,3)$.

Let us return to the theorem. We know that, for two-dimensional Riemannian manifolds $R_{ij} = \frac{1}{2}R\,g_{ij}$. Therefore $\delta S \equiv 0$. It is now easy to derive independence from the metric of the integral

$$S[g_{ij}] = \int_{M^2} R\sqrt{g}\,d^2x.$$

Indeed, according to Lemma 4.1.10 the space of positive definite quadratic forms on a linear space is a convex cone. That is, given two Riemannian metrics $g_{ij}(x)$ and $\tilde{g}_{ij}(x)$ on M the linear combination

$$g_{ij}(x;t) = (1-t)\,g_{ij}(x) + t\,\tilde{g}_{ij}(x), \quad t \in [0,1] \tag{4.16.9}$$

is a Riemannian metric. Consider the value of the functional $S[g_{ij}]$ on the metric (4.16.9). Differentiation with respect to the parameter t yields

$$\frac{d}{dt}\int_M R\sqrt{g}\,d^nx = -\int_M\left(R_{ij} - \frac{1}{2}R\,g_{ij}\right)g^{ik}(x,t)[\tilde{g}_{kl}(x) - g_{kl}(x)]g^{lj}(x;t)\sqrt{\det(g_{pq}(x,t))}\,d^nx$$

due to the Hilbert formula. For a two-dimensional manifold the right hand side vanishes.

We will now assume that the two-dimensional manifold can be embedded into $\mathbb{R}^3$. According to the above arguments it suffices to compute the value of the functional (4.16.1) for the induced metric on M. Consider the *Gauss map*

$$\begin{aligned} \mathbf{n} : M &\to S^2 \\ x &\mapsto \mathbf{n}(x) \end{aligned} \tag{4.16.10}$$

that assigns to a point $x \in M$ a unit normal vector $\mathbf{n}(x)$ at this point. Denote dA_{sphere} the standard area element on the unit sphere S^2. Applying the Theorem 3.4.11 one obtains

$$\int_M \mathbf{n}^* dA_{\text{sphere}} = 4\pi \deg \mathbf{n}$$

since the total area of the unit sphere is equal to 4π.

Lemma 4.16.5 *The pullback of the area element of the unit sphere with respect to the Gauss map is equal to*

$$\mathbf{n}^* dA_{\text{sphere}} = K\, dA \tag{4.16.11}$$

where dA is the area element and K is the Gaussian curvature of the surface.

Proof: Without loss of generality one can assume that the surface is represented as a graph of function $z = f(x, y)$. The area element of the surface is given by the wellknown formula

$$dA = \sqrt{1 + f_x^2 + f_y^2} dx \wedge dy$$

(see Exercise 4.1.15 above). Then the unit normal vector is given by the formula

$$\mathbf{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}.$$

Also the unit sphere

$$u^2 + v^2 + w^2 = 1$$

near the north pole will be written as a graph

$$w = \sqrt{1 - u^2 - v^2}.$$

The area element of the sphere in these coordinates reads

$$dA_{\text{sphere}} = \frac{du \wedge dv}{\sqrt{1 - u^2 - v^2}} = \frac{du \wedge dv}{w}.$$

In these coordinates the Gauss map reads

$$\begin{aligned} u(x, y) &= -\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}} \\ v(x, y) &= -\frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}}. \end{aligned}$$

Also observe a formula for the third coordinate on the sphere

$$w(x, y) = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}.$$

So

$$\mathbf{n}^* dA_{\text{sphere}} = \frac{du(x, y) \wedge dv(x, y)}{w(x, y)} = J\,\sqrt{1 + f_x^2 + f_y^2}\, dx \wedge dy = J\, dA$$

where J is the Jacobian of the Gauss map (4.16.10)

$$J = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

A simple computation of the Jacobian gives

$$J = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

This formula coincides with the expression (4.11.36) for the Gaussian curvature. □

We obtained that the integer number $\chi(M)$ is equal to twice the degree of the Gauss map. The Theorem is proved. □

It can be computed for a particular embedding of a sphere with g handles shown at the picture. In this case the north pole has $g + 1$ preimages. At the upper point of the surface the Gaussian curvature is positive, so the local degree is equal to $+1$. At other g points the Gaussian curvature is negative, so the local degrees at all these points are equal to -1. Thus the degree of the Gauss map for such an embedding is equal to $(1 - g)$. We arrive at the following final form of the Gauss–Bonnet formula for spheres with g handles

$$\frac{1}{2\pi} \int_M K\, dA = 2 - 2g. \tag{4.16.12}$$

The number in the right hand side is called the *Euler characteristic* of the surface M. It is one of the simplest topological invariants of smooth manifolds. The formula (4.16.12) represents the simplest issue of a relationship between differential geometric and topological characteristics of smooth manifolds. There are many other examples of this deep connection. However, the discussion of these examples goes beyond the scope of this course.

4.17 Second variation in the theory of geodesics

In this section we will address the problem of minimality of a *given* geodesic. To this end we will derive a formula for the second variation of the action functional (4.9.3).

Let us first explain the idea for smooth functions on a finite dimensional case. Let $f = f(x)$ be a function on $\mathbb{R}^N$. To find a (local) minimum of this function one has to

- find critical points of this function solving equations

$$\frac{\partial f(x)}{\partial x^i} = 0, \quad i = 1, \dots, N;$$

• for a given critical point x_0 check that the matrix of second derivatives

$$\frac{\partial^2 f(x_0)}{\partial x^i \partial x^j} \tag{4.17.1}$$

is positive definite.

In order to reformulate these two sufficient conditions of a local minimum in a coordinate-free way one can rewrite a definition of a critical point in the form

$$\frac{d}{d\tau} f(x_0 + \tau v)_{\tau=0} = 0 \quad \text{for any vector} \quad v.$$

The symmetric bilinear form associated with the matrix (4.17.1) can be represented as follows

$$d^2 f(x_0)(v_1, v_2) = \frac{\partial^2}{\partial \tau_1 \partial \tau_2} f(x_0 + \tau_1 v_1 + \tau_2 v_2)_{\tau_1=\tau_2=0}. \tag{4.17.2}$$

The following expression for the Taylor expansion of the function $f(x_0 + \tau\, v)$ at the critical point x_0 will be useful

$$f(x_0 + \tau\, v) = f(x_0) + \tau^2 d^2 f(x_0)(v, v) + \mathcal{O}\left(\tau^3\right). \tag{4.17.3}$$

The Hessian is degenerate if there exists a vector $v_1 \neq 0$ such that

$$d^2 f(x_0)(v_1, v_2) = 0 \quad \text{for any vector} \quad v_2. \tag{4.17.4}$$

In this case the first differential of the function f vanishes along the direction of the vector v_1, up to corrections of a higher order

$$df(x_0 + \tau\, v_1) = \mathcal{O}\left(\tau^2\right). \tag{4.17.5}$$

The formula (4.17.5) can be interpreted as an "infinitesimal deformation" of the critical point. In the opposite case of non-degenerateness of the second differential the critical point x_0 is isolated. It is a minimum *iff* the quadratic form associated with (4.17.2) is positive definite

$$d^2 f(x_0)(v, v) > 0 \quad \text{for any} \quad v \neq 0. \tag{4.17.6}$$

We will now consider an infinite dimensional analogue of the above considerations. For a given pair of points x_0, x_1 in a Riemannian manifold M consider the space of smooth curves $\gamma : [a, b] \to M$ such that $\gamma(a) = x_0$, $\gamma(b) = x_1$. The action functional

$$S[\gamma] = \frac{1}{2} \int_a^b \langle \dot\gamma, \dot\gamma \rangle \, dt$$

can be considered as a "function" on the infinite-dimensional space of curves with fixed endpoints. Tangent vectors to this space can be realized as smooth vector fields $v = v(t)$ at the points $\gamma(t)$ vanishing at the endpoints

$$v(a) = v(b) = 0. \tag{4.17.7}$$

With any such a vector field one associates a small deformation $\gamma_\tau(t)$ of the curve of the form

$$\gamma_\tau(t) := \exp_{\gamma(t)}\left(\tau\, v(t)\right), \quad |\tau| < \epsilon \tag{4.17.8}$$

for some positive ϵ (see the previous section for details about exponential map). To simplify notations we will redenote $\gamma + \tau\, v$ the curve (4.17.8).

The formula for the first variation (an analogue of the first differential of a function) of the previous section (see Lemma 4.9.2) says that

$$\delta S[\gamma](v) := \frac{d}{d\tau} S[\gamma + \tau\, v]_{\tau=0} = -\int_a^b \langle \nabla_{\dot\gamma}\dot\gamma, v\rangle\; dt. \tag{4.17.9}$$

Exercise 4.17.1 *Prove the following formula generalizing* (4.17.9) *for piecewise smooth curves*

$$\delta S[\gamma](v) = -\int_a^b \langle \nabla_{\dot\gamma}\dot\gamma, v\rangle\; dt - \sum_k \Delta_{t_k} \langle \dot\gamma, v.\rangle \tag{4.17.10}$$

In this formula Δ_t stands for the jump of a piecewise continuous function at the point t, i.e.,

$$\Delta_t f = f(t+0) - f(t-0). \tag{4.17.11}$$

The summation is taken over all points t_k of discontinuity of $\dot\gamma(t)$.

From the first variation formula it follows, as we already know, that the "critical points" of the action functional are geodesics. We will now show that *minimal geodesics* provide *minima* for the action functional.

Lemma 4.17.2 *Let $\gamma : [0,1] \to M$ be a minimal geodesic between the points $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$. Then, for any piecewise smooth curve $\tilde\gamma : [0,1] \to M$ connecting x_0 with x_1 one has*

$$S[\gamma] \leq S[\tilde\gamma],$$

the equality takes place iff *$\tilde\gamma$ is a minimal geodesic of the same length $s[\tilde\gamma] = s[\gamma]$.*

Proof: We will use the Schwarz inequality

$$\left(\int_0^1 f(t)g(t)\, dt\right)^2 \leq \int_0^1 f^2(t)\, dt \cdot \int_0^1 g^2(t)\, dt$$

valid for arbitrary piecewise continuous functions $f(t)$, $g(t)$, $0 \leq t \leq 1$. The equality takes place *iff* $f(t) \equiv c\, g(t)$ for some constant c. Applying this inequality to the case $f(t) = |\dot\gamma(t)|$, $g(t) \equiv 1$ one obtains

$$(s[\gamma])^2 \leq 2S[\gamma], \tag{4.17.12}$$

the equality takes place *iff* the parameter along the curve γ is proportional to the arc length. Let $\gamma(t)$ be a minimal geodesic from x_0 to x_1 that is, for an arbitrary piecewise smooth curve $\tilde\gamma$ connecting the same points one has $s[\gamma] \leq s[\tilde\gamma]$. Using (4.17.12) we obtain

$$S[\gamma] = \frac{1}{2}(s[\gamma])^2 \leq \frac{1}{2}(s[\tilde\gamma])^2 \leq S[\tilde\gamma].$$

□

Let us proceed with analyzing minimality of geodesics as critical points of the action functional. We will now derive a formula for the *second variation* of the action functional

$$\delta^2 S[\gamma](v_1, v_2) := \frac{\partial^2}{\partial\tau_1 \partial\tau_2} S[\gamma + \tau_1 v_1 + \tau_2 v_2]_{\tau_1=\tau_2=0} \tag{4.17.13}$$

for a geodesic γ and a pair of vector fields $v_1 = v_1(t)$, $v_2 = v_2(t)$ vanishing at the endpoints of γ.

Theorem 4.17.3 *The second variation of the action functional on a geodesic $\gamma : [a, b] \to M$ is given by the following formula*

$$\delta^2 S[\gamma](v_1, v_2) = -\int_a^b \langle J v_1, v_2 \rangle \, dt \tag{4.17.14}$$

where the operator J acting on vector fields along the geodesic is given by the following formula

$$Jv = \nabla^2_{\dot\gamma} v + R(\dot\gamma, v)\dot\gamma. \tag{4.17.15}$$

In the formula (4.17.15) $R(X, Y) : T_x M \to T_x M$ is the curvature operator defined for any pair of vectors at a given point $x \in M$. The second order linear differential operator (4.17.15) is called *Jacobi operator*.

Proof: Using formula (4.17.10) for the first variation obtain

$$\frac{\partial^2}{\partial\tau_1 \partial\tau_2} S[\gamma + \tau_1 v_1 + \tau_2 v_2]_{\tau_2=0} = -\frac{\partial}{\partial\tau_1} \int_a^b \left\langle \nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}, v_2 \right\rangle dt, \quad \text{where} \quad \tilde\gamma = \gamma + \tau_1 v_1.$$

So

$$\frac{\partial^2}{\partial\tau_1 \partial\tau_2} S[\gamma + \tau_1 v_1 + \tau_2 v_2]_{\tau_1=\tau_2=0} = -\int_a^b \nabla_{v_1} \langle \nabla_{\dot\gamma}\dot\gamma, v_2 \rangle \, dt$$
$$= -\int_a^b \left[\langle \nabla_{v_1} \nabla_{\dot\gamma}\dot\gamma, v_2 \rangle + \langle \nabla_{\dot\gamma}\dot\gamma, \nabla_{v_1} v_2 \rangle \right] dt = -\int_a^b \langle \nabla_{v_1} \nabla_{\dot\gamma}\dot\gamma, v_2 \rangle \, dt$$

since $\nabla_{\dot\gamma}\dot\gamma = 0$ on a geodesic. Using the definition of the curvature tensor we obtain

$$\nabla_{v_1}\nabla_{\dot\gamma} - \nabla_{\dot\gamma}\nabla_{v_1} = R(\dot\gamma, v_1).$$

So

$$\delta^2 S[\gamma](v_1, v_2) = -\int_a^b \langle \nabla_{\dot\gamma}\nabla_{v_1}\dot\gamma + R(\dot\gamma, v_1)\dot\gamma, v_1 \rangle \, dt.$$

Using symmetry of the Levi-Civita connection we have

$$\nabla_{v_1}\dot\gamma = \nabla_{\dot\gamma} v_1.$$

We arrive at the needed formula. □

Remark 4.17.4 *Using the symmetry* (4.8.19) *of the Riemann curvature tensor it is easy to check the symmetry of the Jacobi operator*

$$\int_a^b \langle J v_1, v_2 \rangle \, dt = \int_a^b \langle v_1, J v_2 \rangle \, dt.$$

Exercise 4.17.5 *Prove the following generalization of the second variation formula valid for piecewise smooth vector fields on the geodesic* γ

$$\delta^2 S[\gamma](v_1, v_2) = -\int_a^b \langle Jv_1, v_2 \rangle \, dt - \sum \Delta_{t_k} \langle \nabla_{\dot\gamma} v_1, v_2 \rangle \, . \tag{4.17.16}$$

The summation is taken over all discontinuities $t = t_k$ *of the derivative of the vector field* $v_1(t)$.

Definition 4.17.6 *1) A* Jacobi vector field v *on a geodesic* $\gamma : [a, b] \to M$ *is a solution to the* Jacobi equation

$$Jv = 0 \tag{4.17.17}$$

vanishing at the endpoints

$$v|_{\gamma(a)} = v|_{\gamma(b)} = 0.$$

2) The points $x = \gamma(a)$ *and* $y = \gamma(b)$ *are* conjugate *along the geodesics* $\gamma : [a, b] \to M$ *if there exists a nonzero Jacobi field on* γ.

3) The dimension of the space of Jacobi vector fields on γ *is called the* multiplicity *of the conjugate points* $\gamma(a)$ *and* $\gamma(b)$*. It will be denoted* $\mathrm{mult}_\gamma(\gamma(a), \gamma(b))$*. We put* $\mathrm{mult}_\gamma(\gamma(a), \gamma(b)) = 0$ *if the points* $\gamma(a)$*,* $\gamma(b)$ *are notconjugate along* γ.

From the Theorem 4.17.3 we obtain

Corollary 4.17.7 *For a geodesic* $\gamma : [a, b] \to M$ *the bilinear form* $\delta^2 S[\gamma](v_1, v_2)$ *degenerates* iff *the endpoints* $\gamma(a)$*,* $\gamma(b)$ *are conjugate along* γ.

Proof: If $v_1 \neq 0$ is a Jacobi vector field along γ then, using the second variation formula, we obtain

$$\delta^2 S[\gamma](v_1, v_2) = -\int_a^b \langle Jv_1, v_2 \rangle \, dt = 0 \tag{4.17.18}$$

for any vector field v_2. Conversely, assume that, for some $v_1 \neq 0$, (4.17.18) holds true for any vector field v_2. Choose $v_2(t) = \lambda(t) J v_1(t)$ where $\lambda(t) \geq 0$ is a smooth function vanishing at $t = a$ and $t = b$ but different from identical zero. Then

$$\delta^2 S[\gamma](v_1, v_2) = \int_a^b \lambda(t) \langle Jv_1, Jv_1 \rangle \, dt \geq 0,$$

it vanishes only if $Jv_1 = 0$. $\square$

The Jacobi equation (4.17.17) can be considered as a system of $n = \dim M$ second order linear differential equations for a vector-valued function $v(t)$ defined on a geodesic $\gamma(t)$. It can be rewritten as a system of $2n$ first order linear differential equations for the vector-valued functions $v(t)$ and $\dot v(t) := \nabla_{\dot\gamma} v(t)$

$$\left.\begin{array}{rcl} \nabla_{\dot\gamma} v &=& \dot v \\ \nabla_{\dot\gamma} \dot v &=& -R(\dot\gamma, v)\dot\gamma \end{array}\right\} \tag{4.17.19}$$

A solution to this system is uniquely determined by initial conditions

$$\begin{array}{rcl} v(a) & = & v_0 \\ \dot v(a) & = & \dot v_0 \end{array}$$

for a given pair of n-dimensional vectors v_0, $\dot v_0$. Thus the space of solutions to the Jacobi equation has dimension $2n$.

Proposition 4.17.8 *For $(b-a)$ sufficiently small any Jacobi vector field along a geodesic $\gamma : [a,b] \to M$ identically vanishes.*

Proof: Choose a system of local coordinates near the point $\gamma(a)$ such that all Christoffel coefficients vanish at this point (see Corollary 4.9.13 above). In these coordinates

$$\nabla_{\dot\gamma} v|_{t=a} = \frac{dv(a)}{dt} = \dot v_0, \quad \nabla_{\dot\gamma} \dot v|_{t=a} = \frac{d\dot v(a)}{dt} = -R(\dot\gamma_0, v_0)\dot\gamma_0$$

where $\dot\gamma_0 := \dot\gamma|_{t=a}$. Expanding the vector-valued function $v(t) = \left(v^i(t)\right)$ in Taylor series near $t=a$ one obtains

$$v^i(t) = v_0^i + (t-a)\dot v_0^i - \frac12 (t-a)^2 \left[(R(\dot\gamma_0, v_0)\dot\gamma_0)^i + \frac{\partial \Gamma^i_{kl}(\gamma(a))}{\partial x^j} \dot\gamma_0^j \dot\gamma_0^k v_0^l \right] + \mathcal{O}\left((t-a)^3\right).$$

For the choice of initial data $v_0 = 0$ the above expansion specifies to

$$v(t) = (t-a)\dot v_0 + \mathcal{O}\left((t-a)^3\right).$$

For $\dot v_0 \neq 0$ such a vector-function does not vanish for sufficiently small $|t-a|$. $\square$

Corollary 4.17.9 *For a geodesic $\gamma : [a,b] \to M$ consider a linear map of the space of solutions $v = v(t)$ to the Jacobi equation* (4.17.17) *to the space of boundary values*

$$v \mapsto (v(a), v(b)) \in \mathbb{R}^{2n}.$$

For sufficiently small $|b-a|$ this map is an isomorphism of linear spaces.

As a sufficiently small piece of any geodesic is minimal we arrive at

Proposition 4.17.10 *For any point $x_0 \in M$ there exists $\epsilon > 0$ such that, for any geodesic $\gamma : [0,1] \to M$ starting at $x_0 = \gamma(0)$ of length less than ϵ the second variation defines a positive definite quadratic form, i.e.,*

$$\delta^2 S[\gamma](v,v) > 0 \tag{4.17.20}$$

for any nonzero piecewise smooth vector field $v = v(t)$ along γ vanishing at the endpoints

$$v(0) = v(1) = 0.$$

Proof: Since the geodesic between the points $\gamma(0)$ and $\gamma(1)$ of distance $< \epsilon$ for a sufficiently small ϵ is unique and minimal we have

$$S[\gamma + \tau\, v] \geq S[\gamma]$$

for sufficiently small τ, see Lemma 4.17.2 above. The equality takes place only for $v(t) \equiv 0$. Using the expansion

$$S[\gamma + \tau\, v] = S[\gamma] + \tau^2 \delta^2 S[\gamma](v, v) + \mathcal{O}\left(\tau^3\right) \tag{4.17.21}$$

(cf. (4.17.3)) we derive $\delta^2 S[\gamma](v, v) \geq 0$. Due to Corollary 4.17.9 the inequality is strict for a non-zero vector field. $\square$

Let us consider in more details solutions to Jacobi equation on two-dimensional Riemannian manifolds. Let $\gamma : [0, l] \to M^2$ be a geodesic parameterized by arc length, i.e., $|\dot\gamma| = 1$. On the two-dimensional manifold M^2 one can construct an orthogonal vector field $\dot\gamma^\perp$ of unit length along the geodesic. So, at every point $\gamma(t)$ one has an orthonormal frame $\left(\dot\gamma(t), \dot\gamma^\perp(t)\right)$ smoothly depending on $t \in [0, l]$. Represent a solution $v = v(t)$ to the Jacobi equation $Jv = 0$ as a linear combination of these vectors

$$v(t) = \varphi(t)\, \dot\gamma(t) + \psi(t)\, \dot\gamma^\perp(t) \tag{4.17.22}$$

for some smooth functions $\varphi(t)$, $\psi(t)$.

Proposition 4.17.11 *The Jacobi equation for the vector field* (4.17.22) *is equivalent to the following system of linear differential equations*

$$\ddot\varphi = 0 \tag{4.17.23}$$

$$\ddot\psi + K(\gamma(t))\psi = 0 \tag{4.17.24}$$

where $K(\gamma(t))$ is the Gaussian curvature of the manifold at the point $\gamma(t)$.

Proof: We have $\nabla_{\dot\gamma}\dot\gamma = 0$. It is easy to show that also $\nabla_{\dot\gamma}\dot\gamma^\perp = 0$. Thus

$$\nabla_{\dot\gamma} v = \dot\varphi\, \dot\gamma + \dot\psi\, \dot\gamma^\perp, \quad \nabla^2_{\dot\gamma} v = \ddot\varphi\, \dot\gamma + \ddot\psi\, \dot\gamma^\perp.$$

Substituting into the Jacobi equation one obtains

$$\ddot\varphi\, \dot\gamma + \ddot\psi\, \dot\gamma^\perp = -R(\dot\gamma, \varphi\, \dot\gamma + \psi \dot\gamma^\perp)\dot\gamma = -\psi\, R(\dot\gamma, \dot\gamma^\perp)\dot\gamma$$

since $R(\dot\gamma, \dot\gamma) = 0$ due to antisymmetry $R_{jikl} = -R_{ijkl}$ of the Riemann curvature tensor. It remains to prove that

$$R(\dot\gamma, \dot\gamma^\perp)\dot\gamma = K\, \dot\gamma^\perp.$$

Decompose the vector field $R(\dot\gamma, \dot\gamma^\perp)\dot\gamma$ as a linear combination of the basic vector fields

$$R(\dot\gamma, \dot\gamma^\perp)\dot\gamma = \alpha\, \dot\gamma + \beta\, \dot\gamma^\perp.$$

We have

$$\alpha = \left\langle R(\dot\gamma, \dot\gamma^\perp)\dot\gamma, \dot\gamma \right\rangle = 0$$

due to antisymmetry $R_{ijlk} = -R_{ijkl}$. Next,

$$\beta = \left\langle R(\dot\gamma, \dot\gamma^\perp)\dot\gamma, \dot\gamma^\perp \right\rangle.$$

Choose a system of coordinates (x^1, x^2) near the point $\gamma(t)$ such that, in these coordinates

$$\dot\gamma(t) = \frac{\partial}{\partial x^1}, \quad \dot\gamma^\perp(t) = \frac{\partial}{\partial x^2}.$$

In these coordinates

$$R(\dot\gamma(t), \dot\gamma^\perp(t))\dot\gamma(t) = R_{1212}(\gamma(t)) = K(\gamma(t))$$

(see eq. (4.11.34) above). □

We see that the theory of Jacobi vector fields in two dimensions reduces to the study of the Dirichlet boundary value problem for the Sturm–Liouville equation (4.17.24) with the Gaussian curvature as the potential. We will return to this study at the end of the next section. Here we consider two simple examples.

Example 4.17.12 *We already know that the Gaussian curvature of the sphere $S^2 \subset \mathbb{R}^3$ of radius R is equal to $\frac{1}{R^2}$. So, the Jacobi equations* (4.17.23), (4.17.24) *take the form*

$$\begin{array}{rcl} \ddot\varphi & = & 0 \\ \ddot\psi + \frac{1}{R^2}\psi & = & 0. \end{array}$$

Solutions to this system of ODEs vanishing at $t = 0$ have the form

$$\begin{array}{rcl} \varphi & = & a\,t \\ \psi & = & b\,\sin\frac{t}{R} \end{array}$$

for arbitrary constants a, b. The choice $a = 0$, $b \neq 0$ yields a Jacobi vector field vanishing at $t = 0$ and $t = \pi\,R$. Clearly this pair of conjugate points correspond to the pair of opposite poles on the sphere. Furthemore all points of the form $t = \pi\,k\,R$ for any positive integer k will be conjugate with the initial point $t = 0$.

Example 4.17.13 *In a similar way solutions to the Jacobi equations on the pseudosphere of radius R*

$$\begin{array}{rcl} \ddot\varphi & = & 0 \\ \ddot\psi - \frac{1}{R^2}\psi & = & 0 \end{array}$$

vanishing at the initial point $t = 0$ read

$$\varphi = a\,t, \quad \psi = b\,\sinh\frac{t}{R}.$$

Such a vector field for $b \neq 0$ has no other zeroes except for $t = 0$. So, any geodesic on the pseudosphere contains no conjugate points.

A somewhat more intuitive realization of Jacobi vector fields can be obtained in terms of *geodesic variations.* By definition a geodesic variation of a geodesic $\gamma(t)$, $t \in [0,1]$ is a family $\gamma(t,s)$ of geodesics smoothly depending on the parameter $s \in (-\epsilon, \epsilon)$ for sufficiently small $\epsilon > 0$ such that

$$\gamma(t,0) = \gamma(t) \quad \text{(the given geodesic)} \tag{4.17.25}$$

and

$$\gamma(0,s) \equiv \gamma(0), \quad \gamma(1,s) \equiv \gamma(1). \tag{4.17.26}$$

Proposition 4.17.14 *Given a geodesic variation $\gamma(t,s)$, the vector field*

$$v(t) := \frac{d}{ds}\gamma(t,s)_{s=0}$$

is a Jacobi vector field along the geodesic $\gamma(t)$.

Proof: By definition (4.17.25), (4.17.26) the vector field along the geodesic $\gamma(t)$ vanishes at the endpoints. Let us prove that it satisfies Jacobi equation. Denote $\dot\gamma = \frac{d}{dt}\gamma(t,s)$ the velocity vector of the geodesic $\gamma(t,s)$ for a given value of s. We have

$$\nabla_{\dot\gamma}\frac{d}{dt}\gamma(t,s) = 0 \quad \text{for any} \quad s.$$

Differentiating this equation in s and using the definition of the Riemann curvature tensor yields

$$0 = \nabla_s\nabla_t\dot\gamma = \nabla_t\nabla_s\dot\gamma + R(\dot\gamma,\gamma_s)\dot\gamma = \nabla^2_{\dot\gamma}\frac{\partial\gamma}{\partial s} + R(\dot\gamma,\gamma_s)\dot\gamma.$$

Setting $s = 0$ one arrives at the Jacobi equation for the vector field $v = \gamma_s(t,0)$. □

For the example of two-dimensional sphere one can construct a geodesic variation of the big circle geodesic connecting the pair of opposite points rotating the sphere around the line passing through these points. In this way one obtains the Jacobi vector field constructed in the Example 4.17.12.

In a similar way one obtains a $(n-1)$-dimensional space of Jacobi vector fields connecting the pair of opposite points on the standard n-dimensional sphere. Thus the multiplicity of such a pair of conjugate points is greater or equal than $(n-1)$.

Exercise 4.17.15 *Prove that any Jacobi vector field along a geodesic $\gamma(t)$ can be realized by the construction of Proposition 4.17.14 from a geodesic variation of γ.*

4.18 Index theorem

We begin this section with recalling some basic definitions from the theory of quadratic form. Given a quadratic form $Q(x)$, $x \in \mathbb{R}^N$, one can always find a basis such that Q reduces to the diagonal form

$$Q(x) = \frac{1}{2}\left(\lambda_1^2 x_1^2 + \cdots + \lambda_n x_n^2\right), \quad \lambda_i \neq 0.$$

The *negative inertia index* (or, simply, *index*) of the quadratic form is defined as the number of negative squares

$$\operatorname{ind} Q = \#\{\lambda_i < 0\}.$$

A coordinate-free formulation of index of a quadratic form can be given in the following way:

$$\operatorname{ind} Q = \max \dim \left\{ V \subset \mathbb{R}^N \quad \text{such that} \quad Q|_V \quad \text{is negative definite} \right\}. \tag{4.18.1}$$

Index of a critical point x_0 of a smooth function $f(x)$ is defined as the index of the quadratic form $d^2 f(x_0)$

$$\operatorname{ind}_{x_0} f = \operatorname{ind} d^2 f(x_0). \tag{4.18.2}$$

Recall that, if the critical point x_0 is a (local) minimum of the function $f(x)$ then $\operatorname{ind}_{x_0} f = 0$. If $\operatorname{ind}_{x_0} f > 0$ then the critical point x_0 is not a local minimum of the function $f(x)$.

