Scuola Galileiana di Studi Superiori
A. A. 2009/2010
Geometria Differenziale

Boris DUBROVIN

March 15, 2013

Contents

1 Differential geometry in Euclidean space

1.1 Coordinates on Euclidean space

Geometry studies objects consisting of *points* in a *space*. Giving a precise meaning to these names requires significant efforts. So we will begin with introducing simple geometrical structures in Euclidean spaces as a starting point for developing geometry of smooth manifolds.

Definition 1.1.1 *A set X is called a* Euclidean space of dimension n *if a one-to-one correspondence*

$$X \to \mathbb{R}^n \tag{1.1.1}$$

is given.

Recall that $\mathbb{R}^n$ is defined as the set of *all* n-tuples $(x^1, x^2, \dots, x^n)$ of real numbers. Thus the above definition says that to any point $P \in X$ it is assigned a collection of Euclidean coordinates $\left(x^1(P), x^2(P), \dots, x^n(P)\right)$. It is required that

- any n-tuple of real numbers $(x^1, x^2, \dots, x^n)$ are coordinates of a point $P \in X$;
- two points P and Q in X coincide *iff*

$$x^i(P) = x^i(Q), \quad i = 1, 2, \dots, n.$$

The space $\mathbb{R}^n$ itself is a Euclidean space of dimension n. Also in general one can say that points of the n-dimensional Euclidean space X are just n-tuples of real numbers $(x^1, \dots, x^n)$ thus identifying X with $\mathbb{R}^n$. However, in sequel we will deal with different choices of coordinates on the same space. Properties of objects in the space remaining invariant with respect to a certain class of changes of coordinates will be called *geometric*. We will soon be more specific about the main classes of changes of coordinates used in the differential geometry.

The identification (1.1.1) equips X with a natural topology defined on $\mathbb{R}^n$. Namely, given a point $P_0 \in X$ and a positive number ϵ denote

$$B(P_0, \epsilon) = \left\{ P \in X \mid |x^i(P) - x^i(P_0)| < \epsilon \quad \text{for} \quad i = 1, 2, \dots, n \right\}.$$

The subset $B(P_0, \epsilon) \subset X$ is called the ϵ-neighborhood of the point P_0. A subset $U \subset X$ is called *open* if every point of U belongs to U together with its ϵ-neighborhood for a sufficiently small $\epsilon > 0$.

Functions on a n-dimensional Euclidean space

$$f : X \to \mathbb{R} \tag{1.1.2}$$

are represented by functions of n real variables $f(x^1, x^2, \dots, x^n)$. The function (1.1.2) on the topological space X is continuous *iff* $f(x^1, x^2, \dots, x^n)$ is a continuos function in the sense of multivariable calculus.

Exercise 1.1.2 *1) Given a continuous function $f : X \to \mathbb{R}$ prove that the subset*

$$U = \{P \in X \mid f(P) < 0\}$$

is open.

2) For any $k \geq 1$ prove openness of the subset defined by a system of inequalities

$$f_1(P) < 0, \dots, f_k(P) < 0$$

with continuous functions $f_1, \dots, f_k$.

A map of Euclidean spaces

$$f : X \to Y \tag{1.1.3}$$

equipped with coordinates $(x^1, \dots, x^n)$ and $(y^1, \dots, y^m)$ is represented by m real valued functions of n variables

$$y^i = y^i(x^1, \dots, x^n), \quad i = 1, \dots, m. \tag{1.1.4}$$

Such a map is continuous if all the functions (1.1.4) are continuous. The map (1.1.3) is called *homeomorphism* if it is one-to-one and the inverse map

$$\begin{aligned} &f^{-1} : Y \to X \\ &x^j = x^j(y^1, \dots, y^m), \quad j = 1, \dots, n \end{aligned} \tag{1.1.5}$$

is continuous. It is known that homeomorphic Euclidean spaces must have the same dimension $m = n$. We will prove this statement for the *smooth* case.

Smooth functions and smooth maps of Euclidean spaces are defined in a similar way. A function $f : X \to \mathbb{R}$ on a n-dimensional Euclidean space is called $\mathcal{C}^k$-smooth if all partial derivatives

$$\frac{\partial^m f}{\partial x^{i_1} \dots \partial x^{i_m}}$$

of its coordinate representation $f(x^1, \dots, x^n)$ are continuous functions for $m \leq k$. In a similar way one can define $\mathcal{C}^k$-smooth maps of Euclidean spaces. The function is called $\mathcal{C}^\infty$-smooth if it can be continuously differentiated any number of times. We will mainly deal with $\mathcal{C}^\infty$-smooth functions and maps. So the name 'smooth' will be reserved for such functions and maps.

Let

$$f : X \to Y$$

be a smooth map of Euclidean spaces of dimensions n and m respectively.

Definition 1.1.3 *The map f is called* diffeomorphism *if it is one-to-one and the inverse map $f^{-1} : Y \to X$ is smooth. If such a map exists then the Euclidean spaces X and Y are called* diffeomorphic.

Theorem 1.1.4 (Invariance of dimension.) *Diffeomorphic Euclidean spaces must have equal dimensions.*

Proof: Let us first remind the *chain rule* for differentiating the superposition of smooth maps

Lemma 1.1.5 *Let*

$$f : X \to Y, \quad y^j = y^j(x), \quad j = 1, \dots, m, \quad x = (x^1, \dots, x^l) \tag{1.1.6}$$
$$g : Y \to Z, \quad z^i = z^i(y), \quad i = 1, \dots, n, \quad y = (y^1, \dots, y^m) \tag{1.1.7}$$

be smooth maps of Euclidean spaces X, Y, Z of the dimensions l, m and n respectively. Then the superposition $g \circ f$ is a smooth map

$$g \circ f : X \to Z, \quad z^i = z^i\left(y(x)\right), \quad i = 1, \dots, n. \tag{1.1.8}$$

Its partial derivatives can be computed from the following formula

$$\frac{\partial z^i}{\partial x^k} = \sum_{j=1}^m \frac{\partial z^i}{\partial y^j}\frac{\partial y^j}{\partial x^k}, \quad i = 1, \dots, n, \quad k = 1, \dots, l. \tag{1.1.9}$$

This statement is well known from the multivariable calculus.

Let us make two improvements in the notations. First, we introduce the *Jacobi matrix* of a smooth map. For the map (1.1.6) the Jacobi matrix

$$\left(\frac{\partial y}{\partial x}\right) := \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \dots & \frac{\partial y^1}{\partial x^l} \\ \frac{\partial y^2}{\partial x^1} & \dots & \frac{\partial y^2}{\partial x^l} \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ \frac{\partial y^m}{\partial x^1} & \dots & \frac{\partial y^m}{\partial x^l} \end{pmatrix} \tag{1.1.10}$$

has m rows and l columns. Similarly the Jacobi matrix

$$\left(\frac{\partial z}{\partial y}\right) := \begin{pmatrix} \frac{\partial z^1}{\partial y^1} & \dots & \frac{\partial z^1}{\partial y^m} \\ \frac{\partial z^2}{\partial y^1} & \dots & \frac{\partial z^2}{\partial y^m} \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ \frac{\partial z^n}{\partial y^1} & \dots & \frac{\partial z^n}{\partial y^m} \end{pmatrix} \tag{1.1.11}$$

has n rows and m columns. The chain rule says that the Jacobi matrix of the superposition (having n rows and l columns) is equal to the product of the Jacobi matrices (1.1.10) and (1.1.11):

$$\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial x}\right). \tag{1.1.12}$$

The second improvement is often called the *Einstein rule*: we agree to omit the summation sign in (1.1.9) over the twice repeated index j:

$$\frac{\partial z^i}{\partial x^k} = \frac{\partial z^i}{\partial y^j}\frac{\partial y^j}{\partial x^k}, \quad i = 1, \dots, n, \quad k = 1, \dots, l. \tag{1.1.13}$$

In sequel we will use this rule systematically always assuming summation over every pair of repeated indices. Needless to say that the limits of the summation must be known in advance. For example, in the formula (1.1.13) there is a summation over the repeated index j from $j = 1$ to $j = m$.

After such a long digression let us return to the proof of invariance of dimension. Let the spaces X and Y have the dimensions n and m respectively. In the coordinates the map $f : X \to Y$ is represented by n smooth functions

$$y^i = y^i(x), \quad i = 1, \dots, m, \quad x = (x^1, \dots, x^n).$$

The inverse map $f^{-1} : Y \to X$ is represented by m smooth functions

$$x^j = x^j(y), \quad j = 1, \dots, n, \quad y = (y^1, \dots, y^m).$$

The superposition

$$f^{-1} \circ f : X \to X$$

is the identity map $f^{-1} \circ f = \mathrm{id}_X$. That is,

$$x^j\left(y(x)\right) = x^j, \quad j = 1, \dots, n.$$

Applying the chain rule we conclude that

$$\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial x}\right) = \mathbf{1}_n.$$

Here $\mathbf{1}_n$ is the $n \times n$ identity matrix. In a similar way we prove that

$$\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial x}{\partial y}\right) = \mathbf{1}_m.$$

Therefore the Jacobi matrix of the inverse map f^{-1} is the inverse matrix for the Jacobi matrix of f:

$$\left(\frac{\partial x}{\partial y}\right) = \left(\frac{\partial y}{\partial x}\right)^{-1}.$$

As it is well known from linear algebra only square matrices can be invertible[1]. Therefore $m = n$. □

From the proof of the Theorem one can easily derive the following

Corollary 1.1.6 *Let $f : X \to Y$ be a diffeomorphism of Euclidean spaces of dimension n. Then the determinant of the Jacobi matrix*

$$\det\left(\frac{\partial y}{\partial x}\right) \tag{1.1.14}$$

never vanishes.

[1]This can be easily derived from the following estimate for the rank of the product of two matrices:

$$\mathrm{rk}\, AB \leq \min(\mathrm{rk}\, A, \mathrm{rk}\, B).$$

We are now in a position to describe the main class of transformation of coordinates on a Euclidean space. Let two systems of coordinates $(x^1, \dots, x^n)$ and $(y^1, \dots, y^m)$ be two systems of coordinates in a Euclidean space. One has two one-to-one maps

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \mathbb{R}^n_x & & \mathbb{R}^m_y \end{array} \tag{1.1.15}$$

The superposition gives a one-to-one map of Euclidean spaces

$$\begin{gathered} \mathbb{R}^n_x \to \mathbb{R}^m_y \\ \left(x^1(P), \dots, x^n(P)\right) \mapsto \left(y^1(P), \dots, y^m(P)\right) \end{gathered} \tag{1.1.16}$$

for any $P \in X$.

Definition 1.1.7 *The two coordinate systems* (1.1.15) *are called* compatible *if the map* (1.1.16) *is a diffeomorphism.*

Because of invariance of dimension one must have $m = n$ for a pair of compatible coordinate systems. The transformation

$$\left(x^1(P), \dots, x^n(P)\right) \mapsto \left(y^1(P), \dots, y^n(P)\right) \quad \forall P \in X \tag{1.1.17}$$

between two compatible systems of coordinates will be called a *smooth change of coordinates*. The matrix $\left(\frac{\partial y}{\partial x}\right)$ is called the *Jacobi matrix* of the coordinate transformation and its determinant

$$J_{x \to y} := \det \left(\frac{\partial y}{\partial x}\right) \tag{1.1.18}$$

the *Jacobian* of the transformation. According to the Corollary 1.1.6 the Jacobian of the coordinate transformation never vanishes,

$$\det \left(\frac{\partial y}{\partial x}\right) \neq 0.$$

The Jacobian of the inverse transformation is obtained by inversion of the Jacobian (1.1.18)

$$J_{y \to x} = J^{-1}_{x \to y}.$$

Exercise 1.1.8 *Consider a* linear change of coordinates

$$\begin{aligned} y^1 &= t^1_1 x^1 + t^1_2 x^2 + \dots + t^1_n x^n \\ y^2 &= t^2_1 x^1 + t^2_2 x^2 + \dots + t^2_n x^n \\ \dots & \quad \dots \quad \dots \quad \dots \\ y^n &= t^n_1 x^1 + t^n_2 x^2 + \dots + t^n_n x^n \end{aligned} \tag{1.1.19}$$

or, in the short notations explained above

$$y^i = t^i_j x^j, \quad i = 1, \dots, n \tag{1.1.20}$$

(summation over the repeated index j from 1 to n is assumed). Prove that the transformation (1.1.19) *is a smooth change of coordinates* iff *the matrix*

$$T = \begin{pmatrix} t_1^1 & t_2^1 & \dots & t_n^1 \\ \\ t_1^2 & t_2^2 & \dots & t_n^2 \\ \dots & \dots & \dots & \dots \\ t_1^n & t_2^n & \dots & t_n^n \end{pmatrix}$$

does not degenerate,

$$\det T \neq 0.$$

Observe that T coincides with the Jacobi matrix of the transformation,

$$\frac{\partial y^i}{\partial x^j} = t_j^i, \quad i,\, j = 1, \dots, n.$$

In the vector notations the transformation (1.1.19) is given by multiplication of the vector of coordinates by the matrix T:

$$y = Tx$$

$$x = \begin{pmatrix} x^1 \\ x^2 \\ \cdot \\ \cdot \\ \cdot \\ x^n \end{pmatrix}, \quad y = \begin{pmatrix} y^1 \\ y^2 \\ \cdot \\ \cdot \\ \cdot \\ y^n \end{pmatrix}. \tag{1.1.21}$$

The inverse transformation $y \to x$ involves the inverse matrix

$$x = T^{-1} y. \tag{1.1.22}$$

It is often convenient to work in *local coordinates* that establish a one-to-one correspondence of a subset in the Euclidean space with an open domain in $\mathbb{R}^n$. Changes of local coordinates are given by smooth functions defined on open domains in $\mathbb{R}^n$. Like in the global case the Jacobian of a transformation of local coordinates must not vanish.

Example 1. Consider the two-dimensional Euclidean space with the coordinates (x, y). The polar coordinates (r, ϕ) are defined by the formulae

$$\left. \begin{array}{rcl} x & = & r\, \cos\phi \\ y & = & r\, \sin\phi \end{array} \right\}. \tag{1.1.23}$$

Because of 2π-periodicity of the trigonometric functions the pairs (r, ϕ) and $(r, \phi + 2\pi\, k)$ correspond to the same point of the Euclidean plane for any integer k. The transformation $(r, \phi) \mapsto (x, y)$ is (smoothly) invertible on the domain obtained by deleting the ray

$$y = 0, \quad x \geq 0.$$

The polar angle ϕ can be uniquely determined from eqs. (1.1.23) assuming that $0 < \phi < 2\pi$. The polar radius $r = \sqrt{x^2 + y^2}$ takes positive values. Thus the local coordinates (r, ϕ) defined on the above domain take their values on the open half-strip

$$\begin{array}{l} 0 < r \\ 0 < \phi < 2\pi. \end{array}$$

The Jacobian

$$J_{(r,\phi)\to(x,y)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ & \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{pmatrix} = \det \begin{pmatrix} \cos\phi & -r\,\sin\phi \\ & \\ \sin\phi & r\,\cos\phi \end{pmatrix} = r \tag{1.1.24}$$

never vanishes on the punctured plane $x^2 + y^2 > 0$.

Example 2. Cylindrical coordinates (r, ϕ, h) in the three-dimensional Euclidean space are defined by the transformation

$$\left.\begin{array}{rcl} x &=& r\,\cos\phi \\ y &=& r\,\sin\phi \\ z &=& h \end{array}\right\}. \tag{1.1.25}$$

They are local coordinates on the domain obtained by deleting from $\mathbb{R}^3$ the half-plane

$$y = 0, \quad x \geq 0.$$

The Jacobian

$$J_{(r,\phi,h)\to(x,y,z)} = \det \begin{pmatrix} \cos\phi & -r\,\sin\phi & 0 \\ \sin\phi & r\,\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = r \tag{1.1.26}$$

vanishes only on the z-axis $x = y = 0$.

Example 3. Spherical coordinates in the three-dimensional Euclidean space are defined by

$$\left.\begin{array}{rcl} x &=& r\,\cos\theta\,\cos\phi \\ y &=& r\,\cos\theta\,\sin\phi \\ z &=& r\,\sin\theta \end{array}\right\}. \tag{1.1.27}$$

They become local coordinates in the domain obtained by deleting of the half-plane

$$y = 0, \quad x \geq 0$$

and they take their values in

$$\begin{array}{ccccc} 0 & < & r & & \\ 0 & < & \phi & < & 2\pi \\ -\frac{\pi}{2} & < & \theta & < & \frac{\pi}{2}. \end{array} \tag{1.1.28}$$

The Jacobian

$$J_{(r,\phi,\theta)\to(x,y,z)} = \det \begin{pmatrix} \cos\theta\,\cos\phi & -r\,\cos\theta\,\sin\phi & -r\,\sin\theta\,\cos\phi \\ \cos\theta\,\sin\phi & r\,\cos\theta\,\cos\phi & -r\,\sin\theta\,\sin\phi \\ \sin\theta & 0 & r\,\cos\theta \end{pmatrix} = r^2\cos\theta \tag{1.1.29}$$

vanishes only on the z-axis.

At the end of this section we will formulate a useful result that locally gives a converse statement to the Corollary 1.1.6.

Theorem 1.1.9 *Let*

$$y^1 = y^1(x), \dots, y^n = y^n(x), \quad x = (x^1, \dots, x^n) \tag{1.1.30}$$

be n smooth functions defined on a small neighborhood of the point $P_0 = (x_0^1, \dots, x_0^n)$ of a n-dimensional Euclidean space. If the Jacobian

$$J_{x \to y} = \det \left(\frac{\partial y}{\partial x} \right)$$

does not vanish at the point P_0 then the functions (1.1.30) *define a system of* local *coordinates on some neighborhood of this point.*

This is a reformulation of the inverse mapping theorem.

1.2 Linear algebra and Euclidean geometry

Every linear space X of dimension n has a natural structure of Euclidean space. A distinguished class of coordinates on X is obtained in the following way. Let $e_1, \dots, e_n$ be a system of linearly independent vectors of X. Such a system is a *basis* of the linear space, i.e.,

- any vector $x \in X$ can be represented as a linear combination

$$x = x^1 e_1 + x^2 e_2 + \dots + x^n e_n \equiv x^i e_i; \tag{1.2.1}$$

- such a representation is unique:

$$x^i e_i = y^j e_j \quad \Rightarrow \quad x^i = y^i \quad \forall i = 1, \dots, n.$$

The coefficients $x^1, \dots, x^n$ of the linear combination (1.2.1) are called *coordinates* of the vector x with respect to the basis $e_1, \dots, e_n$. Clearly such a definition of coordinates agrees with the one given in the beginning of the previous section (we can use the name 'points' for the vectors of the linear space).

Example. In the standard n-dimensional linear space $\mathbb{R}^n$ there is a distinguished basis

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, \dots, 0), \quad \dots, \quad e_n = (0, 0, \dots, 1). \tag{1.2.2}$$

The coordinates of a vector $x = (x^1, \dots, x^n)$ with respect to the basis (1.2.2) coincide with the numbers $x^1, \dots, x^n$.

Exercise 1.2.1 *Let x and y be two vectors of a linear space. Denote $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ respectively the coordinates of these vectors with respect to a basis $e_1, \dots, e_n$. Prove that the coordinates of the linear combination*

$$\alpha x + \beta y, \quad \alpha, \beta \in \mathbb{R}$$

with respect to the same basis are

$$\alpha x^1 + \beta y^1, \quad \alpha x^2 + \beta y^2, \quad \dots, \quad \alpha x^n + \beta y^n.$$

Definition 1.2.2 *A map $f : X \to Y$ is called* linear *if*

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \quad \forall\ x,\, y \in X, \quad \alpha,\, \beta \in \mathbb{R}.$$

A linear map is called isomorphism of linear spaces *if it is one-to-one.*

Theorem 1.2.3 *Two linear spaces are isomorphic* iff *they have the same dimension.*

Proof: Necessity of the condition of the theorem can be proved as the above theorem about invariance of dimension. As it follows from the Exercise 1.2.1, a choice of a basis in a n-dimensional linear space establishes an isomorphism of this space with $\mathbb{R}^n$ with the standard basis (1.2.2). □

Another basis $f_1, \ldots, f_n$ defines another system of coordinates on the linear space X. Denote $y^1, \ldots, y^n$ the coordinates of the same vector with respect to the new basis

$$x = y^1 f_1 + \cdots + y^n f_n. \tag{1.2.3}$$

Lemma 1.2.4 *The change of coordinates $(x^1, \ldots, x^n) \to (y^1, \ldots, y^n)$ is given by the linear transformation* (1.1.19) *where the entries of the* transition matrix $T = \left(t^i_j\right)$ *are the coordinates of the old basis with respect to the new one:*

$$e_j = t^i_j f_i, \quad j = 1, \ldots, n. \tag{1.2.4}$$

Let us recall again that summation over the repeated index i from $i = 1$ to $i = n$ is assumed in the formula (1.2.4).

Proof: By definitions (1.2.1) and (1.2.3) we have

$$x^j e_j = y^i f_i.$$

Substituting (1.2.4) yields

$$x^j t^i_j f_i = y^i f_i$$

(now we have a double sum in the left hand side!). Because of uniqueness of the coordinates we deduce the following system of equations

$$x^j t^i_j = y^i.$$

This is exactly the transformation rule (1.1.19). □

Exercise 1.2.5 *Verify that the definition* (1.2.4) *of the transition matrix can be represented in terms of multiplication of the row vector $(f_1, \ldots, f_n)$ by the matrix T:*

$$(e_1, \ldots, e_n) = (f_1, \ldots, f_n)\, T. \tag{1.2.5}$$

Use this representation in order to rederive the Lemma 1.2.4 with the help of matrix algebra.

Exercise 1.2.6 *Prove that any linear change of coordinates of the form* (1.1.19) *with an arbitrary nondegenerate matrix T can be realized by a change of a basis in the linear space.*

Remark 1.2.7 *According to our general philosophy one can say that the geometry of a linear space studies those properties that remain invariant with respect to linear changes of coordinates* (1.1.19).

The simplest class of geometrical objects in a linear space are *linear subspaces.* The subset $L \subset X$ is called a linear subspace if, for any pair of vectors x, $y \in L$ and any pair of real numbers α, $\beta \in \mathbb{R}$ the linear combination

$$\alpha x + \beta y$$

belongs to L. Thus L itself is a linear subspace with respect to the operations defined in X. An important result from linear algebra says that any linear subspace of a n-dimensional linear space X is finite-dimensional of the dimension less or equal to n. The following two particular subclasses of linear subspaces appear more often in the geometrical considerations. First, they are one-dimensional subspaces (straight lines). The vectors of these subspaces can be represented in the form

$$x = \lambda f, \quad \lambda \in \mathbb{R}$$

for some nonzero vector f. Another subclass are hyperplanes, that is, linear subspaces of the dimension[2] $n-1$. In coordinates they are conveniently represented as solutions to linear equations

$$\alpha_1 x^1 + \cdots + \alpha_n x^n = 0 \tag{1.2.6}$$

where $\alpha_1, \ldots, \alpha_n$ are real numbers not all of them are equal to zero.

Exercise 1.2.8 *Prove that in the new coordinates defined as in Lemma 1.2.4 the hyperplane* (1.2.6) *can be described by the equation*

$$\beta_1 y^1 + \cdots + \beta_n y^n = 0$$

where the rows of coefficients $\alpha = (\alpha_1, \ldots, \alpha_n)$ *and* $\beta = (\beta_1, \ldots, \beta_n)$ *are related by the transformation*

$$\alpha = \beta T. \tag{1.2.7}$$

At the first glance the transformation rules of coordinates (1.1.21) and the coefficients of equations of hyperplanes (1.2.7) look similar. However these are *different transformation rules* (check it)!

In order to define lengths of intervals of straight lines and also the angles between the lines we have to equip the linear space X with an additional structure of an *inner product.* Recall that this is a function

$$\begin{aligned} X \times X &\to \mathbb{R} \\ (x, y) &\mapsto \langle x, y \rangle \end{aligned} \tag{1.2.8}$$

satisfying the following properties:

[2]We will also use expression 'subspace of codimension one'.

• bilinearity

$$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, x\rangle, \quad \langle z, \alpha x + \beta y\rangle = \alpha\langle z, x\rangle + \beta\langle z, y\rangle$$

for arbitrary vectors x, y, $z \in X$ and arbitrary real numbers α, $\beta \in \mathbb{R}$;

• symmetry

$$\langle y, x\rangle = \langle x, y\rangle \quad \forall x,\, y \in X;$$

• positive definiteness

$$\begin{aligned} &\langle x, x\rangle \geq 0 \\ &\langle x, x\rangle = 0 \quad \text{iff} \quad x = 0. \end{aligned}$$

Recall the *Cauchy–Schwarz inequality*

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle \quad \forall x,\, y \in X \tag{1.2.9}$$

that follows from the above definition of an inner product.

The length $|x|$ of a vector x is defined by

$$|x| = \sqrt{\langle x, x\rangle}. \tag{1.2.10}$$

Two vectors x and y are called *orthogonal* if their inner product is equal to zero. More generally the angle ϕ between two nonzero vectors x, y is defined from the equation

$$\cos\phi = \frac{\langle x, y\rangle}{|x|\,|y|}, \quad 0 \leq \phi \leq \pi. \tag{1.2.11}$$

Observe that such an angle exists due to the Cauchy–Schwarz inequality. In particular the angle between two orthogonal vectors is equal to $\frac{\pi}{2}$.

With the help of an inner product one can define the *distance* $\rho(x, y)$ between two points x, $y \in X$ as the length of the vector $y - x$. In this way we obtain a structure of a *metric space* on X. That means that the distance

$$\rho(x, y) = |x - y| \tag{1.2.12}$$

satisfies the following properties:

• symmetry $\rho(y, x) = \rho(x, y)$ $\forall x,\, y \in X$;

• triangle inequality

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z) \tag{1.2.13}$$

• positivity

$$\begin{aligned} &\rho(x, y) \geq 0 \\ &\rho(x, y) = 0 \quad \text{iff} \quad y = x. \end{aligned} \tag{1.2.14}$$

Let $(X, \langle\ ,\ \rangle)$ be an n-dimensional linear space equipped with an inner product. We will write the coordinate representation for the above objects of Euclidean geometry. They become of a particular simplicity when written in an *orthonormal basis* $e_1, \ldots, e_n$. By definition the basis is called orthonormal if the basic vectors are pairwise orthogonal and all have length one:

$$\langle e_i, e_j \rangle = \delta_{ij}, \quad i, j = 1, \ldots, n. \tag{1.2.15}$$

Here and in sequel we will use the notation δ_{ij} for the Kronecker delta symbol:

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}. \tag{1.2.16}$$

Observe that the coordinates in an orthonormal basis can be expressed via inner products

$$x^i = \langle x, e_i \rangle, \quad i = 1, \ldots, n. \tag{1.2.17}$$

Lemma 1.2.9 *Let $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ be the coordinates of the vectors x and $y \in X$ with respect to an orthonormal basis. Then*

$$\langle x, y \rangle = x^1 y^1 + \cdots + x^n y^n. \tag{1.2.18}$$

Proof: We have

$$\begin{aligned} \langle x, y \rangle &= \langle x^i e_i, y^j e_j \rangle = x^i y^j \langle e_i, e_j \rangle \\ &= x^i y^j \delta_{ij} = x^1 y^1 + \cdots + x^n y^n. \end{aligned}$$

□

Corollary 1.2.10 *The square length of a vector is expressed via its coordinates in an orthonormal basis according to the Pythagorous theorem*

$$|x|^2 = \left(x^1\right)^2 + \cdots + \left(x^n\right)^2, \tag{1.2.19}$$

Justification of existence of orthonormal bases in an arbitrary finite-dimensional Euclidean space $(X, \langle\ ,\ \rangle)$ is obtained from the following well known result from linear algebra.

