Notes on Bethe Ansatz Techniques

Fabio Franchini*
SISSA,
Via Bonomea 265,
34136, Trieste, Italy

May 15, 2011

*E-mail: ffranchi@sissa.it

Contents

Introduction

These notes are the write-up of the lectures I gave for a class on "Introduction to Bethe Ansatz" within the Ph.D. program in Statistical Physics in SISSA (Trieste). They are still a work in progress, probably with many remaining typos. They are intended as a guidance to start the study of this extremely rich subject, by favoring a clear and physical introduction to its fundamental ideas, over many mathematical subtleties that populate its formulation. The emphasis on the physical intuition makes these notes suitable also for the scientist who mostly performs numerical simulations, but what to compare his/her results with exact ones, and to anyone who needs to start reading the literature on Bethe Ansatz. If you are reading these notes and you find mistakes, please let me know so that I can correct them for future readers.

My experience is that the material presented can be covered in some 10 lectures. It is based on a rielaboration of the following sources, whose authors I thank and which I recommend as further readings:

- B. Sutherland, *"Beautiful Models - 70 Years of Exactly Solved Quantum Many-Body Problems"*, World Scientific.

  (Beautiful introduction to the subject in very physical terms. However, its treatment of Bethe Ansatz and its focus are not standard.)

- M. Takahashi, *"Thermodynamics of One-Dimensional Solvable Models"*, Cambridge University Press.

  (Very detailed introduction to the coordinate Bethe ansatz approach and to the thermodynamics of the models, with explicit calculations and derivation that are easy to follow. However, it only touches superficially the most modern developments and it does not cover at all the Algebraic Bethe Ansatz.)

- V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, *"Quantum Inverse Scattering Method and Correlation Functions"*, Cambridge University Press.

  (Probably the most comprehensive, single account of Bethe Ansatz, but relatively hard to read because of its terse mathematical notation and the effort to keep maximum generality in the constructions of the most abstract concepts. It starts with the coordinate Bethe ansatz construction for several, fundamental models. It continues with the algebraic constructions and it introduces the most advanced techniques toward the calculation of correlation functions in integrable models.)

- B.S. Shastry, S.S. Jha, V. Singh, *"Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory"*, Lecture Notes in Physics, Springer-Verlag.

  (Collections of notes of the classes delivered during a summer school on Bethe Ansatz in the 80's. Extremely well done and still quite modern. However, it is hard to find and it typesetting makes it annoying to read –they did not have latex back then–.)

- R.J. Baxter, *"Exactly Solved Models in Statistical Mechanics*, Dover Publications.

  (Beautiful book, hard to read because everything is derived in excruciating detail. But if you have the patience to follow him, at the end of every chapter you have really understood something to a new level. This book is strictly not on Bethe Ansatz, but on two-dimensional classical models. However, the two subjects are very close cousins.)

- C. Gmez, M. Ruiz-Altaba, G. Sierra, *"Quantum Groups in Two-Dimensional Physics"*, Cambridge University Press.

  (Very recent book that tackles the Bethe Ansatz as emerging from quantum group structure. Despite the mathematical nature of the approach, the authors manage to keep the exposition at a very understandable level for physicists.)

- M. Jimbo (ed.), *"Yang-Baxter Equation in Integrable System"*, World Scientific.

  (Collections of seminal papers related to the Yang-Baxter equation.)

- L.D. Faddeev, *"How Algebraic Bethe Ansatz works for integrable model"*, Les-Houches lectures, arXiv:hep-th/9605187v1.

- L.D. Faddeev, *"Algebraic Aspects of Bethe-Ansatz"*, Int. J. Mod. Phys. **A 10**, 1845-1878 (1995) or arXiv:hep-th/9404013.

- A. Kundu, *"Quantum Integrable Systems: Construction, Solution, Algebraic Aspect"*, arXiv:hep-th/9612046.

- H.J. Schulz, *"Fermi liquids and non–Fermi liquids"*, Page 533 in "Proceedings of Les Houches Summer School LXI", ed. E. Akkermans, G. Montambaux, J. Pichard, et J. Zinn-Justin (Elsevier, Amsterdam, 1995) or arXiv:cond-mat/9503150.

- ...

Chapter 1

The XY Model

1.1 Introduction and motivations

The One-Dimensional XY model in a transverse magnetic field is arguably the simplest non-trivial integrable model. Because of this, in the years it has been extensively studied and used to capture the universal behavior of low dimensional systems. In recent years, there has been a renewed interest in the this model and especially in the entanglement of its ground state. This interest is justified in part by the tractability of the problem and in part by the fact laboratory realizations of this system are almost at hand using optical lattice systems and cold Fermi atoms.