Consider now the action functional defined on the infinite-dimensional space $\Omega(M, x_0, x_1)$ of piecewise smooth curves on a Riemannian manifold connecting two points x_0 and x_1. Fix a critical point of this functional, i.e., a geodesic $\gamma : [0,1] \to M$, $\gamma(0) = x_0$, $\gamma(1) = x_1$. The "tangent space" $T_\gamma \Omega(M, x_0, x_1)$ at the point γ by definition consists of all piecewise smooth vector fields $v(t)$ along γ vanishing at the endpoints, $v(0) = v(1) = 0$. The second variation $\delta^2 S[\gamma]$ defines a quadratic form on this space.

Definition 4.18.1 *Index of a geodesic γ is defined as the index of the second variation quadratic form $\delta^2 S[\gamma](v, v)$, $v \in T_\gamma \Omega(M, x_0, x_1)$.*

Our goal is to compute the index of this quadratic form.

Theorem 4.18.2 (Morse) *Index of a given geodesic $\gamma : [0,1] \to M$ is equal to the number of pairs of conjugate points $(\gamma(0), \gamma(t))$ for $0 < t < 1$ counted with their multiplicities*

$$\operatorname{ind} \delta^2 S[\gamma] = \sum_{0<t<1} \operatorname{mult}_\gamma \left(\gamma(0), \gamma(t)\right). \tag{4.18.3}$$

In particular the theorem implies that index of any geodesic is finite.

Proof: We first split the infinite-dimensional space $T_\gamma \Omega := T_\gamma \Omega(M, x_0, x_1)$ into a direct sum of two subspaces such that the first subspace is finite-dimensional and restriction of $\delta^2 S[\gamma]$ onto the second one is positive. To this end choose a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of the segment $[0,1]$ such that every piece $\gamma|_{[t_{i-1}, t_i]}$ is minimal. Denote

1) $T_\gamma \Omega(t_0, t_1, \dots, t_k) \subset T_\gamma \Omega$ the space of broken Jacobi fields $v(t)$ along γ such that $v(0) = v(1) = 0$ and $Jv = 0$ for $t \in [t_{i-1}, t_i]$ for all $i = 1, \dots, k$. Here J is the Jacobi operator (4.17.15). Due to Corollary 4.17.9 we can assume that the vector field $v(t)$ is uniquely determined by the vectors $v(t_0) = 0$, $v(t_1)$, $\dots$, $v(t_{k-1})$, $v(t_k) = 0$. Thus $T_\gamma \Omega(t_0, t_1, \dots, t_k)$ is a finite-dimensional space of dimension $n \cdot (k-1)$ where $n = \dim M$.

2) $T_\gamma^\perp \Omega \subset T_\gamma \Omega$ the space of vector fields $w(t)$ along γ such that $w(t_i) = 0$, $i = 0, 1, \dots, k$. Observe that, if a vector field $w \in T_\gamma^\perp \Omega$ satisfies Jacobi equation $Jw = 0$ then w is identically equal to zero.

Lemma 4.18.3 *The following statements hold true.*

1)
$$T_\gamma\Omega = T_\gamma(t_0, t_1, \dots, t_k) \oplus T_\gamma^\perp\Omega. \tag{4.18.4}$$

2) For arbitrary $v \in T_\gamma(t_0, t_1, \dots, t_k)$, $w \in T_\gamma^\perp\Omega$ one has
$$\delta^2 S[\gamma](v, w) = 0. \tag{4.18.5}$$

3) The restriction of $\delta^2 S[\gamma]$ onto the subspace $T_\gamma^\perp\Omega$ is a positive definite quadratic form.

Proof: For a given vector field $X(t) \in T_\gamma\Omega$ denote $v(t)$ the unique vector field in $T_\gamma(t_0, t_1, \dots)$ such that $v(t_i) = X(t_i)$, $i = 0, \dots, k$. Define $w(t) \in T_\gamma^\perp\Omega$ by
$$w(t) := X(t) - v(t).$$

This justifies the decomposition (4.18.4).

Using the second variation formula (4.17.16) for $v \in T_\gamma\Omega(t_0, t_1, \dots, t_k)$, $w \in T_\gamma^\perp\Omega$ we obtain
$$\delta^2 S[\gamma](v, w) = -\int_0^1 \langle Jv, w\rangle\, dt - \sum_{i=1}^{k-1} \Delta_{t_i} \langle \nabla_{\dot\gamma} v, w\rangle = 0$$
since $Jv = 0$ on any interval (t_{i-1}, t_i) and $w(t_i) = 0$ for $i = 1, \dots, k-1$. This proves (4.18.5).

Let us prove positivity of the restriction of $\delta^2 S[\gamma]$ onto $T_\gamma^\perp\Omega$. The action functional can be represented as a sum
$$S[\gamma] = \sum_{i=1}^k S_i[\gamma], \quad S_i[\gamma] = \frac{1}{2}\int_{t_{i-1}}^{t_i} |\dot\gamma|^2 dt.$$

As the every piece $\gamma|_{[t_{i-1}, t_i]}$ is minimal we have
$$S_i[\gamma + \tau\, w] \geq S_i[\gamma], \quad i = 1, \dots, k$$
for sufficiently small γ. Hence $\delta^2 S[\gamma](w, w) \geq 0$. Let us prove that $\delta^2 S[\gamma](w, w) \neq 0$. In the opposite case choosing an arbitrary vector field $w' \in T_\gamma^\perp\Omega$ and a sufficiently small real parameter c one therefore obtains
$$0 \leq \delta^2 S[\gamma](w + c\, w', w + c\, w') = 2c\, \delta^2 S[\gamma](w, w') + c^2 \delta^2 S[\gamma](w', w').$$

The right hand side of this expression can be nonnegative for an arbitrary sufficiently small value of c only if $\delta^2 S[\gamma](w, w') = 0$. Besides, we already know that $\delta^2 S[\gamma](w, v) = 0$ for any $v \in T_\gamma\Omega(t_0, t_1, \dots, t_k)$. Thus the assumption $\delta^2 S[\gamma](w, w) = 0$ for some nonzero $w \in T_\gamma^\perp\Omega$ implies degeneracy of the bilinear form $\delta^2 S[\gamma]$ on $T_\gamma\Omega$. So $Jw = 0$ hence $w = 0$. This contradiction completes the proof of positivity of the Hessian of action functional on the subspace $T_\gamma^\perp\Omega$. □

Corollary 4.18.4 *Index of the geodesic γ is equal to the index of the restriction of the quadratic form $\delta^2 S[\gamma]$ onto the finite-dimensional subspace $T_\gamma(t_0, t_1, \dots, t_k)$. Hence* $\operatorname{ind} \delta^2 S[\gamma] < \infty$.

Let us consider the restriction $\gamma_{[0,\tau]}$ of the geodesic γ on a subinterval $[0,\tau]$ for $0<\tau\leq 1$. Denote

$$\operatorname{ind}(\tau)=\operatorname{ind}\delta^2 S\left[\gamma_{[0,\tau]}\right]$$

We will now study the dependence of the index on the parameter τ.

Lemma 4.18.5 *For sufficiently small $\tau>0$ the index $\operatorname{ind}(\tau)$ is equal to zero.*

Proof: This readily follows from Proposition 4.17.10. □

Lemma 4.18.6 $\operatorname{ind}(\tau)$ *is an increasing function of τ.*

Proof: If $\tau<\tau'$ then there exists a subspace $V_\tau\subset T_{\gamma_{[0,\tau]}}\Omega$ of dimension $\operatorname{ind}(\tau)$ such that

$$\delta^2 S\left[\gamma_{[0,\tau]}\right](v,v)<0\quad\text{for any nonzero}\quad v\in V_\tau.$$

A nonzero vector field $v\in V_\tau$ can be extended onto the bigger interval $[0,\tau']$ by identical zero on $[\tau,\tau']$. Denote $\bar v\in T_{\gamma_{[0,\tau']}}\Omega$ the extended vector field. We have

$$\delta^2 S\left[\gamma_{[0,\tau']}\right](\bar v,\bar v)=-\int_0^\tau\langle Jv,v\rangle-\Delta_\tau\left\langle\nabla_{\dot\gamma}\bar v,\bar v\right\rangle.$$

The first term in the right hand side is negative,

$$-\int_0^\tau\langle Jv,v\rangle=\delta^2 S\left[\gamma_{[0,\tau]}\right](v,v)<0.$$

The second one vanishes

$$\Delta_\tau\left\langle\nabla_{\dot\gamma}\bar v,\bar v\right\rangle=-\left\langle\nabla_{\dot\gamma}v(\tau),v(\tau)\right\rangle=0$$

since $v(\tau)=0$. In this way we obtain an embedding $V_\tau\subset V_{\tau'}$, $v\mapsto\bar v$ where the quadratic form $\delta^2 S\left[\gamma_{[0,\tau']}\right]$ is negative definite on $V_{\tau'}$. Therefore $\dim V_\tau\leq V_{\tau'}$ hence $\operatorname{ind}(\tau)\leq\operatorname{ind}(\tau')$. □

Lemma 4.18.7 $\operatorname{ind}(\tau)$ *is a left-continuous function, i.e.,*

$$\operatorname{ind}(\tau-\epsilon)=\operatorname{ind}(\tau)$$

for any sufficiently small $\epsilon>0$.

Proof: Let $t_i<\tau<t_{i+1}$. Denote $\mathbf{T}:=T_{\gamma_{[0,\tau]}}\Omega(t_0,t_1,\dots,t_i,\tau)$. This space

$$\mathbf{T}\simeq T_{\gamma(t_1)}M\oplus\cdots\oplus T_{\gamma(t_i)}M$$

of dimension $n\cdot i$ does not depend on $\tau\in(t_i,t_{i+1})$. Denote Q_τ the restriction of the quadratic form $\delta^2 S\left[\gamma_{[0,\tau]}\right]$ onto $\mathbf{T}$. The index $\operatorname{ind}(\tau)$ is equal to the index of the quadratic form Q_τ. Let $V\subset\mathbf{T}$ be a subspace of dimension $\operatorname{ind}(\tau)$ such that

$$Q_\tau|_V<0.$$

Since the quadratic form Q_τ depends continuously on τ one concludes that also

$$Q_{\tau'}|_V < 0$$

for arbitrary τ' sufficiently close to τ. Therefore $\operatorname{ind}(\tau') \geq \operatorname{ind}(\tau)$. But for $\tau' = \tau - \epsilon$ for a small positive ϵ one has $\operatorname{ind}(\tau - \epsilon) \leq \operatorname{ind}(\tau)$ due to the previous Lemma. The two inequalities imply that $\operatorname{ind}(\tau - \epsilon) = \operatorname{ind}(\tau)$. □

We will now describe discontinuity points of $\operatorname{ind}(\tau)$. We will use notations for the space $\mathbf{T}$ and the quadratic form Q_τ introduced in the proof of the previous Lemma.

Lemma 4.18.8 *Denote*

$$\nu = \dim \ker Q_\tau = \operatorname{mult}_\gamma(\gamma(0), \gamma(\tau))$$

Then for sufficiently small $\epsilon > 0$ *one has*

$$\operatorname{ind}(\tau + \epsilon) = \operatorname{ind}(\tau) + \nu.$$

Proof: Let us first prove that

$$\operatorname{ind}(\tau + \epsilon) \leq \operatorname{ind}(\tau) + \nu. \tag{4.18.6}$$

As the negative inertia index of the quadratic form Q_τ on the finite-dimensional space $\mathbf{T}$ is equal to $\operatorname{ind}(\tau)$ and dimension of the kernel of the quadratic form is equal to ν one concludes that the positive inertia index of the quadratic form is equal to $\dim \mathbf{T} - \operatorname{ind}(\tau) - \nu$. Thus there exists a subspace $\tilde{V} \subset \mathbf{T}$ of this dimension such that the restriction of Q_τ onto this subspace is positive definite. By continuity one also have positivity

$$Q_{\tau'}|_{\tilde{V}} > 0$$

for τ' sufficiently close to τ. So

$$\operatorname{ind}(\tau') \leq \dim \mathbf{T} - \dim \tilde{V} \leq \dim \mathbf{T} - (\dim \mathbf{T} - \operatorname{ind}(\tau) - \nu) = \operatorname{ind}(\tau) + \nu.$$

Choosing $\tau' = \tau + \epsilon$ one arrives at the inequality (4.18.6).

Let us now prove the opposite inequality

$$\operatorname{ind}(\tau + \epsilon) \geq \operatorname{ind}(\tau) + \nu. \tag{4.18.7}$$

Choose $\operatorname{ind}(\tau)$ linearly independent vector fields $v_1, \ldots, v_{\operatorname{ind}(\tau)}$ along $\gamma_{[0,\tau]}$ vanishing at the endpoints $\gamma(0)$, $\gamma(\tau)$ such that the symmetric matrix

$$(Q_\tau)_{ij} = \delta^2 S\left[\gamma_{[0,\tau]}\right](v_i, v_j), \quad i,\, j = 1, \ldots, \operatorname{ind}(\tau)$$

is negative definite. One can also choose ν linearly independent Jacobi vector fields $Y_1, \ldots Y_\nu$ vanishing at the endpoints $\gamma(0)$, $\gamma(\tau)$. Observe that the vectors

$$\nabla_{\dot\gamma} Y_j(\tau), \quad j = 1, \ldots, \nu$$

are linearly independent due to independency of vector fields Y_j. Choose ν vector fields X_1, $\ldots$, X_ν along $\gamma_{[0,\tau+\epsilon]}$ vanishing at the endpoints $\gamma(0)$, $\gamma[\tau+\epsilon]$ in such a way that their values at $t=\tau$ satisfy

$$\langle \nabla_{\dot\gamma} Y_j(\tau), X_k(\tau)\rangle = \delta_{jk}, \quad j,\, k=1,\ldots,\nu.$$

Finally, the vector fields v_i, Y_j originally defined on $[0,\tau]$ extend to the interval $[0,\tau+\epsilon]$ by zero on the part $[\tau,\tau+\epsilon]$. The extended vector fields will be denoted by the same letters v_i and Y_j. Using the second variation formula (4.17.16) one can check that

$$\delta^2 S\left[\gamma_{[0,\tau+\epsilon]}\right](Y_j, Y_k) = 0$$

$$\delta^2 S\left[\gamma_{[0,\tau+\epsilon]}\right](v_i, Y_j) = 0$$

$$\delta^2 S\left[\gamma_{[0,\tau+\epsilon]}\right](Y_j, X_k) = \delta_{jk}.$$

For a sufficiently small $c\neq 0$ consider $\mathrm{ind}\,(\tau)+\nu$ vector fields

$$v_1,\ldots,v_{\mathrm{ind}\,(\tau)}, c^{-1}Y_1 - c\,X_1,\ldots, c^{-1}Y_\nu - c\,X_\nu \tag{4.18.8}$$

on the geodesic $\gamma_{[0,\tau+\epsilon]}$. Consider the matrix of the quadratic form $Q_{\tau+\epsilon}$ in this basis. This matrix has the form

$$\begin{pmatrix} Q_\tau(v_i,v_j) & c\,A_{ij} \\ c\,A_{ji} & -\delta_{jk}+c^2 B_{jk}\end{pmatrix} \tag{4.18.9}$$

where

$$A_{ij} = -\delta^2 S\left[\gamma_{[0,\tau+\epsilon]}\right](v_i, X_j),$$

$$B_{jk} = \delta^2 S\left[\gamma_{[0,\tau+\epsilon]}\right](X_j, X_k).$$

As the matrix $Q_\tau(v_i,v_j)$ is negative definite we conclude that the entire matrix (4.18.9) is negative definite for $c=0$. Therefore it remains negative definite also for sufficiently small c.

We have constructed a linear subspace of dimension $\mathrm{ind}\,(\tau)+\nu$ in $T_{\gamma_{[0,\tau+\epsilon]}}\Omega$ spanned by the vectors (4.18.8) such that the quadratic form $\delta^2 S\left[\gamma_{[0,\tau+\epsilon]}\right]$ is negative definite on this space. Hence

$$\mathrm{ind}\,(\tau+\epsilon)\geq \mathrm{ind}\,(\tau)+\nu.$$

Comparing with (4.18.6) we prove the statement of the Lemma. □

We have proved that the monotone increasing function $\mathrm{ind}\,(t)$ has a jump equal to $\mathrm{mult}_\gamma(\gamma(0),\gamma(\tau))$ when the parameter t passes through a conjugate point $t=\tau$; it is continous in other points. This completes the proof of Index Theorem. □

Corollary 4.18.9 *If a geodesic $\gamma:[a,b]\to M$ contains a pair of conjugate points inside the interval (a,b) then it is not minimal.*

We will now apply this condition of non-minimality to two-dimensional connected Riemannian manifolds. Our goal is to prove the following

Theorem 4.18.10 *Let the Gaussian curvature of a two-dimensional Riemannian manifold M^2 satisfy the inequality*

$$K(x) \geq a^2 > 0 \quad \forall\, x \in M^2.$$

Then the distance between arbitrary points x, $y \in M^2$ satisfy

$$\rho(x, y) \leq \frac{\pi}{a}.$$

In particular, if M^2 is geodesically complete then it is compact.

Proof: It suffices to prove that any geodesic of the length greater than π/a contains a pair of conjugate points inside. To this and we will study zeroes of solutions to the Jacobi equation represented as the system (4.17.23), (4.17.24). The first equation can be easily solved, $\varphi = c_0 t + c_1$ for some constants c_0, c_1. To study zeroes of the second equation we will use the following result from the theory of second order linear differential equations.

Lemma 4.18.11 (Sturm). *Let the potential $K(t)$ of the Sturm–Liouville equation*

$$\ddot{\psi} + K(t)\psi = 0 \tag{4.18.10}$$

satisfy inequality

$$K(t) \geq K_0(t)$$

for some smooth function $K_0(t)$. Let $\chi(t)$ be a nontrivial solution to another Sturm–Liouville equation

$$\ddot{\chi} + K_0(t)\chi = 0 \tag{4.18.11}$$

vanishing at the points t_0 and t_1. Then any solution to (4.18.10) *must have a zero in the interval (t_0, t_1).*

Under assumptions of the theorem one can take $K_0(t) = a^2$. The equation (4.18.11) reduces to $\ddot{\chi} + a^2\chi = 0$. Its solution $\chi = \sin at$ has zeroes at $t = 0$ and $t = \pi/a$. Take a nontrivial solution $\psi(t)$ to eq. (4.17.24) vanishing at $t = 0$. According to Sturm lemma it must have another zero $\psi(t_1) = 0$, $t_1 \in \left(0, \frac{\pi}{a}\right)$. So, if the length l of the geodesic $\gamma : [0, l] \to M^2$, $|\dot{\gamma}| = 1$ is greater than π/a then it contains a pair of conjugate points $\gamma(0)$, $\gamma(t_1)$. Hence it is not minimal. Compactness of a geodesically complete two-dimensional manifold with positive Gaussian curvature easily follows from boundedness with the help of Corollary 4.9.24. □

In a similar way one can prove the following

Theorem 4.18.12 *Let M^2 be a two-dimensional connected Riemannian manifold of negative Gaussian curvature $K \leq 0$. Then any geodesic contains no conjugate points. If, in addition, M^2 is geodesically complete and simply connected then it is diffeomorphic to $\mathbb{R}^2$.*

We leave the proof of this theorem as an exercise for the reader (cf. also Examples 4.17.12 and 4.17.13 above).

4.19 Lie groups as Riemannian manifolds

We have already considered the class of linear vector fileds (see Example 1.3.13 above). Namely, for any $n \times n$ matrix A the vector field T_A on $\mathbb{R}^n$ reads

$$T_A^i(x) = -A_k^i x^k$$

(note the sign change with respect to Example 1.3.13). The main property of such vector fields is the following formula

$$[T_A, T_B] = T_{[A,B]}. \tag{4.19.1}$$

The dynamical system associated with a linear vector field T_A is a system of linear differential equations with constant coefficients

$$\dot{x}^i = -A_k^i x^k, \quad i = 1, \dots, n.$$

Its general solution is given by the matrix exponential

$$x(t) = e^{-A\,t} x_0. \tag{4.19.2}$$

Example 4.19.1 *Taking three linear vector fields in* $\mathbb{R}^3$

$$L_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \quad L_y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad L_z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{4.19.3}$$

one obtains a closed Lie algebra

$$[L_x, L_y] = L_z, \quad [L_y, L_z] = L_x, \quad [L_z, L_x] = L_y \tag{4.19.4}$$

isomorphic to the Lie algebra $so(3)$ *of* 3×3 *antisymmetric matrices.*

Let us now consider linear vector fields on the space $\mathbb{R}^{n^2} \simeq Mat(n, \mathbb{R})$ of $n \times n$ square matrices. Namely, for any $X \in Mat(n, \mathbb{R})$ define

$$L_X(A) = AX, \quad A \in Mat(n, \mathbb{R}). \tag{4.19.5}$$

Lemma 4.19.2 *The linear vector fields* (4.19.5) *satisfy*

$$[L_X, L_Y] = L_{[X,Y]}. \tag{4.19.6}$$

We leave the proof as an exercise for the reader.

Linear vector fields (4.19.5) satisfy an important property of *left invariance*. Namely, for a given matrix $g \in Mat(n, \mathbb{R})$ consider the left shift map $Mat(n, \mathbb{R}) \to Mat(n, \mathbb{R})$

$$A \mapsto g\,A. \tag{4.19.7}$$

Due to linearity the differential of this map coincides with the map itself. The action of the differential on the field L_X maps the field to itself

$$g\,L_X(A) = L_X(g\,A). \tag{4.19.8}$$

Remark 4.19.3 *One can define in a similar way* right-invariant *vector fields*

$$R_X(A) = -XA. \tag{4.19.9}$$

Like above one derives a formula for the commutator

$$[R_X, R_Y] = R_{[X,Y]}. \tag{4.19.10}$$

These vector fields satisfy the right invariance property

$$R_X(A)g = R_X(A\,g) \quad \text{for any} \quad g \in Mat(n, \mathbb{R}). \tag{4.19.11}$$

Let us now consider a Lie group $G \subset Mat(n, \mathbb{R})$. Denote $\mathfrak{g} := T_e G$ the tangent space at the unity of the group.

Lemma 4.19.4 *For any $X \in \mathfrak{g}$ the left-invariant vector field L_X is tangent to G.*

Applying the result of Exercise 1.3.16 one obtains

Corollary 4.19.5 *There is a natural Lie algebra structure on the tangent space $\mathfrak{g} := T_e G$ to a Lie group $G \subset Mat(n, \mathbb{R})$ at unity.*

For $X \in \mathfrak{g}$ the exponential $g(t) = e^{t\,X} \in G$ is a one-parameter subgroup, i.e.,

$$g(s+t) = g(s)g(t), \quad g(0) = e, \quad g(-t) = g(t)^{-1}. \tag{4.19.12}$$

A connected Lie group is generated by its one-parameter subgroups.

Example 4.19.6 *For the Lie group $O(n)$ of $n \times n$ orthogonal matrices the Lie algebra coincides with the space of $n \times n$ antisymmetric matrices. This space is usually denoted $so(n)$.*

For $g \in G$ the left shift (4.19.7) is a diffeomorphism. Its differential maps isomorphically the tangent space $T_e G = \mathfrak{g}$ to $T_g G$. Therefore the left-invariant vector fields L_X for $X \in \mathfrak{g}$ exhaust the tangent space at any point of the Lie group, i.e.,

$$T_g G \simeq \mathfrak{g} \quad \text{for any} \quad g \in G \tag{4.19.13}$$

as linear spaces.

Due to the isomorphism (4.19.13) any positive definite symmetric bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ defines a Riemannian metric on the Lie group G by the following rule

$$\langle L_X, L_Y \rangle\,|_g = \langle X, Y \rangle_{\mathfrak{g}} \quad \text{for any} \quad g \in G \quad \text{and arbitrary} \quad X,\, Y \in \mathfrak{g}. \tag{4.19.14}$$

Such a metric is called *left-invariant* metric on G. Alternatively the same bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ defines a *right-invariant* Riemannian metric on G by the formula

$$\langle R_X, R_Y \rangle\,|_g = \langle X, Y \rangle_{\mathfrak{g}} \quad \text{for any} \quad g \in G \quad \text{and arbitrary} \quad X,\, Y \in \mathfrak{g}. \tag{4.19.15}$$

In general the metrics (4.19.14) and (4.19.15) are different.

Definition 4.19.7 *A Riemannian metric on a Lie group G is called* biinvariant *if it is invariant with respect to both left and right shifts.*

Exercise 4.19.8 *Prove that inner automorphisms* $\mathrm{Ad} : G \to G$

$$\mathrm{Ad}_g h \to g\, h\, g^{-1}$$

are isometries of a biinvariant metric on a Lie group.

Proposition 4.19.9 *The metric* (4.19.15) *on a connected Lie group is biinvariant* iff *the bilinear form* $\langle\ ,\ \rangle_{\mathfrak{g}}$ *on* $\mathfrak{g}$ *satisfies*

$$\langle [X, Y], Z\rangle_{\mathfrak{g}} = -\,\langle X, [Y, Z]\rangle_{\mathfrak{g}} \quad \text{for all} \quad X,\, Y,\, Z \in \mathfrak{g}. \tag{4.19.16}$$

Proof: We have to check that the two metrics (4.19.14) and (4.19.15) coincide, i.e.,

$$\langle g\, X, g\, Y\rangle = \langle X\, g, Y\, g\rangle \quad \text{for any} \quad g \in G \quad \text{and arbitrary} \quad X,\, Y \in \mathfrak{g}.$$

It suffices to verify validity of this identity for $g = e^{t\, Z}$, $Z \in \mathfrak{g}$. Differentiating it in t at $t = 0$ one obtains

$$\langle ZX, Y\rangle + \langle X, ZY\rangle = \langle X, YZ\rangle + \langle XZ, Y\rangle.$$

In this equation X, Y, Z are considered as tangent vectors to G at the point e. This gives (4.19.16). Conversely, from (4.19.16) it is easy to derive validity of

$$\langle g(t)X, g(t)Y\rangle = \langle X\, g(t), Y\, g(t)\rangle$$

in all orders in t. □

Definition 4.19.10 *A symmetric bilinear form on a Lie algebra $\mathfrak{g}$ is called* invariant *if it satisfies* (4.19.16).

Exercise 4.19.11 *Let* $\mathfrak{g} = \mathrm{span}\,(e_1, \dots, e_n)$ *be a finite-dimensional Lie algebra. Recall that the structure constants* c_{ij}^k *of the Lie algebra are defined as coordinates of the commutators* $[e_i, e_j]$ *of the basic vectors with respect to the same basis*

$$[e_i, e_j] = c_{ij}^k e_k, \quad i,\, j = 1, \dots, n. \tag{4.19.17}$$

Denote

$$g_{ij} = \langle e_i, e_j\rangle_{\mathfrak{g}} \tag{4.19.18}$$

the Gram matrix of a symmetric bilinear form and put

$$c_{ijk} = c_{ij}^s g_{sk}. \tag{4.19.19}$$

Prove that the bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}}$ *is invariant* iff *the tensor* c_{ijk} *is antisymmetric with respect to arbitrary permutation of indices.*

Let us construct an example of a biinvariant metric on the Lie group $O(n)$ of orthogonal $n \times n$ matrices. Let us first observe that a Euclidean inner product on the space $\mathbb{R}^{n^2} = Mat(n, \mathbb{R})$ can be written in the form

$$\langle X, Y \rangle = \mathrm{tr} X Y^T = \sum_{i,j=1}^{n} X^i_j Y^i_j, \quad X = \left(X^i_j\right), \quad Y = \left(Y^i_j\right). \tag{4.19.20}$$

Here Y^T is the transposed matrix. Observe that for orthogonal matrices $X \in O(n)$ one has

$$\langle X, X \rangle = \mathrm{tr}\mathbf{1} = n.$$

Thus, the orthogonal group $O(n) \subset Mat(n, \mathbb{R})$ belongs to the sphere S^{n^2-1} of radius $\sqrt{n}$.