Theorem 1.2.11 (Gram–Schmidt orthogonalization) *Given an arbitrary basis $e_1, \ldots, e_n$ in $(X, \langle\ ,\ \rangle)$, there exists an upper triangular matrix $T = \left(t^i_j\right)$, $a^i_j = 0$ for $i > j$, and an orthonormal basis $f_1, \ldots, f_n$ such that*

$$(e_1, \ldots, e_n) = (f_1, \ldots, f_n) T. \tag{1.2.20}$$

Such an orthonormal basis is unique up to signs of the vectors $f_1, \ldots, f_n$, i.e., if T' is another upper triangular matrix and $f'_1, \ldots, f'_n$ another orthonormal basis satisfying

$$(e_1, \ldots, e_n) = (f'_1, \ldots, f'_n) T' \tag{1.2.21}$$

then

$$f'_1 = \pm f_1, \quad f'_2 = \pm f_2, \ldots, f'_n = \pm f_n$$

for some choice of the signs $\pm$.

From the Theorem 1.2.11 and Lemma 1.2.9 it follows

Corollary 1.2.12 *Given two Euclidean spaces* $(X, \langle\ ,\ \rangle)$ *and* $(X', \langle\ ,\ \rangle')$ *of the same dimension, there exists an isomorphism of linear spaces*

$$f : X \to X'$$

preserving the inner products, i.e.,

$$\langle f(x), f(y)\rangle' = \langle x, y\rangle \quad \forall\ x,\, y \in X.$$

Proof: A choice of an orthonormal basis in X gives such an isomorphism of X with the standard Euclidean space with the inner product given by the formula (1.2.18). □

1.3 Transformation groups in Euclidean geometry

Let us update the general philosophy according to which those properties of objects in X are called geometric that remain invariant with respect to transformations $f : X \to X$ of a certain class. We will consider only those classes of transformations that form a *group* with respect to a natural superposition of maps

$$\begin{aligned} &f : X \to X, \quad g : X \to X \\ &(f, g) \mapsto f \circ g : X \to X. \end{aligned} \tag{1.3.1}$$

Recall that the superposition of maps is an associative operation:

$$f \circ (g \circ h) = (f \circ g) \circ h \tag{1.3.2}$$

for arbitrary maps $f,\, g,\, h : X \to X$.

The biggest of such groups consists of all bijections. The unit of this group is the identity map id_X. The inverse to a bijection f is the inverse map f^{-1}. This group is of no interest for differential geometry.

More interesting is the subgroup *Diff* (X) of all diffeomorphisms of the Euclidean space X to itself. Much smaller subgroups of *Diff* (X) arise in various problems of linear algebra. Let X be a linear space and $f : X \to X$ a linear map (see Definition 1.2.2). Given a basis $e_1,\, \dots,\, e_n$ in X, the *matrix* $A = \left(a_i^j\right)$ of the linear map f is defined by

$$f(e_i) = a_i^j e_j, \quad i = 1, \dots, n. \tag{1.3.3}$$

In these coordinates the map f acts as follows

$$f(x)^i = a_j^i x^j, \quad i = 1, \dots, n \tag{1.3.4}$$

or

$$x \mapsto Ax. \tag{1.3.5}$$

In the second equation x is the column of coordinates $x^1,\, \dots,\, x^n$.

Lemma 1.3.1 *Change of a basis*

$$e_{i'} = t^i_{i'} e_i, \quad i' = 1, \dots, n \tag{1.3.6}$$

yields the following transformation of the matrix of the linear map f

$$A' = T^{-1} A T \tag{1.3.7}$$

Here $A' = \left(a^{j'}_{i'}\right)$ *is defined by*

$$f(e_{i'}) = a^{j'}_{i'} e_{j'}, \quad i' = 1, \dots, n. \tag{1.3.8}$$

Here and in sequel we will use the following updated version of the notations. We will use different indices i', j' etc. and i, j etc. for labelling the basic vectors, the coordinates etc. of objects represented in different coordinate systems. Thus the basis $e_{1'}$, …, $e_{n'}$ will be different from the basis e_1, …, e_n; the coordinates of a vector $x \in X$ with respect to the basis e_1, …, e_n are denoted x^1, …, x^n as above,

$$x = x^i e_i$$

while the coordinates of the same vector in the basis $e_{1'}$, …, $e_{n'}$ are denoted $x^{1'}$, …, $x^{n'}$,

$$x = x^{i'} e_{i'}.$$

The indices i and i', j and j' will be consider independent even when repeated within the same formula.

Proof: We have

$$a^{j'}_{i'} e_{j'} = f(e_{i'}) = f\left(t^i_{i'} e_i\right) = t^i_{i'} f\left(e_i\right) = t^i_{i'} a^j_i e_j.$$

Substitution of (1.3.6) in the left hand side of the formula gives

$$a^{j'}_{i'} t^j_{j'} e_j = t^i_{i'} a^j_i e_j$$

or, in matrix notations

$$T A' = A T.$$

□

Corollary 1.3.2 *The determinant of the matrix of the linear map* f *does not depend on the choice of the basis in the linear space. The characteristic polynomial*

$$P_A(\lambda) = \det\left(A - \lambda \cdot 1\right) \tag{1.3.9}$$

of the matrix of a linear map does not depend on the choice of a basis in the linear space. In particular he determinant of the matrix of the linear map f *does not depend on the choice of the basis.*

We will call $P_A(\lambda)$ and $\det A = P_A(0)$ respectively the characteristic polynomial and the determinant of the linear map f. Recall that the characteristic polynomial is important in the study of *eigenvalues* of the linear map f. Namely, the nonzero vector $x \in X$ is called an *eigenvector* of the linear map f with the eigenvalue $\lambda_0 \in \mathbb{R}$ if

$$f(x) = \lambda_0 x. \tag{1.3.10}$$

The number λ_0 is an eigenvalue of a linear map f *iff* it is a real root of the characteristic polynomial

$$P_A(\lambda_0) = 0. \tag{1.3.11}$$

A linear map $f : X \to X$ is called *automorphism* if it is invertible. The necessary and sufficient condition for a linear map of a finite dimensional linear space to itself to be an automorphism is given by nonvanishing of the determinant. All automorphisms $f : X \to X$ of this linear space form a group. If n is the dimension of the space X then the group of linear automorphisms is isomorphic to the *general linear group* $GL(n)$ consisting of all nondegenerate $n \times n$ matrices. Matrices with *positive* determinant form a subgroup

$$GL^+(n) \subset GL(n) \tag{1.3.12}$$

of *orientation preserving* automorphisms. Recall that *orientation* in the linear space X is a choice of an equivalence class of bases in X. Two bases $e_1, \ldots, e_n$ and $e_{1'}, \ldots, e_{n'}$ are called equivalent if the transition matrix T defined by (1.3.6) has positive determinant.

The *special linear group*

$$SL(n) \subset GL^+(n) \tag{1.3.13}$$

consists of all $n \times n$ matrices with determinant 1. For $n = 1$ the general linear group is isomorphic to the multiplicative group $\mathbb{R}^*$ of all nonzero real numbers. It is an Abelian group. The special linear group for $n = 1$ consists of one element 1. The groups $GL(n)$ and $SL(n)$ are not Abelian for $n \geq 2$.

An extension of $GL(n)$ is given by *affine transformations*

$$x \mapsto Ax + b. \tag{1.3.14}$$

Here $A \in GL(n)$ is an arbitrary nondegenerate square matrix and b is an arbitrary vector. Any affine transformation is a superposition of a linear automorphism and a translation

$$x \mapsto x + b.$$

Exercise 1.3.3 *Prove that superposition of two affine maps is again an affine map. Also prove that the inverse to an affine map is again an affine map.*

One obtains the *affine group* $Aff(n)$. Translations form an Abelian subgroup in $Aff(n)$. This subgroup is isomorphic to $\mathbb{R}^n$. The affine group itself is not Abelian even for $n = 1$.

Exercise 1.3.4 *Consider the map*

$$\alpha : Aff(n) \to GL(n) \tag{1.3.15}$$

associating the matrix $A \in GL(n)$ *with the affine transformation* (1.3.14). *Prove that the map* (1.3.15) *is a group homomorphism. Prove that the kernel of this homomorphism coincides with the subgroup of translations.*

From the second statement of this Exercise one derives

Corollary 1.3.5 *Translations form a normal subgroup in* $Aff(n)$.

Recall that $H \subset G$ is a normal subgroup in the group G if

$$g^{-1} h\, g \in H \quad \forall\, h \in H, \forall\, g \in G.$$

The properties of objects in a Euclidean space invariant with respect to the action of affine group make part of *affine geometry*.

Exercise 1.3.6 *Prove that the class of affine transformations is invariant with respect to* affine changes of coordinates *of the form* $(x^1, \dots, x^n) \mapsto (x^{1'}, \dots, x^{n'})$

$$x^i = t^i_{i'} x^{i'} + a^i, \quad i = 1, \dots, n \tag{1.3.16}$$

where the $n \times n$ *matrix* $T = \left(t^i_{i'}\right)$ *does not degenerate.*

More rich spectrum of transformation groups arises in the study of spaces $(X, \langle\ ,\ \rangle)$ equipped with a Euclidean inner product.

Definition 1.3.7 *An automorphism of a linear space* $f : X \to X$ *equipped with an inner product is called* orthogonal *if it preserves the inner product*

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \quad \forall\, x,\, y \in X. \tag{1.3.17}$$

It is easy to see that superposition of two orthogonal automorphisms of X is again an orthogonal automorphism. The inverse to an orthognal automorphism is again orthogonal. Indeed, given two arbitrary vectors x', $y' \in X$ one can find $x = f^{-1}(x)$, $y = f^{-1}(y) \in X$ such that

$$x' = f(x), \quad y' = f(y).$$

Since

$$\langle x', y'\rangle \equiv \langle f(x), f(y)\rangle = \langle x, y\rangle \equiv \langle f^{-1}(x'), f^{-1}(y')\rangle,$$

the automorphism f^{-1} is orthogonal. Needless to say that the identity is an orthogonal automorphism. One obtains the group of orthogonal automorphisms of the Euclidean space $(X, \langle\ ,\ \rangle)$, or simply the *orthogonal group* of the space. The matrix realization of the orthogonal group is obtained from the following

Proposition 1.3.8 *Let* $A = \left(a^i_j\right)$ *be the matrix of the linear map* $f : X \to X$ *with respect to an orthonormal basis* $e_1,\, e_2,\, \dots,\, e_n$

$$f(e_j) = a^i_j e_i, \quad j = 1, \dots, n. \tag{1.3.18}$$

The map f *is orthogonal* iff *the matrix* A *is orthogonal, i.e.,*

$$A^{\mathrm{T}} A = 1. \tag{1.3.19}$$

Proof: Given an orthogonal automorphism f, the image $e'_i = f(e_i)$ of an orthonormal basis is again orthonormal:

$$\langle e'_i, e'_j \rangle = \delta_{ij}, \quad i,\, j = 1, \ldots, n.$$

Thus

$$\langle e'_i, e'_j \rangle = \langle a_i^k e_k, a_j^l e_l \rangle = a_i^k a_j^l \delta_{kl} = \sum_{k=1}^{n} a_i^k a_j^k = \delta_{ij}.$$

The last equation coincides with the orthogonality (1.3.19).

Vice versa, the square of the determinant of an orthogonal matrix A is equal to 1 since

$$1 = \det\left(A^{\mathrm{T}} A\right) = (\det A)^2 .$$

So the linear map f is an automorphism. The basis

$$f(e_i) = a_i^k e_k, \quad i = 1, \ldots, n$$

is orthonormal if the basis $e_1, \ldots, e_n$ was so. Let

$$x = x^i e_i, \quad y = y^j e_j$$

be two arbitrary vectors in the Euclidean space. We have

$$\langle f(x), f(y) \rangle = \langle x^i f(e_i), y^j f(y_j) \rangle = x^i y^j \langle f(e_i), f(e_j) \rangle = x^i y^j \delta_{ij} = \langle x, y \rangle.$$

□

Corollary 1.3.9 *The orthogonal group of the space $(X, \langle\ ,\ \rangle)$ is isomorphic to the group $O(n)$ of orthogonal matrices.*

Example 1. For $n = 1$ the orthogonal group consists of two elements ± 1.

Example 2. For $n = 2$ consider the *rotation* by the angle φ about the origin. We have

$$\begin{aligned} f(e_1) &= \cos\varphi\, e_1 + \sin\varphi\, e_2 \\ f(e_2) &= -\sin\varphi\, e_1 + \cos\varphi\, e_2. \end{aligned}$$

This gives the matrix

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{1.3.20}$$

For the orthogonal reflection with respect to the x-axis, $(x, y) \mapsto (x, -y)$ the matrix reads

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3.21}$$

Exercise 1.3.10 *Prove that any orthogonal transformation on the plane is a rotation or a reflection.*

Observe that the determinant of the matrix (1.3.20) is equal to $+1$ while the determinant of the matrix of a reflection is equal to -1.

In general it is clear that the orthogonal transformations with determinant $+1$ form a subgroup in the orthogonal group. It is denoted $SO(n) \subset O(n)$ and called *special orthogonal subgroup*. The transformations from the special orthogonal group are orientation preserving orthogonal transformations. For example the group $SO(2)$ is isomorphic to the group of matrices of the form (1.3.20).

Example 3. For $n = 3$ the rotation by the angle φ about the z-axis has the matrix

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.3.22}$$

(in this example we always use coordinates (x, y, z) in an orthonormal basis). The superposition of such a rotation with the orthogonal reflection with respect to the plane (x, y) has the matrix

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.3.23}$$

Exercise 1.3.11 *Prove that any transformation in* $SO(3)$ *is a rotation about an axis. Prove that any orientation reversing orthogonal transformation in* $\mathbb{R}^3$ *can be represented by the matrix* (1.3.23) *in some orthonormal basis.*

Hint: prove that any transformation in $SO(3)$ has an eigenvector with the eigenvalue 1.

Exercise 1.3.12 *Prove that the cross-product of vectors*

$$\begin{aligned} &\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \\ &(a, b) \mapsto a \times b \\ &a = (x_1, y_1, z_1), \quad b = (x_2, y_2, z_2), \quad a \times b = (y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1) \end{aligned} \tag{1.3.24}$$

is invariant with respect to the action of the group $SO(3)$ *but not invariant with respect to the action of the group* $O(3)$ *on coordinates of the vectors.*

We will now study transformations of Euclidean space considered as a metric space (X, ρ) equipped with the metric (1.2.12).

Definition 1.3.13 *The map* $f : X \to X'$ *of a metric space* (X, ρ) *to a metric space* (X', ρ') *is called* isometry *if*

$$\rho'(f(x), f(y)) = \rho(x, y) \quad \forall\, x, y \in X. \tag{1.3.25}$$

It is clear that superposition of two isometries of a space X to itself is again an isometry. The identity map is an isometry.

For example, any orthogonal automorphism $X \to X$ is an isometry. The translations $x \mapsto x + a$ are isometries too. Thus the transformations of the form

$$x \mapsto Ax + b, \quad A \in O(n), \quad b \in X$$

are isometries. Conversely,

Theorem 1.3.14 *Any isometry of a Euclidean space with the metric* (1.2.12) *to itself is a superposition of an orthogonal transformation and a translation.*

Proof: Let us begin with considering isometries $f : X \to X$ having a fixed point at 0,

$$f(0) = 0.$$

(The general case can be easily reduced to the particular one by considering a superposition with a suitable translation.)

Lemma 1.3.15 *If an isometry f fixes the point* 0 *then the following identity holds true*

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \quad \forall\ x,\ y \in X. \tag{1.3.26}$$

Proof: Since $\rho(x, 0) = |x|$ and $\rho(f(x), 0) = \rho(x, 0)$ we obtain

$$|f(x)| = |x| \quad \forall x \in X. \tag{1.3.27}$$

This gives (1.3.26) for the particular case $y = x$.

Using

$$\rho(f(x), f(y)) = |f(x) - f(y)| = |x - y| = \rho(x, y)$$

we arrive at

$$\langle f(x)-f(y), f(x)-f(y)\rangle = \langle f(x), f(x)\rangle - 2\langle f(x), f(y)\rangle + \langle f(y), f(y)\rangle = \langle x, x\rangle - 2\langle x, y\rangle + \langle y, y\rangle.$$

Applying $\langle f(x), f(x)\rangle = \langle x, x\rangle$, $\langle f(y), f(y)\rangle = \langle y, y\rangle$ one obtains (1.3.26). $\square$

Let us choose an orthonormal basis $e_1, \ldots, e_n$. Due to the Lemma the vectors

$$e_{i'} = f(e_i), \quad i = 1, \ldots, n$$

also form an orthonormal basis. Denote $g : X \to X$ the *linear* orthogonal transformation mapping the basis $e_{1'}, \ldots, e_{n'}$ to $e_1, \ldots, e_n$. The last step in the proof of the Theorem is in proving that the superposition

$$F = g \circ f$$

is an identity map. Indeed, by construction

$$F(e_i) = e_i, \quad i = 1, \ldots, n.$$

Applying Lemma 1.3.15 to the isometry F we derive

$$\langle F(x), e_i\rangle = \langle x, e_i\rangle, \quad i = 1, \ldots, n.$$

Thus the coordinates of the vector $F(x)$ with respect to the basis $e_1, \ldots, e_n$ coincide with the coordinates of the vector x with respect to the same basis (see (1.2.17)). Hence $F = \mathrm{id}$. $\square$

Corollary 1.3.16 *In the orthogonal coordinates in a n-dimensional Euclidean space any isometry can be written in the form*

$$x \mapsto Ax + b, \quad A \in O(n). \tag{1.3.28}$$

Conversely, any transformation of the form (1.3.28) *is an isometry.*

Corollary 1.3.17 *All isometries of a n-dimensional Euclidean space form a group.*

Because of the Corollary 1.2.12 this group depends only on the dimension of the space. It will be denoted $Iso(n)$.

The isometry (1.3.28) will be called *orientation preserving* if $A \in SO(n)$.

Example 1. For $n = 1$ any orientation preserving isometry is a translation $x \mapsto x + b$. An orientation reversing isometry is a reflection with respect to a point. Indeed, such an isometry must have the form

$$x \mapsto -x + b.$$

The point $x = b/2$ is fixed by the transformation. After the change of coordinates

$$x' = x - \frac{b}{2}$$

one obtains the reflection with respect to the origin

$$x' \mapsto -x'.$$

Example 2. For $n = 2$ rotations about a fixed point and translations are examples of orientation preserving isometries. Taking the superposition of the reflection with respect to the axis x with a translation along the same axis

$$(x, y) \mapsto (x + b, -y) \tag{1.3.29}$$

one obtains an orientation reversing isometry (the so-called *glide reflection*).

Exercise 1.3.18 *Prove that any isometry on the plane is a rotation, translation, or a glide reflection.*

Example 3. In the dimension $n = 3$ we have 1) superpositions of rotations about an axis with translations along this axis,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \tag{1.3.30}$$

(*screw axis* symmetry);

2) rotations about an axis followed by a reflection in a plane orthogonal with respect to this axis,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.3.31}$$

(sometimes called *improper rotations*);

3) reflection in a plane followed by a translation along this plane

$$(x, y, z) \mapsto (x + a, y + b, -z) \tag{1.3.32}$$

(*glide planes* symmetry).

Exercise 1.3.19 *Prove that any isometry of the three-dimensional Euclidean space is a screw axis, an improper rotation or a glide plane symmetry.*

One can also consider an *infinitesimal version* of the definition of isometry. Namely, starting as above from the defining identity

$$\langle f(y) - f(x), f(y) - f(x)\rangle = \langle y - x, y - x\rangle$$

let us assume that the point y is close to x,

$$y = x + \Delta x.$$

Expanding

$$\begin{aligned} &f(x + \Delta x) - f(x) = \Delta f(x) + \mathcal{O}\left(|\Delta x|^2\right) \\ &\Delta f(x) = \frac{\partial f(x)}{\partial x^i}\Delta x^i \end{aligned}$$

we rewrite the definition of an isometry f in the form

$$\langle \Delta f(x), \Delta f(x)\rangle = \langle \Delta x, \Delta x\rangle + \mathcal{O}\left(|\Delta x|^3\right) \quad \forall\, x \in X, \quad \forall \,\text{vectors } \Delta x. \tag{1.3.33}$$

Considering the last equation at the leading order approximation one arrives at the following modified version of the definition of an isometry.

Definition 1.3.20 *The smooth map*

$$\begin{aligned} &f : X \to X' \\ &x = (x^1, \dots, x^n) \mapsto f(x) = (y^1, \dots, y^n) \end{aligned}$$

of two Euclidean spaces $(X, \langle\ ,\ \rangle)$ *and* $(X', \langle\ ,\ \rangle')$ *is called isometry if its differential*

$$df(x) = \frac{\partial f(x)}{\partial x^i} dx^i$$

satisfies

$$\langle df(x), df(x)\rangle' = \langle dx, dx\rangle \tag{1.3.34}$$

or, equivalently,

$$\sum_{i=1}^n \left(dy^i(x)\right)^2 = \sum_{k=1}^n (dx^k)^2 \tag{1.3.35}$$

or, one more formulation,

$$\sum_{i=1}^n \frac{\partial y^i(x)}{\partial x^j}\frac{\partial y^i(x)}{\partial x^k} = \delta_{jk} \quad j,\, k = 1, \dots, n \tag{1.3.36}$$

for all $x \in X$.

Observe that the last version (1.3.36) is tantamount to orthogonality of the Jacobi matrix $\left(\frac{\partial y}{\partial x}\right)$ at every point of the space.

It is easy to see that the set of isometries in the sense of new definition is closed under superpositions. The transformations of the form (1.3.28) are still isometries in the sense of the new definition. Conversely, let us prove that the new definition of isometry gives the same class of transformations of the form (1.3.28). For the sake of simplicity let us assume the map $f(x)$ to be analytic, i.e., it can be represented as the sum of a convergent power series that necessarily coincides with its Taylor series.

Theorem 1.3.21 *Let $f : X \to X$ be an isometry of the Euclidean space $(X, \langle\ ,\ \rangle$ to itself. Then it has the form* (1.3.28).

Proof: Taking a superposition with a shift one reduces the proof to the case $f(0) = 0$. Since the matrix

$$A_0 = \left(\frac{\partial y^i(0)}{\partial x^k}\right)$$

is orthogonal, the superposition of f with the orthogonal transformation

$$x \mapsto A_0^{-1} x$$

will be an isometry $x \mapsto \tilde{y} = \tilde{f}(x)$ satisfying

$$\tilde{f}(0) = 0, \quad \frac{\partial \tilde{y}^i(0)}{\partial x^k} = \delta^i_k.$$

It remains to prove that $\tilde{f}$ is the identity map.

Omitting the tildas we arrive at the following system of equations

$$y^i(0) = 0, \quad \frac{\partial y^i(0)}{\partial x^k} = \delta^i_k, \quad \sum_{i=1}^n \frac{\partial y^i(x)}{\partial x^k}\frac{\partial y^i}{\partial x^l} = \delta_{kl}. \tag{1.3.37}$$

Differentiating the last equation in x and setting x to zero one obtains a system of equations

$$a^i_{jk} + a^j_{ik} = 0, \quad i,\, j,\, k = 1, \dots, n \tag{1.3.38}$$

where

$$a^i_{jk} = \frac{\partial^2 y^i(0)}{\partial x^j \partial x^k}.$$

Since the coefficients a^i_{jk} are symmetric in j and k, from (1.3.38) we obtain

$$a^i_{jk} = -a^j_{ik} = a^k_{ji} = -a^i_{jk}.$$

Hence all the second derivatives at the point 0 vanish.

By induction, let us assume that all partial derivatives of the map $f(x)$ of the orders 2, 3, ..., $m-1$ vanish at the origin. Taking partial derivatives of order $(m-1)$ of the last equation in (1.3.37) one arrives at

$$a^i_{j_1 j_2 \dots j_m} + a^{j_m}_{i\, j_2 \dots j_m} = 0$$
$$a^i_{j_1 j_2 \dots j_m} = \frac{\partial^m y^i(0)}{\partial x^{j_1} \partial^{j_2} \dots \partial^{j_m}}.$$

As above from symmetry of partial derivatives $a^i_{j_1 j_2 \dots j_m}$ in j_1, j_2, ..., j_m and antisymmetry in i and j_1 one deduce vanishing of all partial derivatives of order m. So, because of analyticity of the map, it must be equal to identity. □

1.4 Smooth curves in Euclidean space

Let $(X, \langle\ ,\ \rangle)$ be an n-dimensional Euclidean space equipped with a Euclidean inner product. In this section we will use orthogonal coordinates x^1, ..., x^n in X.

Definition 1.4.1 *A* smooth parametrized curve $\gamma \in X$ *is a smooth map*

$$\begin{aligned} &[a,b] \to X \\ &[a,b] \ni t \mapsto x(t) = \big(x^1(t), \dots, x^n(t)\big) \in X. \end{aligned} \tag{1.4.1}$$

The vector

$$v(t) = \dot{x}(t) = \big(\dot{x}^1(t), \dots, \dot{x}^n(t)\big) \tag{1.4.2}$$

is called the velocity vector *of the smooth curve at the point* $\big(x^1(t), \dots, x^n(t)\big)$.

Here

$$\dot{x}^i(t) := \frac{dx^i(t)}{dt}, \quad i = 1, \dots, n.$$

It is clear that the velocity vector is tangent to the smooth curve.

We will always assume that $a < b$.

Definition 1.4.2 *The* length s *of the parametrized curve* (1.4.1) *is equal to the integral of the length of the velocity vector*

$$s = \int_a^b |\dot{x}(t)|\, dt. \tag{1.4.3}$$

The definition can be extended to piecewise smooth curves by taking the sum of all smooth pieces.

Example 1. A segment of a straight line between the points x_0 and x_1 can be written as

$$x^i(t) = x_0^i + t\left(x_1^i - x_0^i\right), \quad i = 1, \dots, n, \quad t \in [0,1].$$

The velocity vector is constant

$$\dot{x}(t) = x_1 - x_0.$$

The length of the segment is equal to the distance between the points

$$s = \int_0^1 |x_1 - x_0|\, dt = |x_1 - x_0| = \rho(x_0, x_1).$$

Example 2. The circle of the radius R on the plane can be represented by the following parametrized curve

$$\begin{aligned} x &= R\cos t \\ y &= R\sin t \\ t &\in [0, 2\pi] \end{aligned} \tag{1.4.4}$$

The velocity vector

$$v(t) = R(-\sin t, \cos t)$$

has constant length. It is orthogonal to the radius vector of the point $(x(t), y(t))$. Also in this case the length is given by the well known formula

$$s = \int_0^{2\pi} R\, dt = 2\pi R.$$

Example 3. For a graph of a smooth function on the plane

$$y = f(x), \quad a \le x \le b \tag{1.4.5}$$

the length is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{1.4.6}$$

It is convenient to introduce the symbol

$$ds^2 = \left(dx^1\right)^2 + \cdots + (dx^n)^2 \tag{1.4.7}$$

for the square of length element. The *restriction* of the quadratic differential on the curve (1.4.1) is defined by

$$ds^2 = \left[\left(\dot{x}^1(t)\right)^2 + \cdots + (\dot{x}^n(t))^2\right] dt^2. \tag{1.4.8}$$

Thus the length of the curve is obtained by restricting the length element on the curve and integrating it along the curve

$$s = \int_\gamma ds \tag{1.4.9}$$

The notation (1.4.7) is convenient in computation of the length of smooth curves in curvilinear coordinates.