Proposition 4.19.12 *The restriction of the Euclidean metric* (4.19.20) *onto the orthogonal group $O(n)$ is an invariant Riemannian metric.*

Proof: It suffices to prove that, restricting the bilinear form (4.19.20) onto the space $so(n)$ of antisymmetric matrices is an invariant symmetric positive definite bilinear form on the Lie algebra $so(n)$. Indeed, for antisymmetric matrices

$$\langle X, Y \rangle = -\mathrm{tr} XY.$$

Using invariance of trace with respect to permutations

$$\mathrm{tr} AB = \mathrm{tr} BA$$

one obtains

$$\langle [X, Y], Z \rangle = -\mathrm{tr} XYZ + \mathrm{tr} YXZ = -\mathrm{tr} YZX + \mathrm{tr} YXZ = \mathrm{tr} Y[X, Z] = -\langle Y, [X, Z] \rangle.$$

$\square$

Exercise 4.19.13 *Prove that the restriction of the Euclidean metric*

$$\langle X, Y \rangle = \Re\, \mathrm{tr} X Y^* = \Re \sum_{i,j=1}^{n} X^i_j \bar{Y}^i_j, \quad X = \left(X^i_j\right), Y = \left(Y^i_j\right) \in Mat(n, \mathbb{C}) \tag{4.19.21}$$

in $\mathbb{R}^{2n^2} = Mat(n, \mathbb{C})$ *onto the unitary group $U(n)$ defines a biinvariant Riemannian metric on $U(n)$.*

Remark 4.19.14 *For any finite-dimensional Lie algebra $\mathfrak{g}$ there is a natural invariant symmetric bilinear form*

$$\langle X, Y \rangle = \mathrm{tr} \left(\mathrm{ad}\, X \cdot \mathrm{ad}\, Y \right) \tag{4.19.22}$$

called Killing form. *Here the* adjoint endomorphism

$$\mathrm{ad}\, X : \mathfrak{g} \to \mathfrak{g}$$

is a linear map defined by

$$\mathrm{ad}\, X(Y) = [X, Y].$$

Its invariance is an easy exercise. According to E.Cartan criterion *the Killing form does not degenerate* iff *the Lie algebra $\mathfrak{g}$ is semisimple. The Killing form of a semisimple Lie algebra is* negative definite iff $\mathfrak{g}$ *is the Lie algerbra of a compact Lie group.*

Exercise 4.19.15 *Prove that the Gram matrix of the Killing form in a basis $e_1, \ldots, e_n$ is expressed via the structure constants of the Lie algebra (see Exercise 4.19.11 above) as follows*

$$\langle e_i, e_j \rangle = c_{ik}^l c_{jl}^k.$$

Exercise 4.19.16 *Prove the following formulae for the Killing forms of some classical Lie algebras*

$$\begin{aligned}
gl(n): &\quad \langle X, Y\rangle = 2n \operatorname{tr} XY - 2 \operatorname{tr} X \operatorname{tr} Y \\
sl(n): &\quad \langle X, Y\rangle = 2n \operatorname{tr} XY \\
so(n): &\quad \langle X, Y\rangle = (n-2) \operatorname{tr} XY \\
su(n): &\quad \langle X, Y\rangle = 2n \operatorname{tr} XY.
\end{aligned}$$

We define a connection on a Lie group G by the formula

$$\nabla_{L_X} L_Y := \frac{1}{2}[L_X, L_Y]. \tag{4.19.23}$$

Theorem 4.19.17 *The connection* (4.19.23) *on a Lie group equipped with a biinvariant Riemannian metric coincides with the Levi-Civita connection on G.*

Proof: With the help of the formula (4.6.12) for the torsion tensor we easily prove that the connection (4.19.23) is symmetric:

$$\nabla_{L_X} L_Y - \nabla_{L_Y} L_X - [L_X, L_Y] = \frac{1}{2}[L_X, L_Y] - \frac{1}{2}[L_Y, L_X] - [L_X, L_Y] = 0.$$

To prove compatibility with the metric it suffices to check that

$$\partial_{L_Z}\langle L_X, L_Y\rangle = \langle \nabla_{L_Z} L_X, L_Y\rangle + \langle L_X, \nabla_{L_Z} L_Y\rangle$$

for arbitrary X, Y, $Z \in \mathfrak{g}$. Indeed, the left hand side is equal to zero since the inner product $\langle L_X, L_Y\rangle$ is constant. Also the right hand side vanishes, indeed

$$\begin{aligned}
&\langle \nabla_{L_Z} L_X, L_Y\rangle + \langle L_X, \nabla_{L_Z} L_Y\rangle = \frac{1}{2}\langle [L_Z, L_X], L_Y\rangle + \frac{1}{2}\langle L_X, [L_Z, L_Y]\rangle \\
&= \frac{1}{2}\langle [Z, X], Y\rangle_{\mathfrak{g}} + \frac{1}{2}\langle X, [Z, Y]\rangle_{\mathfrak{g}} = 0
\end{aligned}$$

due to invariance of the bilinear form $\langle\ ,\ \rangle^{\mathfrak{g}}$. $\square$

Corollary 4.19.18 *The curvature tensor of a biinvariant metric on a Lie group G is given by one of the following two equivalent formulae*

$$\begin{aligned}
R(L_X, L_Y)L_Z &= -\frac{1}{4} L_{[[X,Y],Z]} \\
\langle R(L_X, L_Y)L_Z, L_W\rangle &= -\frac{1}{4}\langle [X, Y], [Z, W]\rangle_{\mathfrak{g}}.
\end{aligned}$$

We end this section with description of geodesics of a biinvariant metric.

Theorem 4.19.19 *One-parameter subgroups $g(t) = e^{tX}$, $X \in \mathfrak{g}$ are geodesics of a biinvariant metric on G.*

Proof: The velocity vector $\dot g(t)$ of the one-parameter subgroup coincides with the vector field L_X. So

$$L_{\dot g}\dot g = \nabla_{L_X} L_X = \frac{1}{2} L_{[X,X]} = 0.$$

□

The statement of the Theorem is a motivation for the name "exponential map" used in section 4.9.

4.20 Differential geometry of complex manifolds

Let us begin with reminding some basics of complex linear algebra. A n-dimensional complex vector space V can be naturally identified with a real space $V_{\mathbb{R}}$ of dimension $2n$. If $z^1, \dots, z^n$ are complex coordinates in V then their real and imaginary parts $x^k = \Re z^k$, $y^k = \Im z^k$ are coordinates in $V_{\mathbb{R}}$. Alternatively it is sometimes convenient to use the combinations

$$z^k = x^k + i\, y^k, \quad \bar z^k = x^k - i\, y^k, \quad k = 1, \dots, n \tag{4.20.1}$$

as *complex-valued coordinates* in $V_{\mathbb{R}}$. (Here and below $i = \sqrt{-1}$.) Clearly such a change of coordinates is invertible

$$x^k = \frac{z^k + \bar z^k}{2}, \quad y^k = \frac{z^k - \bar z^k}{2i}. \tag{4.20.2}$$

Any linear operator $A \in \mathrm{End}_{\mathbb{C}}(V)$ will automatically be a linear operator in $\mathrm{End}_{\mathbb{R}}(V)$. The linear operator of multiplication by i

$$J : V \to V, \quad J\xi = i\,\xi \tag{4.20.3}$$

is of particular importance. The matrix of this operator in the coordinates $x^1, \dots, x^n, y^1, \dots, y^n$ is equal to

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.20.4}$$

(we will often identify the operator J with its matrix).

Definition 4.20.1 *A $\mathbb{R}$-bilinear form $(\ ,\ )$ on $V_{\mathbb{R}}$ is called* Hermitian *if it satisfies the following properties*

$$(\lambda\, x, y) = \lambda(x, y), \quad (x, \lambda\, y) = \bar\lambda(x, y) \quad \forall\, x,\, y, \in V, \quad \forall\, \lambda \in \mathbb{C} \tag{4.20.5}$$

$$(y, x) = \overline{(x, y)}, \quad \forall\, x,\, y \in V. \tag{4.20.6}$$

A Hermitian form is called positive definite *if it also satisfies*

$$(x, x) > 0 \quad \text{for any} \quad x \neq 0. \tag{4.20.7}$$

In a basis $e_1, \ldots, e_n$ an analogue of the Gram matrix is defined

$$h_{k\bar{l}} = (e_k, e_l). \tag{4.20.8}$$

The value of the Hermitian form on a pair of vectors $z = z^k e_k$, $w = w^l e_l$ can be computed by the following formula

$$(z, w) = h_{k\bar{l}} z^k \bar{w}^l. \tag{4.20.9}$$

The matrix

$$H = \begin{pmatrix} h_{1\bar{1}} & \ldots & h_{1\bar{n}} \\ h_{2\bar{1}} & \ldots & h_{2\bar{n}} \\ . & \ldots & . \\ . & \ldots & . \\ h_{n\bar{1}} & \ldots & h_{n\bar{n}} \end{pmatrix}$$

satisfies the property of *Hermitian symmetry*

$$H^T = \bar{H}. \tag{4.20.10}$$

Remark 4.20.2 *The following notation*

$$A^* := \bar{A}^T \tag{4.20.11}$$

is used for Hermitian conjugation *of a matrix. Thus, the Hermitian symmetry property* (4.20.10) *can be written as* $H^* = H$*.*

Any complex Hermitian symmetric matrix H defines a Hermitian form on $\mathbb{C}^n$. Such a Hermitian form is positive definite *iff* all principal minors of H are positive (observe that the principal minors of a Hermitian symmetric matrix are all real).

With a Hermitian form (x, y) one can associate two real bilinear forms taking real and imaginary parts $\Re(x, y)$ and $\Im(x, y)$. Clearly $\Re(x, y)$ is a symmetric bilinear form while $\Im(x, y)$ is an antisymmetric bilinear form

$$\Re(y, x) = \Re(x, y), \quad \Im(y, x) = -\Im(x, y).$$

One of this forms determines another one, in particular,

Lemma 4.20.3 *The real and imaginary parts of a Hermitean form satisfy*

$$\Re(x, y) = \Im(J\, x, y). \tag{4.20.12}$$

Here J is the operator of multiplication by i.

Proof: Due to complex linearity with respect to the first argument one has

$$\Re(i\, x, y) + i\, \Im(i\, x, y) = i\, [\Re(x, y) + i\, \Im(x, y)]\,.$$

This implies

$$\Re(x, y) = \Im(J\, x, y), \quad \Im(x, y) = -\Re(J\, x, y).$$

$\square$

Corollary 4.20.4 *Real and imaginary parts of a positive definite Hermitian forms are a positive definite symmetric bilinear form* $\Re(x,y)$ *and a nondegenerate antisymmetric bilinear form* $\Im(x,y)$.

Proof: The first part of the statement is obvious. To prove the second one one has to check that, if $\Im(x,y)=0$ for any y then $x=0$. Indeed, if $\Im(x,y)=0$ then $\Re(J\,x,y)=0$. Choosing $y=J\,x$ one obtains $\Re(J\,x,J\,x)=0$ hence $J\,x=0$. So $x=0$. $\square$

Example 4.20.5 *Taking* $H=\mathbf{1}$ *one obtains the following Hermitian form*

$$(z_1,z_2)=\sum_{k=1}^{n} z_1^k \bar z_2^k. \tag{4.20.13}$$

In the real coordinates $z_1^k=x_1^k+iy_1^k$, $z_2^k=x_2^k+iy_2^k$ *the real and imaginary parts of the Hermitian form read*

$$\Re(z_1,z_2)=\sum_{k=1}^{n}\left(x_1^k x_2^k+y_1^k y_2^k\right), \quad \Im(z_1,z_2)=-\sum_{k=1}^{n}\left(x_1^k y_2^k-x_2^k y_1^k\right). \tag{4.20.14}$$

A linear operator $A: V\to V$ acting on a complex linear space equipped with a positive definite Hermitian form (,) is called unitary if

$$(Ax,Ay)=(x,y) \quad \forall\, x,\, y\in V. \tag{4.20.15}$$

The matrix of a unitary operator satisfies

$$A^* H\, A = H. \tag{4.20.16}$$

In the particular case $H=\mathbf{1}$ one obtains definition of *unitary matrices*

$$A^* A=\mathbf{1}. \tag{4.20.17}$$

Example 4.20.6 *A unitary matrix of order 1 is just a complex number of modulus one,* $a=e^{i\,\phi}$. *Any unitary matrix of order 2 has the form*

$$\begin{pmatrix} a & -e^{i\phi}\bar b \\ b & e^{i\phi}\bar a\end{pmatrix}, \quad |a|^2+|b|^2=1.$$

Exercise 4.20.7 *For any unitary matrix* A *prove*

$$|\det A|=1. \tag{4.20.18}$$

The subspace of complex linear operators $\mathrm{End}_{\mathbb{C}}(V)\subset \mathrm{End}_{\mathbb{R}}(V)$ can be identified with the centralizer of J. In the simplest case of dimension 1 matrices of complex linear operators have the form

$$A=\begin{pmatrix} a & -b \\ b & a\end{pmatrix}, \quad a,\, b\in\mathbb{R}. \tag{4.20.19}$$

In the coordinates z, $\bar{z}$ the matrix becomes diagonal

$$A \to \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \quad \lambda = a + ib. \tag{4.20.20}$$

A smooth complex valued function $f(x, y) = u(x, y) + iv(x, y)$ can be considered as a map $\mathbb{C} \to \mathbb{C}$. The function is called *holomorphic*[11] if its differential

$$df = f_x dx + f_y dy = u_x dx + u_y dy + i\,(v_x dx + v_y dy)$$

is a complex linear map. In other words, the Jacobi matrix

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{4.20.21}$$

of the map $\mathbb{R}^2 \to \mathbb{R}^2$ must have the form (4.20.19), that is, the *Cauchy–Riemann equations*

$$u_x = v_y, \quad v_x = -u_y \tag{4.20.22}$$

hold true. Representing the differential in complex coordinates

$$df = f_z dz + f_{\bar{z}} d\bar{z}, \quad f_z := \frac{1}{2}\left(f_x - i f_y\right), \quad f_{\bar{z}} := \frac{1}{2}\left(f_x + i f_y\right) \tag{4.20.23}$$

one rewrites the Cauchy–Riemann equations for the function f in the form

$$f_{\bar{z}} = 0. \tag{4.20.24}$$

One can say that holomorphic functions f are those that do not depend on $\bar{z}$. They are usually written as $f = f(z)$.

Exercise 4.20.8 *Prove that the determinant of the Jacobi matrix* (4.20.21) *of a holomorphic function $f(z)$ is given by the formula*

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \left| \frac{\partial f}{\partial z} \right|^2. \tag{4.20.25}$$

In a similar way a smooth function $f : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if it satisfies a system of Cauchy–Riemann equations with respect to every coordinate $z^k = x^k + iy^k$

$$\frac{\partial f}{\partial \bar{z}^k} = 0, \quad k = 1, \dots, n. \tag{4.20.26}$$

Here the operators $\partial/\partial z^k$ and $\partial/\partial \bar{z}^k$ are defined like in (4.20.23)

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right), \quad \frac{\partial}{\partial \bar{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right). \tag{4.20.27}$$

[11] One of the first result of complex analysis says that holomorphic functions can be represented as sums of convergent power series. Because of this they are often called *complex analytic functions*.

Finally, a smooth map $f : U \to \mathbb{C}^m$ of a domain $U \subset \mathbb{C}^n$ is holomorphic if, for its coordinate representation

$$z = \left(z^1, \dots, z^n\right) \mapsto \left(w^1(z), \dots, w^m(z)\right)$$

the Cauchy–Riemann equations hold true for every component

$$\frac{\partial w^j}{\partial \bar{z}^k} = 0, \quad k = 1, \dots, n, \quad j = 1, \dots, m.$$

We are ready to define a class of complex analytic manifolds.

Definition 4.20.9 *A complex analytic manifold of complex dimension n is a smooth $2n$-dimensional manifold M equipped with an atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in I}$,*

$$\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{C}^n = \mathbb{R}^{2n}$$

such that, on the intersections $U_\alpha \cap U_\beta \neq \emptyset$ the transition functions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are holomorphic.

We will denote $z_\alpha^1, \dots, z_\alpha^n$ complex coordinates on the chart U_α. On the intersections $U_\alpha \cap U_\beta$ one has holomorphic transition functions $z_\beta^j\left(z_\alpha^1, \dots, z_\alpha^n\right)$,

$$\frac{\partial z_\beta^j}{\partial \bar{z}_\alpha^k} = 0, \quad j,\, k = 1, \dots, n.$$

Observe that the Jacobian of the transition functions is always positive

$$\det \begin{pmatrix} \frac{\partial z_\alpha}{\partial z_\beta} & \frac{\partial z_\alpha}{\partial \bar{z}_\beta} \\ \frac{\partial \bar{z}_\alpha}{\partial z_\beta} & \frac{\partial \bar{z}_\alpha}{\partial \bar{z}_\beta} \end{pmatrix} = \left| \det \left(\frac{\partial z_\alpha}{\partial z_\beta} \right) \right|^2 > 0 \tag{4.20.28}$$

(cf. Exercise 4.20.8). So, any complex manifold has a natural orientation.

The real and imaginary parts of $z^k = x^k + i y^k$ will be used as local coordinates on M considered as a real manifold $M_\mathbb{R}$. It will also be convenient to use complex combinations

$$z^k = x^k + i y^k, \quad \bar{z}^k = x^k - i y^k$$

as coordinates on $M_\mathbb{R}$. In this way, instead of using the vector fields

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}$$

as a basis in the tangent space to M one can use their complex combinations (4.20.27). Any tangent vector $\xi \in T_P M$ at a point $P \in M$ can be decomposed as

$$\xi = \xi^1 \frac{\partial}{\partial z^1} + \dots + \xi^n \frac{\partial}{\partial z^n} + \xi^{\bar{1}} \frac{\partial}{\partial \bar{z}^1} + \dots + \xi^{\bar{n}} \frac{\partial}{\partial \bar{z}^n} = \xi^k \frac{\partial}{\partial z^k} + \xi^{\bar{k}} \frac{\partial}{\partial \bar{z}^k}. \tag{4.20.29}$$

The complex coordinates ξ^k, $\xi^{\bar{k}}$ satisfy

$$\xi^{\bar{k}} = \overline{\xi^k}, \quad k = 1, \dots, n. \tag{4.20.30}$$

Forgetting about the constraint (4.20.30) one obtains a vector ξ in the complexified tangent space $\xi \in \mathbb{C}T_P M$. It is a complex vector space of complex dimension $2n$. It is naturally decomposed into a sum of two complex n-dimensional subspaces

$$\mathbb{C}T_P M = T_P^{1,0} M \oplus T_P^{0,1} M, \quad T_P^{1,0} M = \operatorname{span}\left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}\right\}, \quad T_P^{0,1} M = \operatorname{span}\left\{\frac{\partial}{\partial \bar z^1}, \dots, \frac{\partial}{\partial \bar z^n}\right\}. \tag{4.20.31}$$

The decomposition (4.20.31) is invariant with respect to changes of local coordinates.

A complex antilinear map

$$\sigma : \mathbb{C}T_P M \to \mathbb{C}T_P M, \quad \sigma(\lambda\,\xi) = \bar\lambda \sigma(\xi) \tag{4.20.32}$$

defined by

$$\sigma\left(\frac{\partial}{\partial z^k}\right) = \frac{\partial}{\partial \bar z^k}, \quad \sigma\left(\frac{\partial}{\partial \bar z^k}\right) = \frac{\partial}{\partial z^k}, \quad k = 1, \dots, n \tag{4.20.33}$$

permutes the subspaces $T^{1,0}$ and $T^{0,1}$. Clearly σ is an involution, $\sigma^2 = \mathrm{id}$. The subspace of σ-invariant vectors, $\sigma(\xi) = \xi$, coincides with the (real) $2n$-dimensional tangent space $T_P M$.

Example 4.20.10 *Complex vector space $\mathbb{C}^n$ is an example of a n-dimensional complex manifold. It is covered with one chart with complex coordinates $z^1, \dots, z^n$. Same is true for any domain in $\mathbb{C}^n$.*

Example 4.20.11 *Complex projective space $\mathbb{C}P^n$ is defined as a quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ over complex rescalings*

$$\mathbb{C}P^n = \left\{ Z = \left(Z^0, Z^1, \dots, Z^n\right) \in \mathbb{C}^{n+1} \setminus \{0\}\right\} / \{Z \sim \lambda\, Z,\ \lambda \in \mathbb{C} \setminus \{0\}\}. \tag{4.20.34}$$

It can be covered by $(n+1)$ charts $U_0, U_1, \dots, U_n$

$$U_k = \left\{ Z = \left(Z^0, Z^1, \dots, Z^n\right) \in \mathbb{C}^{n+1} \,|\, Z^k \neq 0 \right\}. \tag{4.20.35}$$

The local coordinates on the chart U_k are defined as follows

$$\left(z_k^1, \dots, z_k^n\right) = \left(\frac{Z^0}{Z^k}, \dots, \frac{\hat{Z^k}}{Z^k}, \dots, \frac{Z^n}{Z^k}\right) \tag{4.20.36}$$

(the k-th term $\frac{Z^k}{Z^k} = 1$ is omitted). On the intersection $U_k \cap U_l$ one has

$$\left(z_l^1, \dots, z_l^n\right) = \frac{Z^k}{Z^l}\left(z_k^1, \dots, z_k^n\right).$$

It is easy to express the ratio Z^k/Z^l of two non-zero homogeneous coordinates as a holomorphic function of local coordinates. For example, for $k < l$ one has

$$\frac{Z^k}{Z^l} = z_l^{k+1} = \frac{1}{z_k^l}.$$

In the simplest example $n = 1$ one has two charts U_0 and U_1 on $\mathbb{C}P^1$. Any of these two coincides with the complex plane $\mathbb{C}$. Denote z the local coordinate on U_0 and w the local coordinate on U_1. On the intersection $z \neq 0$, $w \neq 0$ the transition function reads

$$w = \frac{1}{z} \quad \text{or} \quad z = \frac{1}{w}.$$

Actually, there is only one point in $\mathbb{C}P^1$ not belonging to the chart U_0, namely, the point $w = 0$ in U_1. When w tends to 0 the coordinate z tends to infinity. So, $\mathbb{C}P^1$ can be identified with the complex plane with one infinite point added

$$\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}. \tag{4.20.37}$$

Topologically it is a two-dimensional sphere S^2. Because of this the complex projective line $\mathbb{C}P^1$ is often called Riemann sphere*.*

Let M be a complex analytic manifold of complex dimension n. We will consider a particular subclass of Riemannian metrics on such a manifold considered as a smooth $2n$-dimensional real manifold $M_{\mathbb{R}}$.

Definition 4.20.12 *We say that a Riemannian metric on $M_{\mathbb{R}}$ is compatible with the complex structure on M if the operator J of multiplication by i is orthogonal,*

$$\langle J\xi_1, J\xi_2 \rangle = \langle \xi_1, \xi_2 \rangle \quad \forall\, \xi_1,\, \xi_2 \in T_P M_{\mathbb{R}} \tag{4.20.38}$$

for any point P in the manifold.

Definition 4.20.13 *A* Hermitian metric *on M is a $\mathbb{C}$-bilinear pairing*

$$T_P^{1,0} M \times T_P^{0,1} M \to \mathbb{C}, \quad \xi,\, \eta \mapsto (\xi, \eta) \tag{4.20.39}$$

smoothly depending on the point $P \in M$ such that

$$(\xi_1, \sigma(\xi_2)) = \overline{(\xi_2, \sigma(\xi_1))} \quad \forall\, \xi_1,\, \xi_2 \in T_P^{1,0} M \tag{4.20.40}$$

and

$$(\xi, \xi) > 0 \quad \text{for any} \quad 0 \neq \xi \in T_P M \tag{4.20.41}$$

Hermitian form on the tangent space $T_P M$ at every point

5 Symplectic manifolds. Poisson manifolds

5.1 Basic definitions. Poisson brackets

Definition 5.1.1 *A symplectic structure on a manifold M is a nondegenerate closed differential 2-form*

$$\begin{aligned} &\omega = \sum_{i<j} \omega_{ij}(x) dx^i \wedge dx^j, \quad \omega_{ji}(x) = -\omega_{ij}(x) \\ &d\omega = 0, \quad \det\left(\omega_{ij}(x)\right) \neq 0 \quad \forall\; x \in M. \end{aligned} \tag{5.1.1}$$

Clearly the dimension of M must be even; in this section it will usualy be denoted $2n$. The (0,2)-tensor $\omega_{ij}(x)$ defines an antisymmetric bilinear form on the tangent spaces

$$\langle X, Y\rangle = \omega_{ij}(x)X^iY^j, \quad i, j \in T_xM \tag{5.1.2}$$

$$\langle Y, X\rangle = -\langle X, Y\rangle.$$

Nondegenerateness of ω means that

$$\text{if} \quad \langle X, Y\rangle = 0 \quad \forall\, Y \in T_xM \quad \text{then } X = 0.$$

Recall that the condition of closedness of the 2-form reads as follows

$$d\omega = 0 \;\Leftrightarrow\; \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{ki}}{\partial x^j} + \frac{\partial \omega_{jk}}{\partial x^i} = 0 \quad \text{for all} \quad i, j, k = 1, \ldots, 2n. \tag{5.1.3}$$

Example 5.1.2 *Consider Euclidean space $\mathbb{R}^{2n}$ with the coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ equipped with the 2-form*

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

The constant matrix of this 2-form has consists of four $n \times n$ blocks

$$\begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} =: J. \tag{5.1.4}$$

This is the standard phase space of classical mechanics. We will often use short notations $q = (q^1, \ldots, q^n)$, $p = (p_1, \ldots, p_n)$,

$$dp \wedge dq = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

More generally, one can consider a $2n$-dimensional linear space W equipped with a nondegenerate antisymmetric bilinear form $\omega(X, Y)$. We will call (W, ω) a symplectic space. *As it is well known from linear algebra for an arbitrary nondegenerate antisymmetric bilinear form there exists a basis $e_1, \ldots, e_n, f^1, \ldots, f^n$ in the space W such that*

$$\omega(e_i, e_j) = \omega(f^i, f^j) = 0, \quad \omega(e_i, f^j) = -\delta_i^j.$$

In this basis the matrix of the bilinear form has the standard form (5.1.4).

Example 5.1.3 *Let Q^n be a smooth manifold. Consider the total space of cotangent bundle*

$$M^{2n} = T^*Q.$$

This is a smooth manifold of the dimension $2n$ with local coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$. Here $q^1, \ldots, q^n$ are local coordinates on the base Q_n while the coordinates $p_1, \ldots, p_n$ on the fiber T_1^*Q over a point $q \in Q^n$ are defined as follows

$$p_i(\xi) = \xi_i \quad \textit{if} \quad \xi = \xi_1 dq^1 + \cdots + \xi_n dq^n.$$

Let us prove that the 1-form

$$\alpha = p_1 dq^1 + \cdots + p_n dq^n \tag{5.1.5}$$

does not depend on the choice of local coordinates $q^1, \ldots, q^n$ on Q^n. For a given a change of coordinates

$$q^i \mapsto q^{i'} = q^{i'}\left(q^1, \ldots, q^n\right)$$

the components of a covector ξ transform as follows

$$\xi_{i'} = \frac{\partial q^i}{\partial q^{i'}} \xi_i$$

(the (0,1)-tensor law). Thus one has the following linear change of the p-coordinates on T^*Q

$$p_{i'} = \frac{\partial q^i}{\partial q^{i'}} p_i.$$

So,

$$p_{i'} dq^{i'} = \frac{\partial q^i}{\partial q^{i'}} p_i \frac{\partial q^{i'}}{\partial q^k} dq^k = p_i dq^i.$$

The differential $d\alpha$ of the 1-form is a well-defined 2-form on T^*Q,

$$\omega = d\alpha = \sum_{i=1}^n dp_i \wedge dq^i. \tag{5.1.6}$$

It defines on T^*Q a structure of symplectic manifold.

We will now define *Poisson bracket* on a symplectic manifold. Denote $\left(\omega^{ij}(x)\right)$ the inverse matrix to $(\omega_{ij}(x))$. Like in the case of inverse of a Riemannian metric, this is a (2,0)-tensor on M^{2n}, i.e., an antisymmetric bilinear form on the cotangent space $T_x^* M^{2n}$. Evaluating this bilinear form on a pair of differentials df, dg, f, $g \in \mathcal{C}^\infty(M)$ defines a new function denoted by

$$\{f, g\} := \omega^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \in \mathcal{C}^\infty(M). \tag{5.1.7}$$

It is called the Poisson bracket of the functions f and g.

For the symplectic manifold of Example 5.1.2 one arrives at the well known formula for the Poisson bracket used in classical mechanics

$$\{f, g\} = \sum_{i=1}^n \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}. \tag{5.1.8}$$

Same formula holds true also for the Poisson bracket of functions on the cotangent bundle T^*Q (see Example 5.1.3 above).

Theorem 5.1.4 *The pairing*

$$\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad f, g \mapsto \{f, g\}$$

defines on $\mathcal{C}^\infty(M)$ a structure of Lie algebra satisfying the following Leibnitz rule

$$\{f\, g, h\} = f\{g, h\} + g\{f, h\} \quad \forall\ f,\ g,\ h \in \mathcal{C}^\infty(M). \tag{5.1.9}$$

We have to prove that the Posson bracket is an antisymmetric bilinear operation satisfying Jacobi identity

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0 \quad \forall\ f,\ g,\ h\in\mathcal{C}^\infty(M). \tag{5.1.10}$$

Bilinearity and antisymmetry, as well as the Leibnitz rule (5.1.9) are obvious. In order to prove validity of the Jacobi identity we will address the following question: given an antisymmetric (2,0)-tensor $\pi^{ij}(x)$ on a manifold M, under what conditions the formula

$$\{f,g\}:=\pi^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \tag{5.1.11}$$

defines on the space of functions $\mathcal{C}^\infty(M)$ a structure of Lie algebra[12]? Observe that the brackets of the coordinate function coincide with the components of the bivector

$$\{x^i,x^j\}=\pi^{ij}(x). \tag{5.1.12}$$

Lemma 5.1.5 *Denote*

$$\Pi(f,g,h):=\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}\in C^\infty(M).$$

the lhs of the Jacobi identity. Then, in local coordinates the following formula holds true

$$\Pi(f,g,h)(x)=\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}\frac{\partial h}{\partial x^k}\Pi(x^i,x^j,x^k).$$

The proof is by a straightforward computation.

Corollary 5.1.6 *The Jacobi identity for the bracket* (5.1.11) *holds true* iff

$$\left\{\{x^i,x^j\},x^k\right\}+\left\{\{x^k,x^i\},x^j\right\}+\left\{\{x^j,x^k\},x^i\right\}\equiv\frac{\partial\pi^{ij}}{\partial x^s}\pi^{sk}+\frac{\partial\pi^{ki}}{\partial x^s}\pi^{sj}+\frac{\partial\pi^{jk}}{\partial x^s}\pi^{si}=0 \tag{5.1.13}$$

$\forall\ i,\ j,\ k=1,\dots,\dim M$.

Exercise 5.1.7 *1) For a symplectic manifold* $\left(M^{2n},\omega\right)$ *the bivector* $\pi^{ij}(x)=\omega^{ij}(x)$ *where* $\left(\omega^{ij}(x)\right)=\left(\omega_{ij}(x)\right)^{-1}$ *satisfies the equations of Corollary 5.1.6*

$$\frac{\partial\pi^{ij}}{\partial x^s}\pi^{sk}+\frac{\partial\pi^{ki}}{\partial x^s}\pi^{sj}+\frac{\partial\pi^{jk}}{\partial x^s}\pi^{si}=0 \quad \forall\ i,\ j,\ k=1,\dots,2n. \tag{5.1.14}$$

2) Prove that a nondegenerate (2,0)-tensor $\pi^{ij}(x)$ *satisfies the Jacobi identity* (5.1.13) iff *the 2-form* $\omega=\sum_{i<j}\omega_{ij}(x)dx^i\wedge dx^j$ *where* $(\omega_{ij}(x))=\left(\pi^{ij}(x)\right)^{-1}$ *is closed.*

One can say that, for a nondegenerate antisymmetric matrix of functions $\pi^{ij}(x)$ the inversion map $\left(\pi^{ij}(x)\right)\mapsto\left(\pi^{ij}(x)\right)^{-1}$ linearizes the nonlinear equations (5.1.14) (see eqs. (5.1.3) above).

Motivating by the previous discussion we introduce

[12] Needless to say that the Leibnitz rule (5.1.9) for the bracket (5.1.11) holds true automatically.

Definition 5.1.8 *A* Poisson structure *on a manifold M is a structure of a Lie algebra on the space of functions $\mathcal{C}^\infty(M)$ satisfying the Leibnitz rule* (5.1.9). *A manifold M equipped with a Poisson structure is called* Poisson manifold.

Exercise 5.1.9 *Prove that a Poisson structure on an arbitrary Poisson manifold has the form* (5.1.11) *for some bivector $\pi^{ij}(x)$.*

Due to the previous results any symplectic manifold $\left(M^{2n}, \omega\right)$ is a Poisson manifold. It satisfies the *non-degeneracy condition*:

$$\{f, g\} = 0 \quad \forall\, g \in \mathcal{C}^\infty(M) \quad \Rightarrow \quad f \quad \text{is (locally) constant.} \tag{5.1.15}$$

Exercise 5.1.10 *Prove that any Poisson manifold satisfying the nondegeneracy condition is a symplectic manifold.*

A simple class of examples of Poisson manifolds can be obtained taking an arbitrary antisymmetric *constant* matrix π^{ij} in the formula (5.1.11) for the Poisson bracket. We will now introduce another important class of examples.

Example 5.1.11 *Let $\mathfrak{g}$ be a Lie algebra of dimension n. Denote $M = \mathfrak{g}^*$ the dual space. Choose a basis $e^1, \ldots, e^n$ in $\mathfrak{g}$. Commutators of the basic elements can be represented as linear combinations of themselves*

$$\left[e^i, e^j\right] = c_k^{ij} e^k. \tag{5.1.16}$$

The coefficients c_k^{ij} of the linear combinations are called structure constants *of the Lie algebra.*

Every vector in $\mathfrak{g}$ defines a linear function on the dual space $\mathfrak{g}^* = M$. In this way the basis in $\mathfrak{g}$ defines a system of coordinates on the dual space. Denote $x^1, \ldots, x^n$ this system of coordinates. The Poisson bracket on $M = \mathfrak{g}^*$ is defined by the following formula

$$\{x^i, x^j\} = c_k^{ij} x^k. \tag{5.1.17}$$

Thus, the components of the bivector $\pi^{ij}(x) = c_k^{ij} x^k$ depend linearly on the coordinates.

Exercise 5.1.12 *Prove that any Poisson manifold equipped with a Poisson bracket M depending linearly on the coordinates is isomorphic to $\mathfrak{g}^*$ for some finite-dimensional Lie algebra $\mathfrak{g}$.*

Hint: observe that linear functions on M form a closed Lie algebra wrt the Poisson bracket.

Example 5.1.13 *Choosing a standard basis in the Lie algebra $so(3)$ one obtains a 3-dimensional Poisson manifold with coordinates x, y, z with Poisson brackets*

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y. \tag{5.1.18}$$

Example 5.1.14 *For linear* inhomogeneous *Poisson brackets*

$$\{x^i, x^j\} = c_k^{ij} x^k + c_0^{ij} \tag{5.1.19}$$

the constant coefficients c_k^{ij} *are still structure constants of a Lie algebra* $\mathfrak{g}$ *while the constants* c_0^{ij} *correspond to a 2-cocycle on the Lie algebra,* $c_0^{ij} = c_0\left(e^i, e^j\right)$*, i.e., a bilinear antisymmetric form on* $\mathfrak{g}$ *satisfying*

$$c_0([a,b],c) + c_0([c,a],b) + c_0([b,c],a) = 0$$

for arbitrary three vectors a*,* b*,* c *in* $\mathfrak{g}$*. The 2-cocycle is called* trivial *if there exists a linear function* ℓ *on* $\mathfrak{g}$ *such that*

$$c_0(a,b) = \ell\left([a,b]\right) \quad \forall\, a,\, b \in \mathfrak{g}.$$

(Prove that the above formula defines a 2-cocycle for an arbitrary linear function ℓ*.) A trivial 2-cocycle in* (5.1.19) *can be killed by the shift*

$$x^i \mapsto x^i + \ell\left(e^i\right).$$

We will now introduce an important class of *Hamiltonian vector fields* on a Poisson manifold (M, π) (and, therefore, on any symplectic manifold (M, ω)).

Definition 5.1.15 *The vector field* X_H *with the components*

$$X_H^i(x) = \pi^{ij}(x)\frac{\partial H(x)}{\partial x^j} \tag{5.1.20}$$

is called the Hamiltonian vector field generated by the Hamiltonian $H \in \mathcal{C}^\infty(M)$*. Equivalently, the first order linear differential operator associated with the Hamiltonian vector field* X_H *acts on an arbitrary smooth function* $f \in \mathcal{C}^\infty(M)$ *by*

$$X_H f = \{f, H\}. \tag{5.1.21}$$

Observe that the dynamical system associated with the Hamiltonian vector field X_H can be written in the following form

$$\dot{x}^i = \{x^i, H(x)\}, \quad i = 1, \ldots, \dim M. \tag{5.1.22}$$

It will be called the *Hamiltonian system* generated by the Hamiltonian H.

Example 5.1.16 *For the standard phase space* $\mathbb{R}^{2n}$ *of classical mechanics the Hamiltonian system coincides with the canonical Hamiltonian equations of motion*

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \end{aligned} \tag{5.1.23}$$

$i = 1, \ldots, n$ *with the Hamiltonian* $H = H(q, p)$*.*

Remark 5.1.17 *In order to treat Hamiltonian systems with time-dependent Hamiltonians* $H = H(q, p, t)$ *it is convenient to introduce* extended phase space $\mathbb{R}^{2n+2}$ *with the coordinates* (q, p, t, E) *with the symplectic structure*

$$\hat{\omega} = dp \wedge dq - dE \wedge dt. \tag{5.1.24}$$

Modify the Hamiltonian:

$$\hat{H} = H - E.$$

Then the Hamiltonian system reads

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \\ \dot{t} &= 1 \\ \dot{E} &= \frac{\partial H}{\partial t}. \end{aligned}$$

On the level surface $\hat{H} = 0$ *invariant wrt the Hamiltonian flow (see below) one obtains the equations of motion along with the identity*

$$\frac{d}{dt} H(q(t), p(t), t) = \frac{\partial H(q, p, t)}{\partial t}$$

well known in classical mechanics.

Example 5.1.18 *A Hamiltonian system on the dual space to the Lie algebra* $so(3)$ *(see Example 5.1.13 above) reads*

$$\begin{aligned} \dot{x} &= \{x, H\} = z\frac{\partial H}{\partial y} - y\frac{\partial H}{\partial z} \\ \dot{y} &= \{y, H\} = x\frac{\partial H}{\partial z} - z\frac{\partial H}{\partial x} \\ \dot{z} &= \{z, H\} = y\frac{\partial H}{\partial x} - x\frac{\partial H}{\partial y} \end{aligned}$$

or, in vector form

$$\dot{\mathbf{r}} = \nabla H \times \mathbf{r}, \quad \mathbf{r} = (x, y, z).$$

In the particular case of a quadratic Hamiltonian $H = \frac{1}{2}\left(a\,x^2 + b\,y^2 + c\,z^2\right)$ *one obtains the Euler equations of free motion of a rotated rigid body*

$$\begin{aligned} \dot{x} &= (b - c)y\,z \\ \dot{y} &= (c - a)z\,x \\ \dot{z} &= (a - b)x\,y. \end{aligned}$$

We finish this section with the following useful statement about Hamiltonian vector fields on *symplectic* manifolds.

Lemma 5.1.19 *Let Y be a vector field on a symplectic manifold (M, ω). Then for any function f on M the following formula holds true*

$$Yf = \omega(Y, X_f). \tag{5.1.25}$$

Proof: We have

$$Yf = Y^k \frac{\partial f}{\partial x^k} = Y^k \delta_k^s \frac{\partial f}{\partial x^s} = Y^k \omega_{kj} \omega^{js} \frac{\partial f}{\partial x^s} = \omega_{kj} Y^k X_f^j = \omega(Y, X_f).$$

$\square$

5.2 Poisson symmetries. Hamiltonian flows as symplectomorphisms

Let (M_1, π_1), (M_2, π_2) be two Poisson manifolds.

Definition 5.2.1 *1) A smooth map $F : M_1 \to M_2$ is called a* Poisson morphism *if the pull-back $F^* : \mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1)$ is a homomorphism of Lie algebras*

$$\{F^* f, F^* g\}_{M_1} = \{f, g\}_{M_2} \quad \forall\ f,\ g \in \mathcal{C}^\infty(M_2).$$

2) A diffeomorphism $F : M \to M$ of a Poisson manifold (M, π) to itself that is a Poisson morphism is called Poisson symmetry*.*

Consider also the infinitesimal version of Poisson symmetries.

Definition 5.2.2 *A vector field $X \in Vect(M)$ on a Poisson manifold (M, π) is called* infinitesimal symmetry *of the Poisson structure if*

$$X\{f, g\} = \{Xf, g\} + \{f, Xg\} \quad \forall\ f,\ g \in \mathcal{C}^\infty(M). \tag{5.2.1}$$

Proposition 5.2.3 *A vector field X^i is an infinitesimal symmetry of the Poisson structure π^{ij} on M* iff

$$Lie_X \pi^{ij} \equiv X^k \frac{\partial \pi^{ij}}{\partial x^k} - \frac{\partial X^i}{\partial x^k} \pi^{kj} - \pi^{ik} \frac{\partial X^j}{\partial x^k} = 0 \quad \forall\ i,\ j = 1, \dots, \dim M. \tag{5.2.2}$$

The proof is straightforward.

The name "infinitesimal symmetry" is motivated by the following

Proposition 5.2.4 *Let $X \in Vect(M)$ be an infinitesimal symmetry on a Poisson manifold (M, π). Denote $g_t : M \to M$ the one-parameter group of diffeomorphisms generated by the vector field X, $|t| < \epsilon$ for some $\epsilon > 0$. Then g_t is a Poisson symmetry of (M, π).*

Proof: For any $x = (x^1, \dots, x^n) \in M$ one has

$$[g_t(x)]^i = x^i + t\, X^i(x) + \mathcal{O}(t^2).$$

So, for any $f \in \mathcal{C}^\infty(M)$ we have

$$g_t^* f(x) = f(x) + t\, X f(x) + \mathcal{O}(t^2).$$

Hence, for a pair of smooth functions one obtains

$$\{g_t^* f_1, g_t^* f_2\} = \{f_1 + t\, X f_1, f_2 + X f_2\} + \mathcal{O}(t^2) = \{f_1, f_2\} + t\,[\{f_1, X f_2\} + \{X f_1, f_2\}] + \mathcal{O}(t^2)$$
$$= \{f_1, f_2\} + t\, X\{f_1, f_2\} + \mathcal{O}(t^2) = g_t^*\{f_1, f_2\}.$$

$\square$

Theorem 5.2.5 *Any Hamiltonian vector field X_H, $H \in \mathcal{C}^\infty(M)$ on a Poisson manifold (M, π) is an infinitesimal symmetry of the Poisson structure π.*

Proof: Using $X_H f = \{f, H\}$, $X_H g = \{g, H\}$ (see eq. (5.1.21) above) along with Jacobi identity we obtain

$$X_H\{f, g\} = \{\{f, g\}, H\} = -\{\{H, f\}, g\} - \{\{g, H\}, f\} = \{X_H f, g\} + \{f, X_H g\}.$$

$\square$

Corollary 5.2.6 *The one-parameter group of diffeomorphisms g_t generated by a Hamiltonian vector field X_H on a Poisson manifold (M, π) consists of Poisson symmetries of (M, π).*

Consider the particular case of Hamiltonian vector fields on a symplectic manifold (M, ω).

Corollary 5.2.7 *For any Hamiltonian vector field X_H, $H \in \mathcal{C}^\infty(M)$ on a symplectic manifold (M, ω) one has*

$$Lie_{X_H} \omega = 0. \tag{5.2.3}$$

Proof: The identity operator id : $T_x M \to T_x M$ is constant along any vector field X: $Lie_X \delta^i_j = 0$. Differentiating the equation

$$\delta^i_j = \omega^{ik} \omega_{kj}$$

along X_H and using $Lie_{X_H} \omega^{ik} = 0$ one derives that also $Lie_{X_H} \omega_{kj} = 0$. $\square$

We will now consider a symplectic analogue of Poisson symmetries.

Definition 5.2.8 *Let $F : M_1 \to M_2$ be a diffeomorphism of two symplectic manifolds (M_1, ω_1) and (M_2, ω_2). It is called* symplectomorphism *if $F^* \omega_2 = \omega_1$.*

For a Hamiltonian vector field X_H on (M, ω) denote $g_t : M \to M$ the one-parameter (local) group of diffeomorphisms generated by X_H. Recall that g_t acts by shifts along trajectories of the Hamiltonian system

$$\dot{x} = \{x, H\}$$

by the time t for sufficiently small $|t|$. It will also be called the *Hamiltonian flow* generated by the Hamiltonian H.

Theorem 5.2.9 *For an arbitrary Hamiltonian H on a symplectic manifold (M, ω) the corresponding Hamiltonian flow g_t defines a (local) one-parameter group of symplectomorphisms $g_t : M \to M$,*

$$g_t^* \omega = \omega.$$

Proof: This immediately follows from Corollary 5.2.7. $\square$

Remarkably, on a symplectic manifold locally also the converse statement holds true.

Theorem 5.2.10 *Let $g_t : M \to M$ be a one-parameter group of symplectomorphisms of the symplectic manifold (M, ω). Denote*

$$X = \frac{d}{dt} g_t(x)|_{t=0}$$

the velocity vector field of g_t. Then locally there exists a function H such that $X = X_H$.

Proof: From the equations

$$0 = Lie_X \omega_{ij} = X^k \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial X^k}{\partial x^i} \omega_{kj} + \omega_{ik} \frac{\partial X^k}{\partial x^j}$$

it follows that the 1-form

$$\xi := \xi_i dx^i \quad \text{where} \quad \xi_i = \omega_{ik} X^k$$

is closed, $d\xi = 0$. Indeed,

$$\frac{\partial \xi_i}{\partial x^j} - \frac{\partial \xi_j}{\partial x^i} = \left(\frac{\partial \omega_{ik}}{\partial x^j} - \frac{\partial \omega_{jk}}{\partial x^i} \right) X^k + \omega_{ik} \frac{\partial X^k}{\partial x^j} + \frac{\partial X^k}{\partial x^i} \omega_{kj} = Lie_X \omega_{ij}$$

where we use the closedness condition (5.1.3) to replace the two terms in the parenthesis with $\partial \omega_{ij} / \partial x^k$. So, due to Poincaré Lemma it locally exists a function H such that $\xi = dH$. Raising the index by the inverse matrix ω^{ks} one obtains $X = X_H$. $\square$

It is easy to see that the obstruction to the global existence of the Hamiltonian H is in the cohomology $H^1(M, \mathbb{R})$. See more details below in the discussion of the Poisson cohomology.

Exercise 5.2.11 *Let $g_t : Q \to Q$ be a one-parameter group of diffeomorphisms of a n-dimensional manifold Q generated by a vector field $X \in Vect(Q)$. Define a lift of this group onto the total space of the cotangent bundle by*

$$G_t : T^*Q \to T^*Q, \quad G_t(x, \xi) = \left(g_t(x), g_{-t}^* \xi \right).$$

1) Prove that G_t is a one-parameter group of symplectomorphisms of T^*Q.

2) Prove that this flow is generated by the Hamiltonian

$$H(x,p) = p_i X^i(x).$$

On a $2n$-dimensional symplectic manifold (M, ω) there is a natural volume element

$$Vol_M = \frac{1}{n!}\,\omega \wedge \omega \wedge \cdots \wedge \omega \quad (n \text{ factors}). \tag{5.2.4}$$

It is easy to see that the $2n$-form (5.2.4) never vanishes. Therefore we arrive at

Proposition 5.2.12 *Any symplectic manifold has a natural orientation.*

Exercise 5.2.13 *On the standard $2n$-dimensional symplectic space $\mathbb{R}^{2n}$ with the symplectic structure $\omega = \sum dp_1 \wedge dq^i$ the volume element* (5.2.4) *takes the form*

$$Vol_{\mathbb{R}^{2n}} = dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2 \wedge \cdots \wedge dp_n \wedge dq^n.$$

From Theorem 5.2.9 it follows

Corollary 5.2.14 (Liouville theorem) *Any Hamiltonian flow on a symplectic manifold is volume preserving.*

Proof: Using Leibnitz rule for Lie derivatives one obtains

$$Lie_{X_H} Vol_M = \frac{1}{n!} Lie_{X_H}\omega \wedge \omega \wedge \cdots \wedge \omega + \cdots + \frac{1}{n!}\,\omega \wedge \omega \wedge \cdots \wedge Lie_{X_H}\omega = 0.$$

$\square$

Remark 5.2.15 *For a nondegenerate antisymmetric $2n \times 2n$ matrix (ω_{ij}) consider the 2-form*

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j$$

on the space $\mathbb{R}^{2n}$ with coordinates $x^1, \ldots, x^{2n}$. The corresponding volume element (5.2.4) *can be represented as follows*

$$\frac{1}{n!}\,\omega \wedge \cdots \wedge \omega = \mathrm{Pf}(\omega) dx^1 \wedge \cdots \wedge dx^{2n} \tag{5.2.5}$$

where $\mathrm{Pf}(\omega)$ is a polynomial in the entries of the matrix ω_{ij}. Such a polynomial is called Pfaffian *of the antisymmetric matrix. For example,*

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$$

$$\mathrm{Pf}\begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & b_3 & -b_2 \\ -a_2 & -b_3 & 0 & b_1 \\ -a_3 & b_2 & -b_1 & 0 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

(prove it!).

Exercise 5.2.16 *Prove that*

$$[\mathrm{Pf}(\omega)]^2 = \det(\omega). \tag{5.2.6}$$

Exercise 5.2.17 *Prove that under a change of a basis in $\mathbb{R}^{2n}$, $\omega \mapsto \omega' = A^T \omega\, A$ for $A \in GL(2n, \mathbb{R})$ the Pfaffian transforms as follows*

$$\mathrm{Pf}\left(A^T \omega\, A\right) = \det A \cdot \mathrm{Pf}(\omega).$$

Let us consider Hamiltonian vector fields on the dual space $M = \mathfrak{g}^*$ to a n-dimensional Lie algebra $\mathfrak{g}$ (see Example 5.1.11 above). More specifically, consider linear Hamiltonians

$$H_a = a_i x^i$$

for some constant coefficients $a_1, \ldots, a_n$. The corresponding Hamiltonian vector field X_{H_a} will be denoted X_a. It is a linear vector field

$$X_a^i(x) = c_k^{ij} a_j x^k, \quad i = 1, \ldots, n$$

associated with the matrix

$$A_k^i(a) = c_k^{ij} a_j, \quad \text{i.e.} \quad X_a^i(x) = A_k^i(a) x^k. \tag{5.2.7}$$

Consider now the following element of the Lie algebra

$$a = a_i e^i \in \mathfrak{g}.$$

It defines a linear operator

$$\mathrm{ad}_a : \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}_a(b) = [b, a]$$

(the so-called *adjoint action* of the Lie algebra on itself). In coordinates one has

$$\left(\mathrm{ad}_a(b)\right)_k = c_k^{ij} b_i a_j = A_k^i(a) b_i \tag{5.2.8}$$

where the matrix $A_k^i(a)$ is as above. Observe that in eq. (5.2.7) one has summation wrt the lower index k of the matrix $A_k^i(a)$ while in eq. (5.2.8) it appears the same matrix but the summation is taken with respect to the upper index i. Therefore one can write eq. (5.2.7) as

$$X_a(x) = \mathrm{ad}_a^*(x) \tag{5.2.9}$$

where

$$\mathrm{ad}_a^* : \mathfrak{g}^* \to \mathfrak{g}^*$$

is the adjoint operator to ad_a.

Recall that the adjoint action of the Lie algebra $\mathfrak{g}$ can be identified with the differential of the adjoint action of the corresponding Lie group G

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}, \quad b \mapsto g\, b\, g^{-1}, \quad g \in G, \quad b \in \mathfrak{g} \tag{5.2.10}$$

where the Lie algebra is identified with the tangent space $T_e G$. So, for a one-parameter subgroup

$$g(t) = e^{t\, a} \in G$$

one has

$$\mathrm{Ad}_{g(t)} b = b + t\,\mathrm{ad}_a(b) + \mathcal{O}(t^2).$$

Therefore the linear Hamiltonian $H_a(x) = a_i x^i$ generate the *coadjoint action* $\mathrm{Ad}^*_{g(t)}$ of the Lie group G on the dual space $\mathfrak{g}^*$ to the Lie algebra.

Taking an arbitrary vector $a \in \mathfrak{g}$ we obtain the coadjoint action of an arbitrary one-parameter subgroup in G generated by the Hamiltonian vector field X_a. So the above arguments together with Corollary 5.2.6 imply the following

Proposition 5.2.18 *The linear Poisson bracket of Example 5.1.11 on the dual space to a Lie algebra $\mathfrak{g}$ is invariant with respect to the coadjoint action of the associated Lie group G.*

5.3 First integrals of Hamiltonian systems

Recall that a smooth function $f \in \mathcal{C}^\infty(M)$ is called first integral of a vector field $X \in Vect(M)$ if $Xf = 0$. The vector field X is tangent to the level surfaces f =const.

Consider a Hamiltonian vector field X_H on a Poisson manifold (M, π).

Lemma 5.3.1 *Derivative of a function f along the Hamiltonian vector field X_H is equal to*

$$X_H f = \{f, H\}. \tag{5.3.1}$$

Proof: Follows immediately from the definitions. □

Corollary 5.3.2 *If the function f Poisson commutes with the Hamiltonian H then it is a first integral of the Hamiltonian vector field X_H.*

In particular, the Hamiltonian H itself is a first integral of X_H. This is the *conservation of energy* statement on Poisson manifolds. In particular we have another

Corollary 5.3.3 *The Hamiltonian vector field X_H on a Poisson manifold is tangent to the level surfaces H=const of the Hamiltonian.*

The following statement about Hamiltonian vector fields on *symplectic* manifolds will be useful in sequel.

Lemma 5.3.4 *A Hamiltonian vector field X_H on a symplectic manifold at any point of the level surface H=const satisfies*

$$\omega(Y, X_H) = 0 \quad \text{for any vector field} \quad Y \quad \text{tangent to the level surface} \quad H = \text{const}. \tag{5.3.2}$$

Proof: According to Lemma 5.1.19 we have

$$\omega(Y, X_H) = Y\,H.$$

As the function H is constant on the level surface, its derivative along any vector field Y tangent to the level surface is equal to zero. □

Theorem 5.3.5 *First integrals of a Hamiltonian system form a Lie subalgebra in the Lie algebra of functions on a Poisson manifold* (M, π).

Proof: Let f, g be two first integrals of a Hamiltonian vector field X_H. From (5.2.1) it follows that

$$X_H\{f, g\} = \{X_H f, g\} + \{f, X_H g\} = 0.$$

□

On a Poisson manifold we have a Lie algebra structure on the space of functions on M defined by the Poisson bracket. We also have a Lie algebra structure on the space of vector fields on M given by the commutator. The following statement establishes a connection between these two structures on a Poisson manifold (M, π).

Theorem 5.3.6 *The map*

$$\mathcal{C}^\infty(M) \to Vect(M), \quad H \mapsto X_H$$

is an (anti)homomorphism of Lie algebras

$$[X_F, X_H] = -X_{\{F,H\}} \quad \forall\ F,\ H \in \mathcal{C}^\infty(M). \tag{5.3.3}$$

Proof: For any function f on M one has

$$X_F f = \{f, F\}.$$

Using (5.2.1) we obtain

$$X_H X_F f = X_H\{f, F\} = \{X_H f, F\} + \{f, X_H F\} = \{\{f, H\}, F\} + \{f, \{F, H\}\}\,.$$

Thus

$$\begin{aligned}&[X_H, X_F]\, f = X_H X_F f - X_F X_H f = \\ &= \{\{f, H\}, F\} + \{f, \{F, H\}\} - \{\{f, F\}, H\} - \{f, \{H, F\}\}\,.\end{aligned}$$

Due to Jacobi identity the first three terms in the rhs give zero. This proves the Theorem.

□

Corollary 5.3.7 *If the Hamiltonians H, F Poisson-commute then the Hamiltonian vector fields X_H, X_F commute.*

Remark 5.3.8 *Actually, for commutativity $[X_H, X_F] = 0$ of Hamiltonian vector fields it suffices that the Poisson bracket $\{H, F\}$=const.*

Exercise 5.3.9 *Prove that two Hamiltonian vector fields on a symplectic manifold commute* iff *the Poisson bracket among the Hamiltonians is a (locally) constant function.*

5.4 Darboux Lemma. Casimirs and symplectic leaves on Poisson manifolds

Darboux lemma claims that, locally all symplectic manifolds of the same dimension are the same, up to a symplectomorphism.

Theorem 5.4.1 (Darboux lemma) *Let $\left(M^{2n}, \omega\right)$ be a symplectic manifold. Then for any point $x_0 \in M^{2n}$ there exists a neighborhood $U \subset M^{2n}$ and local coordinates q^1, ..., q^n, p_1, ..., p_n such that*

$$\omega|_U = \sum_{i=1}^n dp_i \wedge dq^i.$$

Proof: Let p_1 be an arbitrary smooth function on M such that

$$p_1(x_0) = 0, \quad dp_1(x_0) \neq 0.$$

Consider the Hamiltonian vector field X_{p_1}. By assumption it does not vanish at x_0 and, thus also on some neighborhood of this point. In a sufficiently small neighborhood U of x_0 choose a $(2n-1)$-dimensional submanifold N transversal to the vector field,

$$X_{p_1}(y) \neq 0 \quad \forall y \in N \cap U.$$

If U is sufficiently small then, for any $x_1 \in U$ there exists a unique $y \in N \cap U$ such that the trajectory $x(t)$ of the Hamiltonian system

$$\dot{x} = \{x, p_1\}$$

such that $x(0) = y$ reaches x for some t,

$$x(t) = x.$$

Denote $q^1(x) := t$. By construction

$$\{q^1, p_1\} = X_{p_1} q_1 = \frac{d}{dt} t = 1. \tag{5.4.1}$$

We obtain a pair of functions q^1, p_1 on the neighborhood U of x_0 with the canonical Poisson bracket $\{q^1, p_1\} = 1$. If the dimension $n = 1$ then we are done with the proof. Otherwise we proceed by induction.

Define $\tilde{M} \subset U$ by the equations $q^1 = 0$, $p_1 = 0$. It is a smooth submanifold of the dimension $2n-2$. Denote

$$\tilde{\omega} = \omega|_{\tilde{M}}.$$

Let us prove that $\left(\tilde{M}, \tilde{\omega}\right)$ is a symplectic manifold. Clearly $d\tilde{\omega} = 0$. Let us verify nodegenerateness of $\tilde{\omega}$. To this end we will show that, the tangent space $T_y\tilde{M}$ at any point $y \in \tilde{M}$ coincides with the orthogonal complement of the two-dimensional span $\left(X_{p_1}, X_{q^1}\right)$

$$Y \in T_y\tilde{M} \quad \Leftrightarrow \quad \omega(Y, X_{p_1}) = \omega(Y, X_{q^1}) = 0.$$

Take a tangent vector field $Y \in T_y\tilde{M}$. Extend it locally to a vector field on M. At the points of $\tilde{M}$ it satisfies

$$Yp_1 = 0, \quad Yq^1 = 0.$$

Due to the Lemma this implies that

$$\omega(Y, X_{p_1}) = 0, \quad \omega(Y, X_{q^1}) = 0$$

at the points of $\tilde{M}$. Vice versa, these two conditions imply that Y is tangent to $\tilde{M}$.

We have proved that $T_y\tilde{M}$ coincides with the orthogonal complement to the 2-dimensional subspace spanned by X_{p_1}, X_{q^1}. The following statement from linear algebra proves that $\left(\tilde{M}, \tilde{\omega}\right)$ is a $(2n-2)$-dimensional symplectic manifold.

Lemma 5.4.2 *Let V be a subspace in the symplectic space (R^{2n}, ω) such that $\omega|_V$ does not degenerate. Then the restriction of ω on the orthogonal complement*

$$V^\perp := \{Y \in \mathbb{R}^{2n} \mid \omega(Y, X) = 0 \quad \forall\, X \in V\}$$

does not degenerate either.

By induction on a neighbourhood of the point x_0 in $\left(\tilde{M}, \tilde{\omega}\right)$ there exists a system of canonical coordinates $\tilde{q}^2, \dots, \tilde{q}^n, \tilde{p}_2, \dots, \tilde{p}_n$ such that

$$\tilde{\omega} = \sum_{i=2}^{n} d\tilde{p}_i \wedge d\tilde{q}^i.$$

To extend these coordinates on a neighbourhood of $\tilde{M}$ in M we will use shifts along the Hamiltonian flows generated by the functions p_1 and q^1. Observe that the Hamiltonian vector fields X_{p_1} and X_{q^1} are transversal to the submanifold $\tilde{M}$

$$\alpha X_{p_1} + \beta X_{q^1} \in T_y\tilde{M} \quad \Leftrightarrow \quad \alpha = \beta = 0. \tag{5.4.2}$$

Denote g_t and f_s the one-parameter local groups of symplectomorphisms of M generated by X_{p_1} and X_{q^1} respectively. Since $\{q^1, p_1\} = 1$, these one-parameter groups commute

$$g_t \circ f_s = f_s \circ g_t \quad \text{for arbitrary sufficiently small} \quad t,\ s.$$

Due to transversality (5.4.2) for an arbitrary point $x \in M$ sufficiently close to x_0 there exists a unique point $y \in \tilde{M}$ and unique pair of small numbers t, s such that

$$x = g_t\left(f_s(y)\right).$$

Define

$$q^i(x) := \tilde{q}^i(y), \quad p_i(x) := \tilde{p}_i(y), \quad i = 2, \dots, n.$$

We obtain a system of coordinates on a neighbourhood of the point $x_0 \in M$. Let us prove that these coordinates are canonical.