Example. In the polar coordinates (1.1.23) one has

$$ds^2 = dx^2 + dy^2 = (\cos\phi\, dr - r\, \sin\phi\, d\phi)^2 + (\sin\phi\, dr + r\, \cos\phi\, d\phi)^2 = dr^2 + r^2 d\phi^2. \tag{1.4.10}$$

In other words, the length of a smooth curve γ given in a parametric form in the polar coordinates

$$r = r(t), \quad \phi = \phi(t), \quad a \le t \le b$$

is equal to

$$s = \int_\gamma \sqrt{dr^2 + r^2 d\phi^2} = \int_a^b \sqrt{\dot{r}^2(t) + r^2(t)\dot{\phi}^2(t)}\, dt.$$

In a similar way in the cylindrical coordinates (1.1.25) one obtains

$$ds^2 = dr^2 + r^2 d\phi^2 + dh^2. \tag{1.4.11}$$

In the spherical coordinates (1.1.27) the square length element is given by the following formula

$$ds^2 = dr^2 + r^2\left(d\theta^2 + \cos^2\theta\, d\phi^2\right). \tag{1.4.12}$$

We leave the derivation of the formulae (1.4.11), (1.4.12) as an exercise for the reader.

More generally let us derive the formula for the length element

$$ds^2 = (dx^1)^2 + \cdots + (dx^n)^2 \tag{1.4.13}$$

in any *curvilinear* local coordinates

$$\begin{aligned} &x^i = x^i(u), \quad i = 1, \ldots, n, \quad u = (u^1, \ldots, u^n) \in D \subset \mathbb{R}^n \\ &\det\left(\frac{\partial x^i}{\partial u^k}\right) \neq 0 \end{aligned} \tag{1.4.14}$$

defined by a local diffeomorphism in $\mathbb{R}^n$. Define a symmetric $n \times n$ matrix with entries

$$g_{kl}(u) = \sum_{i=1}^{n} \frac{\partial x^i}{\partial u^k} \frac{\partial x^i}{\partial u^l}, \quad k, l = 1, \ldots, n. \tag{1.4.15}$$

The geometrical meaning of the functions $g_{kl}(u)$ is clear: they are equal to the inner products of the vectors $\mathbf{x}_{u^k}(u)$ and $\mathbf{x}_{u^l}(u)$ tangent to the axes of the curvilinear system at the point $u = (u^1, \ldots, u^n)$

$$g_{kl}(u) = \langle \mathbf{x}_{u_k}(u), \mathbf{x}_{u^l}(u) \rangle, \quad k, l = 1, \ldots, n. \tag{1.4.16}$$

Thus the matrix

$$G(u) = (g_{kl}(u))_{1 \leq k, l \leq n} \tag{1.4.17}$$

is the *Gram matrix* of the basis $\mathbf{x}_{u^1}(u), \ldots, \mathbf{x}_{u^n}(u)$. Note that the basis depends on the point $u \in D$.

Theorem 1.4.3 *The Euclidean square length element* (1.4.13) *in the coordinates* (1.4.14) *takes the form*

$$ds^2 = g_{kl}(u) du^k du^l. \tag{1.4.18}$$

Proof: Substituting the differentials

$$dx^i = \frac{\partial x^i}{\partial u^k} du^k$$

into (1.4.13) one obtains

$$ds^2 = \sum_{i=1}^{n} \left(\frac{\partial x^i}{\partial u^k} du^k\right) \left(\frac{\partial x^i}{\partial u^l} du^l\right) = \sum_{i=1}^{n} \frac{\partial x^i}{\partial u^k} \frac{\partial x^i}{\partial u^l} du^k du^l.$$

$\square$

Corollary 1.4.4 *The length of a smooth curve*

$$\gamma : \mathbf{x} = \mathbf{x}(u(t)), \quad u(t) = \left(u^1(t), \ldots, u^n(t)\right)$$

represented in the curvilinear coordinates by smooth functions

$$u^k = u^k(t), \quad k = 1, \ldots, n, \quad t \in [a, b]$$

can be computed by the following formula

$$s = \int_\gamma ds = \int_a^b \sqrt{g_{kl}(u(t))\dot u^k(t)\dot u^l(t)}\, dt. \tag{1.4.19}$$

The angle α between two smooth curves $u_1(t_1) = \left(u_1^1(t_1), \dots, u_1^n(t_1)\right)$ and $u_2(t_2) = \left(u_2^1(t_2), \dots, u_2^n(t_2)\right)$ at the intersection point $u_0 = u_1(t_1^0) = u_2(t_2^0)$ is defined from

$$\cos\alpha = \frac{g_{kl}(u_0)\dot u_1^k(t_1^0)\dot u_2^l(t_2^0)}{\sqrt{g_{kl}(u_0)\dot u_1^k(u_0)\dot u_1^l(u_0)}\sqrt{g_{pq}(u_0)\dot u_2^p(u_0)\dot u_2^q(u_0)}}. \tag{1.4.20}$$

Exercise 1.4.5 *Derive the following formula for the Gram matrix $G = G(u) = (g_{kl}(u))$*

$$G = \left(\frac{\partial x}{\partial u}\right)^{\mathrm{T}} \left(\frac{\partial x}{\partial u}\right). \tag{1.4.21}$$

Recall that $\left(\frac{\partial x}{\partial u}\right)$ is the Jacobi matrix of the transformation of coordinates (1.4.14).

Remark 1.4.6 *Given a symmetric bilinear form $g(x, y)$ on a n-dimensional linear space, one can define the Gram matrix $G = (g_{kl})$ of this form in a basis $e_1, \dots, e_n$ by*

$$g_{kl} = g(e_k, e_l), \quad k, l = 1, \dots, n. \tag{1.4.22}$$

The value of the bilinear form on every pair of vectors

$$x = x^k e_k, \quad y = y^l e_l$$

is given by the formula

$$g(x, y) = g_{kl} x^k y^l. \tag{1.4.23}$$

Clearly the matrix G is symmetric,

$$G^{\mathrm{T}} = G.$$

Conversely, given a symmetric $n \times n$ matrix G the formula (1.4.23) *defines a symmetric bilinear form. One can try to* define *an inner product of vectors in the linear space by*

$$\langle x, y \rangle := g(x, y). \tag{1.4.24}$$

Such an inner product satisfies the properties of bilinearity and symmetry. Under what conditions on the matrix G the positive definiteness also holds true for the inner product (1.4.24)*? The answer is provided by the* Sylvester theorem*: the inner product* (1.4.23), (1.4.24) *is positive definite* iff *all principal minors of the Gram matrix G are positive*

$$g_{11} > 0, \quad \det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} > 0, \dots, \det G > 0. \tag{1.4.25}$$

Remark 1.4.7 *Another natural question: given a symmetric $n \times n$ matrix of smooth* functions $g_{kl}(u)$ *satisfying the Sylvester conditions* (1.4.25) *for any point* $u = (u^1, \ldots, u^n)$*, does there exist a transformation*

$$x^i = x^i(u), \quad i = 1, \ldots, n, \quad \det\left(\frac{\partial x}{\partial u}\right) \neq 0$$

to Euclidean coordinates, i.e., such that

$$g_{kl}(u) du^k du^l = \sum_{i=1}^n \left(dx^i(u)\right)^2 \ ?$$

The answer, in general, is negative. We will see below that the obstacle to local existence of Euclidean coordinates can be described in the framework of the theory of curvature.

Let us return to the properties of the length of curves. Our first statement is independence of the length of a smooth curve from the parametrization.

Theorem 1.4.8 *Let $t = t(\tilde t)$ be a smooth function defined on the segment $[\tilde a, \tilde b]$ satisfying*

$$\begin{aligned} &t(\tilde a) = a, \quad t(\tilde b) = b \\ &\frac{dt}{d\tilde t} > 0. \end{aligned} \tag{1.4.26}$$

Then the lengths of the curves (1.4.1) *and*

$$\begin{aligned} &[\tilde a, \tilde b] \to X \\ &[\tilde a, \tilde b] \ni \tilde t \mapsto x\left(t(\tilde t)\right) = \left(x^1\left(t(\tilde t)\right), \ldots, x^n\left(t(\tilde t)\right)\right) \end{aligned} \tag{1.4.27}$$

coincide.

Proof: The velocity vectors of the curves (1.4.1) and (1.4.27) are proportional at every point:

$$\frac{dx\left(t(\tilde t)\right)}{d\tilde t} = \frac{dx(t)}{dt}\frac{dt}{d\tilde t}.$$

Thus

$$\int_{\tilde a}^{\tilde b} \left|\frac{dx\left(t(\tilde t)\right)}{d\tilde t}\right| d\tilde t = \int_{\tilde a}^{\tilde b} \left|\frac{dx(t)}{dt}\right| \frac{dt}{d\tilde t} d\tilde t = \int_a^b \left|\frac{dx(t)}{dt}\right| dt.$$

□

One can also define the *angle* between two smooth curves

$$\begin{aligned} &x_1(t_1) = \left(x_1^1(t_1), \ldots x_1^n(t_1)\right), \quad x_2(t_2) = \left(x_2^1(t_2), \ldots x_2^n(t_2)\right) \\ &x_1\left(t_1^0\right) = x_2\left(t_2^0\right) =: x_0 \end{aligned}$$

at the intersection point x_0 as the angle α between their tangent vectors at this point

$$v_1^0 = \frac{dx_1(t_1)}{dt_1}\Big|_{t_1 = t_1^0}, \quad v_2^0 = \frac{dx_2(t_2)}{dt_2}\Big|_{t_2 = t_2^0}$$

$$\cos\alpha = \frac{\langle v_1^0, v_2^0\rangle}{|v_1^0|\,|v_2^0|}. \tag{1.4.28}$$

Exercise 1.4.9 *Prove that the polar coordinate lines* $\gamma_r : \{\phi = \phi_0\}$ *and* $\gamma_\phi : \{r = r_0\}$ *on the plane for arbitrary* $r_0 > 0$, ϕ_0 *are orthogonal at every intersection point. In a similar way prove that the coordinate lines in the cylindrical*

$$\gamma_r : \left.\begin{array}{rcl}\phi &=& \phi_0\\ h &=& h_0\end{array}\right\}, \quad \gamma_\phi : \left.\begin{array}{rcl}h &=& h_0\\ r &=& r_0\end{array}\right\}, \quad \gamma_h : \left.\begin{array}{rcl}r &=& r_0\\ \phi &=& \phi_0\end{array}\right\}$$

and spherical

$$\gamma_r : \left.\begin{array}{rcl}\phi &=& \phi_0\\ \theta &=& \theta_0\end{array}\right\}, \quad \gamma_\phi : \left.\begin{array}{rcl}\theta &=& \theta_0\\ r &=& r_0\end{array}\right\}, \quad \gamma_\theta : \left.\begin{array}{rcl}r &=& r_0\\ \phi &=& \phi_0\end{array}\right\}$$

coordinates in $\mathbb{R}^3$ *are pairwise orthogonal at every intersection point.*

The smooth curve (1.4.1) is called *regular* if the velocity vector $\dot{x}(t)$ does not vanish at any point of the curve. On a regular non self-intersecting curve it is convenient to use the length along the curve as a parameter:

$$s(t) = \int_a^t |\dot{x}(t')|\, dt'. \tag{1.4.29}$$

The condition (1.4.26) is satisfied since

$$\frac{ds}{dt} = |\dot{x}(t)| > 0.$$

Lemma 1.4.10 *Modulus of the velocity vector* $v(s) = \frac{dx(s)}{ds}$ *of a curve parametrized by the length is identically equal to one:*

$$\left|\frac{dx(s)}{ds}\right| = 1. \tag{1.4.30}$$

Proof: Differentiating in s the equation

$$s = \int_a^s \left|\frac{dx(s')}{ds'}\right| ds'$$

one obtains

$$1 = \left|\frac{dx(s)}{ds}\right|.$$

□

Lemma 1.4.11 *Let* $x(s)$ *be a smooth curve parametrized by the length. Then the acceleration vector* d^2x/ds^2 *is orthogonal to the velocity vector* dx/ds.

Proof: Using (1.4.30) we obtain

$$0 = \frac{d}{ds}\left\langle \frac{dx}{ds}, \frac{dx}{ds}\right\rangle = 2\left\langle \frac{d^2x}{ds^2}, \frac{dx}{ds}\right\rangle.$$

Hence

$$\ddot{x}(s) \perp \dot{x}(s).$$

□

Definition 1.4.12 *The modulus*

$$k(s) = \left| d^2x(s)/ds^2 \right| \tag{1.4.31}$$

of the acceleration vector of a smooth curve parametrized by the length is called the curvature *of the curve. The vector*

$$n(s) = \frac{1}{k(s)} \frac{d^2x(s)}{ds^2} \tag{1.4.32}$$

defined at the points with nonvanishing curvature is called the principal normal *to the curve.*

Example. The curvature of a straight line is equal to 0. It is easy to prove that, vice versa, the smooth curves with identically vanishing curvature are straight lines. Indeed, if the curvature vanishes then

$$\frac{d^2x(s)}{ds^2} = 0 \quad \Rightarrow \quad x(s) = x_0 + v_0 s$$

with a constant vector v_0.

To compute the curvature of a circle of the radius R let us parametrize it by the length:

$$x(s) = R\cos\frac{s}{R}$$
$$y(s) = R\sin\frac{s}{R}$$

$0 \leq s \leq 2\pi R$. The velocity vector

$$v(s) = \left(-\sin\frac{s}{R}, \cos\frac{s}{R} \right)$$

has the modulus 1, as it should be. The acceleration vector is

$$\left(\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2} \right) = -\frac{1}{R}\left(\cos\frac{s}{R}, \sin\frac{s}{R} \right).$$

So the curvature of the circle is constant and it is equal to

$$k = \frac{1}{R}.$$

The principal normal is the unit vector

$$n(s) = -\left(\cos\frac{s}{R}, \sin\frac{s}{R} \right)$$

opposite to the radius vector of the point.

Let us consider in more details the plane case. Let

$$r(s) = (x(s), y(s)), \quad s \in [0, s_0] \tag{1.4.33}$$

be a smooth curve on the plane parametrized by the length with never vanishing curvature.

Exercise 1.4.13 *For a point $r(s_0) = (x(s_0), y(s_0))$ of the smooth curve (1.4.33) consider the point $O(s_0)$ on the principal normal in the direction of the latter such that*

$$\rho\left(r(s_0), O(s_0)\right) = R(s_0), \quad R(s_0) = \frac{1}{k(s_0)}.$$

Prove that the distance between the points of the circle with the centre at $O(s_0)$ and radius $R(s_0)$ and the points of the curve tends to zero as $\mathcal{O}\left(|s - s_0|^3\right)$ as $s \to s_0$.

Hint: assume, without loss of generality, that $s_0 = 0$. Choose coordinate system in such a way that $r(0) = 0$, $v(0) = (1, 0)$, $n(0) = (0, 1)$. Derive the following Taylor expansion for the radius vector $r(s)$ for $s \to 0$

$$r(s) = \left(s, \frac{1}{2} k\, s^2\right) + \mathcal{O}\left(s^3\right), \quad k = k(0).$$

For small $|s|$ compute the distance between points of this curve and points of the circle

$$\begin{aligned} x &= \qquad \frac{1}{k} \sin ks \\ y &= \frac{1}{k} - \frac{1}{k} \cos ks. \end{aligned}$$

The radius $R(s_0)$ is called the *radius of curvature* of the curve at the point $r(s_0)$ while the point $O(s_0)$ is the *centre of curvature*.

At every point of the curve one has an orthonormal basis

$$\left(v(s), n(s)\right). \tag{1.4.34}$$

The dependence of this basis on the point of the curve is described by

Theorem 1.4.14 (Frenet–Serret formulae on the plane) *The co-moving frame (1.4.33) dependence on s is given by the following equations*

$$\begin{aligned} \frac{d}{ds} v(s) &= \quad k(s)\, n(s) \\ \frac{d}{ds} n(s) &= -k(s)\, v(s). \end{aligned} \tag{1.4.35}$$

Proof: Denote $\phi(s)$ the angle between the vector $v(0)$ and $\phi(s)$. The frame $(v(s), n(s))$ is obtained from $(v(0), n(0))$ by a rotation by the angle $\phi(s)$:

$$(v(s), n(s)) = (v(0), n(0)) \begin{pmatrix} \cos\phi(s) & -\sin\phi(s) \\ \sin\phi(s) & \cos\phi(s) \end{pmatrix}. \tag{1.4.36}$$

Denote

$$G(s) = \begin{pmatrix} \cos\phi(s) & -\sin\phi(s) \\ \sin\phi(s) & \cos\phi(s) \end{pmatrix}$$

the matrix of the rotation (1.4.36). We have

$$\frac{d}{ds}\left(v(s), n(s)\right) = \left(v(0), n(0)\right)\frac{dG(s)}{ds} = \left(v(s), n(s)\right) G^{-1}(s)\frac{dG(s)}{ds}. \tag{1.4.37}$$

The following statement is crucial in completing the proof (it will also used in sequel).

Lemma 1.4.15 *Let $G(s) \in O(n)$ be an orthogonal matrix smoothly depending on the parameter s. Then the matrix*

$$A(s) = G^{-1}(s)\frac{dG(s)}{ds} \tag{1.4.38}$$

is antisymmetric for any s.

Proof: Differentiating the identity $G(s)G^{\mathrm{T}}(s) = 1$ in s one obtains

$$\frac{dG(s)}{ds}G^{\mathrm{T}}(s) + G(s)\frac{dG^{\mathrm{T}}(s)}{ds} = 0.$$

Multiplying this equation by $G^{-1}(s)$ on the left and by $\left(G^{\mathrm{T}}(s)\right)^{-1}$ on the right one obtains

$$G^{-1}(s)\frac{dG(s)}{ds} + \frac{dG^{\mathrm{T}}(s)}{ds}\left(G^{\mathrm{T}}(s)\right)^{-1} = 0.$$

Using

$$\frac{dG^{\mathrm{T}}}{ds} = \left(\frac{dG}{ds}\right)^{\mathrm{T}}, \quad \left(G^{\mathrm{T}}\right)^{-1} = \left(G^{-1}\right)^{\mathrm{T}}$$

along with the general property of the matrix transposition of a product

$$(A\,B)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$$

one obtains

$$A(s) + A^{\mathrm{T}}(s) = 0.$$

□

From the Lemma and (1.4.37) we conclude that

$$\frac{d}{ds}\left(v(s), n(s)\right) = \left(v(s), n(s)\right)\begin{pmatrix} 0 & -a(s) \\ a(s) & 0 \end{pmatrix}.$$

A comparison with the definition of the principal normal yields

$$a(s) = k(s).$$

This completes the proof of the Frenet–Serret formula. □

The curvature of a regular smooth curve for the obvious reasons is invariant with respect to isometries of the ambient space. The following result shows that this is a *complete invariant* for the case $n = 2$.

Theorem 1.4.16 *Let*

$$r_1(s) = (x_1(s), y_1(s)) \quad \text{and} \quad r_2(s) = (x_2(s), y_2(s)), \quad s \in [0, s_0]$$

be two smooth curves on the plane parametrized by their lengths such that their curvatures $k_1(s)$ and $k_2(s)$ coincide for any $s \in [0, s_0]$ and do not vanish. Then there exists an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ mapping one curve onto the other.

Proof: Denote $(v_1(s), n_1(s))$ and $(v_2(s), n_2(s))$ respectively the velocity vectors and the principal normal vectors of the two curves. Applying a suitable isometry one may assume that

$$r_2(0) = r_1(0), \quad v_2(0) = v_1(0), \quad n_2(0) = n_1(0).$$

Let us prove that, after such a choice the curves will coincide.

Due to equality $k_2(s) = k_1(s) =: k(s)$ the vector valued functions $v_{1,2}(s)$ and $n_{1,2}(s)$ satisfy the same system of Frenet–Serret linear differential equations

$$\begin{aligned} \frac{d}{ds} v_{1,2}(s) &= \quad k(s)\, n_{1,2}(s) \\ \frac{d}{ds} n_{1,2}(s) &= -k(s)\, v_{1,2}(s) \end{aligned}$$

with the same initial data $v_{1,2}(0)$, $n_{1,2}(0)$. Due to uniqueness theorem from the theory of ODEs one has

$$v_2(s) = v_1(s), \quad n_2(s) = n_1(s) \quad \forall\, s \in [0, s_0].$$

The last observation to be used says that the curve $r(s)$ can be uniquely reconstructed from its velocity vector $v(s)$ and the initial point $r(0)$ by a quadrature

$$r(s) = r(0) + \int_0^s v(s')\, ds'.$$

□

Exercise 1.4.17 *Prove that for any smooth nonvanishing function $k(s)$, $s \in [0, s_0]$ there exists a smooth curve $r(s)$ parametrized by the length with the curvature $k(s)$.*

Hint: use that solutions to a system of linear differential equations with coefficients smooth on an interval $[a, b]$ exist on the entire interval.

In the three-dimensional case one can associate a co-moving frame $(v(s), n(s), b(s))$ with a smooth curve

$$r(s) = (x(s), y(s), z(s))$$

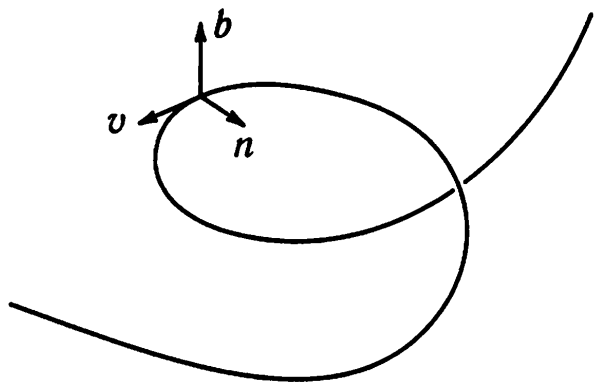
parametrized by the length with a nonvanishing curvature $k(s)$ by adding the vector of *binormal*

$$b(s) = v(s) \times n(s) \tag{1.4.39}$$

obtained as the cross-product of the velocity vector $v(s)$ and the principal normal vector $n(s)$. The dependence of this frame on the length of the curve is given by the following Frenet–Serret formulae

$$\frac{d}{ds}\,(v(s), n(s), b(s)) = (v(s), n(s), b(s)) \begin{pmatrix} 0 & -k(s) & 0 \\ k(s) & 0 & -\kappa(s) \\ 0 & \kappa(s) & 0 \end{pmatrix}. \tag{1.4.40}$$

Here $\kappa(s)$ is another invariant of the curve called *torsion*. It can be defined as the coefficient of proportionality between the vectors db/ds and $n(s)$ taken with the opposite sign. As above one can prove that a pair of functions $(k(s) > 0, \kappa(s))$ determines uniquely the class of equivalence of a space curve up to isometries. We leave the proofs of all these statements as an exercise for the reader.



Frenet–Serret frame of a space curve.

Exercise 1.4.18 *Prove that the space curve with nonvanishing curvature belongs to a plane* iff *the torsion is identically equal to zero.*

Exercise 1.4.19 *Compute the curvature and the torsion of the right-handed*

$$\begin{aligned} x &= r\,\cos t \\ y &= r\,\sin t \\ z &= h\,t \end{aligned}$$

and the left-handed helix

$$\begin{aligned} x &= r\,\cos t \\ y &= -r\,\sin t \\ z &= h\,t \end{aligned}$$

(here $r,\ h > 0$*). Verify that the torsion is positive in the first case but negative in the second one.*

Exercise 1.4.20 *Prove that space curves with constant curvature* $k > 0$ *and torsion* $\kappa \neq 0$ *are helices.*

Exercise 1.4.21 *Let the initial point of a space curve be at the origin and the vectors of the co-moving frame at the initial point coincide with the unit vectors of the axes* x*,* y *and* z *respectively. Derive the following Taylor expansion of the curve for small values of the length*

$$r(s) = \left(s - \frac{1}{6}k^2 s^3, \frac{1}{2}k\,s^2 + \frac{1}{6}k' s^3, \frac{1}{6}k\,\kappa\,s^3 \right) + O\left(s^4\right)$$

where $k = k(0)$*,* $k' = \frac{dk(0)}{ds}$*,* $\kappa = \kappa(0)$*. Use this formula to deriving an interpretation of the sign of torsion for a general smooth curve.*

1.5 Areas and volumes

Let $\Omega \subset \mathbb{R}^n$ be a closed bounded domain in a n-dimensional Euclidean space with orthogonal coordinates $(x^1, \dots, x^n)$. Recall that the *volume* of the domain is defined as the value of multiple integral

$$Vol(\Omega) = \int \cdots \int_\Omega dx^1 \dots dx^n \tag{1.5.1}$$

provided that the integral in question exists. For the particular case $n = 2$ one uses the name 'area' for the integral (1.5.1); it will also be denoted $A(\Omega)$. For the even simpler case $n = 1$ the "volume" of the domain $\Omega = [a, b]$ coincides with the length of the segment.

Let us now derive the formula for the volume in curvilinear coordinates (1.4.14). Recall that the Euclidean length element in the curvilinear coordinates takes the form

$$ds^2 = g_{kl}(u) du^k du^l$$

where the symmetric matrix $G = (g_{kl}(u))$ defined by the formula (1.4.15).

Theorem 1.5.1 *The volume of the domain*

$$\Omega = \{\mathbf{x} = \mathbf{x}(u) \,|\, u \in \mathcal{D} \subset \mathbb{R}^n\}$$

is given by the formula

$$Vol(\Omega) = \int \cdots \int_\mathcal{D} \sqrt{\det(g_{kl}(u))}\, du^1 \dots du^n. \tag{1.5.2}$$

Proof: Doing change of variables in the multiple integral (1.5.1) one arrives at the expression

$$Vol(\Omega) = \int \cdots \int_\mathcal{D} \left|\det\left(\frac{\partial x}{\partial u}\right)\right| du^1 \dots du^n.$$

Thus the proof of the Theorem will follow from

Lemma 1.5.2 *The determinant of of the Gram matrix $G = (g_{kl}(u))$ is equal to the square of the Jacobian of the transformation of the coordinates* (1.4.14)

$$\det(g_{kl}(u)) = \left[\det\left(\frac{\partial x}{\partial u}\right)\right]^2. \tag{1.5.3}$$

Proof: From the Exercise 1.4.5 it follows that

$$\det(g_{kl}(u)) = \det\left[\left(\frac{\partial x}{\partial u}\right)^{\mathrm{T}}\left(\frac{\partial x}{\partial u}\right)\right].$$

Replacing the determinant of the product of matrices by the product of determinants and using that

$$\det\left(\frac{\partial x}{\partial u}\right)^{\mathrm{T}} = \det\left(\frac{\partial x}{\partial u}\right)$$

one arrives at the formula (1.5.3). $\square$

The expression
$$dV = \sqrt{\det(g_{kl}(u))}\, du^1 \dots du^n \tag{1.5.4}$$
is often called the *volume element* (the area element dA for $n = 2$) in the curvilinear coordinates.

Example 1. In the polar coordinates on the plane the square length element is given by
$$ds^2 = dr^2 + r^2 d\phi^2.$$
The Gram matrix is diagonal
$$G = \left(\begin{array}{cc} 1 & 0 \\ 0 & r^2 \end{array} \right).$$
Hence the area element in the polar coordinates reads
$$dA = r\, dr\, d\phi.$$
In the particular case of a circle of radius R
$$\mathcal{D} = \{(r, \phi)\,|\, 0 \le r \le R,\, 0 \le \phi \le 2\pi\}$$
one obtains the well known formula
$$A = \int_0^R \int_0^{2\pi} r\, dr\, d\phi = \pi R^2.$$

Example 2. The expression
$$ds^2 = dr^2 + r^2(d\theta^2 + \cos^2\theta\, d\phi^2)$$
for the square length in the spherical coordinates yields the following formula for the volume element
$$dV = r^2 \cos\theta\, dr\, d\phi\, d\theta.$$
For the particular case of a ball of radius R
$$\mathcal{D} = \left\{(t, \phi, \theta)\,|\, 0 \le R,\ 0 \le \phi \le 2\pi,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$$
one easily arrives at the formula
$$V = \int_0^R \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos\theta\, dr\, d\phi\, d\theta = \frac{4}{3}\pi R^3.$$

1.6 Geometry of surfaces

We will proceed with studying two-dimensional surfaces in the three-dimensional Euclidean space. This seemingly very particular case already possesses many features stimulating development of general notions of differential geometry.

Let (x, y, z) be a system of Euclidean coordinates in the three-dimensional Euclidean space. A smooth *parametrized surface* is defined by a smooth map of a domain D on the plane to the three dimensional Euclidean space. In coordinates it is represented by three smooth functions of two variables

$$\left.\begin{array}{rcl} x & = & x(u,v) \\ y & = & y(u,v) \\ z & = & z(u,v) \end{array}\right\}, \quad (u,v) \in D \subset \mathbb{R}^2. \tag{1.6.1}$$

The notation

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3 \tag{1.6.2}$$

for the radius vector of points of the surface will also be used.

The image of the map (1.6.1) is a set of points $S \subset \mathbb{R}^3$. It will also be often called a *surface* if the choice of a parametrization has been made. We will assume the surface S to be non self-intersecting, i.e.,

$$\mathbf{r}(u_1, v_1) \neq \mathbf{r}(u_2, v_2) \quad \text{for} \quad (u_2, v_2) \neq (u_1, v_1).$$

Under this assumption one can use the parameters (u, v) as *local coordinates* on the surface.

Fixing $v = v_0$ one obtains a curve $\mathbf{r}(u, v_0)$ on the surface. In a similar way, fixing $u = u_0$ one obtain another curve $\mathbf{r}(u_0, v)$ on the surface. The velocity vectors of these curves

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u,v)}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}(u,v)}{\partial v} \tag{1.6.3}$$

are tangent to the surface.

Definition 1.6.1 *The surface* (1.6.1) *is called* regular *if the vectors r_u, r_v are linearly independent at every point of the surface.*

Example 1. With the help of cylindrical coordinates (1.1.25) one obtains a parametric representation $\mathbf{r} = \mathbf{r}(\phi, h)$ of the cylinder of radius R having Oz as the axis:

$$\left.\begin{array}{rcl} x & = & R\cos\phi \\ y & = & R\sin\phi \\ z & = & h \end{array}\right\}. \tag{1.6.4}$$

The tangent vectors

$$\mathbf{r}_\phi = (-R\sin\phi, R\cos\phi, 0), \quad \mathbf{r}_h = (0,0,1)$$

are linearly independent at every point of the surface. The domain $D \subset \mathbb{R}^2$ can be chosen as follows:

$$D = \{(\phi, h) \in \mathbb{R}^2 \,|\, 0 < \phi < 2\pi, \quad -\infty < h < \infty\}.$$

The image of the domain D covers the surface of the cylinder except for the line

$$x = R, \quad y = 0, \quad -\infty < z < \infty.$$

Example 2. One can use spherical coordinates (1.1.27) for parametrization $\mathbf{r} = \mathbf{r}(\phi, \theta)$ of the sphere of radius R centered at the origin:

$$\left.\begin{array}{rcl} x & = & R\cos\phi\cos\theta \\ y & = & R\sin\phi\cos\theta \\ z & = & R\sin\theta \end{array}\right\}. \tag{1.6.5}$$

The tangent vectors

$$\mathbf{r}_\phi = R\cos\theta\,(-\sin\phi, \cos\phi, 0)\,, \quad \mathbf{r}_\theta = R\,(-\cos\phi\sin\theta, -\sin\phi\sin\theta, \cos\theta)$$

are independent outside the poles $\theta = \pm\frac{\pi}{2}$. However, at the poles the vector $\mathbf{r}_\phi$ becomes equal to zero. The coordinates (ϕ, θ) belong to the domain

$$D = \left\{(\phi, \theta) \in \mathbb{R}^2 \,|\, 0 < \phi < 2\pi, \; -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}.$$

The image of the domain D covers the part of the surface of the sphere away from the arc of the $\phi = 0$ meridian

$$\{x = R\cos\theta, \quad y = 0, \quad z = R\sin\theta\}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

Remark 1.6.2 *We want to emphasize that the violation of the regularity property may take place due to a bad parametrization rather than to the shape of the surface. For example, the shape of the sphere near the poles looks as good as at any other point of it. Indeed, choosing a different parametrization one can obtain a regular surface structure also near the poles. We will return below to the problem of reparametrizations on two-dimensional surfaces.*

Example 3. For a graph of a smooth function of two variables

$$z = f(x, y) \tag{1.6.6}$$

one can use x and y as the parameters:

$$\mathbf{r} = (x, y, f(x, y))\,.$$

The tangent vectors of the form (1.6.3) read

$$\mathbf{r}_x = (1, 0, f_x(x, y))\,, \quad \mathbf{r}_y = (0, 1, f_y(x, y))\,. \tag{1.6.7}$$

These vectors are linearly independent. So the graph (1.6.6) is always a regular surface.

Exercise 1.6.3 *Prove that any regular surface in $\mathbb{R}^3$ can be locally represented as a graph $z = f(x, y)$, $y = g(x, z)$, or $x = h(y, z)$ for some smooth functions f, g, or h.*

Hint: represent the regularity condition in the form

$$\operatorname{rk}\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \\ \partial z/\partial u & \partial z/\partial v \end{pmatrix} = 2. \tag{1.6.8}$$

Apply the implicit function theorem to arrive at the needed representation of the surface.

It is sometimes convenient to define surfaces in $\mathbb{R}^3$ by a single equation

$$F(x, y, z) = 0 \tag{1.6.9}$$

for a smooth function of three variables F. The condition of regularity in this case can be formulated as follows.

Definition 1.6.4 *The surface* (1.6.9) *is called regular if the gradient*

$$\operatorname{grad} F(x, y, z) = (F_x, F_y, F_z) \tag{1.6.10}$$

never vanishes at any point of the surface.

The relation of this definition to the previous one is established by

Lemma 1.6.5 *Let* (x_0, y_0, z_0) *be a point of the surface* (1.6.9) *such that*

$$F_z(x_0, y_0, z_0) \neq 0.$$

Then there exists a unique smooth function $f(x, y)$ *defined for sufficiently small*

$$(x - x_0)^2 + (y - y_0)^2$$

such that $f(x_0, y_0) = z_0$ *and, moreover, any solution to the equation* (1.6.9) *in a small neighborhood of the point* (x_0, y_0, z_0) *has the form*

$$(x, y, z = f(x, y))\,.$$

In other words, near the points where the third component of $\operatorname{grad} F$ does not vanish one can represent the surface as a graph of a function $z = f(x, y)$. Similarly, near point where the first or second component of the gradient does not vanish one can represent the surface as a graph of a function $x = g(y, z)$ or $y = h(x, z)$ respectively.

The proof readily follows from the Implicit Function Theorem applied to the equation (1.6.9).

Examples. An affine hyperplane

$$ax + by + cz + d = 0, \quad a^2 + b^2 + c^2 \neq 0$$

is a regular surface. For $c \neq 0$ it is a graph of a linear function

$$z = z(x, y) = -\frac{1}{c}(ax + by + d).$$

Similar representations $x = x(y, z)$ or $y = y(x, z)$ can be used under the assumptions $a \neq 0$ or $b \neq 0$ respectively. The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the hyperboloids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1$$

are all regular surfaces. For the 'minus' sign the hyperboloid consists of two pieces each of them can be written as a graph of a smooth function

$$z = \pm c \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}.$$

For the 'plus' sign only local coordinates exist. The cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

fails to meet the regularity condition at the origin (the latter belongs to the cone but not to the hyperboloids!). Indeed, the gradient $2\left(\frac{x}{a^2}, \frac{y}{b^2}, -\frac{z}{c^2}\right)$ vanishes at the origin.

The image of a plane smooth curve

$$(u = u(t), v = v(t)) \in D \subset \mathbb{R}^2, \quad t \in [a, b] \tag{1.6.11}$$

with respect to the map (1.6.1) gives a smooth curve on the surface

$$\mathbf{r}(t) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))) \in \mathbb{R}^3 \tag{1.6.12}$$

Lemma 1.6.6 *Let the surface* (1.6.1) *be regular. The space curve* (1.6.12) *is regular* iff *the plane curve* (1.6.11) *is so.*

Proof: The velocity vector of the space curve (1.6.12) is equal to

$$\dot{\mathbf{r}}(t) = \mathbf{r}_u \, \dot{u}(t) + \mathbf{r}_v \, \dot{v}(t). \tag{1.6.13}$$

Because of linear independence of the vectors $\mathbf{r}_u$, $\mathbf{r}_v$ vanishing of this vector is equivalent to

$$\dot{u}(t) = \dot{v}(t) = 0.$$

This contradicts regularity of the plane curve (1.6.11). □

The plane

$$T_{(u,v)} = \{\alpha \, \mathbf{r}_u(u, v) + \beta \, \mathbf{r}_v(u, v) \,|\, \alpha, \, \beta \in \mathbb{R}\} \tag{1.6.14}$$

spanned by the tangent vectors $\mathbf{r}_u(u, v)$, $\mathbf{r}_v(u, v)$ is called the *tangent plane* to the surface (1.6.1) at the point (u, v). According to (1.6.13) the tangent vectors of all smooth curves on the surface passing through the point $\mathbf{r}(u, v)$ belong to this plane. Moreover, any vector in the plane (1.6.14) is a tangent vector to some smooth curve on the surface.

Example. Consider the graph of a smooth function $z = f(x, y)$ with the natural parametrization $u = x$, $v = y$ (see above). The plane tangent to the surface at the point $(x_0, y_0, z_0 = f(x_0, y_0)$ consists of all vectors $(x-x_0, y-y_0, z-z_0)$ passing through the tangency point and represented in the form

$$(x - x_0, y - y_0, z - z_0) = \alpha\,(1, 0, f_x(x_0, y_0)) + \beta\,(0, 1, f_y(x_0, y_0)), \quad \alpha,\, \beta \in \mathbb{R}.$$

Eliminating the parameters $\alpha = x - x_0$, $\beta = y - y_0$ one obtains the equation of the tangent plane to the graph known from the course of multivariable calculus

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Exercise 1.6.7 *Prove that the tangent plane to the regular surface defined by equation $F(x, y, z) = 0$ at the point (x_0, y_0, z_0) consists of all vectors passing through the tangency point and orthogonal to the vector* $\operatorname{grad} F(x_0, y_0, z_0)$*:*

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Our strategy is to describe the geometric properties of the parametrized surface (1.6.1) in terms of certain structures to be defined on the domain of definition of the parameters $(u, v) \in D \subset \mathbb{R}^2$. The first step in this direction is given by the Lemma 1.6.6: regular smooth curves on the surface are represented by regular smooth curves of the form $(u(t), v(t))$ on the plane. The velocity vectors (1.6.13) of these curves are uniquely determined by their coordinates $\dot{u}(t)$, $\dot{v}(t)$. Let us now obtain a similar description for the length of these curves as well as for the angle between curves on the surface.

Lemma 1.6.8 *Denote*

$$\begin{aligned} E = E(u, v) &= \langle \mathbf{r}_u(u, v), \mathbf{r}_u(u, v)\rangle \\ F = F(u, v) &= \langle \mathbf{r}_u(u, v), \mathbf{r}_v(u, v)\rangle \\ G = G(u, v) &= \langle \mathbf{r}_v(u, v), \mathbf{r}_v(u, v)\rangle. \end{aligned} \tag{1.6.15}$$

Then the length of the velocity vector of the curve (1.6.12) *on the surface is given by*

$$|\dot{\mathbf{r}}(t)|^2 = E\,(u(t), v(t))\,(\dot{u}(t))^2 + 2F\,(u(t), v(t))\,\dot{u}(t)\dot{v}(t) + G\,(u(t), v(t))\,(\dot{v}(t))^2. \tag{1.6.16}$$

The angle α between two smooth curves on the surface

$$(u_1(t_1), v_1(t_1)) \quad \text{and} \quad (u_2(t_2), v_2(t_2))$$

at the common point

$$(u_0, v_0) = \left(u_1(t_1^0), v_1(t_1^0)\right) = \left(u_2(t_2^0), v_2(t_2^0)\right) \tag{1.6.17}$$

can be defined from

$$\cos\alpha = \frac{E^0 \dot{u}_1^0 \dot{u}_2^0 + F^0\left(\dot{u}_1^0 \dot{v}_2^0 + \dot{v}_1^0 \dot{u}_2^0\right) + G^0 \dot{v}_1^0 v_2^0}{\sqrt{E^0 (\dot{u}_1^0)^2 + 2F^0 \dot{u}_1^0 \dot{v}_1^0 + G^0 (\dot{v}_1^0)^2}\sqrt{E^0 (\dot{u}_2^0)^2 + 2F^0 \dot{u}_2^0 \dot{v}_2^0 + G^0 (\dot{v}_2^0)^2}} \tag{1.6.18}$$

where

$$E^0 = E(u^0, v^0), \quad F^0 = F(u^0, v^0), \quad G^0 = G(u^0, v^0),$$

$$(\dot{u}_1^0, \dot{v}_1^0) = \frac{d}{dt_1}(u_1(t_1), v_1(t_1))_{t_1 = t_1^0}, \quad (\dot{u}_2^0, \dot{v}_2^0) = \frac{d}{dt_2}(u_2(t_2), v_2(t_2))_{t_2 = t_2^0}.$$

Proof: Computing the square length of the vector (1.6.13) one obtains

$$\langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t)\rangle = \langle \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}, \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}\rangle = \langle \mathbf{r}_u, \mathbf{r}_u\rangle \dot{u}^2 + 2\langle \mathbf{r}_u, \mathbf{r}_v\rangle \dot{u}\dot{v} + \langle \mathbf{r}_v, \mathbf{r}_v\rangle \dot{v}^2.$$

This gives the formula (1.6.16). The formula for the angle between the curves can be proved in a similar way. □

The formulae of the Lemma can be represented in the following way. Consider the square length element in the ambient Euclidean space

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Let us restrict the quadratic form onto the tangent plane to the surface. Substituting

$$\begin{aligned} dx &= \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \\ dy &= \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \\ dz &= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv \end{aligned} \tag{1.6.19}$$

the differentials of the functions $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ one obtains

$$ds^2 = E(u, v)du^2 + 2F(u, v)du\, dv + G(u, v)dv^2 \tag{1.6.20}$$

where the coefficients E, F, G are defined by the formulae (1.6.15).

Definition 1.6.9 *The quadratic form* (1.6.20) *is called the* first fundamental form *or the* induced metric *of the parametrized surface* (1.6.1).

The metric properties of the surface (1.6.1) can be modeled on the plain domain D assuming that the inner products of vectors at the point $(u, v) \in D$ have to be computed with the help of the first fundamental form:

$$\langle e_u, e_u\rangle|_{(u,v)} = E(u, v), \quad \langle e_u, e_v\rangle|_{(u,v)} = F(u, v), \quad \langle e_v, e_v\rangle|_{(u,v)} = G(u, v). \tag{1.6.21}$$

Here $e_u = (1, 0)$, $e_v = (0, 1)$ are the unit vectors of the u- and v-axes on the plane.

Exercise 1.6.10 *Prove that the determinant of the* Gram matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \tag{1.6.22}$$

of the quadratic form (1.6.21) *is positive:*

$$EG - F^2 > 0. \tag{1.6.23}$$

From the above considerations one obtains a formula for computation of the length of a curve $(u(t), v(t))$ on a surface in terms of the first fundamental form of the surface.

Corollary 1.6.11 *The length of the curve* $\gamma : \{(u(t), v(t)),\ t \in [a, b]\}$ *on the surface* (1.6.1) *is given by the formula*

$$s = \int_\gamma \sqrt{E\,du^2 + 2F\,du\,dv + G\,dv^2} \tag{1.6.24}$$
$$= \int_a^b \sqrt{E(u(t), v(t))(\dot u(t))^2 + 2F(u(t), v(t))\dot u(t)\dot v(t) + G(u(t), v(t))(\dot v(t))^2}\,dt.$$

Example 1. The first fundamental form of the cylinder (1.6.4) of radius R in the coordinates (ϕ, h) is given by

$$ds^2 = R^2 d\phi^2 + dh^2. \tag{1.6.25}$$

For the sphere (1.6.5) in the coordinates (ϕ, θ) one obtains

$$ds^2 = R^2(d\theta^2 + \cos^2\theta\,d\phi^2). \tag{1.6.26}$$

Example 2. The first fundamental form of the graph of a smooth function $z = f(x, y)$ reads

$$ds^2 = (1 + f_x^2)dx^2 + 2f_x f_y\,dx\,dy + (1 + f_y^2)dy^2. \tag{1.6.27}$$

Example 3. Let us compute the first fundamental form of the regular surface $F(x, y, z) = 0$ at the points where the last component of the gradient does not vanish:

$$F_z(x, y, z) \neq 0.$$

We will use $u = x$, $v = y$ as local coordinates on the surface near such a point. On the surface one has

$$dF(x, y, z) = F_x\,dx + F_y\,dy + F_z\,dz = 0,$$

so

$$dz = -\left(\frac{F_x}{F_z}dx + \frac{F_y}{F_z}dy\right).$$

Substituting this differential into the square length formula one obtains the expression for the first fundamental form

$$ds^2 = dx^2 + dy^2 + dz^2 = \left(1 + \frac{F_x^2}{F_z^2}\right)dx^2 + 2\frac{F_xF_y}{F_z^2}dx\,dy + \left(1 + \frac{F_y^2}{F_z^2}\right)dy^2,$$

that is,

$$E = 1 + \frac{F_x^2}{F_z^2}, \quad F = \frac{F_xF_y}{F_z^2}, \quad G = 1 + \frac{F_y^2}{F_z^2}.$$

One can also define the area of a domain on the surface.

Definition 1.6.12 *The area of a domain* $S_\Omega \subset S$,

$$S_\Omega = \{\mathbf{r}(u, v)\,|\,(u, v) \in \Omega \subset D \in \mathbb{R}^2\}$$

of the parametrized surface (1.6.1) *is defined by*

$$A\,(S_\Omega) = \iint_\Omega \sqrt{EG - F^2}\,du\,dv. \tag{1.6.28}$$

Recall that the determinant $EG - F^2$ of the Gram matrix (1.6.22) is always positive (see Exercise 1.6.10 above).

Example 1. Using the formulae (1.6.27) for the coefficients of the first fundamental form of the graph $z = f(x, y)$ of a smooth function

$$E = 1 + f_x^2, \quad F = f_x f_y, \quad G = 1 + f_y^2$$

one obtains

$$EG - F^2 = 1 + f_x^2 + f_y^2. \tag{1.6.29}$$

Thus the definition (1.6.28) reduces to the formula for the area of the graph of a smooth function $f(x, y)$ defined on the domain Ω in the plain

$$A(S_\Omega) = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy \tag{1.6.30}$$

known from the multivariable calculus.

Exercise 1.6.13 *Prove that the area element on a two-dimensional surface* $\mathbf{r}(u, v)$ *can be written in the form*

$$dA = |\mathbf{r}_u \times \mathbf{r}_\mathbf{v}|\, du\, dv. \tag{1.6.31}$$

Example 2. Consider the domain

$$\theta_0 < \theta < \frac{\pi}{2}, \quad 0 \le \phi \le 2\pi \tag{1.6.32}$$

around the north pole of the sphere of radius R. The domain can be considered as an analogue of a circle from the point of view of spherical geometry (we will explain later *why* this set should be called a circle). Let us first calculate the radius ρ of the circle, i.e., the length of the segment of a meridian

$$\theta_0 \le \theta \le \frac{\pi}{2}, \quad \phi = \phi_0. \tag{1.6.33}$$

This length has to be computed by integrating the length element

$$ds = \sqrt{R^2(d\theta^2 + \cos^2\theta\, d\phi^2)}$$

along the curve (1.6.33). One arrives at the integral

$$\rho = \int_{\theta_0}^{\frac{\pi}{2}} R\, d\theta = R\left(\frac{\pi}{2} - \theta_0\right).$$

Let us now compute the area[3] of the circle. Since

$$E = R^2, \quad F = 0, \quad G = R^2\cos^2\theta, \quad \Rightarrow EG - F^2 = R^4\cos^2\theta$$

[3]Strictly speaking we have to remove the north pole from the domain of integration since it is not a regular point for the chosen parametrization of the sphere by the coordinates (ϕ, θ). However, such a removal will not affect the value of the double integral.

the area of the circle is given by the integral

$$A = \int_0^{2\pi} \int_{\theta_0}^{\frac{\pi}{2}} R^2 \cos\theta \, d\phi \, d\theta = 2\pi R^2 (1 - \sin\theta_0).$$

Using the expression for the radius ρ one can rewrite the formula for the area of the circle of the radius ρ as follows

$$A = 2\pi R^2 \left(1 - \cos\frac{\rho}{R}\right). \tag{1.6.34}$$

In particular, for $\theta_0 = -\frac{\pi}{2}$ the well known formula

$$A = 4\pi R^2$$

for the area of the entire sphere readily follows from (1.6.34). For small radius (i.e., small $\frac{\pi}{2} - \theta_0$) one can expand the area $A = A(\rho)$ in the Taylor series using

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \mathcal{O}\left(x^6\right)$$

to arrive at the approximate expression

$$A(\rho) = \pi\rho^2 - \frac{\pi}{12R^2}\rho^4 + \mathcal{O}\left(\rho^6\right). \tag{1.6.35}$$

We see that the area of a sufficiently small circle of radius ρ on the sphere is smaller than the area of the circle of the same radius on the Euclidean plane. We will return to the formula (1.6.35) later in the discussion of curvature of surfaces.

Exercise 1.6.14 *Prove inequality*

$$A(\rho) \leq \pi\rho^2$$

for all values of the radius $0 \leq \rho \leq \pi R$.

The previous considerations can be applied with no serious modifications to the case of k-dimensional surfaces in the n-dimensional Euclidean space. Their parametric representation is given by n smooth functions of k variables

$$x^i = x^i(u^1, \dots, u^k), \quad i = 1, \dots, n, \quad (u^1, \dots, u^n) \in D \subset \mathbb{R}^k. \tag{1.6.36}$$

We will also use the short notation

$$\mathbf{x}(u) = \left(x^1(u), \dots, x^n(u)\right), \quad u = (u^1, \dots, u^k) \tag{1.6.37}$$

for the radius vector of the points of the surface. As above we impose the topological condition saying that locally, in a sufficiently small ball in $\mathbb{R}^n$ the representation (1.6.36) of points of the surface is unique. The surface is called *regular* if the tangent vectors

$$\mathbf{x}_{u^1}(u), \dots, \mathbf{x}_{u^k}(u) \tag{1.6.38}$$

are linearly independent at every point of the surface, i.e., the rank of the $n \times k$ Jacobi matrix of the functions (1.6.36) is equal to k:

$$\mathrm{rk}\begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \dots & \frac{\partial x^1}{\partial u^k} \\ \dots & \dots & \dots \\ \frac{\partial x^n}{\partial u^1} & \dots & \frac{\partial x^n}{\partial u^k} \end{pmatrix} = k. \tag{1.6.39}$$

Clearly the regularity implies $k \leq n$; only the case $k < n$ is of interest since for $k = n$ the map (1.6.36) gives just new local coordinates in $\mathbb{R}^n$. The k-dimensional space spanned by the vectors (1.6.38) is called the *tangent space* to the surface at the point $\mathbf{x}(u)$. The velocity vector of the smooth curve

$$u(t) = \left(u^1(t), \ldots, u^k(t)\right), \quad t \in [a, b] \tag{1.6.40}$$

on the surface is equal to

$$\dot{\mathbf{x}}(t) = \dot{u}^\alpha \mathbf{x}_{u^\alpha}(u(t)). \tag{1.6.41}$$

Here and in sequel we will use Greek indices for the coordinates on the surface. The summation over the Greek indices from 1 to k=dimension of the surface will be assumed.

The restriction of the Euclidean square length to the surface

$$\begin{gathered} ds^2 = (dx^1(u))^2 + \cdots + (dx^n(u))^2 = g_{\alpha\beta}(u) du^\alpha du^\beta \\ g_{\alpha\beta}(u) = \langle \mathbf{x}_{u^\alpha}(u), \mathbf{x}_{u^\beta}(u) \rangle = \sum_{i=1}^n \frac{\partial x^i(u)}{\partial u^\alpha} \frac{\partial x^i(u)}{\partial u^\beta}, \quad \alpha,\, \beta = 1, \ldots, k \end{gathered} \tag{1.6.42}$$

is called the *first fundamental form* or the *induced metric* on the surface. Knowledge of the induced metric allows one to calculate the lengths of curves on the surface and angles between tangent vectors at any point of the surface. Namely, in order to compute the inner product of two tangent vectors at the same point $u = (u^1, \ldots, u^k)$ of the surface

$$\dot{u}_1^\alpha \mathbf{x}_{u^\alpha}(u) \quad \text{and} \quad \dot{u}_2^\alpha \mathbf{x}_{u^\alpha}(u)$$

represented by their coordinates $\dot{u}_1 = (\dot{u}_1^1, \ldots, \dot{u}_1^k)$ and $\dot{u}_2 = (\dot{u}_2^1, \ldots, \dot{u}_2^k)$ in the basis (1.6.38) of the tangent space one has to use the matrix of the first fundamental form as the Gram matrix of the inner product:

$$\langle \dot{u}_1, \dot{u}_2 \rangle|_u = g_{\alpha\beta}(u) \dot{u}_1^\alpha \dot{u}_2^\beta. \tag{1.6.43}$$

In particular the length of a curve

$$\gamma : \{u^\alpha = u^\alpha(t),\ \alpha = 1, \ldots, k,\ t \in [a, b]\}$$

on the surface is given by the integral

$$\int_\gamma ds = \int_a^b \sqrt{g_{\alpha\beta}(u(t)) \dot{u}^\alpha \dot{u}^\beta}\, dt. \tag{1.6.44}$$

The k-dimensional volume of a domain

$$S_\Omega : \{\mathbf{x}(u) \,|\, u \in \Omega \subset D \subset \mathbb{R}^k\}$$

can be defined by a multiple integral

$$Vol\,(S_\Omega) = \int \cdots \int_\Omega \sqrt{\det g_{\alpha\beta}(u)}\, du^1 \ldots du^k. \tag{1.6.45}$$

Exercise 1.6.15 *Prove that for $k = 1$ the formula* (1.6.45) *gives the old expression* (1.4.3) *for the length of the curve $\mathbf{x} = \mathbf{x}(u)$, $u \in D = (a, b)$.*

So far all the definitions of lengths, angles, areas have been given for surfaces with e *given* parametrization. We have now to verify independence of these definitions on the choice of parametrization. Let us first do it for two-dimensional surfaces in the three-dimensional Euclidean space.

Let a diffeomorphism

$$\begin{aligned} & D' \to D \\ & (u', v') \mapsto (u = u(u', v'), v = v(u', v')) \end{aligned} \tag{1.6.46}$$

be given. It provides the surface (1.6.1) with a new parametrization

$$\mathbf{r} = \mathbf{r}\left(u(u', v'), v(u', v')\right), \quad (u', v') \in D' \subset \mathbb{R}^2. \tag{1.6.47}$$

Lemma 1.6.16 *If the surface* (1.6.1) *is regular then the surface* (1.6.47) *is so, and vice versa.*

Proof: Using chain rule one obtains

$$\begin{aligned} \mathbf{r}_{u'} &= \frac{\partial u}{\partial u'}\mathbf{r}_u + \frac{\partial v}{\partial u'}\mathbf{r}_v \\ \mathbf{r}_{v'} &= \frac{\partial u}{\partial v'}\mathbf{r}_u + \frac{\partial v}{\partial v'}\mathbf{r}_v \end{aligned} \tag{1.6.48}$$

or, in matrix form,

$$(\mathbf{r}_{u'}, \mathbf{r}_{v'}) = (\mathbf{r}_u, \mathbf{r}_v) \begin{pmatrix} \partial u/\partial u' & \partial u/\partial v' \\ \partial v/\partial u' & \partial v/\partial v' \end{pmatrix}. \tag{1.6.49}$$

Since the Jacobi matrix

$$J = \begin{pmatrix} \partial u/\partial u' & \partial u/\partial v' \\ \partial v/\partial u' & \partial v/\partial v' \end{pmatrix} \tag{1.6.50}$$

does not degenerate, the transformation (1.6.49) maps a basis $(\mathbf{r}_u, \mathbf{r}_v)$ in the tangent space to the surface into another basis in the same tangent space. □

Lemma 1.6.17 *The coefficients of the first fundamental form*

$$ds^2 = E'(u', v')du'^2 + 2F'(u', v')du'dv' + G'(u', v')dv'^2 \tag{1.6.51}$$

of the surface with the new parametrization are related with the coefficients of the old first fundamental form

$$ds^2 = E(u, v)du^2 + 2F(u, v)du\, dv + G(u, v)dv^2 \tag{1.6.52}$$

by means of the following transformation rule

$$\begin{pmatrix} E' & F' \\ F' & G' \end{pmatrix} = J^{\mathrm{T}} \begin{pmatrix} E & F \\ F & G \end{pmatrix} J. \tag{1.6.53}$$

Here J is the Jacobi matrix (1.6.50).