First, by construction the functions p_i and q^i for $i \geq 2$ are invariant with respect to the Hamiltonian flows g_t, f_s generated by p_1 and q^1 respectively. Hence

$$\{p_1, p_i\} = \{p_1, q^i\} = 0, \quad \{q^1, p_i\} = \{q^1, q^i\} = 0 \quad \text{for} \quad i \geq 2. \tag{5.4.3}$$

From vanishing of these Poisson brackets it follows that the Hamiltonian vector fields X_{p_i} and X_{q^i} for $i \geq 2$ are invariant wrt the flows g_t, f_s. So, the values of the two-form ω on these vector fields at the points $y \in \tilde{M}$ and $x = g_t f_s(y) \in M$ coincide. Therefore

$$\{p_i, p_j\}_x = \{p_i, p_j\}_y, \quad \{p_i, q^j\}_x = \{p_i, q^j\}_y, \quad \{q^i, q^j\}_x = \{q^i, q^j\}_y \quad \text{for} \quad i,\, j = 2, \dots, n$$

and $x = g_t f_s(y)$.

Since the functions p_1, q^1 are first integrals of the vector fields X_{p_i}, X_{q^i}, $i \geq 2$, these vector fields are tangent to the level surface $\tilde{M}$. Thus the matrix of the 2-form ω in the basis X_{q^1}, $\dots$, X_{q^n}, X_{p_1}, $\dots$, X_{p_n} decomposes in two blocks of sizes 2×2 and $(2n-2) \times (2n-2)$ where the latter coincides with the matrix of $\tilde{\omega}$ on $\tilde{M}$. Therefore the vector fields $X_{p_i}|_{\tilde{M}}$, $X_{q^i}|_{\tilde{M}}$, $i \geq 2$ are Hamiltonian vector fields on $(\tilde{M}, \tilde{\omega})$ with the Hamiltonians $\tilde{p}_i$, $\tilde{q}^i$ respectively. Due to the inductive construction these are canonical coordinates on $\tilde{M}$. So

$$\{q^i, p_j\} = \delta^i_j, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0, \quad i,\, j = 2, \dots, n. \tag{5.4.4}$$

From (5.4.1), (5.4.3), (5.4.4) it follows that q^1, $\dots$, q^n, p_1, $\dots$, p_n are canonical local coordinates on M. $\square$

Similar arguments can be applied to local classification of Poisson manifolds (M, π) of constant rank.

Exercise 5.4.3 *Let π be an antisymmetric bilinear form on a N-dimensional vector space V. Denote $2n$ the rank of the matrix of the bilinear form. Assuming that $2n < N$ denote $W \subset V$ the kernel of the bilinear form, $\dim W = N - 2n$. Prove that π induces a well-defined antisymmetric bilinear form on the quotient space V/W that is* nondegenerate*.*

Theorem 5.4.4 *Let the matrix $\pi^{ij}(x)$ of the bivector π on the Poisson manifold (M, π) have a constant rank $2n < \dim M$ on a neighbourhood of a point $x_0 \in M$. Denote $k := \dim M - 2n$.*

1) Locally near x_0 there exist k independent[13] functions $c^1(x)$, $\dots$, $c^k(x)$ such that

$$\{f, c^1\} = \cdots = \{f, c^k\} = 0 \quad \forall\; f \in \mathcal{C}^\infty(M). \tag{5.4.5}$$

2) Near x_0 there locally exist functions q^1, $\dots$, q^n, p_1, $\dots$, p_n on M with canonical Poisson brackets

$$\{q^i, p_j\} = \delta^i_j, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0, \quad i,\, j = 1, \dots, n. \tag{5.4.6}$$

3) The functions q^1, $\dots$, q^n, p_1, $\dots$, p_n, $c^1(x)$, $\dots$, $c^k(x)$ give a system of local coordinates on M near the point x_0.

[13] Recall that the functions $c^1(x)$, $\dots$, $c^k(x)$ are called independent if their differentials span a k-dimensional subspace in $T^*_x M$ at every $x \in M$.

The crucial point in the proof of the Theorem is in the following

Lemma 5.4.5 *Let X_1, ..., X_N be smooth vector fields on the manifold M satisfying the following conditions:*

(1) $\dim \operatorname{span}(X_1, \dots, X_N) \equiv m$*;*

(2) There exist smooth functions $c_{ij}^s(x)$ such that

$$[X_i, X_j] = \sum_{s=1}^{N} c_{ij}^s(x) X_s, \quad i,\, j = 1, \dots, N.$$

Then locally there exist independent functions $f_1(x)$, ..., $f_k(x)$ such that

$$m + k = \dim M$$

and

$$X_1 f_a = \cdots = X_N f_a = 0, \quad a = 1, \dots, k.$$

The functions $c^1(x)$, ..., $c^k(x)$ Poisson-commuting with everything are called *Casimir functions* or simply *Casimirs* on the Poisson manifold. From the above statement it follows that every smooth common level surface of the Casimirs

$$M_{\mathbf{c}_0} := \{x \in M \mid c^1(x) = c_0^1, \dots, c^k(x) = c_0^k\}$$

has a natural symplectic structure. In this way one obtains a structure of *symplectic foliation* of a Poisson manifold of constant rank. The level surfaces of the form $M_{\mathbf{c}_0}$ are called *symplectic leaves* of the foliation.

Example 5.4.6 *The Poisson bracket* (5.1.18) *on the dual space to the Lie algebra so*(3) *has constant rank 2 away from the origin. One can choose the function*

$$c(x, y, z) = x^2 + y^2 + z^2$$

as the Casimir. The symplectic leaves are two-dimensional spheres.

Exercise 5.4.7 *For a smooth function $f(x, y, z)$ on $\mathbb{R}^3$ define a Poisson bracket by*

$$\begin{aligned} \{x, y\} &= f_z(x, y, z) \\ \{y, z\} &= f_x(x, y, z) \\ \{z, x\} &= f_y(x, y, z). \end{aligned}$$

Here the subscripts denote the partial derivatives. Prove that the above formulae define a Poisson bracket on $\mathbb{R}^3$. Morever, the symplectic leaves of this bracket are (nonsingular) level surfaces $f(x, y, z) = $ const.

Example 5.4.8 *A function $c(x)$ on the dual space to a Lie algebra $\mathfrak{g}$ is a Casimir of the linear Poisson bracket* (5.1.17) *if it satisfies the linear differential equation*

$$0 = \{c(x), a(x)\} = c_k^{ij} \frac{\partial c}{\partial x^i} a_j x^k, \quad a(x) = a_j x^j$$

for any vector $a \in \mathfrak{g}$. That means that $c(x)$ takes constant values along the trajectories of the Hamiltonian vector field $X_a^i = c_k^{ij} a_j x^k$. Since the vector fields of the form X_a generate the coadjoint action of the associated Lie group G (see the end of Section 5.2 above), we conclude that the Casimirs of the linear Poisson bracket (5.1.17) *are functions invariant with respect to the coadjoint action,*

$$c\left(\mathrm{Ad}_g^*(x)\right) = c(x) \quad \forall\, g \in G.$$

Therefore the symplectic leaves of the linear Poisson bracket on $\mathfrak{g}^*$ are orbits of the coadjoint action of the associated Lie group.

Remark 5.4.9 *An arbitrary foliation $M = \cup_{\phi_0} M_{\phi_0}$ of a codimension m in a Poisson manifold (M, π) represented locally in the form*

$$M_{\phi_0} = \{x \mid \phi^1(x) = \phi_0^1, \dots, \phi^m(x) = \phi_0^m\}$$

will be called cosymplectic if the $m \times m$ matrix $\{\phi^a, \phi^b\}$ does not degenerate on the leaves. In this situation a new Poisson structure $\{\ ,\ \}_D$ can be defined on M s.t. the functions $\phi^a(x)$ are Casimirs of $\{\ ,\ \}_D$. This is the Dirac bracket given explicitly by the formula

$$\{f, g\}_D = \{f, g\} - \sum_{a,b} \{f, \phi^a\}\{\phi^a, \phi^b\}^{-1}\{\phi^b, g\}. \tag{5.4.7}$$

It can be restricted in an obvious way to produce a Poisson structure on every leaf. The restriction map

$$(\mathcal{C}^\infty(M), \{\ ,\ \}) \to (\mathcal{C}^\infty(M_{\phi_0}), \{\ ,\ \}_D)$$

is a homomorphism of Lie algebras.

5.5 Poisson cohomology and supermanifolds

The notion of Poisson cohomology of $(M, \pi\})$ was introduced by Lichnerowicz. Before we proceed to precise definitions let us consider a particular case. Consider a Poisson manifold (M, π). A bivector $\delta\pi = \left(\delta\pi^{ij}(x)\right)$ is called an *infinitesimal deformation* of the Poisson structure if the family of brackets

$$\{f, g\}_\epsilon = \left(\pi^{ij} + \epsilon \cdot \delta\pi^{ij}\right) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

depending on a parameter ϵ satisfies Jacobi identity modulo terms of order $\mathcal{O}\left(\epsilon^2\right)$. In local coordinates this condition reads as follows

$$\pi^{is}\frac{\partial}{\partial x^s}\delta\pi^{jk} + \delta\pi^{is}\frac{\partial}{\partial x^s}\pi^{jk} + \pi^{ks}\frac{\partial}{\partial x^s}\delta\pi^{ij} + \delta\pi^{ks}\frac{\partial}{\partial x^s}\pi^{ij} + \pi^{js}\frac{\partial}{\partial x^s}\delta\pi^{ki} + \delta\pi^{js}\frac{\partial}{\partial x^s}\pi^{ki} = 0 \quad \forall\, i,\ j,\ k.$$

Lemma 5.5.1 *For any vector field $X \in Vect(M)$ on (M, π) the bivector*

$$\delta\pi^{ij} = \mathcal{L}_X \pi^{ij} \tag{5.5.1}$$

defines an infinitesimal deformation of the Poisson structure.

Clearly in this case the deformation of the Poisson structure is induced by the infinitesimal change of coordinates

$$x^i \mapsto x^i + \epsilon \cdot X^i(x) + \mathcal{O}(\epsilon^2)$$

generated by the vector field X. Such infinitesimal deformations will be called *trivial.*

Infinitesimal deformations of (M, π) form a linear space. There is a subspace of trivial infinitesimal deformations. The quotient

$$\{\text{infinitesimal deformations of } (M, \pi)\}/\{\text{trivial infinitesimal deformations}\}$$

classifies deformations of the Poisson structure π on M modulo changes of coordinates. This quotient is denoted $H^2(M, \pi)$ and called the second Poisson cohomology of (M, π).

Example 5.5.2 *Let π be a non-degenerate Poisson structure on the $2n$-dimensional ball $M = B^{2n}$. Then $H^2(M, \pi) = 0$. This follows from Darboux lemma.*

For a general definition of Poisson cohomology we need to use the *Schouten–Nijenhuis bracket.* Denote

$$\Lambda^k = H^0(M, \Lambda^k TM)$$

the space of multivectors on M. The Schouten–Nijenhuis bracket is a bilinear pairing $a, b \mapsto [a, b]$,

$$\Lambda^k \times \Lambda^l \to \Lambda^{k+l-1}$$

uniquely determined by the properties of supersymmetry

$$[b, a] = (-1)^{kl}[a, b], \quad a \in \Lambda^k, \ b \in \Lambda^l \tag{5.5.2}$$

the graded Leibnitz rule

$$[c, a \wedge b] = [c, a] \wedge b + (-1)^{lk+k} a \wedge [c, b], \quad a \in \Lambda^k, \ c \in \Lambda^l \tag{5.5.3}$$

and the conditions $[f, g] = 0, \ f, g \in \Lambda^0 = \mathcal{F}$,

$$[v, f] = v^i \frac{\partial f}{\partial x^i}, \quad v \in \Lambda^1 = Vect(M), \quad f \in \Lambda^0 = \mathcal{F},$$

$[v_1, v_2]$ = commutator of vector fields for $v_1, v_2 \in \Lambda^1$. In particular for a vector field v and a multivector a

$$[v, a] = Lie_v a.$$

Exercise 5.5.3 *For two bivectors $h = (h^{ij})$ and $f = (f^{ij})$ their Schouten–Nijenhuis bracket is the following trivector*

$$[h, f]^{ijk} = \frac{\partial h^{ij}}{\partial x^s} f^{sk} + \frac{\partial f^{ij}}{\partial x^s} h^{sk} + \frac{\partial h^{ki}}{\partial x^s} f^{sj} + \frac{\partial f^{ki}}{\partial x^s} h^{sj} + \frac{\partial h^{jk}}{\partial x^s} f^{si} + \frac{\partial f^{jk}}{\partial x^s} h^{si}. \tag{5.5.4}$$

Observe that the l.h.s. of the Jacobi identity (5.1.13) reads

$$\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} = \frac{1}{2}[h, h]^{ijk}.$$

The Schouten–Nijenhuis bracket satisfies the graded Jacobi identity

$$(-1)^{km}[[a,b],c]+(-1)^{lm}[[c,a],b]+(-1)^{kl}[[b,c],a]=0, \quad a\in\Lambda^k,\ b\in\Lambda^l,\ c\in\Lambda^m. \tag{5.5.5}$$

It follows that, for a Poisson bivector h the map

$$\partial:\Lambda^k\to\Lambda^{k+1},\ \partial a=[h,a] \tag{5.5.6}$$

is a differential, $\partial^2=0$. The cohomology of the complex $(\Lambda^*,\partial\pi)$. We will denote it

$$H^*(M,\pi)=\oplus_{k\geq 0}H^k(M,\pi).$$

In particular, $H^0(M,\pi)$ coincides with the ring of Casimirs of the Poisson bracket, $H^1(M,\pi)$ is the quotient of the Lie algebra of infinitesimal symmetries

$$v\in Vect(M),\ Lie_v\{\ ,\ \}=0$$

over the subalgebra of Hamiltonian vector fields, $H^2(M,\pi)$ is the quotient of the space of infinitesimal deformations of the Poisson bracket by those obtained by infinitesimal changes of coordinates (i.e., by those of the form $Lie_v\{\ ,\ \}$ for a vector field v).

On a symplectic manifold (M,π) Poisson cohomology coincides with the de Rham one. The isomorphism is established by "lowering the indices": for a cocycle $a=(a^{i_1\dots i_k})\in\Lambda^k$ the k-form

$$\sum_{i_1<\dots<i_k}\omega_{i_1\dots i_k}dx^{i_1}\wedge\dots\wedge dx^{i_k}, \quad \omega_{i_1\dots i_k}=h_{i_1j_1}\dots h_{i_kj_k}a^{j_1\dots j_k}$$

is closed.

We will now introduce an equivalent approach to Poisson cohomology that uses the language of supermanifolds. Coordinates on a supermanifold are of two types: even and odd. Even coordinates commute pairwise while odd coordinates anticommute. We will now introduce a particular example of a supermanifold. Let M be an arbitrary smooth manifold of the dimension N. There are standard coordinates x^1, ..., x^N, p_1, ..., p_N on the total space of cotangent bundle T^*M. Here x^i are local coordinates on the base M and p_j are coordinates on the fibers T^*_xM for $x\in M$. Define a supermanifold ΠT^*M with even coordinates x^1, ..., x^N,

$$x^jx^i=x^ix^j$$

and odd coordinates θ_1, ..., θ_N,

$$\theta_j\theta_i=-\theta_i\theta_j.$$

One can say that the supermanifold ΠT^*M is obtained from T^*M by changing the parity along the fibers. As usual, a smooth change of coordinates on the base

$$x^i\mapsto x^{i'}(x)$$

induces a linear change of odd coordinates

$$\theta_i\mapsto\theta_{i'}=\frac{\partial x^i}{\partial x^{i'}}\theta_i. \tag{5.5.7}$$

The variables x^i and θ_j commute pairwise,

$$x^i\theta_j=\theta_jx^i.$$

The space of functions on ΠT^*M has the following description

$$\mathcal{F}(\Pi T^*M) = \left\{ \hat{a} = \sum_k \frac{1}{k!} a^{i_1 \dots i_k}(x) \theta_{i_1} \dots \theta_{i_k} \right\},$$

(note that the sum is necessarily finite due to the anticommutativity of the θ-variables) where the coefficients $a^{i_1 \dots i_k}(x)$ are antisymmetric wrt permutations of indices. Due to the transformation rule (5.5.7) these coefficients are k-vectors on M. Thus

$$\mathcal{F}(\Pi T^*M) \cong \Gamma(\Lambda TM). \tag{5.5.8}$$

As the variables x^i commute with everything, the derivatives $\dfrac{\partial}{\partial x^i}$ as operators on $\mathcal{F}(\Pi T^*M)$ are defined in a natural way. Definition of derivatives wrt odd variables requires more attention. Define *left derivative* $\frac{\partial}{\partial \theta_i}$ as follows.

1) If the monomial $\theta_{i_1} \dots \theta_{i_k}$ does not contain θ_i then

$$\frac{\partial}{\partial \theta_i} (\theta_{i_1} \dots \theta_{i_k}) = 0.$$

2) In the opposite case, using the anticommutativity move the factor θ_i on the first position in the monomial, then erase it.

3) Extend this definition linearly on $\mathcal{F}(\Pi T^*M)$,

$$\frac{\partial}{\partial \theta_i} \left(\sum_k \frac{1}{k!} a^{i_1 \dots i_k}(x) \theta_{i_1} \dots \theta_{i_k} \right) = \sum_k \frac{1}{k!} a^{i_1 \dots i_k}(x) \frac{\partial}{\partial \theta_i} (\theta_{i_1} \dots \theta_{i_k}).$$

Example 5.5.4

$$\frac{\partial}{\partial \theta_1} \theta_1 \theta_2 = \theta_2, \qquad \frac{\partial}{\partial \theta_1} \theta_2 \theta_1 = -\theta_2.$$

The algebra $\mathcal{F}(\Pi T^*M)$ has a natural gradation defined by degree in the odd variables. Define a *superbracket*

$$\mathcal{F}(\Pi T^*M) \times \mathcal{F}(\Pi T^*M) \to \mathcal{F}(\Pi T^*M)$$

on a pair of homogeneous elements P, $Q \in \mathcal{F}(\Pi T^*M)$ by

$$\{P, Q\} = \frac{\partial P}{\partial \theta_s} \frac{\partial Q}{\partial x^s} + (-1)^{|P|} \frac{\partial P}{\partial x^s} \frac{\partial Q}{\partial \theta_s}, \tag{5.5.9}$$

where $|P| := \deg P$.

Proposition 5.5.5 *The superbracket* (5.5.9) *satisfies graded commutativity*

$$\{Q, P\} = (-1)^{|P| \cdot |Q|} \{P, Q\}, \tag{5.5.10}$$

graded Leibnitz rule

$$\{R, P\,Q\} = \{R, P\}\, Q + (-1)^{|P|(|Q|+1)} P \{R, Q\} \tag{5.5.11}$$

and graded Jacobi identity

$$(-1)^{|P||R|}\{\{P, Q\}, R\} + (-1)^{|R||Q|}\{\{R, P\}, Q\} + (-1)^{|Q||P|}\{\{Q, R\}, P\}. \tag{5.5.12}$$

Example 5.5.6 *Let* $\pi = \frac{1}{2}\pi^{ij}(x)\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$ *be a bivector on a manifold* M. *Denote*

$$\hat\pi = \frac{1}{2}\pi^{ij}(x)\theta_i\theta_j \in \mathcal{F}\left(\Pi T^* M\right)$$

the corresponding function on the supermanifold ΠT^*M. *Then*

$$\begin{aligned}\{\hat\pi,\hat\pi\} &= \pi^{sk}\theta_k\,\frac{\partial \pi^{ij}}{\partial x^s}\theta_i\theta_j\\ &= \sum_{i<j<k}\left(\frac{\partial\pi^{ij}}{\partial x^s}\pi^{sk} + \frac{\partial\pi^{ki}}{\partial x^s}\pi^{sj} + \frac{\partial\pi^{jk}}{\partial x^s}\pi^{si}\right)\theta_i\theta_j\theta_k.\end{aligned}$$

This expression is equal to zero iff *the bivector* π *defines a Poisson structure on the manifold* M.

Exercise 5.5.7 *If* $\hat\rho = \frac{1}{2}\rho^{ij}\theta_i\theta_j$ *is another bivector then*

$$\begin{aligned}&\{\hat\pi,\hat\rho\}\\ &= \sum_{i<j<k}\left(\frac{\partial\pi^{ij}}{\partial x^s}\rho^{sk} + \frac{\partial\pi^{ki}}{\partial x^s}\rho^{sj} + \frac{\partial\pi^{jk}}{\partial x^s}\rho^{si} + \frac{\partial\rho^{ij}}{\partial x^s}\pi^{sk} + \frac{\partial\rho^{ki}}{\partial x^s}\pi^{sj} + \frac{\partial\rho^{jk}}{\partial x^s}\pi^{si}\right)\theta_i\theta_j\theta_k.\end{aligned}$$

Let (M,π) be a Poisson manifold. Denote $\mathcal{F}^k = \mathcal{F}^k\left(\Pi T^*M\right)$ the component of degree k of the space of functions on the supermanifold ΠT^*M. Define an operator $\partial\colon \mathcal{F}^k\to\mathcal{F}^{k+1}$ by the formula

$$\partial P = \{\hat\pi, P\}. \tag{5.5.13}$$

Using $\{\hat\pi,\hat\pi\}=0$ along with the graded Jacobi identity (5.5.12) we derive

Proposition 5.5.8 *Square of the operator* (5.5.13) *is equal to zero*

$$\partial^2 = 0.$$

Therefore the Poisson structure on the manifold M induces a structure of a complex of vector spaces on

$$\bigoplus_k \mathcal{F}^k\left(\Pi T^*M\right) = \mathcal{F}\left(\Pi T^*M\right).$$

Cohomology of this complex are called the Poisson cohomology of (M,π)

$$H^k(M,\pi) := H^k\left(\mathcal{F}\left(\Pi T^*M\right),\partial\right).$$

The following simple statement provides a necessary and sufficient condition for triviality of 1- and 2-cocycles on a Poisson manifold (M,π) with trivial topology $M\simeq$ ball and constancy of the rank of π.

Exercise 5.5.9 *Let* $\pi = (\pi^{ij}(x))$ *be a Poisson structure of a constant rank* $2n < N = \dim M$ *on a sufficiently small* N*-dimensional ball* U.

1). 1-cocycle $v = (v^i(x))$ *is trivial in* $H^1(U,\pi)$ iff *the vector field* v *is tangent to the symplectic leaves.*

2). 2-cocycle $\rho = (\rho^{ij}(x))$ *is trivial in* $H^2(U,\pi)$ iff

$$\rho(dc', dc'') = 0 \tag{5.5.14}$$

for an arbitrary pair of Casimirs of the bracket π.

5.6 Symplectic reduction

Let (M, ω) be a symplectic manifold of dimension $2n$ and $H \in \mathcal{C}^\infty(M)$ a Hamiltonian. Assume that the Hamiltonian vector field $X_H(x)$ never vanishes for any $x \in M$. Moreover, assume that there exists a codimension one submanifold $M' \subset M$ such that any integral curve of X_H has a unique point $x' \in M'$ of intersection with M' and $X_H(x')$ is transversal to $T_{x'}M'$. In this case one can consider M' as the space of integral trajectories of the vector field X_H or, equivalently, as the quotient of M over the action of the one-parameter group of diffeomorphisms generated by X_H. Functions on such a quotient can be identified with those functions on M being constant on the integral trajectories. Therefore

$$\mathcal{C}^\infty(M') = \{f \in \mathcal{C}^\infty(M) \,|\, \{f, H\} = 0\}\,.$$

Since the centralizer of H in the Lie algebra $\mathcal{C}^\infty(M)$ is a Lie subalgebra, one obtains a natural Poisson structure on the quotient. Let us call it the *reduced Poisson bracket.*

Lemma 5.6.1 *Any Casimir of the reduced Poisson bracket has locally the form $f(H(x))$ for an arbitrary smooth function $f(H)$.*

Proof: Consider the inverse matrix $\omega^{ij}(x)$ as a nondegenerate antisymmetric bilinear form on the cotangent space T_x^*M,

$$\langle a, b\rangle_x = \omega^{ij}(x) a_i b_j.$$

Denote $v = dH(x) \in T_x^*M$ and introduce the orthogonal complement

$$V = \{u \in T_x^*M \,|\, \langle u, v\rangle_x = 0\} \subset T_x^*M.$$

Due to nondegeneracy one has $\dim V = 2n-1$. The subspace V is spanned by differentials of functions $f \in \mathcal{C}^\infty(M)$ commuting with H: $\{f, H\} = 0$. Obviously it also contains the vector $v \neq 0$ itself. For similar reasons the orthogonal complement of V is one-dimensional,

$$\{w \in T_x^*M \,|\, \langle w, u\rangle = 0 \;\forall\, u \in V\} = \mathrm{span}(v).$$

This implies the statement of Lemma. □

Corollary 5.6.2 *Rank of the reduced Poisson bracket is equal to $2n-2$.*

According to the constructions of Section 5.4 the symplectic foliation on the $(2n-1)$-dimensional manifold M' consists of the level surfaces of the Casimir $H(x)$ of the reduced Poisson bracket. More precisely, denote

$$M_h := \{x \in M \,|\, H(x) = h\}$$

a level surface of the Hamiltonian. Assume that h is a regular value of the function $H(x)$. Then $M_h \subset M$ is a smooth submanifold of codimension 1. The corresponding $(2n-2)$-dimensional symplectic leaf of the reduced Poisson bracket is obtained by intersection

$$M_{\mathrm{red}}(h) := M' \cap M_h \tag{5.6.1}$$

assuming that the submanifolds M' and M_h intersect transversally. A somewhat shorter description says that, points of $M_{\rm red}(h)$ are integral curves of the Hamiltonian vector field belonging to the level surface $H(x) = h$.

Denote $\omega_{\rm red}(h)$ the symplectic structure induced on the symplectic leaf (5.6.1). The pair $(M_{\rm red}(h), \omega_{\rm red}(h))$ is called *reduced symplectic manifold.*

The above construction plays the central role in the procedure of reduction of Hamiltonian systems with symmetries. Consider another Hamiltonian $F \in \mathcal{C}^\infty(M)$ commuting with H. Then shifts along trajectories of the Hamiltonian vector field X_H define a *symmetry* of the Hamiltonian vector field X_F. That means that these shifts map trajectories of X_F to trajectories of X_F. The *symplectic reduction* allows one to reduce the problem of integration of the dynamical system

$$\dot x = \{x, F\}$$

with $2n$ degrees of freedom to a reduced dynamical system with $(2n-2)$ degrees of freedom. Indeed, the Hamiltonian flow generated by X_F maps trajectories of X_H to trajectories of X_H. Moreover it is tangent to the level surface M_h for any h. Therefore it defines a dynamical system on $M_{\rm red}(h)$. It is Hamiltonian wrt the symplectic structure $\omega_{\rm red}(h)$.

Let us consider a more general situation of symplectic reductions over Lie groups of symmetries of Hamiltonian systems. Let

$$G \times M \to M, \quad (g, x) \mapsto g \cdot x$$

$$(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \quad \forall\, g_1,\, g_2 \in G$$

be a smooth (left) action of a connected Lie group G by symplectomorphisms on a symplectic manifold (M, ω). For any one-parameter subgroup

$$\left\{ g_t^a = e^{t\,a} \right\} \subset G, \quad a \in \mathfrak{g}$$

one obtains a flow on M

$$x \mapsto g_t^a \cdot x$$

preserving the symplectic structure. Denote

$$X_a(x) = \frac{d}{dt}\left(g_t^a \cdot x\right)_{t=0} \tag{5.6.2}$$

the velocity vector of the flow. From

$$e^{t\,a} e^{t\,b} = \mathbf{1} + t(a+b) + \mathcal{O}(t^2)$$

we conclude that the vector fields X_a depend linearly on $a \in \mathfrak{g}$. Moreover, using the well known identity for one-parameter subgroups

$$e^{t\,a} e^{t\,b} e^{-t\,a} e^{-t\,b} = \mathbf{1} + t^2 [a, b] + \mathcal{O}(t^2)$$

valid for any pair of elements a, $b \in \mathfrak{g}$ one obtains that

$$[X_a, X_b] = X_{[a,b]}. \tag{5.6.3}$$

Locally the vector field X_a is generated by some Hamiltonian $H_a(x)$. Assume that these Hamiltonians are defined *globally* on M. So one has Hamiltonian vector fields of the form (5.6.2)

$$X_a = X_{H_a} \quad \forall\, a \in \mathfrak{g}.$$

that generate the symplectic action of the Lie group G.

The Hamiltonians $H_a(x)$ are determined up to additive constants. Using this freedom and linearity of X_a in a we can assume, without loss of generality that also the Hamiltonians depend linearly on $a \in \mathfrak{g}$. Due to the identity (5.6.3) one has

$$\{H_a, H_b\} = H_{[a,b]} + \kappa(a,b), \quad a,\, b \in \mathfrak{g} \tag{5.6.4}$$

for some constant $\kappa(a,b)$. Clearly $\kappa(a,b)$ is a bilinear antisymmetric function that is a 2-cocycle on the Lie algebra $\mathfrak{g}$, i.e., it satisfies the identity

$$\kappa\left([a,b],c\right) + \kappa\left([c,a],b\right) + \kappa\left([b,c],a\right) = 0 \quad \forall\, a,\, b,\, c \in \mathfrak{g}$$

(cf. Example 5.1.14 above).