Proof: We have

$$
\begin{aligned}
E' = \langle \mathbf{r}_{u'}, \mathbf{r}_{u'} \rangle &= \langle \frac{\partial u}{\partial u'}\mathbf{r}_u + \frac{\partial v}{\partial u'}\mathbf{r}_v, \frac{\partial u}{\partial u'}\mathbf{r}_u + \frac{\partial v}{\partial u'}\mathbf{r}_v \rangle \\
&= \left(\frac{\partial u}{\partial u'}\right)^2 \langle \mathbf{r}_u, \mathbf{r}_u \rangle + 2\frac{\partial u}{\partial u'}\frac{\partial v}{\partial u'}\langle \mathbf{r}_u, \mathbf{r}_v \rangle + \left(\frac{\partial v}{\partial u'}\right)^2 \langle \mathbf{r}_v, \mathbf{r}_v \rangle \\
&= \left(\frac{\partial u}{\partial u'}\right)^2 E + 2\frac{\partial u}{\partial u'}\frac{\partial v}{\partial u'}F + \left(\frac{\partial v}{\partial u'}\right)^2 G.
\end{aligned}
$$

In a similar way we prove that

$$
\begin{aligned}
F' &= \frac{\partial u}{\partial u'}\frac{\partial u}{\partial v'}E + \left(\frac{\partial u}{\partial u'}\frac{\partial v}{\partial v'} + \frac{\partial v}{\partial u'}\frac{\partial u}{\partial v'}\right)F + \frac{\partial v}{\partial u'}\frac{\partial v}{\partial v'}G \\
G' &= \left(\frac{\partial u}{\partial v'}\right)^2 E + 2\frac{\partial u}{\partial v'}\frac{\partial v}{\partial v'}F + \left(\frac{\partial v}{\partial v'}\right)^2 G.
\end{aligned}
$$

These formulae coincide with (1.6.53). □

Corollary 1.6.18 *Consider the smooth curve*

$$u' = u'(t), \quad v' = v'(t), \quad t \in [a', b'] \tag{1.6.54}$$

on the parametrized surface (1.6.47) *along with its image*

$$u(t) := u\left(u'(t), v'(t)\right), \quad v(t) := v\left(u'(t), v'(t)\right), \quad t \in [a, b] \tag{1.6.55}$$

on the parametrized surface (1.6.2). *Then the lengths of the velocity vectors of the two curves computed with the help of the quadratic forms* (1.6.51) *and* (1.6.52) *are equal*

$$
\begin{aligned}
&E'(u'(t), v'(t))\left(\dot{u}'(t)\right)^2 + 2F'(u'(t), v'(t))\dot{u}'(t)\dot{v}'(t) + G'(u'(t), v'(t))\left(\dot{v}'(t)\right)^2 \\
&= E(u(t), v(t))\left(\dot{u}(t)\right)^2 + 2F(u(t), v(t))\dot{u}(t)\dot{v}(t) + G(u(t), v(t))\left(\dot{v}(t)\right)^2
\end{aligned} \tag{1.6.56}
$$

for any $t \in [a, b]$. *In particular, the lengths of the curves* (1.6.54) *and* (1.6.55) *coincide.*

Proof: For a given $t \in [a, b]$ the derivatives $(\dot{u}'(t), \dot{v}'(t))$ and $(\dot{u}(t), v(t))$ give coordinates of the same velocity vector

$$\dot{u}'(t)\,\mathbf{r}_{u'}(u'(t), v'(t)) + \dot{v}'(t)\,\mathbf{r}_{v'}(u'(t), v'(t)) = \dot{u}(t)\,\mathbf{r}_u(u(t), v(t)) + \dot{v}(t)\,\mathbf{r}_v(u(t), v(t))$$

with respect to two bases in the tangent plane to the surface. Restricting the quadratic form $ds^2 = dx^2 + dy^2 + dz^2$ on this vector one obtains respectively the left- and the right-hand sides of (1.6.56). □

In a similar way one can prove that the angles between tangent vectors to the curves on the parametrized surface do not depend on the choice of parametrization. We leave the precise formulation and the proof as an exercise for the reader.

Corollary 1.6.19 *Let the diffeomorphism* (1.6.46) *map the domain* $\Omega' \subset \mathbb{R}^2_{(u',v')}$ *onto the domain* $\Omega \subset \mathbb{R}^2_{(u,v)}$. *Then the areas of the domains* $S_{\Omega'}$ *and* S_Ω *on the parametrized surfaces* (1.6.1) *and* (1.6.47) *coincide:*

$$\iint_{\Omega'} \sqrt{E'G' - F'^2}\, du'dv' = \iint_{\Omega} \sqrt{EG - F^2}\, du\, dv. \tag{1.6.57}$$

Proof: From the transformation law (1.6.53) it follows that

$$\det \begin{pmatrix} E' & F' \\ F' & G' \end{pmatrix} = \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} (\det J)^2 .$$

Recall that

$$\det J = \det \begin{pmatrix} \partial u/\partial u' & \partial u/\partial v' \\ \partial v/\partial u' & \partial v/\partial v' \end{pmatrix}$$

is the Jacobian of the coordinate transformation. So

$$\sqrt{E'G' - F'^2} = \sqrt{EG - F^2}\ |\det J| .$$

Thus the equality (1.6.57) follows from the theory of changes of variables in a double integral.

□

In the general case of a k-dimensional surface (1.6.36) in a n-dimensional Euclidean space a change of the parametrization is given by a diffeomorphism

$$u^\alpha = u^\alpha(u'), \quad u' = \left(u^{1'}, \ldots, u^{k'}\right), \quad \det\left(\frac{\partial u}{\partial u'}\right) \neq 0. \tag{1.6.58}$$

Such a change induces a change of a basis in the tangent space

$$\mathbf{x}_{u^{\alpha'}} = \frac{\partial u^\alpha}{\partial u^{\alpha'}} \mathbf{x}_{u^\alpha}. \tag{1.6.59}$$

Recall that in the last formula α and α' are independent indices; a summation in α from $\alpha = 1$ to $\alpha = k$ is assumed in the right-hand side of (1.6.59). The index α' is fixed in the both sides of the eq. (1.6.59); it takes values from $\alpha' = 1$ to $\alpha' = k$.

Similarly to the Lemma 1.6.17 one can prove

Lemma 1.6.20 *The coefficients*

$$g_{\alpha\beta}(u) = \langle \mathbf{x}_{u^\alpha}(u), \mathbf{x}_{u^\beta}(u)\rangle \quad \text{and} \quad g_{\alpha'\beta'}(u') = \left\langle \mathbf{x}_{u^{\alpha'}}(u'), \mathbf{x}_{u^{\beta'}}(u')\right\rangle, \quad u = u(u')$$

of the first fundamental forms of the surface (1.6.36) *with respect to two parametrizations are related by the following transformation rule*

$$g_{\alpha'\beta'}(u') = \frac{\partial u^\alpha}{\partial u^{\alpha'}} \frac{\partial u^\beta}{\partial u^{\beta'}} g_{\alpha\beta}(u). \tag{1.6.60}$$

Proof: Using chain rule

$$\mathbf{x}_{u^{\alpha'}} = \frac{\partial u^\alpha}{\partial u^{\alpha'}} \mathbf{x}_{u^\alpha}.$$

one has

$$\begin{aligned} g_{\alpha'\beta'}(u') &= \left\langle \mathbf{x}_{u^{\alpha'}}(u'), \mathbf{x}_{u^{\beta'}}(u') \right\rangle = \left\langle \frac{\partial u^\alpha}{\partial u^{\alpha'}} \mathbf{x}_{u^\alpha}, \frac{\partial u^\beta}{\partial u^{\beta'}} \mathbf{x}_{u^\beta} \right\rangle \\ &= \frac{\partial u^\alpha}{\partial u^{\alpha'}} \frac{\partial u^\beta}{\partial u^{\beta'}} \langle \mathbf{x}_{u^\alpha}(u), \mathbf{x}_{u^\beta}(u) \rangle = \frac{\partial u^\alpha}{\partial u^{\alpha'}} \frac{\partial u^\beta}{\partial u^{\beta'}} g_{\alpha\beta}(u). \end{aligned}$$

□

Observe that the transformation rule (1.6.60) of the Gram matrix of the metric induced on the surface can be represented in the matrix form as follows

$$G'(u') = \left(\frac{\partial u}{\partial u'} \right)^{\mathrm{T}} G(u) \left(\frac{\partial u}{\partial u'} \right) \tag{1.6.61}$$

where

$$G'(u') = \left(g_{\alpha'\beta'}(u') \right), \quad G(u) = \left(g_{\alpha\beta}(u) \right).$$

Lemma 1.6.21 *Under a change* (1.6.58) *of parametrization on the surface the coordinates of tangent vectors transform as follows*

$$\dot{u}^\alpha = \frac{\partial u^\alpha}{\partial u^{\alpha'}} \dot{u}^{\alpha'}. \tag{1.6.62}$$

Proof: This nothing but the chain rule for the functions $u^\alpha = u^\alpha(u'(t))$. □

Lemma 1.6.22 *The inner product* (1.6.43) *of tangent vectors does not depend on the choice of parametrization on the surface.*

Proof: Using the transformation rules (1.6.62) and (1.6.60) we obtain

$$\dot{u}_1^{\alpha'} \dot{u}_2^{\beta'} g_{\alpha'\beta'}(u') = \frac{\partial u^{\alpha'}}{\partial u^\alpha} \dot{u}_1^\alpha \frac{\partial u^{\beta'}}{\partial u^\beta} \dot{u}_2^\beta g_{\alpha'\beta'}(u') = \dot{u}^\alpha \dot{u}^\beta g_{\alpha\beta}(u), \quad u = u(u').$$

□

Summarizing, we arrive at

Theorem 1.6.23 *The lengths of curves* (1.6.44)*, angles between the curves, and volumes* (1.6.45) *on a k-dimensional parametrized surface in the n-dimensional Euclidean space do not depend on the choice of parametrization of the surface.*

1.7 Geometry in Minkowski space. Lorentz transformations

Let X be a n-dimensional linear space and

$$\begin{aligned} & X \times X \to \mathbb{R} \\ & (x, y) \mapsto g(x, y) \end{aligned} \tag{1.7.1}$$

a bilinear form on it. Recall that from bilinearity it follows that

$$g(0, y) = g(y, 0) = 0 \quad \forall\, y \in X.$$

The bilinear form (1.7.1) is called *nondegenerate* if it satisfies the following condition:

$$\text{if} \quad g(x, y) = 0 \quad \forall\, y \in X \quad \text{then} \quad x = 0. \tag{1.7.2}$$

Let $e_1, \ldots, e_n$ be a basis in the linear space X. Like above we introduce the Gram matrix of the bilinear form (1.7.1) by

$$g_{ij} = g(e_i, e_j), \quad i,\, j = 1, \ldots, n. \tag{1.7.3}$$

The value of the bilinear form on the vectors $x = x^i e_i$ and $y = y^j e_j$ is given by the already familiar formula

$$g(x, y) = g_{ij} x^i y^j. \tag{1.7.4}$$

Proposition 1.7.1 *The bilinear form* (1.7.4) *satisfies the nondegeneracy condition* iff *the Gram matrix $G = (g_{ij})$ does not degenerate*

$$\det G \neq 0.$$

Proof: Let us first prove

Lemma 1.7.2 *A bilinear form $(x, y) \mapsto g(x, y)$ on a n-dimensional linear space does not degenerate* iff *the system of n linear equations*

$$g(x, e_1) = 0, \ldots, g(x, e_n) = 0 \tag{1.7.5}$$

has only the trivial solution $x = 0$.

Proof: Specializing $g(x, y) = 0$ at $y = e_1, \ldots, y = e_n$ one obtains the system (1.7.5). Conversely, since

$$g(x, y) = g(x, e_j)\, y^j \quad \text{for} \quad y = y^j e_j$$

validity of the system (1.7.5) implies that $g(x, y) = 0$ for any $y \in X$. □

We can now complete the proof of the Proposition. The system (1.7.5) can be represented by n linear equations for the coordinates of the vector $x = x^i e_i$:

$$0 = \langle x, e_j \rangle = g_{ij} x^i, \quad j = 1, \ldots, n.$$

According to the Lemma this system must have only trivial solution $x^1 = x^2 = \cdots = x^n = 0$. This requirement is equivalent to nondegeneracy of the coefficient matrix of the system. □

Definition 1.7.3 *A* pseudo-Euclidean inner product *on a linear space X is a symmetric nondegenerate bilinear form*

$$\begin{aligned}
&X \times X \to \mathbb{R} \\
&(x, y) \mapsto g(x, y) \\
&\left.\begin{aligned} g(\alpha\, x + \beta\, y, z) = \alpha\, g(x, z) + \beta\, g(y, z) \\ g(z, \alpha\, x + \beta\, y) = \alpha\, g(z, x) + \beta\, g(z, y) \end{aligned}\right. \quad \forall\, x,\, y,\, z \in X, \quad \forall\, \alpha,\, \beta \in \mathbb{R} \\
&g(y, x) = g(x, y) \quad \forall x,\, y \in X \\
&g(x, y) = 0 \quad \forall\, y \in X \quad \text{iff} \quad x = 0.
\end{aligned} \tag{1.7.6}$$

In coordinates a pseudo-Euclidean inner product is represented by the formula (1.7.4) with an arbitrary symmetric nondegenerate matrix $G = (g_{ij})$.

Example 1. A diagonal matrix with ± 1 on the diagonal

$$G = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix} \tag{1.7.7}$$

(p times $+1$ and q times -1) gives a bilinear form

$$g(x, y) = x^1 y^1 + \cdots + x^p y^p - x^{p+1} y^{p+q} - \cdots - x^{p+q} y^{p+q}. \tag{1.7.8}$$

This form does not degenerate *iff* $p + q = n$. For the case $p = n$, $q = 0$ one obtains a positive definite inner product $\langle x, y \rangle = g(x, y)$; in the case $p = 0$, $q = n$ the inner product $\langle x, y \rangle := -g(x, y)$ is positive definite.

Exercise 1.7.4 *Prove that the Gram matrices $G = (g_{ij}) = (g(e_i, e_j))$ and $G' = (g_{i'j'}) = (g(e_{i'}, e_{j'})$ of the bilinear form* (1.7.4) *with respect to two bases*

$$e_{i'} = t^i_{i'} e_i$$

are related by the transformation

$$g_{i'j'} = t^i_{i'} t^j_{j'} g_{ij} \tag{1.7.9}$$

or, equivalently

$$G' = T^{\mathrm{T}} G\, T. \tag{1.7.10}$$

Here $T = \left(t^i_{i'}\right)$ is the transition matrix.

Using transformations of the form (1.7.9) (or, equivalently, of the form (1.7.10)) one can reduce the matrix of the bilinear form to a suitable simple form. From the Lagrange theorem about quadratic forms $g(x, x)$ one readily derives the following statement.

Theorem 1.7.5 *Given a pseudo-Euclidean inner product $g(x, y)$ on a n-dimensional linear space X, there exists a basis in the space such that the bilinear form in this basis becomes equal to* (1.7.8) *with $p + q = n$. The coordinates with respect to such a basis are called* pseudo-orthogonal. *The numbers p and q are called the* inertia indices *of the symmetric bilinear form. They do not depend on the choice of a system of pseudo-orthogonal coordinates.*

A pair (X, g) consisting of a linear space X with a pseudo-Euclidean inner product $g(x, y)$ will be called a *pseudo-Euclidean space*. The pair of inertia indices (p, q) of the quadratic form will be called the *signature*[4] of the pseudo-Euclidean space. Recall that $p+q$=dimension of the space. From the Theorem 1.7.5 it follows that all pseudo-Euclidean spaces of a given signature (p, q) are isomorphic to the standard one with the inner product (1.7.8).

The 4-dimensional pseudo-Euclidean space of signature $(1, 3)$ is called the *Minkowski* space. It plays an important role in the geometric description of special relativity. The points of this space are *events* characterized by three spatial coordinates x, y, z and the time t in an inertial reference frame. The pseudo-orthogonal coordinates are usually denoted (x^0, x^1, x^2, x^3) where $x^1 = x$, $x^2 = y$, $x^3 = z$ and the coordinate x^0 has the form

$$x^0 = c\,t. \tag{1.7.11}$$

Here c is a constant called the *speed of light* ($c \simeq 300\,000$ km/sec). The associated vector space is also often called Minkowski space. It is denoted $\mathbb{R}^{1,3}$ and consists of *four-vectors*. The inner product of vectors $a = (a^0, a^1, a^2, a^3)$ and $b = (b^0, b^1, b^2, b^3)$ in the Minkowski space reads

$$g(a, b) = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3. \tag{1.7.12}$$

Observe that the "square length" $g(a, a)$ of a vector $a \in \mathbb{R}^{1,3}$ can be positive, negative, or zero.

Definition 1.7.6 *A vector $a \in \mathbb{R}^{1,3}$ is called*

$$\begin{array}{ll} \text{timelike if} & g(a,a) > 0 \\ \text{spacelike if} & g(a,a) < 0 \\ \text{null if} & g(a,a) = 0. \end{array}$$

For example the unit vector $(1, 0, 0, 0)$ of the x^0-axis is timelike, the unit vectors $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ of the x^1-, x^2- and x^3-axes are spacelike. The null vectors form a cone

$$(a^0)^2 = (a^1)^2 + (a^2)^2 + (a^3)^2 \tag{1.7.13}$$

called *light cone* in the special relativity.

It is convenient to also consider analogues of Minkowski space of dimensions two and three in order to simplify the pictures.

[4] Also the difference $p - q$ is often called the signature of the quadratic form. For the case of nondegenerate quadratic forms the pair of inertia indices is determined by the difference along with the dimension of the space.

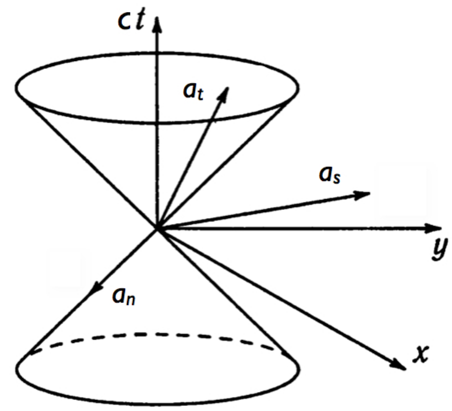


Fig. 2. Light cone in the three-dimensional Minkowski space with the coordinates (ct, x, y). The vector a_t is timelike, the vector a_s is spacelike and the vector a_n is null.

To a smooth curve

$$x^1 = x^1(t),\ x^2 = x^2(t),\ x^3 = x^3(t) \tag{1.7.14}$$

describing motion of a particle in the usual three-dimensional space one can associate the *world line*

$$x^0 = c\,t,\ x^1 = x^1(t),\ x^2 = x^2(t),\ x^3 = x^3(t) \tag{1.7.15}$$

of the particle in the Minkowski space. In particular the world line of a particle remaining at rest at a fixed point (x^1, x^2, x^3) of the three-space is timelike.

The first main postulate of special relativity says that for all times t the velocity four-vector

$$\left(\dot{x}^0 = c, \dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t)\right)$$

of the curve (1.7.15) is timelike for the motion of massive particles; it is null for the motion of massless particles. This implies that the absolute value of the 3-velocity must be less than the speed of light for massive particles

$$\sqrt{(\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2} < c \tag{1.7.16}$$

while the massless particles propagate with the speed of light.

A parametrized[5] smooth curve

$$\gamma :\ x(t) = \left(x^0(t), x^1(t), x^2(t), x^3(t)\right), \quad t \in [a, b]$$

[5]Here the parameter t can have no relationship with the physical time.

is called timelike if the tangent vector

$$\dot{x}(t) = \left(\dot{x}^0(t), \dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t)\right)$$

is timelike at every point. One can define an analogue of the length of such a curve by the integral

$$\tau = \frac{1}{c}\int_a^b \sqrt{(\dot{x}^0(t))^2 - (\dot{x}^1(t))^2 - (\dot{x}^2(t))^2 - (\dot{x}^3(t))^2}\, dt. \tag{1.7.17}$$

Like in the Euclidean case one can establish independence of the "length" (1.7.17) from the parametrization. For the workd line (1.7.15) of a massive particle the quantity τ is called *proper time* of the particle. In the particular case of a particle staying at a fixed point of the three-space the world line has the form

$$x^0 = c\,t,\ x^1 = x_0^1,\ x^2 = x_0^2,\ x^3 = x_0^3, \quad t_1 \leq t \leq t_2.$$

The proper time of such a particle coincides with the time interval

$$\tau = t_2 - t_1.$$

For a particle moving with a constant speed $\mathbf{v} = (v^1, v^2, v^3)$ the interval of the proper time τ on the world line

$$x^0 = c\,t,\ x^1 = v^1 t + x_0^1,\ x^2 = v^2 t + x_0^2,\ x^3 = v^3 t + x_0^3$$

is proportional to the interval of the laboratory time t with the coefficient $\sqrt{1 - \frac{v^2}{c^2}}$ where $v = |\mathbf{v}|$. Indeed,

$$\tau = \frac{1}{c}\int_{t_1}^{t_2} \sqrt{c^2 - v^2}\, dt = \sqrt{1 - \frac{v^2}{c^2}}(t_2 - t_1).$$

Thus, the proper time of a moving particle runs more slowly than the proper time of a particle at rest.

For two points x, y in the Minkowski space the square length of the vector $x - y$ is called the *space-time interval* between the events x and y. The second main postulate of special relativity requires that changes of an inertial reference frame must keep invariant the space-time interval. Such transformation is an analogue of isometries of Euclidean spaces discussed above. Let us first consider such isometries in the general setting of pseudo-Euclidean spaces.

A map $f : X \to X'$ of two pseudo-Euclidean spaces (X, g) and (X', g') of signatures (p, q) and (p', q') respectively is called *isometry* if

$$g'(f(x) - f(y), f(x) - f(y)) = g(x - y, x - y) \quad \forall\, x,\, y \in X. \tag{1.7.18}$$

For example, an affine transformation

$$f(x) = Ax + b \tag{1.7.19}$$

will be an isometry of the pseudo-Euclidean space (X, g) to itself *iff* the matrix A satisfies

$$A^{\mathrm{T}} G\, A = G \tag{1.7.20}$$

where the Gram matrix G has the form (1.7.7). Here b is an arbitrary vector in X.

A matrix A satisfying (1.7.20) is called *pseudo-orthogonal*. It does not degenerate since

$$(\det A)^2 = 1.$$

Thus all matrices in (X, g) form a group called *pseudo-orthogonal group* of the signature (p, q) and denoted $O(p, q)$. Here (p, q) is the signature of the pseudo-Euclidean space (X, g). The pseudo-orthogonal matrices with determinant 1 form a subgroup $SO(p, q) \subset O(p, q)$.

Example. The matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(1,1)$$

must satisfy the system

$$\begin{aligned} a^2 - c^2 &= 1 \\ d^2 - b^2 &= 1 \\ ab - cd &= 0. \end{aligned}$$

The set of solutions consists of four families

$$\begin{gathered} \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}, \quad \begin{pmatrix} -\cosh\psi & -\sinh\psi \\ -\sinh\psi & -\cosh\psi \end{pmatrix}, \\ \begin{pmatrix} \cosh\psi & -\sinh\psi \\ \sinh\psi & -\cosh\psi \end{pmatrix}, \quad \begin{pmatrix} -\cosh\psi & \sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix}. \end{gathered} \tag{1.7.21}$$

Here $\psi \in \mathbb{R}$ is an arbitrary parameter. The matrices A of the first and the second families belong to the subgroup $SO(1, 1)$ since they have determinant $\det A = +1$; the matrices from the third and fourth families have determinant $\det A = -1$. The transformations of the first and third types preserve the direction of times since they have $a > 0$. The transformations of the second and fourth types revert the direction of time.

Let us consider the transformation of the first type in the two-dimensional Minkowski space with the pseudo-orthogonal coordinates $(x^0 = c\,t, x^1 = x)$. The transformation defines a change of pseudo-orthogonal coordinates

$$\begin{gathered} (x^0, x^1) = (c\,t, x) \mapsto (x^{0'}, x^{1'}) = (c\,t', x') \\ \left. \begin{array}{rcl} c\,t &=& c\,t' \cosh\psi + x' \sinh\psi \\ x &=& c\,t' \sinh\psi + x' \cosh\psi \end{array} \right\} \end{gathered} \tag{1.7.22}$$

or, inverting,

$$\left. \begin{array}{rcr} c\,t' &=& c\,t\,\cosh\psi - x\,\sinh\psi \\ x' &=& -c\,t\,\sinh\psi + x\,\cosh\psi \end{array} \right\}.$$

The above formulae say that a particle remaining at rest at any point $x' = x'_0$ in the new reference frame (x', t') will move in the old reference frame (x, t)

$$x = c\,t\,\tanh\psi + \frac{x'_0}{\cosh\psi}$$

with a constant speed

$$v = c \tanh\psi. \tag{1.7.23}$$

The speed v can be interpreted as the velocity of the new reference frame with respect to the old one Using the identity

$$\cosh^2\psi - \sinh^2\psi = 1$$

we find

$$\begin{aligned}\cosh\psi &= \frac{1}{\sqrt{1-\tanh^2\psi}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\\ \sinh\psi &= \frac{\tanh\psi}{\sqrt{1-\tanh^2\psi}} = \frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}}.\end{aligned} \tag{1.7.24}$$

Substituting we finally arrive at the following expression for the *Lorentz boost*

$$\left.\begin{array}{rcl} t' & = & \frac{t-\frac{v\,x}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \\ x' & = & \frac{x-v\,t}{\sqrt{1-\frac{v^2}{c^2}}} \end{array}\right\} \tag{1.7.25}$$

describing the transformation of the space-time coordinates in the frame moving along the x-axis with the constant velocity v. If the velocity v is small with respect to the speed of light

$$|v| << |c|$$

then the Lorentz transformation (1.7.25) becomes equal to the Galilean transformation of Newtonian mechanics

$$\left.\begin{array}{rcl} t' & \simeq & t \\ x' & \simeq & x - v\,t \end{array}\right\}. \tag{1.7.26}$$

In a way similar to the Theorem 1.3.14 one can prove

Theorem 1.7.7 *1) Two isometric pseudo-Euclidean spaces must have the same signatures*

$$(p', q') = (p, q).$$

2) Any isometry $f : X \to X$ of a pseudo-Euclidean space (X, g) to itself must have the form (1.7.19)*,* (1.7.20) *in a system of pseudo-orthogonal coordinates.*

We leave the proof of this theorem as an exercise for the reader.

Corollary 1.7.8 *All isometries of a pseudo-Euclidean space (X, g) form a group Iso(X, g).*

Translations

$$x \mapsto x + b$$

form a normal subgroup in the group $Iso(X, g)$. The quotient over the subgroup of translations is isomorphic to the group of isometries keeping a given point $O \in X$ fixed. Such a group can be realized by transformations

$$x \mapsto Ax, \quad A \in O(p, q). \tag{1.7.27}$$

Here, as above, (p, q) is the signature of the pseudo-Euclidean space (X, g).

The group of isometries of the four-dimensional Minkowski space-time is called *Poincaré group*. It is generated by translations, orthogonal transformations in the three-dimensional space and Lorentz boosts. The subgroup of isometries preserving the origin is called *Lorentz group*. One can identify the kinematics of special relativity with the geometry of the Minkowski 4-space invariant with respect to the action of Poincaré group.

At the end of this section we will consider geometry of *spacelike* surfaces in pseudo-Euclidean spaces. A regular parametrized surface

$$x = x(u) \in X, \quad u = (u^1, \ldots, u^k) \in D \subset \mathbb{R}^k \tag{1.7.28}$$

in (X, g) is called spacelike if all tangent vectors

$$a^1 x_{u^1}(u) + \cdots + a^k x_{u^k}(u) \quad \forall\, a^1, \ldots, a^k \in \mathbb{R}$$

are spacelike at every point of the surface.

Definition 1.7.9 *The restriction of the quadratic differential form*

$$-ds^2 = -\left[(dx^1)^2 + \cdots + (dx^p)^2 - (dx^{p+1})^2 - \cdots - (dx^{p+q})^2\right]$$

on the spacelike surface (1.7.28) *is called the* first fundamental form *or the* induced metric *of the spacelike surface*

$$-ds^2 = -\left[(dx^1(u))^2 + \cdots + (dx^p(u))^2 - (dx^{p+1}(u))^2 - \cdots - (dx^{p+q}(u))^2\right] = g_{\alpha\beta}(u)du^\alpha du^\beta \tag{1.7.29}$$

$$g_{\alpha\beta}(u) = -\sum_{i=1}^{p} \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} + \sum_{i=p+1}^{p+q} \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}, \quad \alpha,\, \beta = 1, \ldots, k.$$

It is easy to check that the above definitions do not depend on the choice of parametrization of the surface. The induced metric defines a positive definite quadratic form on the tangent space at every point of the surface.

Let us consider an example of a two-dimensional spacelike surface in the three-dimensional Minkowski space. Put $c = 1$ and denote (t, x, y) the pseudo-orthogonal coordinates in the Minkowski space. Consider an analogue of the sphere of radius R:

$$t^2 - x^2 - y^2 = R^2. \tag{1.7.30}$$

From the point of view of "normal" 3-space such a "sphere" is a two-sheets hyperboloid. The upper sheet can be represented as the graph of function

$$t = \sqrt{R^2 + x^2 + y^2}.$$

Choosing x and y as the parameters one finds the basic tangent vectors in the form

$$\left(\frac{x}{\sqrt{R^2+x^2+y^2}}, 1, 0\right), \quad \left(\frac{y}{\sqrt{R^2+x^2+y^2}}, 0, 1\right).$$

The pseudo-Euclidean Gram matrix of these vectors is negative definite

$$\begin{pmatrix} \frac{x^2}{R^2+x^2+y^2} - 1 & \frac{x\,y}{R^2+x^2+y^2} \\ \\ \frac{x\,y}{R^2+x^2+y^2} & \frac{y^2}{R^2+x^2+y^2} - 1 \end{pmatrix} < 0.$$

Hence (1.7.30) is a spacelike surface.

In order to compute the induced metric on the surface let us introduce the *pseudo-spherical coordinates* (r, ϕ, ψ) in the three-dimensional space-time

$$\begin{aligned} t &= r\,\cosh\psi \\ x &= r\,\cos\phi\sinh\psi \\ y &= r\,\sin\phi\sinh\psi. \end{aligned}$$

Since

$$t^2 - x^2 - y^2 = r^2$$

the pseudo-spherical coordinates cover only part of the space-time, namely, the inner part

$$t^2 - x^2 - y^2 > 0$$

of the light cone. It is easy to see that the space-time interval $ds^2 = dt^2 - dx^2 - dy^2$ in the pseudo-spherical coordinates becomes takes the following form

$$ds^2 = dr^2 - r^2(d\psi^2 + \sinh^2\psi\, d\phi^2). \tag{1.7.31}$$

Restricting $(-ds^2)$ to the hyperboloid $r = \pm R$ one obtains the induced metric in the coordinates (ϕ, ψ)

$$R^2(d\psi^2 + \sinh^2\psi\, d\phi^2). \tag{1.7.32}$$

Let us compute the radius ρ and the area $A = A(\rho)$ of the circle domain on the hyperboloid defined by

$$0 \le \psi \le \psi_0, \quad 0 \le \phi \le 2\pi$$

cut from the hyperboloid by the horizontal plane $t = R\,\cos\psi_0$. The radius ρ can be computed as the length of the line

$$\phi = \text{const}, \quad 0 \le \psi \le \psi_0.$$

This gives

$$\rho = R\int_0^{\psi_0} d\psi = R\,\psi_0.$$

Since the area element for the metric (1.7.32) is equal to

$$dA = R^2 \sinh\psi\, d\phi\, d\psi,$$

the area of the disk is equal to

$$A = R^2 \int_0^{2\pi} d\phi \int_0^{\psi_0} \sinh\psi \, d\psi = 2\pi R^2(\cosh\psi_0 - 1).$$

Substituting

$$\psi_0 = \frac{\rho}{R}$$

yields the formula for the area of the circle of radius ρ on the hyperboloid (1.7.30) in the Minkowski space

$$A(\rho) = 2\pi R^2 \left(\cosh\frac{\rho}{R} - 1\right). \tag{1.7.33}$$

For small ρ using the Taylor expansion

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \mathcal{O}(x^6)$$

one obtains

$$A(\rho) = \pi\rho^2 + \frac{\pi}{12R^2}\rho^4 + \mathcal{O}(\rho^6). \tag{1.7.34}$$

Thus the area of the circle of a small radius ρ on the hyperboloid in the Minkowski space is bigger than the area of the circle of the same radius on the plane.

1.8 Curvature of surfaces

The curvature of surfaces can be characterized by the curvature of certain curves on the surface. Let us introduce the tools useful for computing these curvatures.

Let

$$\mathbf{r} = \mathbf{r}(u, v) \tag{1.8.1}$$

be a regular smooth two-dimensional surface in a three-dimensional Euclidean space. Define the unit *normal* vector at the point $\mathbf{r}(u, v)$

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}. \tag{1.8.2}$$

The vector $\mathbf{n}$ is orthogonal to $\mathbf{r}_u$ and $\mathbf{r_v}$ and, hence, it is orthogonal to the tangent plane $T_{(u,v)}$ to the surface.

Introduce functions

$$\begin{aligned} b_{11}(u, v) &= \langle \mathbf{r}_{uu}, \mathbf{n} \rangle \\ b_{12}(u, v) &= \langle \mathbf{r}_{uv}, \mathbf{n} \rangle \\ b_{22}(u, v) &= \langle \mathbf{r}_{vv}, \mathbf{n} \rangle. \end{aligned} \tag{1.8.3}$$

Definition 1.8.1 *The quadratic form*

$$b_{11}(u, v)du^2 + 2b_{12}(u, v)du\, dv + b_{22}(u, v)dv^2 \tag{1.8.4}$$

is called the second fundamental form *of the surface* (1.8.1).

A geometric meaning of the value of the second fundamental form on tangent vectors to the surface becomes clear from the following statement.

Lemma 1.8.2 *Let $(u(t), v(t))$ be a smooth curve on the surface. The normal component of the acceleration vector $\ddot{\mathbf{r}}$ at a point $(u = u(t), v = v(t))$ is equal to the value of the second fundamental form on the velocity vector $(\dot{u}, \dot{v})$ at this point*

$$\langle \ddot{\mathbf{r}}, \mathbf{n} \rangle = b_{11}(u, v)\dot{u}^2 + 2b_{12}(u, v)\dot{u}\,\dot{v} + b_{22}(u, v)\dot{v}^2. \tag{1.8.5}$$

Proof: In the expression

$$\ddot{\mathbf{r}} = \mathbf{r}_{uu}\dot{u}^2 + 2\mathbf{r}_{uv}\dot{u}\dot{v} + \mathbf{r}_{vv}\dot{v}^2 + \mathbf{r}_u\ddot{u} + \mathbf{r}_v\ddot{v}$$

the last two terms are orthogonal to $\mathbf{n}$. Hence

$$\langle \ddot{\mathbf{r}}, \mathbf{n} \rangle = \langle \mathbf{r}_{uu}, \mathbf{n} \rangle \dot{u}^2 + 2\langle \mathbf{r}_{uv}, \mathbf{n} \rangle \dot{u}\dot{v} + \langle \mathbf{r}_{vv}, \mathbf{n} \rangle \dot{v}^2.$$

$\square$

Denote ν the principal normal to the curve

$$\mathbf{r}(u(t), v(t))$$

on the surface.

Theorem 1.8.3 *The curvature of a smooth curve on the surface* (1.8.1) *multiplied by the cosine of the angle between the principal normal to the curve and the normal to the surface is equal to the ratio of values of the second and first fundamental forms on the velocity vector of the curve*

$$k\,\langle \nu, \mathbf{n} \rangle = \frac{b_{11}(u, v)\dot{u}^2 + 2b_{12}(u, v)\dot{u}\dot{v} + b_{22}(u, v)\dot{v}^2}{g_{11}(u, v)\dot{u}^2 + 2g_{12}(u, v)\dot{u}\dot{v} + g_{22}(u, v)\dot{v}^2}. \tag{1.8.6}$$

Proof: Recall that the principal normal to the curve is the normalized vecor of acceleration

$$\frac{d^2\mathbf{r}}{ds^2} = k\,\nu, \quad k > 0, \quad |\nu| = 1.$$

Applying the Lemma one obtains

$$k\,\langle \nu, \mathbf{n} \rangle = b_{11}(u, v)\left(\frac{du}{ds}\right)^2 + 2b_{12}(u, v)\frac{du}{ds}\frac{dv}{ds} + b_{22}(u, v)\left(\frac{dv}{ds}\right)^2.$$

This proves the formula (1.8.6) for the curves parametrized by length since, in that case, the denominator in (1.8.6) is equal to 1. Since both sides of (1.8.6) do not depend on the parametrization of the curve, the formula holds trues also for an arbitrary parametrization.

$\square$

Let us consider the curve obtained by intersecting the surface by the plane passing through the normal $\mathbf{n}$. It is called the *normal section*. It is a plane curve; its principal normal vector ν is collinear with $\mathbf{n}$. Denote τ a unit tangent vector to the surface belonging to the normal plane. It coincides with the velocity vector of the normal section passing through $\mathbf{n}$ and τ. We obtain

Corollary 1.8.4 *The absolute value of the second fundamental form on a unit tangent vector τ to the surface is equal to the curvature of the normal section passing through τ and $\mathbf{n}$.*

Let us slightly modify the definition of the curvature for the case of a plane section of an oriented surface: it will coincide with the old one if the direction of the principal normal to the curve coincides, $\nu = \mathbf{n}$, with the direction of the normal to the surface; in the opposite case, $\nu = -\mathbf{n}$, the new curvature will be equal to the negative old one. With such a definition the result of the Corollary for the curvature $k = k(\tau)$ of a plane section passing through the unit tangent vector $\tau = (\tau^1, \tau^2)$ at a point (u, v) can be represented in the following form:

$$k(\tau) = b_{11}(u,v)(\tau^1)^2 + 2b_{12}(u,v)\tau^1\tau^2 + b_{22}(u,v)(\tau^2)^2 \tag{1.8.7}$$

$$g_{11}(u,v)(\tau^1)^2 + 2g_{12}(u,v)\tau^1\tau^2 + g_{22}(u,v)(\tau^2)^2 = 1. \tag{1.8.8}$$

Example. On the sphere of radius R all normal sections are circles of the same radius R. The curvature of these circles is equal to $1/R$. With the choice of the orientation on the surface by ordering the spherical coordinates $u = \phi$, $v = \theta$ the curvature of any normal section is equal to $-1/R$. Hence the second fundamental form of the sphere in the spherical coordinates reads

$$-R(d\theta^2 + \cos^2\theta\, d\phi^2).$$

In order to get more clear idea about dependence of the curvature of a normal section on the direction τ at a given point of the surface let us study the minima and maxima of the function $k(\tau)$. This problem is tantamount to finding the maxima/minima of the function (1.8.7) of two variables τ^1, τ^2 constrained by the equation (1.8.8). In order to simplify notations let us redenote

$$x := \tau^1, \quad y := \tau^2.$$

We will also omit writing explicitly the dependence of the coefficients of the first and second fundamental forms on u and v.

We arrive at the following constraint maximum/minimum problem:

$$b_{11}x^2 + 2b_{12}xy + b_{22}y^2 \to \max/\min \tag{1.8.9}$$

$$g_{11}x^2 + 2g_{12}xy + g_{22}y^2 = 1. \tag{1.8.10}$$

To resolve this problem let us consider the following auxiliary function

$$F = b_{11}x^2 + 2b_{12}xy + b_{22}y^2 - \lambda\left(g_{11}x^2 + 2g_{12}xy + g_{22}y^2 - 1\right). \tag{1.8.11}$$

One has to find the stationary points of $F = F(x, y, \lambda)$ from the system

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial \lambda} = 0.$$

The last equation is nothing but the constraint (1.8.10). The first two, after division by 2 yield a linear homogeneous system

$$\begin{aligned} b_{11}x + b_{12}y &= \lambda(g_{11}x + g_{12}y) \\ b_{12}x + b_{22}y &= \lambda(g_{12}x + g_{22}y) \end{aligned}$$

or, in the matrix form,

$$B\,X = \lambda\,G\,X \tag{1.8.12}$$

where

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.8.13}$$

Recall that the matrices B and G are both symmetric and, moreover, the matrix G is positive definite.

We arrive at the theory of invariants of pairs of quadratic forms with the Gram matrices B and G. Let us briefly explain the main points of this theory in a linear space of an arbitrary dimension n

$$b(x, y) = b_{ij} x^i y^j, \quad g(x, y) = g_{ij} x^i y^j.$$

Definition 1.8.5 *A nonzero vector X satisfying the linear homogeneous system* (1.8.12) *is called* an eigenvector *of a pair of quadratic forms with the* eigenvalue λ.

Lemma 1.8.6 *The eigenvalues of a pair of quadratic forms satisfy the* characteristic equation

$$\det(B - \lambda G) = 0. \tag{1.8.14}$$

Proof: The linear homogeneous system (1.8.12) has a nonzero solution *iff* its determinant vanishes. □

Lemma 1.8.7 *The eigenvalues of a pair of quadratic forms do not depend on the choice of the basis in the space.*

Proof: Changing the basis transforms the Gram matrices of the quadratic forms to

$$B' = T^{\mathrm{T}} B\,T, \quad G' = T^{\mathrm{T}} G\,T.$$

So the new characteristic equation

$$\det(B' - \lambda\,G') = \det\left[T^{\mathrm{T}}(B - \lambda\,G)\,T\right] = (\det T)^2 \det(B - \lambda\,G) \tag{1.8.15}$$

is proportional to the old one. □

In order to complete the theory of normal forms of a pair of bilinear forms we will use the connection between self-adjoint operators and symmetric bilinear forms in a Euclidean space. Recall that a linear operator

$$A : X \to X \tag{1.8.16}$$

on a Euclidean space $(X, \langle\ ,\ \rangle)$ is called *self-adjoint* if it satisfies

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \forall\, x,\, y. \tag{1.8.17}$$

The bilinear form

$$b(x, y) := \langle x, Ay\rangle \tag{1.8.18}$$

is symmetric *iff* the operator A is self-adjoint. Given the matrix (a^i_j) of the operator in a basis $e_1, \ldots, e_n$,

$$Ae_j = a^i_j e_i, \quad j = 1, \ldots, n \tag{1.8.19}$$

and the Gram matrix of the inner product in the same basis

$$\langle e_i, e_j\rangle = g_{ij}, \quad i,\, j = 1, \ldots, n \tag{1.8.20}$$

one can can compute the Gram matrix of the bilinear form $b(x, y)$:

$$b_{ij} \equiv b(e_i, e_j) = g_{ik} a^k_j, \quad i,\, j = 1, \ldots \tag{1.8.21}$$

or, in the matrix form

$$B = G\,A. \tag{1.8.22}$$

Inverting one reconstructs the operator A by

$$A = G^{-1} B \tag{1.8.23}$$

or, in the index notations

$$a^i_j = g^{ik} b_{kj}, \quad i,\, j = 1, \ldots, n \tag{1.8.24}$$

where g^{ij} are the entries of the matrix inverse to $G = (g_{ij})$

$$G = (g_{ij}), \quad G^{-1} = \left(g^{ij}\right). \tag{1.8.25}$$

They say that the matrix of the bilinear form $b(x, y)$ is obtained from the matrix of the operator A by *lowering the index* (see (1.8.21)) while the inverse procedure (1.8.24) of reconstructing the operator from the bilinear form is called *raising of indices*.

Let us return to the eigenvalues and eigenvectors of a pair of quadratic forms. They coincide with the eigenvalues and eigenvectors of the self-adjoint linear operator A. At this point it is crucial that the quadratic form g defining the inner product in the space is positive definite. Under this assumption the following theorem is fundamental in the theory of self-adjoint operators.

Theorem 1.8.8 *Let*

$$A : X \to X$$

be a self-adjoint operator in a n-dimensional Euclidean space $(X, \langle\ ,\ \rangle)$. Then there exists an orthonormal basis $e_1, \ldots, e_n$ in X consisting of eigenvectors of the operator A

$$\begin{aligned} Ae_i &= \lambda_i e_i, \quad i = 1, \ldots, n \\ \langle e_i, e_j\rangle &= \delta_{ij}. \end{aligned}$$

Applying this theorem to the self-adjoint operator (1.8.23) we arrive at the following

Corollary 1.8.9 *Let B and G be two symmetric $n\times n$ matrices, and the matrix G is positive definite. Then*

1) the characteristic equation (1.8.14) *has n real roots $\lambda_1, \ldots, \lambda_n$.*

2) There exists a system of coordinates in the linear space such that the quadratic forms b and g take the following diagonal form

$$b(x,x) = \lambda_1 (x^1)^2 + \cdots + \lambda_n (x^n)^2$$
$$g(x,x) = (x^1)^2 + \cdots + (x^n)^2. \tag{1.8.26}$$

Exercise 1.8.10 *Prove that*

$$\lambda_1 \lambda_2 \dots \lambda_n = \frac{\det B}{\det G}. \tag{1.8.27}$$

Let us come back to the curvature of normal sections of a surface in $\mathbb{R}^3$. We have proved that the characteristic equation (1.8.14) has two real roots λ_1 and λ_2 depending on the point of the surface and, moreover, at a given point there exist two tangent vectors e_1, e_2 such that

$$b(e_1,e_1) = \lambda_1, \quad b(e_2,e_2) = \lambda_2, \quad b(e_1,e_2) = b(e_2,e_1) = 0 \tag{1.8.28}$$

$$g(e_1,e_1) = g(e_2,e_2) = 1, \quad g(e_1,e_2) = g(e_2,e_1) = 0. \tag{1.8.29}$$

The second line means that the tangent vectors e_1 and e_2 are orthogonal as vectors in the three-dimensional Euclidean space and, moreover, they have unit length.

Definition 1.8.11 *The linear operator $A = G^{-1}B$ is called the* shape operator *of the surface. The numbers $k_1 = \lambda_1$ and $k_2 = \lambda_2$ are called the* principal curvatures *of the surface at a given point. The directions of the vectors e_1 and e_2 are called the* principal directions *at the same point.*

We will now see that the principal curvatures give the maximal and minimal values of curvatures normal sections we were looking after.

Theorem 1.8.12 (Euler formula) *Let φ be the angle between a unit tangent vector τ and e_1. Then the curvature k of the normal section of the surface passing through τ and the normal* **n** *is equal to*

$$k = k_1^2 \cos^2\varphi + k_2 \sin^2\varphi. \tag{1.8.30}$$

Proof: In the basis e_1, e_2 the first and the second fundamental forms become equal to

$$g(\tau,\tau) = \quad (\tau^1)^2 \quad + \quad (\tau^2)^2$$
$$b(\tau,\tau) = k_1(\tau^1)^2 + k_2(\tau^2)^2$$

$$\tau = \tau^1 e_1 + \tau^2 e_2.$$

In this basis the vector τ reads

$$\tau = \cos\varphi\, e_1 + \sin\varphi\, e_2.$$

For the curvature of normal section passing through τ one obtains

$$k = \frac{k_1(\tau^1)^2 + k_2(\tau^2)^2}{(\tau^1)^2 + (\tau^2)^2} = k_1 \cos^2\varphi + k_2 \sinh^2\varphi.$$

$\square$

Corollary 1.8.13 *Let the principal curvatures at a given point of the surface satisfy*

$$k_2 \le k_1.$$

Then the curvature k of an arbitrary normal section passing through the same point satisfies

$$k_2 \le k \le k_1.$$

Definition 1.8.14 *The product of principal curvatures*

$$K = k_1 k_2 \tag{1.8.31}$$

is called the Gaussian curvature *of the surface at a given point. The mean value*

$$H = \frac{k_1 + k_2}{2} \tag{1.8.32}$$

is called the mean curvature *at the point.*

From the result of Exercise 1.8.10 it follows that the Gaussian curvature is equal to the ration of the determinants of the second and the first fundamental forms

$$K = \frac{\det B}{\det G}. \tag{1.8.33}$$

Example 1. For the sphere of radius R with the standard orientation the Gaussian curvature is equal to $K = 1/R^2$ and the mean curvature is $H = -1/R$.

Example 2. Let the surface in the Euclidean space be represented as a graph of a function

$$z = f(x, y).$$

The tangent vectors have the already familiar form

$$\mathbf{r}_x = (1, 0, f_x), \quad \mathbf{r}_y = (0, 1, f_y).$$

Computing their cross-product we obtain the unit normal vector

$$\mathbf{n} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}. \tag{1.8.34}$$

So the coefficients of the second fundamental form are equal to

$$b_{11} = \langle \mathbf{r}_{xx}, \mathbf{n} \rangle = \frac{f_{xx}}{\sqrt{1 + f_x^2 + f_y^2}}$$
$$b_{12} = \langle \mathbf{r}_{xy}, \mathbf{n} \rangle = \frac{f_{xy}}{\sqrt{1 + f_x^2 + f_y^2}}$$
$$b_{22} = \langle \mathbf{r}_{yy}, \mathbf{n} \rangle = \frac{f_{yy}}{\sqrt{1 + f_x^2 + f_y^2}}.$$

Computing the determinant

$$\det B = b_{11}b_{22} - b_{12}^2$$

and dividing by the determinant of the first fundamental form (1.6.27)

$$\det G = 1 + f_x^2 + f_y^2$$

(see (1.6.29)) one obtains

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}. \tag{1.8.35}$$

One observes that, at a stationary point

$$f_x = f_y = 0$$

the Gaussian curvature is positive near a point of a maximum/minimum where the graph is convex but it is negative near a saddle point where the Hessian $f_{xx}f_{yy} - f_{xy}^2$ is negative.

Let us now compute the mean curvature of the graph surface. Inverting the matrix G

$$G^{-1} = \frac{1}{\det G}\begin{pmatrix} 1 + f_y^2 & -f_x f_y \\ -f_x f_y & 1 + f_x^2 \end{pmatrix}$$

and computing the trace of the matrix of the shape operator $G^{-1}B$ one obtains

$$\begin{aligned}
H &= \frac{1}{2}\operatorname{tr} G^{-1}B = \frac{1}{(\det G)^{3/2}}\left[(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}\right] \\
&= \frac{1}{2}\left[\frac{f_{xx} + f_{yy}}{(\det G)^{1/2}} - \frac{f_x(f_x f_{xx} + f_y f_{xy})}{(\det G)^{3/2}} - \frac{f_y(f_x f_{xy} + f_y f_{yy})}{(\det G)^{3/2}}\right] \\
&= \frac{1}{2}\left[\frac{f_{xx} + f_{yy}}{(\det G)^{1/2}} - \frac{f_x(\det G)_x}{2(\det G)^{3/2}} - \frac{f_y(\det G)_y}{2(\det G)^{3/2}}\right] \\
&= \frac{1}{2}\left[\frac{\partial}{\partial x}\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}} + \frac{\partial}{\partial y}\frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}}\right] = \frac{1}{2}\operatorname{div}\frac{\operatorname{grad} f}{\sqrt{1 + f_x^2 + f_y^2}}.
\end{aligned}$$

Let us now consider the important particular case of *minimal surfaces* having zero mean curvature. Clearly, the Gaussian curvature of such a surface must be negative since the principal curvatures k_1 and k_2 have opposite signs.

Assuming that the minimal surface is represented as a graph of function $z = f(x, y)$ one obtains the following PDE for the function $f = f(x, y)$

$$\operatorname{div}\frac{\operatorname{grad} f}{\sqrt{1 + f_x^2 + f_y^2}} = 0 \tag{1.8.36}$$

or, equivalently

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0. \tag{1.8.37}$$

This equation describes the shape of soap films that, in the absence of external forces tend to minimize their area. Indeed, let us consider the area of a small piece of the surface

$$A[f] = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy \tag{1.8.38}$$

as a *functional* of the shape function f. Here Ω is a sufficiently small domain on the (x, y)-plane. A necessary condition to minimize the value of the functional is that, under an arbitrary small variation of the function f,

$$f(x, y) \mapsto f(x, y) + \delta f(x, y)$$

the variation of the functional must satisfy

$$A[f + \delta f] - A[f] = \mathcal{O}(\|\delta f\|^2). \tag{1.8.39}$$

Here the function $\delta f(x, y)$ must vanish together with its derivatives on the boundary of the domain Ω; the definition of the norm $\|\delta f\|$ will be clear from subsequent calculations. In other words, the equation (1.8.39) says that f is a "stationary point" of the "function" $A[f]$ on the infinite-dimensional space of functions $f = f(x, y)$.

Let us prove that stationary point condition (1.8.39) reduces to equation (1.8.38). The left hand side of this condition can be written in the following way

$$A[f + \delta f] - A[f] = \iint_\Omega \left[\sqrt{1 + (f_x + \delta f_x)^2 + (f_y + \delta f_y)^2} - \sqrt{1 + f_x^2 + f_y^2}\right] dx\, dy.$$

The part of the increment linear in δf can be obtained by expanding the above expression in Taylor series

$$A[f + \delta f] - A[f] = \iint_\Omega \left[\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}}\, \delta f_x + \frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}}\, \delta f_y\right] dx\, dy + \mathcal{O}(\|\delta f\|^2).$$

Thus the stationarity condition (1.8.39) can be recast into the form

$$\iint_\Omega \left[\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}}\, \delta f_x + \frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}}\, \delta f_y\right] dx\, dy = 0 \tag{1.8.40}$$

for an arbitrary function $\delta f(x, y)$ vanishing on the boundary of the domain Ω. Applying in two different ways the Fubini theorem

$$\iint_\Omega \quad dx\, dy = \int \quad dx \int \quad dy = \int \quad dy \int \quad dx$$

to the two parts of the double integral and integrating by parts one reduces the equation (1.8.40) to

$$\begin{aligned} &\iint_\Omega \left[\frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}}\, \delta f_x + \frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}}\, \delta f_y\right] dx\, dy \\ &= -\iint_\Omega \left[\frac{\partial}{\partial x} \frac{f_x}{\sqrt{1 + f_x^2 + f_y^2}} + \frac{\partial}{\partial y} \frac{f_y}{\sqrt{1 + f_x^2 + f_y^2}}\right] \delta f\, dx\, dy = 0. \end{aligned} \tag{1.8.41}$$

Since $\delta f(x, y)$ is an arbitrary function one obtains the equation for the stationary points of the area functional $A[f]$ written in the form

$$H = 0$$

where H is the mean curvature of the surface.