The last assumption about the action of the Lie group is that the 2-cocycle is trivial, i.e., there exists a linear function ℓ on the Lie algebra such that

$$\kappa(a,b) = \ell\left([a,b]\right).$$

Then one can shift the Hamiltonians by constants

$$H_a \mapsto H_a + \ell(a)$$

in order to arrive at the commutation relations

$$\{H_a, H_b\} = H_{[a,b]}. \tag{5.6.5}$$

After all these preliminaries we can give the following

Definition 5.6.3 *A smooth action of a connected Lie group G on a symplectic manifold (M, ω) is called* Hamiltonian *if the generator* (5.6.2) *of any one-parameter subgroup $g_t^a = e^{t\,a}$ is a Hamiltonian vector field with the Hamiltonian $H_a(x)$ depending linearly on $a \in \mathfrak{g}$ and satisfying the commutation relations* (5.6.5)*. The functions $H_a(x)$ will be called* Hamiltonian generators *of the action of the Lie group.*

Example 5.6.4 *Let the group G act on a n-dimensional manifold Q,*

$$G \times Q \to Q, \quad (g,q) \mapsto g \cdot q.$$

We can lift this action on the symplectic manifold $M = T^*Q$ by

$$g \cdot (q, \xi) = \left(g \cdot q, g^{-1*}\xi\right). \tag{5.6.6}$$

We claim that this action is Hamiltonian. The needed family of Hamiltonians $H_a(q,p)$ are given by the following formula

$$H_a(q,p) = \alpha(v_a(q)) = p_i v_a^i(q) \tag{5.6.7}$$

where $\alpha = p_i dq^i$ is the standard 1-form on T^*Q and

$$v_a(q) = \frac{d}{dt}\left(e^{t\,a}\cdot q\right)_{t=0} \in Vect(Q)$$

is the generator of the action on Q of the one-parameter group $\left\{g_t^a = e^{t\,a}\right\} \subset G$.

We leave as an exercise to verify linearity in $a \in \mathfrak{g}$ of the Hamiltonians (5.6.7) *as well as validity of the commutation relation* (5.6.5)*.*

Let us consider a Hamiltonian action of a connected Lie group G on a symplectic manifold (M,ω) generated by Hamiltonians $H_a(x)$. Due to linearity in $a \in \mathfrak{g}$ one obtains a map

$$\mu : M \to \mathfrak{g}^*, \quad \langle \mu(x), a\rangle = H_a(x) \quad \forall\, a \in \mathfrak{g}. \tag{5.6.8}$$

Definition 5.6.5 *The map* (5.6.8) *is called the* moment map *for the Hamiltonian action of G on M.*

Theorem 5.6.6 *Let $\mu : M \to \mathfrak{g}^*$ be the moment map for the Hamiltonian action of a connected Lie group G on (M,ω). Then $\forall\, g \in G$ the following diagram is commutative*

$$\begin{array}{ccc} M & \xrightarrow{g\cdot} & M \\ \mu\big\downarrow & & \mu\big\downarrow \\ \mathfrak{g}^* & \xrightarrow[\mathrm{Ad}^*_{g^{-1}}]{} & \mathfrak{g}^* \end{array}$$

The proof readily follows from the following

Lemma 5.6.7 *For any $g \in G$ one has*

$$H_a(g\cdot x) = H_{\mathrm{Ad}_{g^{-1}}(a)}(x). \tag{5.6.9}$$

Proof: It suffices to verify the statement of the Lemma for a one-parameter subgroup $g_t = e^{t\,b}$ for an arbitrary vector $b \in \mathfrak{g}$. One has

$$\frac{d}{dt} H_a\left(g_t^b \cdot x\right)_{t=0} = X_b H_a(x) = \{H_a, H_b\} = H_{[a,b]} = H_{-\mathrm{ad}_b(a)}.$$

This is the infinitesimal version of the identity (5.6.9). □

Let $H \in \mathcal{C}^\infty(M)$ be a G-invariant Hamiltonian on (M,ω),

$$H(g\cdot x) = H(x) \quad \forall\, g \in G, \quad \forall x \in M. \tag{5.6.10}$$

We say that G acting on (M,ω) is a *group of symmetries* of the Hamiltonian system generated by H.

Lemma 5.6.8 *Let the symplectic manifold (M, ω) be equipped with a Hamiltonian action of a connected Lie group G being a group of symmetries of the Hamiltonian H. Then*

(1) H Poisson commutes with the Hamiltonian generators of the group action

$$\{H, H_a\} = 0 \quad \forall\, a \in \mathfrak{g}.$$

(2) The Hamiltonian flow X_H maps orbits of the G-action to orbits of the same action.

Proof: Let $g_t = e^{t\,a}$ for some $a \in \mathfrak{g}$ be a one-parameter subgroup in G. Using symmetry (5.6.10) of the Hamiltonian we obtain

$$0 = \frac{d}{dt} H(g_t \cdot x) = X_a H = \{H, H_a\}.$$

This proves the first part of Lemma.

Due to the commutativity the flow generated by the Hamiltonian H maps solutions to the system

$$\dot{x} = \{x, H_a\}$$

to other solutions of the same system, for any $a \in \mathfrak{g}$. Such solutions are nothing but orbits of the one-parameter subgroup $g_t = e^{t\,a}$. This proves the second part of Lemma. $\square$

Corollary 5.6.9 *For a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) and for any G-invariant Hamiltonian H the level surface of the moment map*

$$M_{\mu_0} := \{x \in M \,|\, \mu(x) = \mu_0\} \tag{5.6.11}$$

for any $\mu_0 \in \mathfrak{g}^*$ is invariant wrt the flow generated by H.

For a given $\mu_0 \in \mathfrak{g}^*$ denote

$$G_{\mu_0} := \{g \in G \,|\, \mathrm{Ad}_g^*(\mu_0) = \mu_0\} \tag{5.6.12}$$

the stabilizer of the point $\mu_0 \in \mathfrak{g}^*$. It is a Lie subgroup in G. From Theorem 5.6.6 it readily follows that this subgroup acts on the level surface M_{μ_0}.

The flow generated by the G-invariant Hamiltonian H leaves invariant the level surface M_{μ_0} and it reshuffles the orbits of the stabilizer G_{μ_0} belonging to this level surface. The main idea of symplectic reduction is to define the reduced phase space $M_{\mathrm{red}}(\mu_0)$ as the space of these orbits

$$M_{\mathrm{red}}(\mu_0) := M_{\mu_0}/G_{\mu_0}. \tag{5.6.13}$$

This can be done under the following additional assumptions:

(i) $\mu_0 \in \mathfrak{g}^*$ must be a regular value of the moment map. Then the level surface (5.6.11) will be a smooth submanifold in M.

(ii) The stabilizer $G_{\mu_0} \subset G$ is a compact subgroup.

(iii) Elements $g \neq e$ of this subgroup act on the level surface M_{μ_0} without fixed points.

Theorem 5.6.10 *Let (M,ω) be a symplectic manifold equipped with a Hamiltonian action of a Lie group G and let $\mu_0 \in \mathfrak{g}^*$ satisfy the above conditions. Then*

1) the reduced phase space (5.6.13) *carries a natural symplectic structure $\omega_{\rm red}(\mu_0)$.*

2) The flow generated by a G-invariant Hamiltonian $H \in \mathcal{C}^\infty(M)$ can be restricted onto the level surface (5.6.13)*. After the restriction one obtaines a Hamiltonian flow on $(M_{\rm red}(\mu_0), \omega_{\rm red}(\mu_0))$.*

The above construction is called *Marsden–Weinstein symplectic reduction.*

Example 5.6.11 *Consider in particular the zero level of the moment map. Let $0 \in \mathfrak{g}^*$ be a regular value of μ. Then $\mu^{-1}(0)$ is a smooth manifold. Assume that G acts freely and properly on the zero level $\mu^{-1}(0)$ of the moment map. Then also the quotient $\mu^{-1}(0)/G$ is a smooth manifold. This quotient has a natural symplectic structure $\omega_{\rm red}$ inherited from ω. The resulting symplectic manifold is denoted $M/\!\!/G$. It is called the* symplectic quotient *of (M,ω) over the group action G. Observe that* $\dim M/\!\!/G = \dim M - 2\dim G$. *The construction of symplectic quotient is closely related with the Geometric Invariant Theory (GIT).*

5.7 Evolution PDEs as infinite-dimensional Hamiltonian systems

Let us start with a list of main formulae illustrating analogies between basic definitions of the theory of finite-dimensional Hamiltonian systems (the first column of the table) and their infinite-dimensional counterparts (partial differential equations, PDEs). We begin with the case of systems of evolution PDEs with one spatial variable $x \in S^1$.

	ODEs	PDEs
Functions	$H(u)$	$H = \int h(u; u_x, u_{xx}, \dots)\, dx$
Differentials	$dH = \sum \frac{\partial H}{\partial u^i} du^i$	$\delta H = \int \frac{\delta H}{\delta u^i(x)} \delta u^i(x)\, dx$
		$\frac{\delta H}{\delta u^i(x)} = \sum (-1)^s \partial_x^s \frac{\partial h}{\partial u^{i,s}}, \quad u^{i,s} := \partial_x^s u^i$
Vector fields	$\dot u^i = F^i(u)$	$u^i_t = F^i(u; u_x, u_{xx}, \dots)$
Poisson brackets	$\{f,g\} = \sum \{u^i, u^j\} \frac{\partial f}{\partial u^i} \frac{\partial g}{\partial u^j}$	$\{F,G\} = \iint \frac{\delta F}{\delta u^i(x)} \{u^i(x), u^j(y)\} \frac{\delta G}{\delta u^j(y)} dx dy$
Hamiltonian vector fields	$\dot u^i = \{u^i, H\}$	$u^i_t = \{u^i(x), H\}$
		$= \int \{u^i(x), u^j(y)\} \frac{\delta H}{\delta u^j(y)}\, dy$
Super-bracket	$\hat\pi = \sum \{u^i, u^j\} \theta_i \theta_j$	$\hat\pi = \iint \{u^i(x), u^j(y)\} \theta_i(x) \theta_j(y)\, dx\, dy$
	$\{\hat\pi, \hat\pi\} = \sum \frac{\partial \hat\pi}{\partial \theta_i} \frac{\partial \hat\pi}{\partial u^i}$	$\{\hat\pi, \hat\pi\} = \int \frac{\delta \hat\pi}{\delta \theta_i(x)} \frac{\delta \hat\pi}{\delta u^i(x)}\, dx$

The main technical constructions will be

- Formal variational calculus;

- Local Poisson brackets.

Let M be a smooth n-dimensional manifold. The infinite-dimensional "manifold" we will be dealing with the loop space

$$\mathcal{L}(M) = \{S^1 \to M\}. \tag{5.7.1}$$

It will be defined in terms of the ring of functions on it.

For a given chart $U \subset M$ with coordinates $u^1, \ldots, u^n$ denote$\mathcal{A}(U)$ the space of *differential polynomials*, i.e., polynomials in independent variables $u^{i,s}$, $i = 1, \ldots, n$, $s = 1, 2, \ldots$

$$f(x; u; u_x, u_{xx}, \ldots) := \sum_{m \geq 0} f_{i_1 s_1; \ldots; i_m s_m}(x; u) u^{i_1, s_1} \ldots u^{i_m, s_m} \tag{5.7.2}$$

where the coefficients $f_{i_1 s_1; \ldots; i_m s_m}(x; u)$ are smooth functions on $S^1 \times U$. Here the alternative notations are often used

$$u^i_x = u^{i,1}, \quad u^i_{xx} = u^{i,2}, \ldots$$

We emphasize that no polynomiality in $u = \left(u^1, \ldots, u^n\right)$ is assumed.

On the intersection of two charts $\left(U, u^1, \ldots, u^n\right)$ and $\left(U', u^{1'}, \ldots, u^{n'}\right)$ we have graded polynomial changes

$$u^{i'}_x = \frac{\partial u^{i'}}{\partial u^i} u^i_x$$

$$u^{i'}_{xx} = \frac{\partial u^{i'}}{\partial u^i} u^i_{xx} + \frac{\partial^2 u^{i'}}{\partial u^i \partial u^j} u^i_x u^j_x$$

etc. The differential polynomials (5.7.2) are required to be invariant wrt such changes. In this way we obtain a globally well-defined space $\mathcal{A}(M)$. According to this definition every differential polynomial $f(x; u; u_x, u_{xx}, \ldots) \in \mathcal{A}(M)$ can be considered as a function of the jet space $J^N(M)$ of the manifold M for some $N \geq 0$. The variables $u^i_x = u^{i,1}$, $u^i_{xx} = u^{i,2}, \ldots$ will be called *jet variables*.

Introduce an operator $\partial_x : \mathcal{A}(M) \to \mathcal{A}(M)$ by

$$\partial_x f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u^i} u^{i,1} + \cdots + \frac{\partial f}{\partial u^{i,s}} u^{i,s+1} + \ldots. \tag{5.7.3}$$

Define spaces

$$\mathcal{A}_{0,0} = \mathcal{A}/\mathbb{R}, \quad \mathcal{A}_{0,1} = \mathcal{A}_{0,0}\, dx,$$

and an operator

$$d : \mathcal{A}_{0,0} \to \mathcal{A}_{0,1}, \quad df := \partial_x f\, dx. \tag{5.7.4}$$

Denote

$$\Lambda_0 = \mathcal{A}_{0,1}/d\mathcal{A}_{0,0} \tag{5.7.5}$$

the quotient and define an operator

$$d : \mathcal{A}_{0,0} \to \mathcal{A}_{0,1}, \quad df := \partial_x f\, dx. \tag{5.7.6}$$

Elements of the quotient

$$\Lambda_0 = \mathcal{A}_{0,1}/d\mathcal{A}_{0,0} \tag{5.7.7}$$

will be written as "integrals over a circle" S^1

$$\bar f := \int f(x; u; u_x, u_{xx}, \dots) dx \in \Lambda_0. \tag{5.7.8}$$

They will also be called *local functionals* with densities f.

Define a symmetric bilinear form on $\mathcal{A}(M)$ by

$$< f, g >= \int f\, g\, dx. \tag{5.7.9}$$

Its nondegeneracy follows from

Lemma 5.7.1 *If $\int f g\, dx = 0$ for any $g \in \mathcal{A}(M)$ then $f \in \mathcal{A}$ is equal to zero.*

The full *ring* $\mathcal{F} = \mathcal{F}(\mathcal{L}(M))$ of functions on the formal loop space coincides with the completed symmetric tensor algebra of local functionals Λ_0

$$\mathcal{F} = \mathbb{R} \oplus \Lambda_0 \oplus \hat S^2 \Lambda_0 \oplus \hat S^3 \Lambda_0 \hat\oplus \dots \tag{5.7.10}$$

Elements of $\hat S^k \Lambda_0$ are represented as multiple integrals of differential polynomials of k copies of the variables $u^i(x_1)$, ..., $u^i(x_k)$, $u^i_x(x_1)$, ..., $u^i_x(x_k)$ etc.

$$\int f(x_1, \dots, x_k; u(x_1), \dots, u(x_k); u_x(x_1), \dots, u_x(x_k), \dots) dx_1 \dots dx_k \in \hat S^k \Lambda_0$$

(they are often called *k-local functionals*).

Define the *variational bicomplex*

$$\begin{array}{ccccccccc}
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
0 & \to & \mathcal{A}_{2,0} & \stackrel{d}{\to} & \mathcal{A}_{2,1} & \stackrel{\pi}{\to} & \Lambda_2 & \to & 0 \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
0 & \to & \mathcal{A}_{1,0} & \stackrel{d}{\to} & \mathcal{A}_{1,1} & \stackrel{\pi}{\to} & \Lambda_1 & \to & 0 \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
0 & \to & \mathcal{A}_{0,0} & \stackrel{d}{\to} & \mathcal{A}_{0,1} & \stackrel{\pi}{\to} & \Lambda_0 & \to & 0 \\
 & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the entries $\mathcal{A}_{k,l}$ are elements of the total degree $k + l$ in the Grassmann algebra with generators $\delta u^{i,s}$ and dx of the degree l in dx. The horizontal differentials are defined by

$$d : \mathcal{A}_{k,0} \to \mathcal{A}_{k,1}, \quad d\omega = dx \wedge \partial_x \omega, \quad \partial_x \delta u^{i,s} = \delta u^{i,s+1}. \tag{5.7.11}$$

Elements of the factor

$$\Lambda_k = \mathcal{A}_{k,1} / d\mathcal{A}_{k,0}$$

are called *(local) k-forms* on the loop space. k-forms will also be written as integrals

$$\int dx \wedge \omega \in \Lambda_k, \quad \omega = \frac{1}{k!} \omega_{i_1 s_1; \dots; i_k s_k} \delta u^{i_1, s_1} \wedge \dots \wedge \delta u^{i_k, s_k} \in \mathcal{A}_{k,0}. \tag{5.7.12}$$

It is assumed that the coefficients

$$\omega_{i_1 s_1; \dots; i_k s_k} \in \mathcal{A}$$

are antisymmetric wrt permutations of pairs of indices

$$i_p, s_p \leftrightarrow i_q, s_q.$$

Example 5.7.2 *Any 1-form is represented as*

$$\phi = \int dx \wedge \phi_i \delta u^i \in \Lambda_1. \tag{5.7.13}$$

Let us proceed to defining the vertical arrows of the bicomplex. For a monomial

$$\omega = f\, \delta u^{i_1, s_1} \wedge \cdots \wedge \delta u^{i_k, s_k}$$

put

$$\delta\omega = \sum_{t \geq 0} \frac{\partial f}{\partial u^{j,t}} \delta u^{j,t} \wedge \delta u^{i_1, s_1} \wedge \cdots \wedge \delta u^{i_k, s_k}, \tag{5.7.14}$$

and

$$\delta dx = 0.$$

Due to anticommutativity

$$\delta d = -d\delta$$

the action of δ is well-defined also on the quotient Λ_k.

Example 5.7.3 *Action of δ on Λ_0 is given by*

$$\delta \int f\, dx = \int dx \wedge \left(\sum_s (-1)^s \partial_x^s \frac{\partial f}{\partial u^{i,s}} \right) \delta u^i \in \Lambda_1 \tag{5.7.15}$$

(the Euler–Lagrange *differential).*

We will use the following notations

$$\frac{\delta \bar{f}}{\delta u^i(x)} := \sum_s (-1)^s \partial_x^s \frac{\partial f}{\partial u^{i,s}} \tag{5.7.16}$$

for components of the 1-form. Here $\bar{f} = \int f\, dx$.

Theorem 5.7.4 *For M=*ball *the rows and columns of the variational bicomplex are exact.*

Example 5.7.5 *A differential polynomial f is (locally) a total x-derivative* iff

$$\frac{\delta \bar{f}}{\delta u^i(x)} = 0, \quad i = 1, \dots, n.$$

The following statement is useful in the study of deformations of infinite-dimensional Poisson brackets.

Example 5.7.6 (Inverse problem of caluclus of variations.) *A system of ODEs*

$$\omega_1(u; u_x, u_{xx}, \dots) = 0, \dots, \omega_n(u; u_x, u_{xx}, \dots) = 0$$

can be locally represented as the system of Euler–Lagrange equations

$$\omega_i = \frac{\delta \bar{f}}{\delta u^i(x)}, \quad i = 1, \dots, n$$

for some functional

$$\bar{f} = \int f(u; u_x, \dots)\, dx$$

iff *the following equations*

$$\frac{\partial \omega_i}{\partial u^{j,s}} = \sum_{t \geq s} (-1)^t \begin{pmatrix} t \\ s \end{pmatrix} \partial_x^{t-s} \frac{\partial \omega_j}{\partial u^{i,t}} \tag{5.7.17}$$

hold true for any $i,\ j = 1, \dots, n,\ s = 0,\ 1,\ \dots$ (the Helmholtz criterion*).*
(Hint: spell out the conditions of closedness

$$\delta\omega = 0 \in \Lambda_2$$

of a 1-form

$$\omega = \omega_i\, dx \wedge \delta u^i \in \Lambda_1;$$

then use exactness of the variational bicomplex.

Example 5.7.7 *A 2-form*

$$\omega = \int dx\ \left(\omega_{i;\, j\, s} \delta u^i \wedge \delta u^{j,s}\right)$$

is closed, $\delta\omega = 0 \in \Lambda_3$, iff

$$\left(\sum_{m=s}^{t+s} \sum_{r=0}^{m-s} + \sum_{m \geq t+s+1} \sum_{r=0}^{t} \right) (-1)^m \begin{pmatrix} m \\ r\, s \end{pmatrix} \partial_x^{m-r-s} \frac{\partial \omega_{j;k\, t-r}}{\partial u^{i,m}} + \frac{\partial \omega_{i;j\, s}}{\partial u^{k,t}} - \frac{\partial \omega_{i;k\, t}}{\partial u^{j,s}} = 0 \tag{5.7.18}$$

for any $i,\ j,\ k = 1, \dots n,\ s = 0,\ 1,\ 2, \dots$.

Corollary 5.7.8 *Any solution to the equations (5.7.18) satisfying*

$$\omega_{i;j\, s} = \sum_{t \geq s} (-1)^{t+1} \begin{pmatrix} t \\ s \end{pmatrix} \partial_x^{t-s} \omega_{j;i\, t}$$

(antisymmetry) can be locally represented in the form

$$\omega_{i;j\, s} = \frac{1}{2} \left(\frac{\partial \phi_i}{\partial u^{j,s}} + \sum_{t \geq s} (-1)^{t+1} \binom{t}{s} \partial_x^{t-s} \frac{\partial \phi_j}{\partial u^{i,t}} \right).$$

5.8 Lagrangian submanifolds, generating functions and Hamilton–Jacobi equation

Let V be a linear subspace in the symplectic space $(\mathbb{R}^{2n}, \omega)$.

Definition 5.8.1 *The subspace V is called* isotropic *if*

$$\omega(X, Y) = 0 \quad \forall\, X,\, Y \in V.$$

Lemma 5.8.2 *Dimension of an arbitrary isotropic subspace in $(\mathbb{R}^{2n}, \omega)$ is less or equal than n.*

Proof: For a linear k-dimensional subspace $V \subset \mathbb{R}^{2n}$ consider its orthogonal complement

$$V^{\perp} := \{X \in \mathbb{R}^{2n} \mid \omega(X, Y) = 0 \quad \forall\, Y \in V.\}$$

Nondegeneracy of the bilinear form ω implies that $\dim V^{\perp} = 2n - k$. The isotropic subspace belongs to its orthogonal complement, $V \subset V^{\perp}$. Therefore $k \leq 2n - k$, $\Rightarrow k \leq n$. $\square$

Definition 5.8.3 *An n-dimensional isotropic subspace in a $2n$-dimensional symplectic space is called* Lagrangian subspace*.*

Example 5.8.4 *In the standard symplectic phase space $(\mathbb{R}^{2n}, dp \wedge dq)$ the coordinate subspace P with coordinates $(p_1, \dots, p_n)$ is a Lagrangian subspace. Another example is given by the coordinate subspace Q with coordinates $(q^1, \dots, q^n)$. More generally, for any subset $I \subset \{1, 2, \dots, n\}$ denote L_I the coordinate subspace with coordinates*

$$\left(p_{i_1}, \dots, p_{i_k}, q^{j_1}, \dots q^{j_{n-k}}\right) \in L_I, \quad i_1, \dots, i_k \in I, \quad j_1, \dots, j_{n-k} \in \{1, 2, \dots, n\} \setminus I. \tag{5.8.1}$$

Clearly it is a Lagrangian subspace in $\mathbb{R}^{2n}$.

Consider now the case of an arbitrary symplectic manifold (M^{2n}, ω).

Definition 5.8.5 *A submanifold $L \subset M^{2n}$ is called* isotropic *if $\omega|_L = 0$. An isotropic submanifold is called* Lagrangian *if $\dim L = n$.*

Clearly the tangent space $T_x L$ to an isotropic submanifold $L \subset M$ for any $x \in L$ is an isotropic subspace in $(T_x M, \omega|_{T_x M})$. A similar claim works for tangent spaces to a Lagrangian submanifold.

The following alternative definition of a Lagrangian submanifold will also be useful. Recall that the symplectic form ω can be locally represented as differential of a 1-form, $\omega = d\alpha$.

Definition 5.8.6 *Let $L \subset M$ be a n-dimensional submanifold of a $2n$-dimensional symplectic manifold (M, ω). It is Lagrangian* iff

$$\oint_{\gamma} \alpha = 0 \tag{5.8.2}$$

for an arbitrary sufficiently small closed contour $\gamma \subset L$.

Lemma 5.8.7 *The definitions 5.8.5 and 5.8.6 are equivalent.*

Proof: Let L be a Lagrangian submanifold in the sense of the first definition. Restricting the equation $\omega = d\alpha$ onto L we obtain

$$d\left(\alpha|_L\right) = 0.$$

Due to Stokes theorem integrals of the closed 1-form $\alpha|_L$ over small[14] closed contours $\gamma \subset L$ are all equal to zero.

Conversely, choose a point $x_0 \in L$ and consider a function

$$S(x) = \int_{x_0}^{x} \alpha \tag{5.8.3}$$

for $x \in L$ sufficiently close to x_0. The integral is taken along an arbitrary sufficiently small path on L connecting x_0 with x. Because of (5.8.2) this integral does not depend on the choice of the integration path, so the function $S(x)$ is well defined. One has

$$\alpha|_L = dS(x) \quad \Rightarrow \quad \omega|_L = d\left(\alpha|_L\right) = 0.$$

$\square$

Consider a particular case of Lagrangian submanifolds in the standard phase space $(\mathbb{R}^{2n}, dp\wedge dq)$ represented as a graph

$$L = \{(q,p) \mid p_i = f_i(q),\ i = 1, \dots, n\}. \tag{5.8.4}$$

Equivalently one can say that the projection of the Lagrangian manifold L onto the coordinate subspace $Q = (q^1, \dots, q^n)$ is a local diffeomorphism.

Proposition 5.8.8 *For a sufficiently small piece of Lagrangian manifold of the form* (5.8.4) *there exists a function $S(q)$ such that*

$$f_i(q) = \frac{\partial S(q)}{\partial q^i}, \quad i = 1, \dots, n.$$

Proof: Put

$$S(q) = \int_{q_0}^{q} (p\,dq)|_L \tag{5.8.5}$$

for an arbitrary point $(q, f(q)) \in L$. Using the second version 5.8.6 of the definition of Lagrangian submanifold we conclude that the integral does not depend on the choice of integration path on L. So the function $S(q)$ on L is well defined. Differentiating the integral wrt the upper limit we conclude that

$$p_i = \frac{\partial S(q)}{\partial q^i}, \quad i = 1, \dots, n. \tag{5.8.6}$$

$\square$

[14] It suffices to consider closed contours homotopic to the trivial one consisting of one point.

Definition 5.8.9 *The function* $S(q)$ *in the representation* (5.8.6) *is called* the generating function *of the Lagrangian submanifold* L.

Example 5.8.10 *Consider a Lagrangian subspace* $L \subset \mathbb{R}^{2n}$ *in the standard symplectic phase space* $(\mathbb{R}^{2n}, dp \wedge dq)$. *Assume that the projection*

$$L \to Q = \{(q^1, \ldots, q^n)\}$$

is one-to-one. Then the generating function of L *is quadratic*

$$L = \left\{ p_i = \frac{\partial S(q)}{\partial q^i} \right\}, \quad S(q) = \frac{1}{2} S_{ij} q^i q^j \tag{5.8.7}$$

for some symmetric $n \times n$ *matrix* (S_{ij}).

The following statement will be important in understanding of the geometric origin of the Hamilton–Jacobi equation.

Theorem 5.8.11 *Let* H *be an arbitrary smooth function on a symplectic manifold* (M^{2n}, ω). *Assume that* $E \in \mathbb{R}$ *is a regular value of the function* H, *so the level surface* $H^{-1}(E) \subset M$ *is a smooth submanifold of the dimension* $2n-1$. *Consider a Lagrangian submanifold* $L \subset M$ *belonging to the level surface,*

$$H|_L \equiv E.$$

Then any integral curve of the Hamiltonian system

$$\dot{x} = \{x, H\}$$

that has a common point with L *belongs entirely to* L.

Proof: According to Corollary 5.3.3 the Hamiltonian vector field X_H is tangent to the level surface of the Hamiltonian. Assume that the integral curve of the Hamiltonian system passes through the point $x \in L$ but the tangent vector $X_H(x)$ does not belong to T_xL. Choose a basis $Y_1, \ldots, Y_n$ in the tangent space T_xL to the Lagrangian submanifold. By definition $\omega(Y_i, Y_j) = 0$. From Lemma 5.3.4 it follows that $\omega(Y, X_H) = 0$ for any vector tangent to the level surface of the Hamiltonian. Thus $\omega(Y_i, X_H) = 0$, $i = 1, \ldots, n$. In this way we obtain a $(n+1)$-dimensional isotropic subspace in T_xM spanned by the vectors $Y_1, \ldots, Y_n, X_H$. Such a contradiction completes the proof of the Theorem. □

Thus a Lagrangian submanifold L belonging to the level surface of the Hamiltonian is fibered into trajectories of the Hamiltonian system. It can be constructed therefore by choosing a submanifold $L_0 \subset L$ of codimension one transversal to the Hamiltonian vector field and, then transporting L_0 along the trajectories of the Hamiltonian flow.

The following Example explains connection of the setting of the Theorem with the (truncated) *Hamilton–Jacobi equation*.

Example 5.8.12 *Let L be a Lagrangian submanifold in the standard symplectic phase space $(\mathbb{R}^{2n}, dp \wedge dq)$ belonging to the non-singular level surface $H(q,p) = E$. Assume that L has the form $p = \frac{\partial S_0(q)}{\partial q}$ for some function $S_0(q)$. Then the generating function satisfies the truncated Hamilton–Jacobi equation*

$$H\left(q, \frac{\partial S_0(q)}{\partial q}\right) = E. \tag{5.8.8}$$

From above considerations it immediately follows

Corollary 5.8.13 *Consider a $(n-1)$-dimensional isotropic submanifold L_0 in the $(2n-1)$-dimensional level surface $H(q,p) = E$. Assume that (1) the projection of L_0 to the coordinate q-space $Q = \{(q^1, \dots, q^n)\}$ is injective, and (2) that the Hamiltonian vector field X_H is transversal to L_0 at the points of L_0. Then the submanifold*

$L = \{(x_0, x(t)) \mid x_0 \in L_0,\ x(t)$ is the solution to the system $\dot{x} = \{x, H\}$ such that $x(0) = x_0\}$

spanned with trajectories $x(t)$ of the Hamiltonian vector field for sufficiently small $|t|$ is a Lagrangian submanifold in $(\mathbb{R}^{2n}, dp \wedge dq)$ belonging to the level surface of the Hamiltonian. Every such Lagrangian submanifold can be obtained by this construction.