1.9 Extrinsic and intrinsic geometry of surfaces

The vectors $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{n}$ are linearly independent at any point (u, v) of the surface. Let us consider the dependence of the frame $(\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})$ on the point. The derivatives of each vector of the frame can be represented as linear combinations of the same vectors. Some of coefficients of the linear combination are already known: for example, the coefficient of $\mathbf{n}$ in the decomposition of $\mathbf{r}_{uu}$ is equal to $b_{11} = b_{11}(u, v)$. We also know that the derivatives $\mathbf{n}_u$ and $\mathbf{n}_v$ are orthogonal to $\mathbf{n}$ since $|\mathbf{n}| = 1$ (see Lemma 1.4.11 above). We arrive at a representation

$$\begin{aligned}
\mathbf{r}_{uu} &= \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + b_{11}\mathbf{n}\\
\mathbf{r}_{uv} &= \Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + b_{12}\mathbf{n}\\
\mathbf{r}_{vv} &= \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + b_{22}\mathbf{n}\\
\mathbf{n}_u &= -a^1_1\mathbf{r}_u - a^2_1\mathbf{r}_v\\
\mathbf{n}_v &= -a^1_2\mathbf{r}_u - a^2_2\mathbf{r}_v
\end{aligned} \tag{1.9.1}$$

where the functions $\Gamma^k_{ij} = \Gamma^k_{ij}(u, v)$, $a^l_i = a^l_i(u, v)$ are defined as coefficients of the above decomposition. One can also use the definition of the coefficients Γ^k_{ij} and a^l_i of the expansion (1.9.1) written in the form

$$\partial_j \mathbf{r}_i = \Gamma^l_{ij}\mathbf{r}_l + b_{ij}\mathbf{n} \tag{1.9.2}$$

$$\partial_i \mathbf{n} = -a^l_i \mathbf{r}_l, \quad i = 1,\, 2. \tag{1.9.3}$$

where we adjust the notations as follows: we put $u^1 = u$, $u^2 = v$, redenote

$$\partial_i := \frac{\partial}{\partial u^i}, \quad i = 1,\, 2$$

the partial derivatives $\partial/\partial u$ and $\partial/\partial v$. We also use the short notation

$$\mathbf{r}_i = \mathbf{r}_{u^i}, \quad i = 1,\, 2.$$

Lemma 1.9.1 *The coefficients Γ^k_{ij} can be expressed via the entries of the first fundamental form and its derivatives, namely,*

$$\Gamma^k_{ij} = \frac{1}{2} g^{ks}\left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s}\right), \quad i,\, j,\, k = 1,\, 2. \tag{1.9.4}$$

Proof: Differentiating the coefficients of the first fundamental form one obtains

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial u^k} &= \frac{\partial}{\partial u^k}\langle \mathbf{r}_i, \mathbf{r}_j\rangle = \langle \mathbf{r}_{ik}, \mathbf{r}_j\rangle + \langle \mathbf{r}_i, \mathbf{r}_{jk}\rangle\\
&= \langle \Gamma^l_{ik}\mathbf{r}_l + b_{ik}\mathbf{n}, \mathbf{r}_j\rangle + \langle \mathbf{r}_i, \Gamma^l_{jk}\mathbf{r}_l + b_{jk}\mathbf{n}\rangle = \Gamma^l_{ik} g_{lj} + \Gamma^l_{jk} g_{li}.
\end{aligned} \tag{1.9.5}$$

Denote

$$\Gamma_{ij,k} := g_{kl}\Gamma^l_{ij}.$$

Observe that

$$\Gamma^k_{ij} = g^{ks}\Gamma_{ij,s}.$$

The equation (1.9.5) can be rewritten as follows

$$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ik,j} + \Gamma_{jk,i}. \tag{1.9.6}$$

Cyclic permutations of indices i, j, k yield two more equations

$$\begin{aligned} \frac{\partial g_{ki}}{\partial u^j} &= \Gamma_{kj,i} + \Gamma_{ij,k} \\ \frac{\partial g_{jk}}{\partial u^i} &= \Gamma_{ji,k} + \Gamma_{ki,j}. \end{aligned}$$

Adding these two equations and subtracting (1.9.6) gives

$$2\Gamma_{ij,k} = \frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}. \tag{1.9.7}$$

Multiplying the last equation by g^{ks} and dividing by 2 one obtains (1.9.4), after redenoting the indices $s \leftrightarrow k$. □

Lemma 1.9.2 *The coefficients* a^i_j *coincide with the matrix entries of the shape operator* (1.8.23), *i.e.,*

$$a^i_j = g^{is} b_{sj}, \quad i, j = 1, 2. \tag{1.9.8}$$

Proof: A derivation of the orthogonality condition

$$\langle \mathbf{r}_i, \mathbf{n} \rangle = 0$$

yields

$$\begin{aligned} 0 &= \langle \mathbf{r}_{ij}, \mathbf{n} \rangle + \langle \mathbf{r}_i, \partial_j \mathbf{n} \rangle \\ &= \langle \Gamma^k_{ij} \mathbf{r}_k + b_{ij} \mathbf{n}, \mathbf{n} \rangle + \langle \mathbf{r}_i, -a^k_j \mathbf{r}_k \rangle = b_{ij} - a^k_j g_{ki}. \end{aligned}$$

Thus

$$b_{ij} = a^k_j g_{ki}$$

or, in the matrix form

$$B = GA$$

where $A = (a^j_i)$. Multiplying by G^{-1} we complete the proof of Lemma. □

We arrive at the following linear differential equations describing the dependence of the frame $(\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})$ on the point of the surface

$$\begin{aligned} \frac{\partial}{\partial u} (\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}) &= (\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})\, M_u \\ \frac{\partial}{\partial v} (\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}) &= (\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})\, M_v \end{aligned} \tag{1.9.9}$$

where the matrices M_u and M_v can be read from eqs. (1.9.1)

$$M_u = \begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & -a_1^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & -a_1^2 \\ b_{11} & b_{12} & 0 \end{pmatrix}$$
$$M_v = \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{22}^1 & -a_2^1 \\ \Gamma_{12}^2 & \Gamma_{22}^2 & -a_2^2 \\ b_{12} & b_{22} & 0 \end{pmatrix}. \tag{1.9.10}$$

The matrices $M_u = M_u(u,v)$ and $M_v = M_v(u,v)$ satisfy an important *compatibility condition*[6].

Lemma 1.9.3 *The matrix valued functions M_u and M_v satisfy*

$$\frac{\partial M_v}{\partial u} - \frac{\partial M_u}{\partial v} + [M_u, M_v] = 0. \tag{1.9.11}$$

Proof: Differentiating the first equation in (1.9.9) in v and the second one in u one must obtain the same result

$$\frac{\partial}{\partial v}\frac{\partial}{\partial u}(\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}) = \frac{\partial}{\partial v}\left[(\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})M_u\right] = (\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})\left[\frac{\partial M_u}{\partial v} + M_v M_u\right]$$
$$\frac{\partial}{\partial u}\frac{\partial}{\partial v}(\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}) = \frac{\partial}{\partial u}\left[(\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})M_v\right] = (\mathbf{r}_u, \mathbf{r}_v, \mathbf{n})\left[\frac{\partial M_v}{\partial u} + M_u M_v\right].$$

This implies (1.9.11). □

The following general statement about a pair of systems of linear differential equations satisfying the compatibility condition holds true.

Proposition 1.9.4 *Let two $n \times n$ matrix valued functions $M_u = M_u(u,v)$, $M_v = M_v(u,v)$, $(u,v) \in D \subset \mathbb{R}^2$ satisfy the compatibility condition* (1.9.11)*. Then for a given point $(u_0, v_0) \in D$ and arbitrary initial data $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ there exists a unique vector function $\mathbf{x}(u,v) = (x_1(u,v), \ldots, x_n(u,v))$ satisfying a pair of systems of linear differential equations*

$$\frac{\partial \mathbf{x}}{\partial u} = \mathbf{x}\, M_u$$
$$\frac{\partial \mathbf{x}}{\partial v} = \mathbf{x}\, M_v$$

such that $\mathbf{x}(u_0, v_0) = \mathbf{x}^0$. Conversely, if such existence holds true then the coefficient matrices M_u, M_v satisfy the compatibility condition (1.9.11).

[6]The eq. (1.9.11) is also called *zero curvature condition*. We will not use this terminology here to avoid confusion with many other appearances of the word 'curvature'.

Proof: Let $\mathbf{y} = \mathbf{y}(u)$ be the solution to the system

$$\frac{\partial \mathbf{y}}{\partial u} = \mathbf{y}\, M_u(u, v_0)$$

specified by the initial data

$$\mathbf{y}(u_0) = \mathbf{x}^0.$$

Denote $\mathbf{x} = \mathbf{x}(u, v)$ the solution to the system

$$\frac{\partial \mathbf{x}}{\partial v} = \mathbf{x}\, M_v(u, v) \tag{1.9.12}$$

with the initial data depending on u as on a parameter

$$\mathbf{x}(u, v_0) = \mathbf{y}(u).$$

Since the initial data $\mathbf{y}(u)$ depends smoothly on u the solution $\mathbf{x}(u, v)$ is a smooth function of two variables. Let us prove that the function $\mathbf{x}(u, v)$ also satisfies the first system. Indeed,

$$\frac{\partial}{\partial v}\left(\frac{\partial \mathbf{x}}{\partial u} - \mathbf{x}\, M_u\right) = \frac{\partial}{\partial v}\frac{\partial \mathbf{x}}{\partial u} - \frac{\partial \mathbf{x}}{\partial v}\, M_u - \mathbf{x}\,\frac{\partial M_u}{\partial v} = \frac{\partial}{\partial u}\frac{\partial \mathbf{x}}{\partial v} - \mathbf{x}\, M_v M_u - \mathbf{x}\,\frac{\partial M_u}{\partial v}$$
$$= \frac{\partial \mathbf{x}}{\partial u}\, M_v + \mathbf{x}\,\frac{\partial M_v}{\partial u} - \mathbf{x}\, M_v M_u - \mathbf{x}\,\frac{\partial M_u}{\partial v} = \left(\frac{\partial \mathbf{x}}{\partial u} - \mathbf{x}\, M_u\right) M_v.$$

Thus the vector function

$$\tilde{\mathbf{x}} = \frac{\partial \mathbf{x}}{\partial u} - \mathbf{x}\, M_u$$

satisfy the same linear homogeneous system (1.9.12). At $v = v_0$ one obtains the trivial initial data

$$\tilde{\mathbf{x}}(u, v_0) = \frac{\partial \mathbf{y}}{\partial u} - \mathbf{y}\, M_u(u, v_0) = 0.$$

Due to uniqueness of solutions to (1.9.12) one must have $\tilde{\mathbf{x}}(u, v) \equiv 0$. Therefore the function $\mathbf{x}(u, v)$ satisfies the two linear systems and, by construction it also satisfies the initial condition $\mathbf{x}(u_0, v_0) = \mathbf{x}^0$. The proof of the converse statement repeats the proof of the Lemma 1.9.3. $\square$

We are now in a position to prove the first main result of this section.

Theorem 1.9.5 *If two surfaces $\mathbf{r} = \mathbf{r}(u, v)$ and $\mathbf{r} = \mathbf{r}'(u, v)$, $(u, v) \in D \subset \mathbb{R}^2$ have the same first and second fundamental forms*

$$g'_{ij}(u, v) = g_{ij}(u, v), \quad b'_{ij}(u, v) = b_{ij}(u, v), \quad i, j = 1, 2 \tag{1.9.13}$$

then there exists an isometry of the ambient Euclidean space $\mathbb{R}^3 \to \mathbb{R}^3$ transforming one surface to another

$$A\mathbf{r}(u, v) + b = \mathbf{r}'(u, v) \quad \forall\, (u, v) \in D, \quad A^{\mathrm{T}} A = 1. \tag{1.9.14}$$

Proof: Let us fix a point (u_0, v_0). One has two bases $(\mathbf{r}_u(u_0, v_0), \mathbf{r}_v(u_0, v_0), \mathbf{n}(u_0, v_0))$ and $(\mathbf{r}'_u(u_0, v_0), \mathbf{r}'_v(u_0, v_0), \mathbf{n}'(u_0, v_0))$ for the two surfaces. By assumption the Gram matrices of these two bases coincide

$$\begin{pmatrix} \langle \mathbf{r}_u(u_0,v_0), \mathbf{r}_u(u_0,v_0)\rangle & \langle \mathbf{r}_u(u_0,v_0), \mathbf{r}_v(u_0,v_0)\rangle & \langle \mathbf{r}_u(u_0,v_0), \mathbf{n}(u_0,v_0)\rangle \\ \langle \mathbf{r}_v(u_0,v_0), \mathbf{r}_u(u_0,v_0)\rangle & \langle \mathbf{r}_v(u_0,v_0), \mathbf{r}_v(u_0,v_0)\rangle & \langle \mathbf{r}_v(u_0,v_0), \mathbf{n}(u_0,v_0)\rangle \\ \langle \mathbf{n}(u_0,v_0), \mathbf{r}_u(u_0,v_0)\rangle & \langle \mathbf{n}(u_0,v_0), \mathbf{r}_v(u_0,v_0)\rangle & \langle \mathbf{n}(u_0,v_0), \mathbf{n}(u_0,v_0)\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \mathbf{r}'_u(u_0,v_0), \mathbf{r}'_u(u_0,v_0)\rangle & \langle \mathbf{r}'_u(u_0,v_0), \mathbf{r}'_v(u_0,v_0)\rangle & \langle \mathbf{r}'_u(u_0,v_0), \mathbf{n}'(u_0,v_0)\rangle \\ \langle \mathbf{r}'_v(u_0,v_0), \mathbf{r}'_u(u_0,v_0)\rangle & \langle \mathbf{r}'_v(u_0,v_0), \mathbf{r}'_v(u_0,v_0)\rangle & \langle \mathbf{r}'_v(u_0,v_0), \mathbf{n}'(u_0,v_0)\rangle \\ \langle \mathbf{n}'(u_0,v_0), \mathbf{r}'_u(u_0,v_0)\rangle & \langle \mathbf{n}'(u_0,v_0), \mathbf{r}'_v(u_0,v_0)\rangle & \langle \mathbf{n}'(u_0,v_0), \mathbf{n}'(u_0,v_0)\rangle \end{pmatrix}$$

$$= \begin{pmatrix} g_{11}(u_0,v_0) & g_{12}(u_0,v_0) & 0 \\ g_{21}(u_0,v_0) & g_{22}(u_0,v_0) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence there exists an isometry $\mathbf{r} \mapsto A\mathbf{r} + b$ shifting the point $\mathbf{r}(u_0, v_0)$ to $\mathbf{r}'(u_0, v_0)$ and moving the first basis to the second one. Let us prove that, after such an isometry the surfaces will coincide. Indeed, the two bases as functions of u, v satisfy a compatible pair of linear systems of the form (1.9.9). Due to the assumptions of the Theorem and because of the lemmata 1.9.1 and 1.9.2 the coefficients of these linear systems coincide. Since the initial conditions are the same we conclude that

$$\mathbf{r}'_u(u,v) = A\mathbf{r}_u(u,v), \quad \mathbf{r}'_v(u,v) = A\mathbf{r}_v(u,v), \quad \mathbf{n}'(u,v) = A\mathbf{n}(u,v)$$

for all (u, v). Since $\mathbf{r}'(u_0, v_0) = A\mathbf{r}(u_0, v_0) + b$ we arrive at (1.9.14). □

The two fundamental forms of a surface in a three-dimensional Euclidean space satisfy a complicated system of constraints following from (1.9.11). They have the form

$$\left(\frac{\partial \Gamma^t_{ij}}{\partial u^k} - \frac{\partial \Gamma^t_{ik}}{\partial u^j} + \Gamma^s_{ij}\Gamma^t_{ks} - \Gamma^s_{ik}\Gamma^t_{js} \right) g_{tl} = b_{ij}b_{kl} - b_{ik}b_{jl}, \quad i,\ j,\ k,\ l = 1,\ 2 \tag{1.9.15}$$

and

$$\frac{\partial b_{ij}}{\partial u^k} - \frac{\partial b_{ik}}{\partial u^j} = \Gamma^s_{ik} b_{sj} - \Gamma^s_{ij} b_{sk}, \quad i,\ j,\ k = 1,\ 2. \tag{1.9.16}$$

The first eq. (1.9.15) is called *Gauss equations* while (1.9.16) is called *Peterson–Mainardi–Codazzi*[7] *equations*. Let us give an alternative derivation of these equations without using matrix form of the compatibility condition. A differentiation of (1.9.2) in u^k gives

$$\begin{aligned} \partial_k \partial_j \mathbf{r}_i &= \partial_k \Gamma^l_{ij} \mathbf{r}_l + \Gamma^l_{ij} \partial_k \mathbf{r}_l + \partial_k b_{ij} \mathbf{n} + b_{ij} \partial_k \mathbf{n} \\ &= \partial_k \Gamma^l_{ij} \mathbf{r}_l + \Gamma^l_{ij} \left(\Gamma^s_{kl} \mathbf{r}_s + b_{kl} \mathbf{n} \right) + \partial_k b_{ij} \mathbf{n} - b_{ij} a^l_k \mathbf{r}_l. \end{aligned}$$

In the double sum $\Gamma^l_{ij}\Gamma^s_{kl}\mathbf{r}_s$ we can exchange the notations for the summation indices $l \leftrightarrow s$:

$$\Gamma^l_{ij}\Gamma^s_{kl}\mathbf{r}_s = \Gamma^s_{ij}\Gamma^l_{ks}\mathbf{r}_l.$$

In this way we arrive at

$$\partial_k \partial_j \mathbf{r}_i = \left(\partial_k \Gamma^l_{ij} + \Gamma^s_{ij}\Gamma^l_{ks} - b_{ij} a^l_k \right) \mathbf{r}_l + \left(\Gamma^l_{ij} b_{lk} + \partial_k b_{ij} \right) \mathbf{n}.$$

[7] In the chronological order.

Due to equality of mixed derivatives the last expression must be symmetric in k and j. Writing such a symmetry explicitely and collecting the coefficients of the basic vectors $\mathbf{r}_l$ and $\mathbf{n}$ we obtain

$$\partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^s_{ij}\Gamma^l_{ks} - \Gamma^s_{ik}\Gamma^l_{js} = b_{ij} a^l_k - b_{ik} a_j = g^{ls}(b_{ij}b_{ks} - b_{ik}b_{js}) \tag{1.9.17}$$

and (1.9.16). Now we redenote $l \to t$ in (1.9.17), multiply both sides by g_{lt} and perform summation in t. Due to the identity $g_{lt}g^{ts} = \delta^s_l$ we arrive at (1.9.15).

The following *Bonnet theorem* says that system of Gauss and Codazzi equations suffices for local reconstruction of the surface.

Theorem 1.9.6 *Let the symmetric matrix valued function $b_{ij}(u,v)$ and a symmetric positive definite matrix valued function $g_{ij}(u)$ satisfy the Gauss and Peterson–Mainardi–Codazzi equations* (1.9.15), (1.9.16) *where the functions $\Gamma^k_{ij} = \Gamma^k_{ij}(u,v)$ are defined by eqs.* (1.9.4). *Then there exists a sufficiently small piece of a surface in the three-dimensional Euclidean space with the given first and second fundamental forms g_{ij} and b_{ij}.*

The proof of this Theorem uses the same scheme we used in the proof of the Proposition 1.9.4. It will not be given here.

Corollary 1.9.7

$$K = (g_{11}g_{22} - g_{12}^2)^{-1} g_{2t}\left(\frac{\partial \Gamma^t_{11}}{\partial u^2} - \frac{\partial \Gamma^t_{12}}{\partial u^1} + \Gamma^s_{11}\Gamma^t_{2s} - \Gamma^s_{12}\Gamma^t_{1s}\right). \tag{1.9.18}$$

Proof: Specializing the values of indices

$$i = j = 1, \quad k = l = 2$$

one obtains in the right hand side of (1.9.15)

$$b_{11}b_{22} - b_{12}^2 = K\,(g_{11}g_{22} - g_{12}^2).$$

This proves (1.9.18). □

The formula (1.9.18) can be represented in the following way. For any vector field tangent to the surface

$$\mathbf{z} = a(u,v)\mathbf{r}_u + b(u,v)\mathbf{r}_v$$

define the *covariant derivatives* $\nabla_u \mathbf{z}$ and $\nabla_v \mathbf{z}$ by

$$\nabla_u \mathbf{z} = \mathrm{pr}\,\frac{\partial \mathbf{z}}{\partial u}, \quad \nabla_v \mathbf{z} = \mathrm{pr}\,\frac{\partial \mathbf{z}}{\partial v}. \tag{1.9.19}$$

Here pr is the projection of a vector onto the tangent plane to the surface at the point (u,v)

$$\mathrm{pr}\,\mathbf{x} = \mathbf{x} - \langle \mathbf{x}, \mathbf{n}\rangle\,\mathbf{n}. \tag{1.9.20}$$

Exercise 1.9.8 *Prove that*

$$\nabla_{u^i}\mathbf{r}_{u^j} = \Gamma_{ij}^k \mathbf{r}_{u^k}, \quad i,\, j = 1,\, 2. \tag{1.9.21}$$

Thus, the covariant derivatives $\nabla_u \mathbf{z}$, $\nabla_v \mathbf{z}$ of any tangent vector field

$$\mathbf{z} = a^i(u,v)\mathbf{r}_i$$

can be computed, in the same basis $\mathbf{r}_u$, $\mathbf{r}_v$ via the coefficients $a^i(u,v)$ and their derivatives and the coefficients of the first fundamental form and their derivatives.

Exercise 1.9.9 *Prove that the formula* (1.9.18) *can be written in the form*

$$K = \frac{\langle (\nabla_v \nabla_u - \nabla_u \nabla_v)\mathbf{r}_u, \mathbf{r}_v \rangle}{\det G}. \tag{1.9.22}$$

Corollary 1.9.10 (Gauss' **Theorema Egregium**) *The Gaussian curvature of a surface can be determined by measurements of lengths of curves and angles between curves on the surface.*

Another reformulation of *Theorema Egregium* can be done as follows. A map

$$(u,v) \mapsto (u'(u,v), v'(u,v)) \tag{1.9.23}$$

of a surface $\mathbf{r}(u,v)$ to another surface $\mathbf{r}'(u',v')$ is called *isometry* if it maps the induced metric of the first surface to the induced metric of the second one (cf. the infinitesimal version (1.3.35) of the definition of isometries of Euclidean spaces).

Definition 1.9.11 *A geometric characteristic of a two-dimensional surfaces in the three-dimensional Euclidean space is called* extrinsic *if it is invariant with respect to isometries of the space. The characteristic is called* intrinsic *if it remains invariant with respect to isometries of the surfaces.*

For example, the Gaussian and mean curvatures of the surface are extrinsic invariants of the surface. According to Gauss' theorem the Gaussian curvature is an intrinsic invariant. Clearly any intrinsic characteristic of surfaces is also extrinsic, but not vice versa.

Example 1. Consider the cylinder of radius R,

$$\mathbf{r} = (R\cos\phi, R\sin\phi, h).$$

The induced metric on the cylinder written in the cylindrical coordinates (ϕ, h) has the form

$$ds^2 = R^2 d\phi^2 + dh^2.$$

The map

$$(\phi, h) \mapsto (u = R\,\phi, v = h) \tag{1.9.24}$$

establishes an isometry of the cylinder with the Euclidean plane with the standard metric

$$ds^2 = du^2 + dv^2.$$

The curvatures of the normal sections of the cylinder by the planes $z = 0$ and $y = 0$ are equal to $-\frac{1}{R}$ and 0 respectively. It is easy to see that these are the principal curvatures of the surface. So

$$K = 0, \quad H = -\frac{1}{2R}.$$

Thus the isometry (1.9.24) does not preserve the mean curvature.

Observe that curves in Euclidean spaces do not have intrinsic invariants. Indeed, introducing the length parameter on a regular curve one establishes a local isometry of the curve with the straight line.

According to *Theorema Egregium* vanishing of Gaussian curvature of the metric is a necessary condition for existence of an isometry of the surface with the Euclidean plane. The proof of sufficiency of this condition is outlined in the Exercise 1.10.12 below.

Example 2. Let us consider the induced metric

$$ds^2 = R^2(d\psi^2 + \sinh^2\psi\, d\phi^2) \tag{1.9.25}$$

on the "pseudo-sphere"

$$t^2 - x^2 - y^2 = R^2$$

in the three-dimensional Minkowski space. Denote

$$u^1 = \phi, \quad u^2 = \psi.$$

An easy calculation gives

$$\Gamma^2_{11} = -\frac{1}{2}\sinh 2\psi, \quad \Gamma^1_{12} = \Gamma^1_{21} = \coth\psi,$$

other Christoffel coefficients vanish. Hence

$$g_{2t}\left(\frac{\partial\Gamma^t_{11}}{\partial u^2} - \frac{\partial\Gamma^t_{12}}{\partial u^1} + \Gamma^s_{11}\Gamma^t_{2s} - \Gamma^s_{12}\Gamma^t_{1s}\right) = g_{22}\left(\frac{\partial\Gamma^2_{11}}{\partial u^2} - \Gamma^1_{12}\Gamma^2_{11}\right) = -R^2\sinh^2\psi.$$

Dividing by $\det G = R^4\sinh^2\psi$ one obtains the Gaussian curvature of the metric (1.9.25)

$$K = -\frac{1}{R^2}. \tag{1.9.26}$$

We see that the "pseudo-sphere" in the Minkowski space is an analogue of the sphere in the Euclidean space: the Gaussian curvature of this surface is constant, but the constant is *negative*. In particular it follows that the pseudo-sphere is not isometric to the Euclidean plane.

In differential geometry an important object is the *isometry group* of a given space with a metric. For Euclidean spaces this group has been studied in Section 1.3. It is isomorphic to the group of affine transformations $x \mapsto Ax + b$ with an orthogonal matrix A.

In the three-dimensional Euclidean space the transformations of orthogonal group $O(3)$ leave invariant the sphere of a radius R with the center at the origin. Clearly this action defines an isometry of the sphere to itself. The subgroup $SO(3) \subset O(3)$ also acts on the sphere and , hence it defines a subgroup of isometries of the sphere.

Exercise 1.9.12 *Prove that the Lorentz group $SO_\uparrow(1,2)$ acts by isometries on the pseudo-sphere.*

There is another important notion of *isometric deformations*. One says that the surface $\mathbf{r}^1(u,v)$ is obtained from the surface $\mathbf{r}^0(u,v)$ by an isometric deformation if there exists a family of surfaces

$$\begin{aligned}&\mathbf{r} = \mathbf{r}(u,v;t), \quad t \in [0,1]\\ &\mathbf{r}(u,v;0) = \mathbf{r}^0(u,v), \quad \mathbf{r}(u,v;1) = \mathbf{r}^1(u,v)\end{aligned}$$

depending on the parameter t such that the first fundamental form of the surfaces of the family does not depend on t.