Example 5.8.14 *Consider now a $(n+1)$-dimensional Lagrangian submanifold L in the extended phase space $(\mathbb{R}^{2n+2}, dp \wedge dq - dE \wedge dt)$ (see Remark 5.1.17 above) belonging to the level surface $\hat{H} = 0$ of the Hamiltonian $\hat{H} = H(q,p) - E$. Assume that the projection of L to the coordinate space $(q^1, \dots, q^n, t)$ is a local diffeomorphism. Like in the previous Example we prove that the Lagrangian submanifold is spanned by trajectories of the Hamiltonian system*

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q,p)}{\partial q}, \quad \dot{t} = 1.$$

The generating function $S(q,t)$ of such a Lagrangian submanifold satisfies

$$p_i = \frac{\partial S(q,t)}{\partial q^i}, \quad E = -\frac{\partial S(q,t)}{\partial t}.$$

The imposed condition $L \subset \{\hat{H} = 0\}$ spells out as the Hamilton–Jacobi equation *for the function $S = S(q,t)$*

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0. \tag{5.8.9}$$

Corollary 5.8.15 *Let $L_0 = \left\{q = \frac{\partial S_0(q)}{\partial q}\right\}$ be a Lagrangian submanifold in the phase space $\left(\mathbb{R}^{2n}, dp \wedge dq\right)$. For sufficiently small $|t|$ consider the family L_t of Lagrangian manifolds obtained by translations of L_0 along trajectories of the Hamiltonian system $\dot{x} = \{x, H\}$. Then*

1) the family $L_t \subset \mathbb{R}^{2n}$ spans a $(n+1)$-dimensional Lagrangian submanifold $L \subset \mathbb{R}^{2n+2}$ belonging to the level surface $\hat{H} = 0$ with the generating function

$$S(q,t) = S_0(q) + \int_0^t [pdq - H(q,p)dt] \tag{5.8.10}$$

where the integration is taken along the integral trajectory $(q(t), p(t))$ *of the Hamiltonian flow starting from the point* $x_0 = (q, \partial S_0(q)/\partial q) \in L_0$ *to a point in* L_t.

2) Any solution $S(q,t)$ *to the Hamilton–Jacobi equation* (5.8.9) *can be obtained in this way.*

Remark 5.8.16 *We have proved that the problem of solving the Hamilton–Jacobi PDEs* (5.8.9) *(or the truncated version* (5.8.8)*) can be reduced to integrating the corresponding system of Hamiltonian ODEs. However, the technique based on the Hamilton–Jacobi equation proved to be very powerful in solving Hamiltonian systems. The main point is the following statement, due to Jacobi.*

Theorem 5.8.17 *Let* $S(q, Q)$ *be a solution to the truncated Hamilton–Jacobi equation depending on* n *independent parameters* $Q = (Q^1, \ldots, Q^n)$,

$$H\left(q, \frac{\partial S(q,Q)}{\partial q}\right) = E(Q). \tag{5.8.11}$$

Assume validity of the following nondegeneracy condition

$$\det\left(\frac{\partial^2 S(q,Q)}{\partial q^i\,\partial Q^j}\right) \neq 0,$$

so that the system of equations

$$p_i = \frac{\partial S(q,Q)}{\partial q^i}, \quad i = 1, \ldots, n$$

can be locally resolved by smooth functions $Q^1(q,p), \ldots, Q^n(q,p)$. *Then these functions are pairwise commuting first integrals of the Hamiltonian system* $\dot{x} = \{x, H\}$.

Proof: *Introduce functions* $P_1, \ldots, P_n$ *by*

$$P_i = -\frac{\partial S(q,Q)}{\partial Q^i}, \quad i = 1, \ldots, n.$$

From the nondegeneracy condition it follows that $P_1, \ldots, P_n, Q^1, \ldots, Q^n$ *is a system of local coordinates. By definition one has*

$$pdq - PdQ = dS(q,Q) \quad \Rightarrow \quad dp \wedge dq = dP \wedge dQ.$$

So, the new coordinates are also canonical.

One has a Lagrangian fibration

$$L_Q := \left\{p = \frac{\partial S(q,Q)}{\partial q}\right\}$$

over the n*-dimensional coordinate space* $Q^1, \ldots, Q^n$. *The Hamilton–Jacobi equation* (5.8.11) *says that the restriction*

$$H\,|_{L_Q}$$

depends only on Q *but not on* P. *So, after the canonical transformation* $(q,p) \mapsto (Q,P)$ *one obtains*

$$H(q,p),\ Q^1(q,p), \ldots, Q^n(q,p) \mapsto E(Q),\ Q^1, \ldots, Q^n.$$

□

The family L_t of Lagrangian submanifolds in the phase space obtained by moving L_0 along the trajectories of the Hamiltonian flow $\dot{x} = \{x, H\}$ can be defined also for not necessarily small values of the time parameter t. However, it can happen that, after some moment t_0 the projection of L_t on the coordinate q-space will not be a diffeomorphism. In other words for $t > t_0$ the Lagrangian submanifold L_t will not be representable as a graph of the form $p_i = f_i(q)$. In order to describe this process in more details we will study below singularities of projections of Lagrangian submanifolds. This study will also be related to topology of Lagrangian Grassmannian.

5.9 Symplectic group

Definition 5.9.1 *The symplectic group $Sp(n)$ is the group of linear symplectomorphisms*

$$A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

of the symplectic phase space $\left(\mathbb{R}^{2n}, \omega = dp \wedge dq\right)$, *i.e.*

$$\omega\left(Ax, Ay\right) = \omega\left(x, y\right) \quad \forall\; x,\, y \in \mathbb{R}^{2n}. \tag{5.9.1}$$

Denote the $2n \times 2n$ matrix of the linear map A by the same letter A. The definition (5.9.1) can be rewritten in the matrix form

$$A^T J\, A = J \tag{5.9.2}$$

where J is the standard antisymmetric matrix of the 2-form $dp \wedge dq$ (see eq.(5.1.4) above).

Exercise 5.9.2 *Prove that the matrix A satisfying* (5.9.2) *has* $\det A = 1$.

Hint: use the Liouville theorem.

Example 5.9.3 *For $n = 1$ the group $Sp(1)$ coincides with the group $SL(2)$ of 2×2 matrices with determinant 1.*

Consider the Lie algebra $sp(n)$ of the group $Sp(n)$. By definition it consists of symplectic transformations close to identity,

$$A = \mathbf{1} + \delta A + \text{higher order terms}, \quad \delta A^T J + J\, \delta A = 0.$$

The corresponding linear vector field

$$X(x) = \delta A\, x$$

must be a symmetry of the symplectic structure. Thus it is a Hamiltonian vector field with quadratic Hamiltonian

$$H(x) = \frac{1}{2} Q_{ij} x^i x^j = \frac{1}{2}\left(a_{ij} q^i q^j + 2 b_i^j q^i p_j + c^{ij} p_i p_j\right) \tag{5.9.3}$$

where $n \times n$ matrices $a = (a_{ij})$ and $c = (c^{ij})$ are symmetric. The $2n \times 2n$ matrix $Q = (Q_{ij})$ of the quadratic Hamiltonian reads

$$Q = \begin{pmatrix} a & b^T \\ b & c \end{pmatrix}.$$

The matrix $\delta A \in sp(n)$ is equal to

$$\delta A = \begin{pmatrix} b & c \\ -a & -b^T \end{pmatrix}.$$

Let δB be another infinitesimal symplectic transformation generated by a quadratic Hamiltonian $F(x)$. From Theorem 5.3.6 it follows that the commutator $[\delta A, \delta B]$ is a Hamiltonian vector field generated by the quadratic Hamiltonian $-\{H, F\}$.

We arrive at the following

Theorem 5.9.4 *The Lie algebra $sp(n)$ of the symplectic group is isomorphic to the space of quadratic Hamiltonians* (5.9.3) *with the Lie algebra structure given by the Poisson bracket with negative sign.*

So, the dimension of the Lie algebra $sp(n)$ and, hence, of the group $Sp(n)$ is equal to

$$\dim Sp(n) = n(2n+1).$$

We will now obtain another useful realization of the symplectic group. Define a complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$ by introducing complex coordinates $z = q + ip$. Observe that the matrix of the operator of multiplication by $i = \sqrt{-1}$ coincides with the matrix J of the symplectic form. The group $GL(n, \mathbb{C})$ of complex invertible linear transformations consists of those invertible linear maps $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ that commute with the operator J of multiplication by i.

Define also a Euclidean inner product on $\mathbb{R}^{2n}$ by

$$ds^2 = \sum_{i=1}^{n} \left({dq^i}^2 + {dp_i}^2 \right).$$

Linear transformations preserving the Euclidean structure form the group $O(2n)$.

So, we have three structures on $\mathbb{R}^{2n}$: a symplectic form, a complex structure, and a Euclidean structure.

Lemma 5.9.5 *A linear transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserving two of the above structures preserves also the third one.*

Proof: We have to consider combinations of the following three conditions for the matrix $A \in Mat(2n, \mathbb{R})$

$$A \in Sp(n) \Leftrightarrow A^T J\, A = J \tag{5.9.4}$$

$$A \in Gl(n, \mathbb{C}) \Leftrightarrow AJ = JA \text{ and } \det A \neq 0 \tag{5.9.5}$$

$$A \in O(2n) \Leftrightarrow A^T A = \mathbf{1}. \tag{5.9.6}$$

1) (5.9.4)+(5.9.5) imply

$$J = A^T J\, A = A^T A\, J \Rightarrow A^T A = \mathbf{1}.$$

2) (5.9.4)+(5.9.6) imply

$$J = A^T A\, J = A^T J\, A.$$

3) (5.9.5)+(5.9.6) imply

$$A^T = A^{-1} \Rightarrow A^T J\, A = A^{-1} J\, A = A^{-1} A\, J = J.$$

□

Corollary 5.9.6 *One has the following group isomorphisms*

$$O(2n) \cap GL(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap Sp(n) = Sp(n) \cap O(2n) = U(n). \tag{5.9.7}$$

Recall that the unitary group $U(n)$ is defined as the subgroup of $GL(n, \mathbb{C})$ consisting of linear transformations preserving the Hermitean form $dz\, d\bar{z}$. Observe that the real part of the Hermitean form coincides with the above Euclidean structure while its imaginary part coincides with the symplectic form.

Example 5.9.7 *For $n = 1$ the symplectic group consists of 2×2 unimodular matrices (see above). Complex linear transformations are multiplications by complex numbers $a + ib$. Their matrices are $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. In the intersection of these two families we obtain the matrices of the form $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$. This is a unitary transformation of the one-dimensional complex space given by multiplication by $e^{i\phi}$.*

5.10 Lagrangian Grassmannian

Definition 5.10.1 *The set $\Lambda(n)$ of all Lagrangian subspaces in the symplectic space $(\mathbb{R}^{2n}, dp \wedge dq)$ is called the* Lagrangian Grassmannian*.*

Example 5.10.2 *For $n = 1$ any line on the plane passing through the origin is Lagrangian. So $\Lambda(1) = \mathbb{R}\mathbf{P}^1 \simeq S^1$.*

We will now introduce a structure of a smooth manifold on $\Lambda(n)$. We have already seen that Lagrangian subspaces $L \subset \mathbb{R}^{2n}$ projectable onto the coordinate Lagrangian q-subspace $Q = \{(q^1, \ldots, q^n)\}$ can be represented as

$$L = \{p_i = \frac{\partial S(q)}{\partial q^i}, \quad i = 1, \ldots, n\}, \quad S(q) = \frac{1}{2} S_{ij} q^i q^j.$$

Thus, one can use coefficients $S_{ij} = S_{ji}$ of the quadratic generating function as coordinates on the subset of $\Lambda(n)$ of Lagrangian subspaces intersecting transversally the coordinate Lagrangian subspace $P = \{(p_1, \ldots, p_n)\}$. Following this idea we will construct an atlas of 2^n charts on the Lagrangian submanifold using the following geometrical

Lemma 5.10.3 *For any Lagrangian subspace $L \in \Lambda(n)$ there exists a subset $I \subset \{1, 2, \ldots, n\}$ such that $L \cap L_I = 0$.*

Recall (see Example 5.8.4 above) that L_I is a coordinate Lagrangian subspace in $\mathbb{R}^{2n}$ with the coordinates

$$L_I \ni \left(q^{i_1}, \ldots, q^{i_k}, p_{j_1}, \ldots, p_{j_{n-k}}\right), \quad i_1, \ldots, i_k \in I, \quad j_1, \ldots, j_{n-k} \in J := \{1, 2, \ldots, n\} \setminus I.$$

Proof: Consider the intersection $T = L \cap P$ of L with the coordinate Lagrangian subspace P. Denote $k = \dim T$. If $k = 0$ then we are done: take $I =$ the empty set. Otherwise choose a coordinate $(n-k)$-dimensional subspace $V \subset P$ with the coordinates $(p_{j_1}, \ldots, p_{j_{n-k}})$ intersecting transversally T:

$$T \cap V = 0.$$

Denote $J = \{j_1, \ldots, j_{n-k}\}$ and take the complement $I = \{1, 2, \ldots, n\} \setminus J$. Let us prove that $L \cap L_I = 0$.

Denote W the k-dimensional subspace with the coordinates $\left(q^{i_1}, \ldots, q^{i_k}\right)$ where $\{i_1, i_2, \ldots, i_k\} = I$. By definition

$$L_I = V \oplus W.$$

Assume that $X \in L \cap L_I$. Then, since $X \in L$ we have

$$\omega(X, L) = 0 \quad \text{hence} \quad \omega(X, T) = 0.$$

Next, since $X \in L_I$ we have

$$\omega(X, L_I) = 0 \quad \text{hence} \quad \omega(X, V) = 0.$$

By construction $P = T \oplus V$, therefore $\omega(X, P) = 0$. Since P is a Lagrangian subspace we deduce that $X \in P$. Therefore $X \in P \cap L = T$, $X \in P \cap L_I = V$. But $T \cap V = 0$, so $X = 0$.

$\square$

According to the Lemma the Lagrangian Grassmannian can be covered by 2^n charts $\mathcal{U}_I$ where I is an arbitrary subset of $\{1, 2, \ldots, n\}$ defined by

$$L \in \mathcal{U}_I \quad \Leftrightarrow \quad L \cap L_I = 0.$$

Equivalently, the projection of $L \in \Lambda(n)$ onto the coordinate Lagrangian subspace L_J, $J = \{1, 2, \ldots, n\} \setminus I$ is an isomorphism of linear spaces. Local coordinates on the chart $\mathcal{U}_I$ can be chosen taken coefficients of a quadratic generating function $S(x)$, $x \in L_J$. We leave as an exercise to verify that, on the intersections $\mathcal{U}_{I_1} \cap \mathcal{U}_{I_2}$ the transition functions are smooth.

As the number of subsets in $\{1, 2, \ldots, n\}$ is equal to 2^n, we obtain an atlas of 2^n charts on the $\frac{n(n+1)}{2}$-dimensional manifold $\Lambda(n)$.

We will now give an alternative description of the Lagrangian Grassmannian representing it as a homogeneous space of the unitary group $U(n)$. Recall that the unitary group consists of $n \times n$ complex matrices satisfying

$$\bar{U}^T U = \mathbf{1}.$$

Any real symmetric matrix satisfies this condition *iff* it is orthogonal. We obtain a natural embedding $O(n) \subset U(n)$. Also recall that, according to Lemma 5.10.3 the unitary group is a subgroup in the symplectic group. Therefore it acts on the set of Lagrangian subspaces.

Lemma 5.10.4 $\Lambda(n) \simeq U(n)/O(n)$.

Proof: Fix a Lagrangian subspace $L_0 \in \Lambda(n)$. We want to obtain any other Lagrangian subspace L by acting on L_0 with a unitary transformation $A \in U(n) \subset Sp(n)$. Choose an orthonormal, with respect to the Euclidean structure on $\mathbb{R}^{2n}$, basis τ_0 in L_0 and an orthonormal basis τ in L. The pairs $(\tau_0, J\tau_0)$ and $(\tau, J\tau)$ give us two orthonormal bases in $\mathbb{R}^{2n}$. Consider the orthogonal transformation $A \in O(2n)$ mapping the first basis to the second one. By construction it commutes with the operator J of multiplication by i, that is $A \in GL(n, \mathbb{C})$. So $A \in O(2n) \cap GL(n, \mathbb{C}) = U(n)$.

We obtain an action of the unitary group on the Lagrangian Grassmannian. Clearly the action is transitive. Let us describe the stabilizer of L_0 with respect to this action of $U(n)$ on $\Lambda(n)$. If the Lagrangian subspaces L and L_0 coincide then τ and τ_0 are two orthonormal bases in the n-dimensional space, so the linear transformation A belongs to $O(n)$. $\square$

Corollary 5.10.5 $\Lambda(n)$ *is a connected compact manifold.*

From Lemma 5.10.4 we obtain a fibration $U(n) \to \Lambda(n)$ with the fiber $O(n)$. For example, for $n = 1$ $\Lambda(1) = \mathbb{R}\mathbf{P}^1 \simeq S^1$, $U(1) = S^1$, $O(1) = \{\pm 1\}$. The fibration $U(1) \to S^1$ is a two-sheet covering over the circle.

We will now construct an important fibration of the Lagrangian Grassmannian $\Lambda(n)$ over the circle. It is given by the square of determinant

$$\det{}^2 : \Lambda(n) = U(n)/O(n) \to S^1. \tag{5.10.1}$$

The map is well defined since (1) determinant of a unitary matrix is a complex number with absolute value 1 and (2) determinant of an orthogonal matrix is equal to ± 1, so the map (5.10.1) does not depend on the choice of a representative in the coset $\in U(n)/O(n)$.

Let us describe the fiber of the map $\det^2$ over a given point $e^{i\phi} \in S^1$. It suffices to consider the full preimage of the point $1 \in S^1$. Denote this preimage by

$$S\Lambda(n) := \{L \in \Lambda(n) \mid \det{}^2(L) = 1\}.$$

The subgroup $SU(n) \subset U(n)$ of unitary matrices with determinant 1 acts transitively on $S\Lambda(n)$; the stabilizer of any point in $S\Lambda(n)$ is isomorphic to $SO(n)$. Therefore the fibers of the $\det^2$-map can be identified with the quotient $SU(n)/SO(n)$. By the way, it implies that the fibers are connected.

We will now prove that the $\det^2$ map provides a generator in the cohomology group $H^1(\Lambda(n), \mathbb{Z})$.

Lemma 5.10.6 $\pi_1(\Lambda(n)) = \mathbb{Z}$.

Proof: Consider a commutative diagram of fibrations

$$\begin{array}{ccccc}
SO(n) & \to & O(n) & \overset{\det}{\to} & S^0 = \{\pm 1\} \\
\downarrow & & \downarrow & & \downarrow \\
SU(n) & \to & U(n) & \overset{\det}{\to} & S^1 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle e^{2i\phi}} \\
S\Lambda(n) & \to & \Lambda(n) & \overset{\det^2}{\to} & S^1
\end{array}$$

Applying the long exact sequence of the fibration[15] in the first column

$$\cdots \to \pi_1(SO(n)) \to \pi_1(SU(n)) \to \pi_1(S\Lambda(n)) \to \pi_0(SO(n)) \to 0$$

one deduces from $\pi_1(SU(n)) = 1$, $\pi_0(SO(n)) = 1$ that $S\Lambda(n)$ is simply-connected. From the exact sequence of the fibration in the last line

$$\cdots \to \pi_1(S\Lambda(n)) \to \pi_1(\Lambda(n)) \overset{\det^2}{\to} \pi_1(S^1) \to \pi_0(S\Lambda(n)) \to 0$$

we obtain, using $\pi_1(S\Lambda(n)) = \pi_0(S\Lambda(n)) = 1$ and $\pi_1(S^1) = \mathbb{Z}$ that $\pi_1(\Lambda(n)) = \mathbb{Z}$. □

The map $\det^2 : \Lambda(n) \to S^1$ defines a 1-cocycle $\alpha \in H^1(\Lambda(n), \mathbb{Z})$. Value of this cocycle on a closed loop $\gamma : S^1 \to \Lambda(n)$ is the degree of the through map

$$S^1 \overset{\gamma}{\to} \Lambda(n) \overset{\det^2}{\to} S^1. \tag{5.10.2}$$

Example 5.10.7 *For a given $L \in \Lambda(n)$ consider the closed curve $\gamma(\theta) = e^{i\theta}L$ in $\Lambda(n)$, $0 \le \theta \le \pi$ (observe that multiplication by -1 belongs to the subgroup $O(n)$). We have $\det\left(e^{i\theta} \cdot \mathbf{1}\right) = e^{in\theta}$, so $\det^2\left(e^{i\theta} \cdot \mathbf{1}\right) = e^{2in\theta}$. Thus the value of the cocycle α on γ equals n.*

For subsequent considerations it will be useful to construct local sections of the fibration $U(n) \to \Lambda(n)$ over every coordinate chart $\mathcal{U}_I$. Let us do it for the particular chart with $I = \emptyset$. That is, the Lagrangian subspaces belonging to this chart are graphs of the form

$$L = \{p = Sq\} \quad \text{where} \quad S = (S_{ij}) \quad \text{is a symmetric} \quad n \times n \quad \text{matrix.}$$

Define a complex $n \times n$ matrix U by

$$U = \frac{1 + i\,S}{1 - i\,S}. \tag{5.10.3}$$

The formula makes sense since $1 - iS$ is an invertible matrix. Indeed, all eigenvalues of the symmetric matrix S are real.

Lemma 5.10.8 *(i) The matrix U satisfies the following properties.*

(1) $U \in U(n)$.

(2) $U^T = U$.

(3) $\det(U + 1) \neq 0$.

(ii) For any matrix U satisfying the above three conditions define the matrix

$$S = i\frac{1 - U}{1 + U}. \tag{5.10.4}$$

[15]Recall that for a fibration $p : E \to B$ with the base B, the fiber $p^{-1}(\mathrm{pt}) =: F$ and the total space of fibration E one has a long exact sequence of homotopy groups

$$\cdots \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \cdots \to \pi_0(E) \to 0.$$

It satisfies

(4) *It has real entries.*

(5) $S^T = S$.

(6) *The matrix* $\sqrt{U} := \frac{1+iS}{\sqrt{1+S^2}}$ *is symmetric, unitary and it satisfies* $\left(\sqrt{U}\right)^2 = U$.

(7) *The matrix* $\sqrt{U}$ *maps the Lagrangian subspace* $L_\emptyset$ *to* $L = \{p = Sq\}$.

Proof: Applying the Hermitean conjugation $U^* = \bar{U}^T$ to the matrix (5.10.3) obtain

$$U^* = \frac{1 - iS}{1 + iS} = U^{-1}.$$

Symmetry of the matrix U is obvious. To prove the third statement we observe that $(U + 1)^{-1} = \frac{1}{2}(1 - iS)$.

Proof of the formula (5.10.4) as well as of the properties (4)–(6) is straightforward. In order to verify the last statement of the Lemma first observe that the real $n \times n$ matrix $\sqrt{1+S^2}$ maps $L_\emptyset$ to itself. So, it suffices to check that the matrix $1 + iS$ maps $L_\emptyset$ to L. Indeed, the real and imaginary parts of the vector $z = (1 + iS)q$ are

$$\Re z = q, \quad \Im z = Sq.$$

Hence $z \in L$. $\square$

Corollary 5.10.9 *The value of the* $\det^2$ *map at the point* $L = \{p = Sq\}$ *in* $\mathcal{U}_\emptyset \subset \Lambda(n)$ *is equal to*

$$\det{}^2(L) = \det \frac{1 + i\,S}{1 - i\,S}. \tag{5.10.5}$$

5.11 Maslov index

Our next goal is to represent the cocyle $\alpha \in H^1(\Lambda(n), \mathbb{Z})$ by intersection with a cycle of codimension 1.

For a given integer $k \geq 0$ and a given Lagrangian subspace $L_0 \in \Lambda(n)$ consider the subset

$$\Lambda^k(n, L_0) = \{L \in \Lambda(n) \mid \dim L \cap L_0 = k\}. \tag{5.11.1}$$

Lemma 5.11.1 $\Lambda^k(n, L_0)$ *is an open submanifold in* $\Lambda(n)$ *of codimension* $\frac{k(k+1)}{2}$.

Proof: Consider the set of symmetric $n \times n$ matrices of corank k. Let us prove that this set has codimension $\frac{k(k+1)}{2}$ in the space of all symmetric matrices.

Assume that there exists a nondegenerate principal $(n - k) \times (n - k)$ minor

$$\left(S_{i_p i_q}\right)_{1 \leq p,\, q \leq n-k}.$$

Write

$$S = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \quad A^T = A, \quad C^T = C$$

where the $(n-k) \times (n-k)$ block A has nonzero determinant. Eliminating the B^T-block we obtain

$$\left(\begin{array}{cc} A^{-1} & 0 \\ -B^T A^{-1} & \mathbf{1} \end{array} \right) \left(\begin{array}{cc} A & B \\ B^T & C \end{array} \right) = \left(\begin{array}{cc} \mathbf{1} & A^{-1}B \\ 0 & C - B^T A^{-1} B \end{array} \right).$$

This matrix has corank k *iff* the symmetric $k \times k$ matrix $C - B^T A^{-1} B$ is equal to zero. This imposes $\frac{k(k+1)}{2}$ equations on S.

The above calculation proves the statement of the Lemma for the case L_0 is the coordinate q-subspace. For any other choice the proof is similar. □

The concrete realization of the cycle $\Lambda^1(n)$ as a subset in the Lagrangian Grassmannian clearly depends on the choice of the subspace L_0. Choose

$$L_0 = \{q_1 = \cdots = q_n = 0\}. \tag{5.11.2}$$

Exercise 5.11.2 *For the choice* (5.11.2) *prove that* $\mathcal{U}_\emptyset \cap \Lambda^1(n) = \emptyset$.

We will now choose a subset of n charts of the form $\mathcal{U}_I$ covering $\Lambda^1(n)$. For any $k = 1, \ldots, n$ denote L_k the coordinate Lagrangian subspace with coordinates

$$L_k = \{(q_1, \ldots, \hat{q}_k, \ldots, q_n, p_k)\}. \tag{5.11.3}$$

Here and below a hat means that this coordinate is omitted from the list. For the corresponding coordinate chart $\mathcal{U}_I$, $I = \{1, 2, \ldots, \hat{k}, \ldots, n\}$ the short notation

$$\mathcal{U}_k := \mathcal{U}_{\{1,2,\ldots,\hat{k},\ldots,n\}}$$

will be used. Recall that this chart consists of Lagrangian subspaces projectable onto L_k.

Lemma 5.11.3 *Let $L(t)$, $|t| < \epsilon$ for a sufficiently small ϵ be a smooth curve in $\Lambda(n)$ having a unique intersection point with $\Lambda^1(n)$ at $t = 0$. Then there exists $k \in \{1, 2, \ldots, n\}$ such that, for small $|t|$ the curve belongs to the chart $\mathcal{U}_k$, i.e., it can be represented in the form*

$$\begin{aligned} p_i &= p_i\left(q_1, \ldots, \hat{q}_k, \ldots, q_n, p_k; t\right) = \sum_{j \neq k} R_{ij}(t) q_j + R_{ik}(t) p_k, \quad i \neq k \\ q_k &= q_k\left(q_1, \ldots, \hat{q}_k, \ldots, q_n, p_k; t\right) = \sum_{j \neq k} R_{kj}(t) q_j + R_{kk}(t) p_k \end{aligned} \tag{5.11.4}$$

for some smooth functions $R_{ij}(t)$.

Proof: Consider the case of a family of Lagrangian subspaces $L(t)$ belonging to the chart $\mathcal{U}_{\{1,2,\ldots,n\}}$. They can be represented in the form

$$q_i = \sum_{j=1}^n S_{ij}(t) p_j, \quad i = 1, \ldots, n \tag{5.11.5}$$

for some symmetric matrix $S_{ij}(t)$ smoothly depending on t. $L(t_0)$ belongs to Λ^1 for some t_0, i.e., by definition, $\dim L(t_0) \cap L_0 = 1$ *iff* the rank of the matrix $S(t_0) = (S_{ij}(t_0))$ is equal to $n-1$. Then there exists a nonzero vector $v = (v_1, \dots, v_n)$ in the kernel of $S_{ij}(t_0)$,

$$S(t_0)v = 0.$$

The vector v is determined uniquely up to a nonzero factor. We will derive the representation of the form (5.11.4) under the condition $v_k \neq 0$. It suffices to consider the particular case $v_n \neq 0$. Then the $(n-1) \times (n-1)$ minor

$$S_{ij}(t_0), \quad i,\, j = 1, \dots, n-1$$

does not degenerate. Write

$$S = \begin{pmatrix} A & b \\ b^T & c \end{pmatrix}$$

where $A = (S_{ij}(t))_{1 \leq i,\, j \leq n-1}$, $b = (b_1, \dots, b_n)^T$, $b_i = S_{in}(t)$, $c = S_{nn}(t)$. The matrix A does not degenerate. So, eqs. (5.11.5) can be resolved for p_1, ..., p_{n-1}, q_n. This yields a representation of the Lagrangian subspace $L(t)$ as a graph of a (linear) function on L_n:

$$\begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_{n-1} \\ q_n \end{pmatrix} = \begin{pmatrix} A^{-1} & -A^{-1}b \\ & \\ & \\ & \\ & \\ b^T A^{-1} & c - b^T A^{-1} b \end{pmatrix} \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_{n-1} \\ p_n \end{pmatrix}$$

This completes the proof of Lemma for $L(t)$ belonging to the chart $\mathcal{U}_{\{1,2,\dots,n\}}$. For other charts the proof is analogous. □

Corollary 5.11.4 *Intersection of the curve* (5.11.4) *with* $\Lambda^1(n)$ *is determined by the equation*

$$\frac{\partial q_k}{\partial p_k} \equiv R_{kk}(t) = 0. \tag{5.11.6}$$

The curve $\gamma(t)$ *is transversal to the cycle* $\Lambda^1(n)$ *at the point* $\gamma(t_0) \in \Lambda^1(n)$ iff $t = t_0$ *is a simple zero of* $R_{kk}(t)$

Proof: Let $L(t)$ be represented in the form (5.11.4). For the points in the intersection $L \cap L_0$ we have

$$p_i = R_{ik}(t)p_k, \quad i \neq k, \quad 0 = R_{kk}(t)p_k.$$

For $R_{kk}(t) \neq 0$ the set $L(t) \cap L_0$ consists of one point 0. This proves the Corollary. □

According to the above statements the intersection points of the curve $\gamma(t)$ with the part of the cycle $\Lambda^1(n)$ belonging to $\mathcal{U}_k$ are determined by the equation

$$\frac{\partial q_k}{\partial p_k} = 0 \quad \text{where} \quad q_k = q_k\left(q_1, \dots, \hat{q}_k, \dots, q_n, p_k\right).$$

If the intersection is transversal then this derivative changes sign at the intersection point.