Example 3. Consider the following family of surfaces depending on the parameter t

$$\begin{aligned}x(u,v;t) &= \quad \cos t\, \sinh v\, \sin u + \sin t\, \cosh v\, \cos u\\ y(u,v;t) &= -\cos t\, \sinh v\, \cos u + \sin t\, \cosh v\, \sin u\\ z(u,v;t) &= u\, \cos t + v\, \sin t.\end{aligned} \tag{1.9.27}$$

For $t = 0$ the surface becomes the left-handed helicoid (see Exercise 1.10.9 below), for $t = \frac{\pi}{2}$ it becomes catenoid (see Exercise 1.10.7 below) and for $t = \pi$ it becomes a right-handed helicoid. An easy computation yields that the first fundamental form of the family of surfaces depending on the parameter t does not depend on t:

$$ds^2 = \cosh^2 v\,(du^2 + dv^2). \tag{1.9.28}$$

So, all the surfaces (1.9.27) are isometric for all values of the parameter t. However the second fundamental form of these surfaces does depend on the parameter t (see Exercise 1.10.10 below) and, hence they cannot be obtained one from another by an isometry of $\mathbb{R}^3$ for different values of the parameter. Animation of the isometric deformation of a catenoid to helicoids can be found in

1.10 Exercises to Chapter 1

Exercise 1.10.1 *Prove that the rank of the Jacobi matrix of a smooth map $f : X \to Y$ of Euclidean spaces at a given point $P \in X$ does not depend on the choice of (local) coordinates.*

Exercise 1.10.2 *Prove that the area of a closed non-selfintersecting curve*

$$\begin{aligned}x &= x(\alpha)\\ y &= y(\alpha)\\ (x(0), y(0)) &= (x(2\pi), y(2\pi)).\end{aligned}$$

where $x(\alpha)$, $y(\alpha)$ are smooth 2π-periodic functions can be computed by the following integral

$$A = \int_0^{2\pi} x(\alpha)\, dy(\alpha).$$

Exercise 1.10.3 *Derive the following formula for the curvature of an arbitrary regular smooth curve* $(x(t), y(t))$ *on the plane*

$$k(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}. \tag{1.10.1}$$

In the particular case of graph of a smooth function (1.4.5) *prove that*

$$k = \frac{|f''|}{\left(1 + f'^2\right)^{3/2}}. \tag{1.10.2}$$

Exercise 1.10.4 *A closed smooth curve* $\gamma : (x = x(s), y = y(s))$ *on the plane with never vanishing curvature is called an* oval *if it has at most two intersection points with any straight line. Denote* $k = k(s)$ *the curvature of this curve as function of the length.*

1) Prove that for such a curve

$$\oint_\gamma \frac{dk}{ds} ds = \oint_\gamma x(s) \frac{dk}{ds} ds = \oint_\gamma y(s) \frac{dk}{ds} ds = 0. \tag{1.10.3}$$

Hint*: use Frenet–Serret formulae written in the form*

$$\begin{aligned} \frac{d^2 x(s)}{ds^2} &= -k(s) \frac{dy}{ds} \\ \frac{d^2 y(s)}{ds^2} &= \quad k(s) \frac{dx}{ds}. \end{aligned}$$

A point of the curve is called vertex *if* $\frac{dk(s)}{ds} = 0$ *at this point.*

2) Prove that there are at least four vertices on any oval.

Hint*: assuming the curvature not to be constant denote* $P_{\min}$ *and* $P_{\max}$ *the points of the curve where the curvature attains its maximum and minimum values. Consider the line*

$$ax + by + c = 0$$

passing through these two points. Assuming the curvature has no other maxima/minima on the curve prove that the integral

$$\oint_\gamma (ax(s) + by(s) + c) \frac{dk(s)}{ds} ds$$

cannot be equal to zero. This contradicts to (1.10.3).

3) Find vertices on an ellipse.

Exercise 1.10.5 *Denote*

$$\alpha(s) = \int_0^s k(s)\, ds. \tag{1.10.4}$$

Prove that the plane curve

$$\mathbf{r}(s) = \int_0^s (\cos \alpha(s), \sin \alpha(s))\; ds$$

has the curvature $k(s)$ *as a function of the length parameter* s.

Exercise 1.10.6 *Prove that the angle α defined in (1.10.4) can be used in order to parametrize a closed smooth non-selfintersecting curve*

$$\begin{aligned} x &= x(\alpha) \\ y &= y(\alpha) \\ (x(0), y(0)) &= (x(2\pi), y(2\pi)). \end{aligned}$$

with never vanishing curvature.

Denote $k = k(\alpha)$ the curvature of this curve. It is a smooth 2π-periodic function. Consider the Fourier expansion of the function $1/k(\alpha)$

$$\frac{1}{k(\alpha)} = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\alpha + b_m \sin m\alpha) \tag{1.10.5}$$

where, as it follows from the general theory of Fourier series

$$a_m = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos m\alpha}{k(\alpha)} d\alpha, \quad b_m = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin m\alpha}{k(\alpha)} d\alpha.$$

1) Prove that the length L of the closed curve is given by

$$L = \pi\, a_0.$$

2) Prove that $a_1 = b_1 = 0$.

Hint: use the result of Exercise 1.10.5.

3) Prove that the area A of the domain bounded by the curve is equal to

$$A = \frac{\pi a_0^2}{4} - \frac{\pi}{2} \sum_{m=2}^{\infty} \frac{a_m^2 + b_m^2}{m^2 - 1}.$$

Hint: use the results of Exercises 1.10.2 and 1.10.5.

4) Derive the isoperimetric inequality

$$4\pi\, A \le L^2. \tag{1.10.6}$$

Prove that the equality takes place iff *the curve is a circle.*

Exercise 1.10.7 *Derive the following formulae for the curvature and torsion of a parametrized space curve $\mathbf{r} = \mathbf{r}(t)$*

$$\begin{aligned} k &= \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3} \\ \kappa &= \frac{\langle \dot{\mathbf{r}} \times \ddot{\mathbf{r}}, \dddot{\mathbf{r}} \rangle}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2}. \end{aligned} \tag{1.10.7}$$

Exercise 1.10.8 *Prove that the graph*

$$z = \log \frac{\cos x}{\cos y}, \quad -\frac{\pi}{2} < x, \, y < \frac{\pi}{2}$$

*is a minimal surface (*Scherk surface*).*

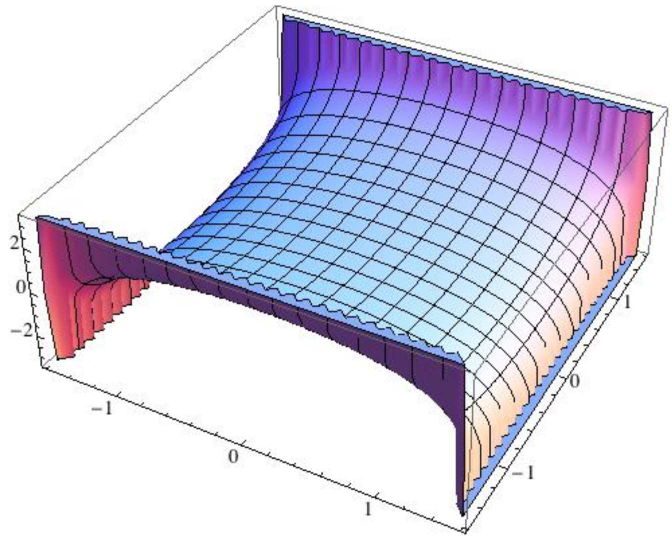


Fig. 4. Scherk minimal surface

Exercise 1.10.9 *Compute the first and the second fundamental forms of the* helicoid

$$\mathbf{r} = (u\,\cos v, u\,\sin v, a\,v). \tag{1.10.8}$$

Here a is a constant. Prove that the mean curvature H is identically equal to zero.

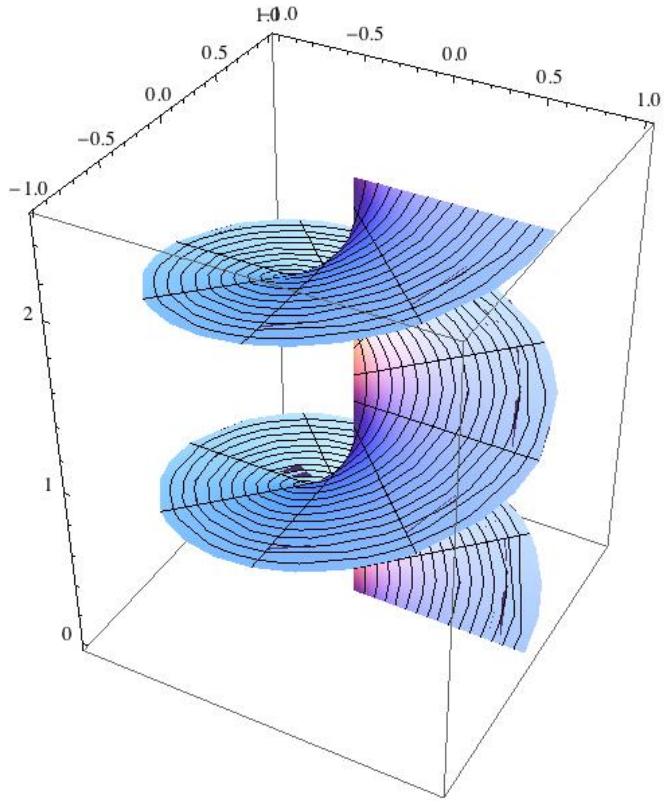


Fig. 5. Helicoid is a minimal surface

Exercise 1.10.10 *Prove that the second fundamental form of the family of surfaces* (1.9.27) *is equal to*

$$\sin t\,(dv^2 - du^2) + 2\cos t\,du\,dv.$$

Prove that the principal curvatures of any surface of the family are

$$k_{1,2} = \pm\frac{1}{\cosh^2 v}.$$

Exercise 1.10.11 *A* surface of revolution *is obtained by rotation of the curve*

$$x = \rho(z)$$

around the axis Oz.

1) Find a parametric representation of the surface.

2) Compute the first and the second fundamental forms of the surface.

3) Prove that the Gaussian and mean curvatures of the surface are equal to

$$K = -\frac{\rho''}{\rho\,(1+\rho'^2)^2}, \quad H = \frac{1}{2}\frac{\rho\,\rho'' - (1+\rho'^2)}{\rho\,(1+\rho'^2)^{3/2}}. \tag{1.10.9}$$

4) Prove that the surface of revolution with vanishing mean curvature is a catenoid, *i.e., a surface obtained by rotation of the "catenary curve"*

$$x = \frac{1}{a}\cosh(az + b)$$

around the axis z.

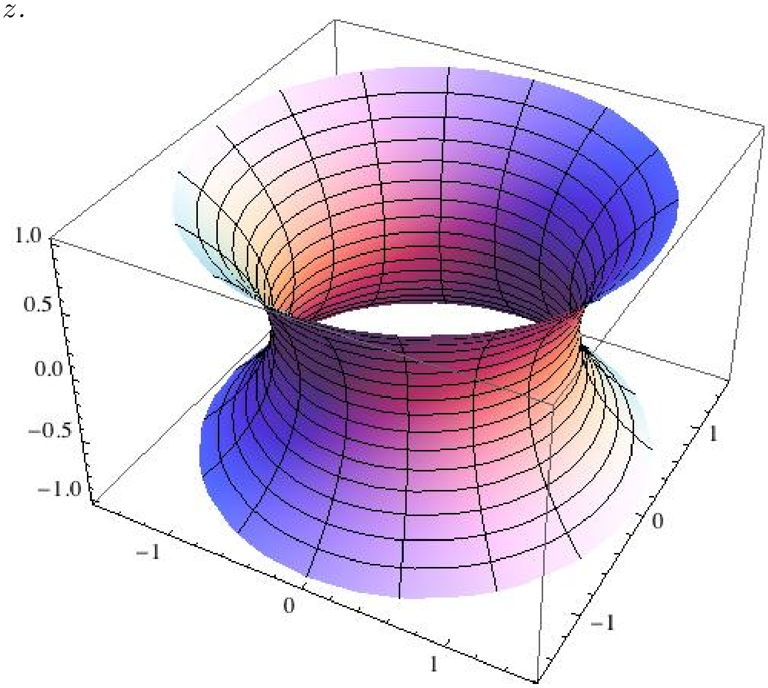


Fig. 6. Catenoid is the only one minimal surface of revolution

5) Prove that a surface of revolution with vanishing Gaussian curvature must be a cylinder $\rho(z) = \text{const}$.

6) Prove that the surfaces of revolution with constant Gaussian curvature $K \neq 0$ can be determined by an elliptic quadrature

$$\int \frac{\sqrt{K(\rho^2 - c)}}{\sqrt{1 - K(\rho^2 - c)}}\, d\rho = z - z_0. \tag{1.10.10}$$

Here c, z_0 are integration constants.

7) For constant negative Gaussian curvature $K = -1/R^2$ derive the following Beltrami surface *obtained by rotation of the curve*

$$|z - z_0| = R\left[\log(R^2 + R\sqrt{R^2 - x^2}) - \log R\, x\right] - \sqrt{R^2 - x^2}, \quad 0 < x \le R \tag{1.10.11}$$

around the axis Oz.

Hint: choose the particular value $c = R^2$ of the integration constant in (1.10.10).

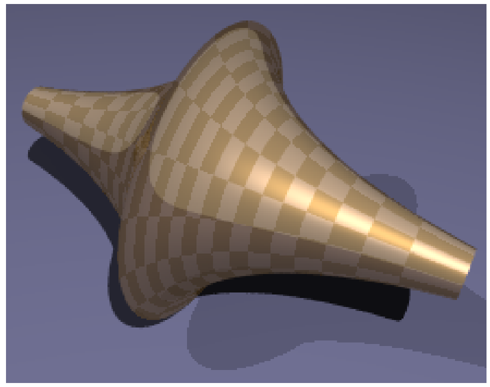


Fig. 7. Beltrami surface of revolution of constant negative curvature

Observe that the Beltrami surface has a singularity on the circle

$$x^2 + y^2 = R^2, \quad z = z_0.$$

Exercise 1.10.12 *Let the Gaussian curvature of the metric $ds^2 = g_{ij} du^i dv^i$ identically vanish.*

1) Prove that the system of equations

$$\frac{\partial^2 f}{\partial u^i \partial u^j} = \Gamma^k_{ij} \frac{\partial f}{\partial u^k}, \quad i, j = 1, 2 \tag{1.10.12}$$

for the function $f = f(u, v)$ has a unique solution with arbitrary initial data at a given point (u_0, v_0)

$$f(u_0, v_0) = f^0, \quad \frac{\partial f(u_0, v_0)}{\partial u^i} = f^0_i, \quad i = 1, 2.$$

Hint: rewrite (1.10.12) as a first order system for the vector valued function (f, f_u, f_v). Check that vanishing of the Gaussian curvature provides validity of the compatibility conditions for this system.

2) Choose two solutions $f^1(u, v)$, $f^2(u, v)$ to the system (1.10.12) *with the initial data $f^1(u_0, v_0) = f^2(u_0, v_0) = 0$, and the initial values of the derivatives are chosen in such a way that*

$$g^{ij}(u_0, v_0) f^p_i(u_0, v_0) f^q_j(u_0, v_0) = \delta^{pq}, \quad p, q = 1, 2. \tag{1.10.13}$$

Prove that these functions can be locally chosen as a new system of coordinates on the surface

$$\tilde{u} = f^1(u, v), \quad \tilde{v} = f^2(u, v).$$

Prove that in these coordinates the metric takes the Euclidean form

$$ds^2 = d\tilde{u}^2 + d\tilde{v}^2.$$

Exercise 1.10.13 *Let the induced metric on the surface have a diagonal form*

$$ds^2 = h_1^2 (du^1)^2 + h_2^2 (du^2)^2, \quad h_i = h_i(u^1, u^2), \quad i = 1, 2. \tag{1.10.14}$$

Prove that the Gauss formula (1.9.18) *reduces to*

$$K = -\frac{1}{h_1 h_2}\left[\left(\frac{h_{1,2}}{h_2}\right)_2 + \left(\frac{h_{2,1}}{h_1}\right)_1\right]. \tag{1.10.15}$$

Here

$$h_{1,2} = \frac{\partial h_1}{\partial u^2}, \quad h_{2,1} = \frac{\partial h_2}{\partial u^1},$$

the subscripts $(\ )_1$ *and* $(\ )_2$ *denote partial derivatives in* u^1 *and* u^2 *respectively.*

Exercise 1.10.14 *A curve on the surface is called a* curvature line *if its velocity vector is an eigenvector of the shape operator* $A = G^{-1}B$*:*

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}\begin{pmatrix} \dot u \\ \dot v \end{pmatrix} = k\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}\begin{pmatrix} \dot u \\ \dot v \end{pmatrix}, \quad k = k_1 \quad \text{or} \quad k = k_2. \tag{1.10.16}$$

Let the principal curvatures k_1, k_2 *of a surface be distinct at a point* (u_0, v_0).

1) Prove that there exist two families of curvature lines near the point (u_0, v_0) *that can be determined from the differential equation*

$$a_1^2\, du^2 + (a_2^2 - a_1^1) du\, dv - a_2^1\, dv^2 = 0 \tag{1.10.17}$$

or, equivalently, under assumption $a_1^2 \neq 0$,

$$2a_1^2\, du + \left[a_2^2 - a_1^1 \pm (k_1 - k_2)\right] dv = 0. \tag{1.10.18}$$

Here $a_j^i = a_j^i(u, v)$ *are the matrix entries of the shape operator* A.

2) Let

$$f_+(u, v) = C_+, \quad f_-(u, v) = C_-$$

be first integrals of eqs. (1.10.40) *with gradients non-vanishing at the point* (u_0, v_0). *Prove that the functions*

$$\tilde u = f_+(u, v)$$
$$\tilde v = f_-(u, v)$$

are independent coordinates on the surface near the point (u_0, v_0). *Prove that, in these coordinates both the first and the second fundamental forms of the surface become diagonal.*

Exercise 1.10.15 *A curve on a surface of negative Gaussian curvature is called* asymptotic *if its velocity vector* $(\dot u^1, \dot u^2)$ *is a null-vector of the second fundamental form*

$$b_{ij}\dot u^i \dot u^j = 0 \tag{1.10.19}$$

at every point.

1) Prove that there are two asymptotic curves passing through any point of a surface of negative curvature.

2) Let the Gaussian curvature of the surface be equal to $K = -1$. *Prove that one can use the length parameters* p, q *on the asymptotic lines as local coordinates on the surface. Prove that, in these coordinates the metric becomes equal to*

$$ds^2 = dp^2 + 2\cos\varphi\, dp\, dq + dq^2 \tag{1.10.20}$$

where $\varphi = \varphi(p, q)$ *is the angle between the asymptotic lines while the second fundamental form reduces to*

$$2\sin\varphi\, dp\, dq. \tag{1.10.21}$$

3) Compute the Christoffel coefficients of the metric (1.10.42)

$$\begin{aligned}
&\Gamma^1_{11} = \varphi_p \cot\varphi, \quad \Gamma^2_{11} = -\frac{\varphi_p}{\sin\varphi}\\
&\Gamma^1_{12} = \Gamma^2_{12} = 0\\
&\Gamma^1_{22} = -\frac{\varphi_q}{\sin\varphi}, \quad \Gamma^2_{22} = \varphi_q \cot\varphi.
\end{aligned}$$

Prove that the Gauss equations for the first and second fundamental forms (1.10.42), (1.10.43) *reduce to the* Sine-Gordon equation

$$\varphi_{pq} = \sin\varphi. \tag{1.10.22}$$

Exercise 1.10.16 *Coordinates* (u, v) *on a surface are called* isothermal *if the induced metric is proportional to the metric of Euclidean plane:*

$$ds^2 = e^{2\varphi}(du^2 + dv^2), \quad \varphi = \varphi(u, v). \tag{1.10.23}$$

1) Introducing complex coordinates

$$z = u + iv, \quad \bar z = u - iv$$

rewrite the metric (1.10.23) *in the complex form*

$$ds^2 = e^{2\varphi} dz\, d\bar z.$$

2) Consider changes of coordinates

$$(u, v) \mapsto (u', v')$$

defined by holomorphic functions

$$z = u + iv \mapsto z' = u' + iv' = f(z), \quad \frac{\partial f}{\partial \bar z} = 0. \tag{1.10.24}$$

Prove that the new coordinates (u', v') *are also isothermal,*

$$ds^2 = e^{2\varphi'}(du'^2 + dv'^2)$$

and

$$\varphi = \varphi' + \log\left|\frac{df}{dz}\right| \equiv \varphi' + \frac{1}{2}\left(\log\frac{df}{dz} + \log\overline{\frac{df}{dz}}\right). \tag{1.10.25}$$

3) Prove that a surface equipped with a system of isothermal coordinates has a constant curvature K iff *the function φ satisfies the* Liouville equation

$$\Delta\varphi = -K\, e^{2\varphi}. \tag{1.10.26}$$

Here

$$\Delta\varphi = \varphi_{uu} + \varphi_{vv}$$

is the Laplace operator.

4) Recast the Liouville equation into the following form

$$\frac{\partial^2\varphi}{\partial z \partial \bar{z}} = -\frac{K}{4} e^{2\varphi}. \tag{1.10.27}$$

Here we use the standard notations for the complex derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right).$$

Derive directly from (1.10.27) *that the Liouville equation is invariant under transformations* (1.10.24), (1.10.25).

5) For the surfaces of zero Gaussian curvature prove that the metric can be reduced to the Euclidean form by transformations of the form (1.10.24), (1.10.25)

Hint: use the representation $\varphi = \frac{1}{2}(g(z) + \overline{g(z)})$ of any harmonic function $\Delta\varphi = 0$ in terms of an analytic function $g(z)$.

6) Derive the following formulae for the v-independent solutions to the Liouville equation, $\varphi = \varphi(u)$

$$e^{2\varphi} = \frac{a^2 R^2}{\cosh^2(au + b)}, \quad K = \frac{1}{R^2} > 0 \tag{1.10.28}$$

$$e^{2\varphi} = \frac{a^2 R^2}{\sinh^2(au + b)}, \quad K = -\frac{1}{R^2} < 0 \tag{1.10.29}$$

where $a \neq 0$, b are integration constants. Reduce the metrics (1.10.28), (1.10.29) *to the form*

$$ds^2 = \frac{4R^2}{(1 + |w|^2)^2} dw\, d\bar{w}, \quad K > 0 \tag{1.10.30}$$

$$ds^2 = \frac{4R^2}{(1 - |w|^2)^2} dw\, d\bar{w}, \quad K > 0 \tag{1.10.31}$$

by the substitution $w = e^{az+b}$. Derive the following family of solutions to Liouville equation

$$\varphi(u, v) = \log \frac{2R|f'(z)|}{1 \pm |f(z)|^2}, \quad K = \pm\frac{1}{R^2} \tag{1.10.32}$$

where $f(z)$ is an arbitrary holomorphic function of $z = u + iv$.

The subsequent arguments can be used to prove that the formulae (1.10.32) give the general solution to the Liouville equation.

7) Let φ be a solution to the Liouville equation. Prove that

$$\omega := \varphi_{zz} - \varphi_z^2 \tag{1.10.33}$$

is a holomorphic function of z

$$\frac{\partial \omega}{\partial \bar z} = 0.$$

8) Prove that, under holomorphic changes of coordinates (1.10.24) *the function $\omega(z) \mapsto \omega'(z') = \varphi'_{z'z'} - (\varphi'_{z'})^2$ that can be computed from the following transformation law*

$$\omega'(z') = f_z^{-2}\left[\omega(z) - \frac{1}{2}\left(\frac{f_{zzz}}{f_z} - \frac{3}{2}\frac{f_{zz}^2}{f_z^2}\right)\right]. \tag{1.10.34}$$

9) Prove that the solutions to the Schwarzian equation

$$\frac{f_{zzz}}{f_z} - \frac{3}{2}\frac{f_{zz}^2}{f_z^2} = 2\omega(z) \tag{1.10.35}$$

can be represented by a ratio

$$f = \frac{\psi_1}{\psi_2}$$

of two linearly independent solutions to the following second order linear differential equation

$$\frac{d^2\psi}{dz^2} + \omega(z)\,\psi = 0. \tag{1.10.36}$$

The above arguments show that any metric of constant Gaussian curvature in isothermal coordinates can be reduced to one with vanishing $\omega'(z') = \varphi'_{z'z'} - (\varphi'_{z'})^2 = 0$. In sequel we will omit the primes.

10) Let φ be a solution to the Liouville equation satisfying $\varphi_{zz} - \varphi_z^2 = 0$. Prove that the real valued function $e^{-\varphi}$ satisfies

$$\frac{\partial^2}{\partial z^2}e^{-\varphi} = \frac{\partial^2}{\partial \bar z^2}e^{-\varphi} = 0$$

and, hence, the metric must have the form

$$ds^2 = e^{2\varphi}dz\,d\bar z = \frac{dz\,d\bar z}{(az\bar z + b\,\bar z + \bar b\,z + c)^2}. \tag{1.10.37}$$

Here a and c are real constants, b is a complex constant. Compute the curvature of this metric and prove that

$$K = 4(ac - |b|^2).$$

11) Doing a fractional-linear transformation

$$w = \frac{\alpha\, z' + \beta}{\gamma\, z' + \delta}$$

reduce the metric to one of the following canonical forms

$$\begin{aligned} ds^2 &= \frac{4R^2 dw\, d\bar{w}}{(1+|w|^2)^2}, \quad & K &= \frac{1}{R^2} > 0 \\ ds^2 &= dw\, d\bar{w}, & K &= 0 \\ ds^2 &= \frac{4R^2 dw\, d\bar{w}}{(1-|w|^2)^2}, & K &= -\frac{1}{R^2} < 0. \end{aligned}$$

Exercise 1.10.17 *A curve on the surface is called a* curvature line *if its velocity vector is an eigenvector of the shape operator $A = G^{-1}B$:*

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = k \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}, \quad k = k_1 \quad \text{or} \quad k = k_2. \tag{1.10.38}$$

Let the principal curvatures k_1, k_2 of a surface be distinct at a point (u_0, v_0).

1) Prove that there exist two families of curvature lines near the point (u_0, v_0) that can be determined from the differential equation

$$a_1^2\, du^2 + (a_2^2 - a_1^1) du\, dv - a_2^1\, dv^2 = 0 \tag{1.10.39}$$

or, equivalently, under assumption $a_1^2 \neq 0$,

$$2a_1^2\, du + \left[a_2^2 - a_1^1 \pm (k_1 - k_2)\right] dv = 0. \tag{1.10.40}$$

Here $a_j^i = a_j^i(u, v)$ are the matrix entries of the shape operator A.

2) Let

$$f_+(u, v) = C_+, \quad f_-(u, v) = C_-$$

be first integrals of eqs. (1.10.40) *with gradients non-vanishing at the point (u_0, v_0). Prove that the functions*

$$\begin{aligned} \tilde{u} &= f_+(u, v) \\ \tilde{v} &= f_-(u, v) \end{aligned}$$

are independent coordinates on the surface near the point (u_0, v_0). Prove that, in these coordinates both the first and the second fundamental forms of the surface become diagonal.

Exercise 1.10.18 *A curve on a surface of negative Gaussian curvature is called* asymptotic *if its velocity vector $(\dot{u}^1, \dot{u}^2)$ is a null-vector of the second fundamental form*

$$b_{ij}\dot{u}^i\dot{u}^j = 0 \tag{1.10.41}$$

at every point.

1) Prove that there are two asymptotic curves passing through any point of a surface of negative curvature.

2) Let the Gaussian curvature of the surface be equal to $K = -1$. Prove that one can use the length parameters p, q on the asymptotic lines as local coordinates on the surface. Prove that, in these coordinates the metric becomes equal to

$$ds^2 = dp^2 + 2\cos\varphi\, dp\, dq + dq^2 \tag{1.10.42}$$

where $\varphi = \varphi(p, q)$ is the angle between the asymptotic lines while the second fundamental form reduces to

$$2 \sin\varphi \, dp \, dq. \tag{1.10.43}$$

3) Compute the Christoffel coefficients of the metric (1.10.42)

$$\begin{aligned}
&\Gamma^1_{11} = \varphi_p \cot\varphi, \quad \Gamma^2_{11} = -\frac{\varphi_p}{\sin\varphi} \\
&\Gamma^1_{12} = \Gamma^2_{12} = 0 \\
&\Gamma^1_{22} = -\frac{\varphi_q}{\sin\varphi}, \quad \Gamma^2_{22} = \varphi_q \cot\varphi.
\end{aligned}$$

Prove that the Gauss equations for the first and second fundamental forms (1.10.42), (1.10.43) *reduce to the* Sine-Gordon equation

$$\varphi_{pq} = \sin\varphi. \tag{1.10.44}$$