Definition 5.11.5 *The positive/negative side of the cycle $\Lambda^1(n)$ is defined by the condition*

$$\frac{\partial q_k}{\partial p_k} < 0 \;/\; \frac{\partial q_k}{\partial p_k} > 0$$

respectively.

Example 5.11.6 *Let us first consider the case $n = 1$. Consider the circle*

$$q = \cos\theta$$
$$p = \sin\theta$$

on the (q,p)-plane. The Lagrangian Grassmannian $\Lambda(1)$ can be obtained by identifying the opposite points on the circle. Thus

$$(q,p) = (\cos\theta, \sin\theta)\,, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \tag{5.11.7}$$

can be considered as a closed curve on $\Lambda(1)$. Let us look at the intersection of this curve with the cycle $\Lambda^1 = \Lambda^1(1, L_0)$. Choose $L_0 = \{p = 0\}$. Then the cycle Λ^1 is the point $(1,0) \sim (-1,0)$. The curve (5.11.7) intersects Λ^1 at one point $\theta = 0$. Near this point the curve can be represented as a graph $q = q(p)$. One has

$$\frac{dq}{dp} = \frac{dq/d\theta}{dp/d\theta} = -\tan\theta.$$

So, the curve (5.11.7) is on the negative side $dq/dp > 0$ of the cycle Λ^1 for $\theta < 0$ and on the positive side for $\theta > 0$. At $\theta = 0$ the derivative dq/dp has a simple zero. Thus the intersection index of the curve (5.11.7) with the cycle Λ^1 is equal to $+1$.

Let us now compute the value of the cocycle defined by the degree of the $\det^2$ *map on the curve (5.11.7). In the complex coordinates the curve reads*

$$q + ip = e^{i\theta}.$$

So

$$\overset{2}{\det}(\theta) = e^{2i\theta}.$$

To compute the degree of the map $S^1 \to S^1$,

$$\theta \mapsto e^{2i\theta}$$

it suffices to count the algebraic number of preimages of the point 1. Within the range $\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ the only point in the preimage is $\theta = 0$. It is easy to see that the map is orientation-preserving near $\theta = 0$. Thus the degree of the $\det^2$ *map on the closed loop (5.11.7) is equal to $+1$. As $H_1(\Lambda(n), \mathbb{Z}) \simeq \pi_1(\Lambda(n)) = \mathbb{Z}$ we conclude that, for $n = 1$ the value of the* $\det^2$ *cocycle on any closed oriented loop in $\Lambda(1)$ coincides with the intersection index of the loop with the cycle Λ^1. In other words, the* $\det^2$ *cocycle is dual to the cycle $\Lambda^1(1)$.*

Let us proceed to the general case. For a given $L \in \Lambda^1(n)$ consider the curve $\gamma(\theta) = e^{i\theta} L$. For $\theta = 0$ it crosses the cycle $\Lambda^1(n)$.

Lemma 5.11.7 *The curve $\gamma(\theta)$ intersects transversally the cycle $\Lambda^1(n)$.*

Proof: Let us first compute the curve $\gamma(\theta)$ for $L \in \mathcal{U}_\emptyset$, $L = \{p = Sq\} = \{(1 + iS)q\}$. We have

$$e^{i\theta}L = (\cos\theta - \sin\theta\, S)q + i(\sin\theta + \cos\theta\, S)q = (\cos\theta - \sin\theta\, S)\left[q + i\frac{\sin\theta + \cos\theta\, S}{\cos\theta - \sin\theta\, S}\, q\right].$$

So, the curve $\gamma(\theta)$ for sufficiently small $|\theta|$ belongs to the chart $\mathcal{U}_\emptyset$ and the corresponding symmetric matrix $S(\theta)$ reads

$$S(\theta) = \frac{\sin\theta + \cos\theta\, S}{\cos\theta - \sin\theta\, S}. \tag{5.11.8}$$

The velocity vector of this curve at the initial point

$$\frac{d}{d\theta}S(\theta)|_{\theta=0} = 1 + S^2$$

is a symmetric positive definite matrix.

A similar calculation can be repeated for the Lagrangian subspace L in $\Lambda^1(n)$ assuming that L is projectable onto one of the coordinate Lagrangian subspaces. Introduce the canonical coordinates $Q_1, \ldots, Q_n, P_1, \ldots, P_n$, $dp \wedge dq = dP \wedge dQ$, on the set of Lagrangian subspaces projectable onto L_k by

$$\begin{aligned} Q_i &= q_i, \quad i \neq k, \quad Q_k = -p_k \\ P_i &= p_i, \quad i \neq k, \quad P_k = q_k. \end{aligned}$$

In these coordinates the Lagrangian subspace L reads $L = \{P = SQ\}$ for some symmetric matrix S. Repeating the above calculation we obtain again that the velocity vector of the curve $\gamma(\theta) = e^{i\theta}L$ at the point $\gamma(0) \in \Lambda^1(n)$ corresponds to a positive definite symmetric matrix $1 + S^2$. According to Lemma 5.11.3 the velocity vector is not transversal to $\Lambda^1(n)$ *iff* its diagonal entry $\frac{d}{d\theta}S_{kk}(\theta)|_{\theta=0} = \left(1 + S^2\right)_{kk}$ is equal to zero. But all diagonal entries of a positive definite symmetric matrix are positive. This contradiction completes the proof of the Lemma. $\square$

Theorem 5.11.8 *The oriented cycle $\Lambda^1(n) \subset \Lambda(n)$ is dual to the cocycle* $\det^2 \in H^1(\Lambda(n), \mathbb{Z})$.

Proof: Consider the closed curve $\gamma(\theta) = e^{i\theta}L$, $0 \leq \theta \leq \pi$ in $\Lambda(n)$. We already know that the value of the cocycle $\det^2$ on this curve is equal to n (see Example 5.10.7 above). It remains to prove that the number of intersection points of this curve with the cycle $\Lambda^1(n)$ is also equal to n.

Let the Lagrangian subspace $L \in \Lambda^0(n)$ have the form $L = \{q + iSq\}$ for a symmetric matrix S. Denote $U = \frac{1+iS}{1-iS} \in U(n)$ the corresponding symmetric unitary matrix satisfying $\det(1 + U) \neq 0$ such that

$$S = S(U) = i\frac{1 - U}{1 + U}$$

(see Lemma 5.10.8 above). For the rotated subspace $e^{i\theta}L$ the corresponding symmetric matrix $S(\theta)$ reads

$$S(\theta) = \frac{\sin\theta + \cos\theta\, S}{\cos\theta - \sin\theta\, S} = i\,\frac{1 - e^{2i\theta}U}{1 + e^{2i\theta}U}.$$

This Lagrangian subspace fails to belong to $\Lambda^0(n)$ *iff*

$$\det\left(1+e^{2i\theta}U\right)=0. \tag{5.11.9}$$

Let us compute the number of solutions to this equation wrt θ.

Denote $e^{i\phi_1}$, ..., $e^{i\phi_n}$ the eigenvalues of the matrix U. By assumption they satisfy

$$|\phi_k|<\pi, \quad k=1,\dots,n.$$

Without loss of generality we can assume that all the eigenvalues are distinct. Equation (5.11.9) reduces to

$$\theta\equiv\frac{\pi-\phi_k}{2}\bmod\pi.$$

There are exactly n solutions to this equation on the interval $(0,\pi)$. Like in Example 5.11.6 it is easy to see that all intersection points have to be counted with $+1$ sign. That is, the intersection index of the curve with the oriented cycle $\Lambda^1(n)$ is equal to n.

On another side, value of the $\det^2$ cocycle on the curve $e^{i\theta}L$ is equal to the degree of the following map $\theta\to S^1$, $0\le\theta\le\pi$

$$\det\frac{1+iS(\theta)}{1-iS(\theta)}=\det\left(e^{2i\theta}U\right).$$

This degree is also equal to n.

We have proved that the intersection index of the curve γ with the cycle $\Lambda^1(n)$ is equal to the value of the basic cocycle

$$\det{}^2\in H^1(\Lambda(n),\mathbb{Z}).$$

This means that the cycle $\Lambda^1(n)$ is dual to the cocycle $\det^2$. □

5.12 Applications to quasiclassical asymptotics of solutions to Schrödinger equation

The Schrödinger equation for the wave function $\psi=\psi(\mathbf{x},t;\hbar)$ is a linear partial differential equation depending on a small parameter $\hbar$. It is one of the main object in quantum mechanics. Here we consider the simplest case of the Schrödinger equation for a particle of mass m in the n-dimensional space with a potential $U(\mathbf{x})$

$$i\hbar\frac{\partial\psi}{\partial t}=\hat H\psi, \quad \hat H:=-\frac{\hbar^2}{2m}\Delta+U\left(\mathbf{x}\right) \tag{5.12.1}$$

Here

$$\Delta=\sum_{k=1}^n\frac{\partial^2}{\partial x_k^2}$$

is the Laplace operator. The stationary version of the Schrödinger equation

$$\hat H\psi=E\psi \tag{5.12.2}$$

depending on a parameter E will also be under consideration.

The study of behaviour of solutions to Schrödinger equation in the limit $\hbar \to 0$ is an important point in the analysis of correspondence between classical and quantum mechanics. Locally the asymptotic solutions are supposed to have the form

$$\psi = e^{\frac{i}{\hbar} S} \tag{5.12.3}$$

where S admits an asymptotic expansion in positive powers of $\hbar$

$$S \sim S_0(x,t) + \hbar\, S_1(x,t) + \hbar^2 S_2(x,t) + \dots \tag{5.12.4}$$

For the leading term the Schrödinger equations (5.12.1) and (5.12.2) reduce to the Hamilton–Jacobi equation

$$\frac{\partial S_0}{\partial t} + H\left(\mathbf{x}, \frac{\partial S_0}{\partial x}\right) = 0 \tag{5.12.5}$$

and the truncated Hamilton–Jacobi equation

$$H\left(\mathbf{x}, \frac{\partial S_0}{\partial x}\right) = E \tag{5.12.6}$$

respectively. Here the Hamiltonian[16] is given by

$$H\left(\mathbf{x}, \mathbf{p}\right) = \frac{\mathbf{p}^2}{2m} + U(\mathbf{x}). \tag{5.12.7}$$

It is often called *classical Hamiltonian* while the operator $\hat{H}$ is obtained by its *quantization*.

As we already know, solutions to the truncated Hamilton–Jacobi equation are generating functions of Lagrangian submanifolds in the symplectic space $\left(\mathbb{R}^{2n}, dp \wedge dx\right)$ belonging to the energy level surface $H(\mathbf{x}, \mathbf{p}) = E$. A geometric interpretation of solutions to the full Hamilton–Jacobi equation takes us to the study of families of Lagrangian submanifolds transported by the Hamiltonian flow

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}.$$

Such a geometric interpretation suggests that the quasiclassical asymptotics (5.12.3), (5.12.4) work only before arriving at singularities of projections of the Lagrangian submanifolds. It turns out that the description of the *global* structure of the quasiclassical solutions involves Maslov index of the Lagrangian submanifolds.

We will begin with considering the simplest case of quasiclassical asymptotics of stationary Schrödinger equation for $n = 1$,

$$-\frac{\hbar^2}{2}\psi'' + u(x)\psi = E\,\psi \tag{5.12.8}$$

(we put $m = 1$). To derive the structure of the asymptotic expansion (5.12.3), (5.12.4) it is convenient to do the substitution

$$\psi = e^{\frac{i}{\hbar}\int^x \sigma\, dx}.$$

[16] In this section we will use the notations $(x^1, \dots, x^n) = \mathbf{x}$ for the canonical coordinates in the Hamiltonian formalism.

For the function σ obtain a Riccati equation

$$-i\,\hbar\,\sigma' + \sigma^2 + 2(u - E) = 0. \tag{5.12.9}$$

Look for a solution in the form of a power series in $\hbar$

$$\sigma = \sigma_0 + \hbar\,\sigma_1 + \hbar^2\sigma_2 + \dots$$

This gives

$$\sigma_0 = \pm\sqrt{2(E-u)}$$

$$\sigma_1 = -\frac{i}{4}\frac{u'}{E-u} = \frac{i}{4}\frac{d}{dx}\log(E-u)$$

etc. This implies the following structure of the quasiclassical solution

$$\psi_\pm = (E - u(x))^{-1/4}\exp\left[\pm\frac{i}{\hbar}\int^x \sqrt{2(E-u(x))}\,dx\right](1 + \mathcal{O}(\hbar)). \tag{5.12.10}$$

We see that the function S_0 from (5.12.3), (5.12.4) in this example has the form

$$S_0 = \pm\int^x \sqrt{2(E-u(x))}\,dx.$$

So, it satisfies the truncated Hamilton–Jacobi equation

$$H\left(x, S_0'(x)\right) = E, \quad \text{where} \quad H(x,p) = \frac{p^2}{2} + u(x).$$

Observe that, on the energy level surface

$$\frac{p^2}{2} + u(x) = E \tag{5.12.11}$$

one has

$$p = \pm\sqrt{2(E-u(x))}.$$

Thus the phase function S_0 coincides with the generating function of the Lagrangian curve $H = E$

$$S_0 = \int^x p\,dx. \tag{5.12.12}$$

We will us assume that the potential $u(x)$ is a bounded from below smooth function of $x \in \mathbb{R}$ going sufficiently fast to $+\infty$ for $x \to \pm\infty$. The quasiclassical solution (5.12.10) has different properties in two regions

$$u(x) < E, \quad \text{oscillatory behaviour}$$

and

$$u(x) > E, \quad \text{exponential growth/decay.}$$

A natural question to be addressed now is in matching of the two asymptotic solutions near a *turning point* x_0 such that $u(x_0) = E$. At this point the projection of the Lagrangian curve (5.12.11) onto the x-axis becomes singular.

Consider the case of a largest turning point x_0, i.e., $u(x) > E$ for $x > x_0$. Assume that $u(x) - E$ has a simple zero at $x = x_0$. For $x > x_0$ choose the exponentially decaying solution

$$\psi_{\text{right}} = (u(x) - E)^{-1/4} \exp\left[-\frac{1}{\hbar}\int_{x_0}^{x} \sqrt{2(u(x) - E)}\right] (1 + \mathcal{O}(\hbar)). \tag{5.12.13}$$

On the left of the turning point we have two oscillatory solutions (5.12.10). We want to find coefficients $C_\pm$ such that (5.12.13) and

$$\psi_{\text{left}} = C_+\psi_+ + C_-\psi_- \tag{5.12.14}$$

were the asymptotic expansions of the *same* solution to the Schrödinger equation (5.12.8) in the regions $x > x_0$ and $x < x_0$ respectively.

Proposition 5.12.1 *If the asymptotic expansions* (5.12.13), (5.12.14) *correspond to the same solution of eq.* (5.12.8) *then*

$$C_\pm = e^{\pm \frac{\pi i}{4}}.$$

We will outline the idea of derivation of the above proposition. It is based on calculation of the asymptotic behaviour of solutions to the Schrödinger equation near the turning point x_0. Write

$$u(x) - E = a_0(x - x_0) + a_1(x - x_0)^2 + \dots, \quad a_0 \neq 0$$

the Taylor expansion near the turning point. Assume that x_0 is the right end point of the interval $u(x) \leq E$, so $a_0 > 0$. After a change of the independent variable

$$x \mapsto \bar{x}, \quad x - x_0 = \lambda\,\bar{x}, \quad \lambda = \frac{\hbar^{2/3}}{(2a_0)^{1/3}}$$

the equation (5.12.8) becomes

$$\frac{d^2\psi}{d\bar{x}^2} - \left[\bar{x} + \mathcal{O}\left(\hbar^{2/3}\right)\right]\psi = 0.$$

This suggests that, modulo small corrections the solutions to the Schrödinger equation near a turning point can be approximated by solutions to *Airy equation*

$$y'' - x\,y = 0. \tag{5.12.15}$$

Solutions to eq. (5.12.15) are entire functions of the complex variable $x \in \mathbb{C}$. They can be represented by a contour integral

$$y(x) = \int_\gamma e^{-\frac{\lambda^3}{3} + \lambda x} d\lambda \tag{5.12.16}$$

where the integration contour γ goes to infinity such that

$$\Re\left(-\frac{\lambda^3}{3} + \lambda x\right) \to 0, \quad x \to \infty, \quad x \in \gamma$$

To this end the tails of the contour γ must go to infinity in such a way that

$$\Re(\lambda^3)|_\gamma \to +\infty.$$

For different choices of the contour we obtain different solutions. They depend on the homology class of γ in the relative homology $H_1(\mathbb{C}, \{\Re(\lambda^3) >> 0\})$. In particular, integrating along the imaginary axis we obtain, after a change of the integration variable $\lambda = i\,t$ and multiplication of (5.12.16) by a suitable constant the solution

$$\mathrm{Ai}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-\frac{\lambda^3}{3}+\lambda\, x} d\lambda = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + x\, t\right) dt \tag{5.12.17}$$

called *Airy function.*

Let us describe the asymptotic behaviour of Airy function for large $|x|$. They can be obtained using the method of *steepest descent.* First, consider the case $x \to +\infty$. The phase $-\frac{1}{3}\lambda^3 + x\,\lambda$ has a critical points at $\lambda = \pm\sqrt{x}$. The direction of the steepest descent goes in the vertical direction. Move the integration from the imaginary axis to the line $\Re(\lambda) = -\sqrt{x}$. Denote s the parameter along this line,

$$\lambda = -\sqrt{x} + i\, s.$$

After the substitution

$$-\frac{\lambda^3}{3} + \lambda\, x = -\frac{2}{3}x^{3/2} - \sqrt{x}\, s^2 + \frac{i\, s^3}{3}$$

one obtains

$$\mathrm{Ai}(x) \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{2}{3}x^{3/2} - \sqrt{x}\, s^2} ds \simeq \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}, \quad x \to +\infty. \tag{5.12.18}$$

The case $x \to -\infty$ can be treated in a similar way. In this case one has two critical points of the phase

$$\lambda = i\sqrt{|x|} \quad \text{and} \quad \lambda = -i\sqrt{|x|}.$$

The steepest descent directions make angle $\mp\frac{\pi}{4}$ with the real line. Integrate along the broken line

$$\lambda(s) = \begin{cases} \sqrt{|x|} + s\, e^{\frac{\pi i}{4}}, & -\infty < s \le 0 \\ \sqrt{|x|} + s\, e^{-\frac{\pi i}{4}}, & 0 \le s < \infty \end{cases}$$

we obtain, after simple calculations

$$\mathrm{Ai}(x) \simeq \frac{1}{\sqrt{\pi}} \frac{\cos\left(\frac{2}{3}|x|^{3/2} - \frac{\pi}{4}\right)}{|x|^{1/4}}. \tag{5.12.19}$$

Coming back to the problem of matching of asymptotic expansions of solutions to Schrödinger equation (5.12.8) in the regions $x > x_0$, $x \sim x_0$, $x < x_0$ recall that the independent variable in the Airy equation is related to the "physical" coordinate by a shift and rescaling

$$\bar{x} = (2a_0)^{1/3}\frac{x - x_0}{\hbar^{2/3}}.$$

So, the asymptotics (5.12.18), (5.12.19) of the Airy solution $\psi(x) = \mathrm{Ai}(\bar{x})$ in the physical coordinates become

$$\psi \sim \begin{cases} C\dfrac{\exp\left(-\frac{2\sqrt{2a_0}}{3\hbar}(x-x_0)^{3/2}\right)}{2(x-x_0)^{1/4}}, & x > x_0 \\ \\ C\dfrac{\cos\left(\frac{2\sqrt{2a_0}}{3\hbar}|x-x_0|^{3/2} - \frac{\pi}{4}\right)}{|x-x_0|^{1/4}}, & x < x_0 \end{cases} \tag{5.12.20}$$

where

$$C = (2a_0)^{\frac{1}{12}} \hbar^{\frac{1}{6}}.$$

But, for $x > x_0$ one has

$$\int_{x_0}^{x} \sqrt{2(u(x) - E)}\, dx = \int_{x_0}^{x} \sqrt{2a_0(x - x_0) + \dots}\, dx \simeq \frac{2}{3}\sqrt{2a_0}(x - x_0)^{3/2}.$$

So, the solution ψ_{right} (see eq. (5.12.13) above) for x near x_0 behaves as

$$\psi_{\text{right}}(x) \sim a_0^{-\frac{1}{4}} \frac{e^{-\frac{2\sqrt{2a_0}}{3\hbar}(x-x_0)^{3/2}}}{(x - x_0)^{1/4}}, \quad x > x_0.$$

In a similar way on the left of x_0, $x \sim x_0$ from (5.12.14) one obtains

$$\psi_{\pm}(x) \sim a_0^{-\frac{1}{4}} \frac{e^{\pm i \frac{2\sqrt{2a_0}}{3\hbar}|x-x_0|^{3/2}}}{|x - x_0|^{1/4}}, \quad x < x_0.$$

Multiplying the solution $\psi(x) = \text{Ai}\left((2a_0)^{1/3} \frac{x-x_0}{\hbar^{2/3}}\right)$ by a suitable constant we see that all the three asymptotic expansions match one another if the coefficients $C_{\pm}$ are equal to $e^{\pm \frac{\pi i}{4}}$.

Observe that, a similar calculation at the left turning point x_0 produces the same result, as in this case, $a_0 < 0$, the solution

$$\psi_{\text{left}}(x) \sim (-a_0)^{-\frac{1}{4}} \frac{e^{-\frac{2\sqrt{-2a_0}}{3\hbar}(x_0-x)^{3/2}}}{(x_0 - x)^{1/4}}, \quad x < x_0$$

exponentially decays at $x \to -\infty$ and the oscillatory solutions have the form

$$\psi_{\pm}(x) \sim (-a_0)^{-\frac{1}{4}} \frac{e^{\pm i \frac{2\sqrt{-2a_0}}{3\hbar}|x-x_0|^{3/2}}}{|x - x_0|^{1/4}}, \quad x > x_0.$$

Consider now the global behaviour of the quasiclassical solution

$$\psi \sim \frac{1}{\sqrt{p}}\, e^{\frac{i}{\hbar}\int p\, dx}.$$

Consider the simplest case of a potential $u(x)$ with one minimum at $x = x_{\min}$ and monotone descreasing/increasing on the left/on the right of $x_{\min}$. Denote $x_0 < x_1$ the two turning points $u(x) = E$. The quasiclassical solution exponential decays outside the interval $[x_0, x_1]$. Inside the interval it oscillates. From the above computations it follows that

$$\psi \sim \frac{c_0}{\sqrt{p}} \cos\left(\frac{1}{\hbar}\int_{x_0}^{x} p\, dx - \frac{\pi}{4}\right), \quad x > x_0$$

and

$$\psi \sim \frac{c_1}{\sqrt{p}} \cos\left(\frac{1}{\hbar}\int_{x_1}^{x} p\, dx - \frac{\pi}{4}\right), \quad x < x_1$$

for some constants c_0, c_1. These two asymptotics agree on the interval $[x_0, x_1]$ *iff*

$$c_0 \cos\left(\frac{1}{\hbar}\int_{x_0}^{x_1} - \frac{\pi}{4}\right) = c_1 \cos\frac{\pi}{4}$$

that implies

$$\frac{1}{\hbar}\int_{x_0}^{x_1} p\,dx - \frac{\pi}{2} = \pi\, n, \quad c_1 = (-1)^n c_0.$$

The last equation can be rewritten in the form of the *Bohr–Sommerfeld quantization* condition

$$\frac{1}{2\pi}\oint p\,dx = \hbar\left(n + \frac{1}{2}\right). \tag{5.12.21}$$

Exercise 5.12.2 *Prove that the quasiclassical wave function satisfying* (5.12.21) *has* n *zeroes on the interval* (x_0, x_1).

The quantity

$$I(E) := \frac{1}{2\pi}\oint p\,dx \tag{5.12.22}$$

is the *action* variable of the Hamiltonian system

$$\dot{x} = p, \quad \dot{p} = -u'(x)$$

evaluated on the closed Lagrangian curve

$$\frac{p^2}{2} + u(x) = E.$$

The quantization condition (5.12.21) selects as discrete subset $E = E_n$,

$$I(E_n) = \hbar\left(n + \frac{1}{2}\right) \tag{5.12.23}$$

of the Lagrangian curves that correspond to quasiclassical asymptotics of the eigenvalues of the Schrödinger operator $\hat{H}$. The roots of eq. (5.12.23) provide a good approximation for actual eigenvalues at the limit $E \to \infty$.

The quantization condition (5.12.21) admits a generalization to the multidimensional case. Quasiclassical eigenfunctions of the truncated Schödinger equation (5.12.2) are associated with n-dimensional Lagrangian submanifolds $M \subset \mathbb{R}^{2n}$ belonging to the level surfaces of the Hamiltonian

$$H|_L \equiv E.$$

An analogue of the Bohr–Sommerfeld quantization condition imposes a series of restrictions onto the Lagrangian submanifold M. They have the form

$$\frac{1}{2\pi\hbar}\oint_C p\,dx \equiv \frac{1}{4}\alpha(C)\,(\mathrm{mod}\,\mathbb{Z}) \quad \forall\, C \in H_1(M, \mathbb{Z}). \tag{5.12.24}$$

Here $\alpha \in H^1(M, \mathbb{Z})$ is the pullback, with respect to the analogue of the Gauss map

$$M \to \Lambda(n)$$

of the Maslov class denoted by the same letter $\alpha \in H^1(\Lambda(n), \mathbb{Z})$. We see that this condition depends only on the class of α in $H^1(M, \mathbb{Z}/4\mathbb{Z})$.

In the particular case $n = 1$ the general quantization condition (5.12.24) reduces to the classical Bohr–Sommerfeld condition (5.12.21). Indeed, let the cycle C coincide with the level curve $\frac{p^2}{2} + u(x) = E$ going counterclockwise. Then the full preimage of the Maslov cycle $\Lambda^1(1) \in H_0(\Lambda(1) = S^1, \mathbb{Z})$ consists of two points $(x_0, 0)$ and $(x_1, 0)$. According to the orientation introduced above both points are counted with the multiplicity $+1$. Therefore $\alpha(C) = 2$, so we arrive at the quantization condition (5.12.21).

Let us now briefly discuss quasiclassical solutions to the time-dependent Schrödinger equation (5.12.1). The initial data for these solutions consist of

- a Lagrangian submanifold $L_0 \subset \left(\mathbb{R}^{2n}, dp \wedge dq\right)$;
- a 1/2-density A_0 on this submanifold.

We will consider only the particular case of initial Lagrangian submanifolds represented as graphs of functions of q,

$$L_0 = \{p = \frac{\partial S_0}{\partial q}\}.$$

In the q-coordinates the 1/2-density is represented by a function $A_0(q)$. The semiclassical initial data for the Schrödinger equation then has the form

$$\psi_0(q) \sim A_0(q) e^{\frac{i}{\hbar} S_0(q)}. \tag{5.12.25}$$

The solution to the semiclassical Cauchy problem (5.12.1), (5.12.25) will be described in terms of the family L_t of Lagrangian manifolds obtained from L_0 by shifting along trajectories of the Hamiltonian flow

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

Denote g_t the one-parameter group of symplectomorphisms

$$(q, p) \to g_t(q, p)$$

generated by the Hamiltonian vector field. Denote

$$Q = g_t\left(q, \frac{\partial S_0(q)}{\partial q}\right) = Q(q)$$

the restriction of g_t onto L_0.

The shifted Lagrangian submanifold

$$L_t = g_t(L_0)$$

is not necessarily projectable onto the q-space. Let Q be not a critical value of the projection of L_t onto the q-space. That is, for a given Q there are few points of the form $(Q, P_j) \in L_t$, and the q-projection is a local diffeomorphism near every point (Q, P_j).

Every such point (Q, P_j) is the end point of an integral curve $\gamma_j(\theta)$ starting at a point $x_j = (q_j, p_j) \in L_0$,

$$\gamma_j(0) = (q_j, p_j), \quad \gamma_j(t) = (Q, P_j), \quad j = 1, 2, \ldots$$

Assume nonvanishing of the Jacobians

$$\det\left(\frac{\partial Q}{\partial q_j}\right) \neq 0, \quad j = 1, 2, \ldots \tag{5.12.26}$$

Under this assumption the q-coordinates near the point $(q_j, p_j) \in L_0$ can be used as local coordinates on L_t near the point (Q, P_j).

The family of Lagrangian submanifolds L_t span a $(n+1)$-dimensional Lagrangian submanifold in the extended phase space

$$\hat{L} = \bigcup_{0 \le \theta \le t} L_\theta \subset \left(\mathbb{R}^{2n+2}, dp \wedge dq - dE \wedge dt\right).$$

Denote

$$\hat{\gamma}_j(\theta) = (\gamma_j(\theta), \theta, H(q(\theta), p(\theta))$$

the lift of the curve $\gamma_j(\theta)$ to the extendend phase space. As we already know the generating function of the Lagrangian submanifold L_t near the point (Q, P_j) can be represented by the integral

$$S_j(Q,t) = S_0(q_j) + \int_0^t [p\, dq - H(q,p) d\theta] \tag{5.12.27}$$

where the integration is taken along the integral curve $\hat{\gamma}_j(\theta)$. Finally denote μ_j the sum of Maslov indices of all turning points on the curve $\hat{\gamma}_j(\theta)$, $0 \le \theta \le t$.

Theorem 5.12.3 *(J.Keller; V.Maslov; V.Arnold) Solution to the quasiclassical initial value problem* (5.12.1), (5.12.25) *has the form*

$$\psi(Q,t) = \sum_j A_0(q_j) \left|\det\left(\frac{\partial q_j}{\partial Q}\right)\right|^{1/2} e^{\frac{i}{\hbar} S_j(Q,t) - \frac{\pi\, i}{2}\mu_j}. \tag{5.12.28}$$