In fact, the XY model has always been a great test study, because, despite its apparent simplicity, it has a rich two dimensional phase diagram characterized, at zero temperature, by two Quantum Phase Transitions (QPT): one of them belongs to the universality class of the critical Heisenberg spin (XX model) and the other is the phase transition of the One-Dimensional Quantum Ising model.

The model has been studied and solved in a series of papers and its fundamental correlation functions were calculated in [1]. More complicated correlators like the Emptiness Formation Probability [2, 3] and the Von Neumann entropy [4, 5] were calculated more recently . Essentially all the correlation functions of the model can be expressed as determinants of matrices with a special structure, known as Toeplitz matrices [6]. The asymptotic behavior of Toeplitz determinants can be studied using fairly sophisticated mathematical techniques or just by relying on known theorems, such as the Szegö Theorem, the Fisher-Hartwig conjecture, Widom's theorem and so on ...[7]

The Hamiltonian of the XY model can be written as

$$H = \frac{J}{2}\sum_{j=1}^{N}\left[\left(\frac{1+\gamma}{2}\right)\sigma_j^x\sigma_{j+1}^x + \left(\frac{1-\gamma}{2}\right)\sigma_j^y\sigma_{j+1}^y + h\,\sigma_j^z\right], \tag{1.1}$$

where σ_j^α, with $\alpha = x, y, z$, are the Pauli matrices which describe spin operators on the j-th lattice site of the spin chain. This Hamiltonian was firstly introduced and solved in the case of zero magnetic field by Lieb, Schultz and Mattis in [8] and in [9, 10] by Miemeijer with a finite external field.

The phase diagram of this model is parametrized by the *anisotropy parameter* γ and by the *external magnetic field* h, directed along the transverse z-axis. We take these parameters to be dimensionless and from now on we set the energy-scale defining parameter $J = -1$ (For now we shall consider the *ferromagnetic* case only.). The model has obvious symmetries for $\gamma \to -\gamma$ and $h \to -h$, so we will concentrate only on the portion of the phase diagram where $\gamma \geq 0$ and $h \geq 0$ and we will find two Quantum Phase Transitions (QPT) where the theory becomes gapless. These QPT are located on the line $\gamma = 0$, the isotropy line, and at the critical magnetic field $h = 1$.

There are several important subspaces of this phase-diagram. For $\gamma = 0$,

$$H = -\frac{1}{2}\sum_{j=1}^{N}\left[\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y + h\ \hat{S}_j^z\right], \tag{1.2}$$

the hamiltonian reduces to the isotropic XX model, i.e. the $\Delta = 0$ limit of the critical Heisenberg XXZ chain:

$$H = -\frac{1}{2}\sum_{j=1}^{N}\left[\hat{S}_j^x\hat{S}_{j+1}^x + \hat{S}_j^y\hat{S}_{j+1}^y + \Delta\ \hat{S}_j^z\hat{S}_{j+1}^z\right] - h\sum_{j=1}^{N}\hat{S}_j^z, \tag{1.3}$$

where $\hat{S}_j^\alpha = \sigma_j^\alpha/2$ are the spin operators. For $\gamma = 1$, we recover the **One-Dimensional Quantum Ising model**:

$$H = -\sum_{j=1}^{N}\hat{S}_j^x\hat{S}_{j+1}^x - h\sum_{j=1}^{N}\hat{S}_j^z. \tag{1.4}$$

There are other noticeable subspaces of the XY model, but we will show them as we encounter them.

Two competing universality classes exist in the XY model: the one of the isotropic XX model and the one of the Ising chain. We identify that the phase transition at the critical magnetic field $h = 1$ is an Ising transition. This is a transition from a doubly degenerated ground state (for $h < 1$) to a single ground state system (for $h > 1$). This is in analogy, with the classical two-dimensional Ising model, where it is well known that the critical temperature separates a region of vanishing order parameter at high temperatures from a region of spontaneously broken Z_2 symmetry at low temperatures where the order parameter can assume two opposite finite values. Since the mapping between the two-dimensional classical system and the one-dimensional quantum case is exact[1], the same kind of transition takes place at the critical magnetic field $h = 1$ of the Ising model and in the bulk of the XY model by extension, since the universality class is the same away from the point $\gamma = 1$.

In fact, the non-vanishing order parameter in the XY model for $h < 1$ is the magnetization along the x-axis and as long as no magnetic field is applied along the x-direction resolving the degeneracy, both a positive and a negative value for the order parameter are to be expected.

[1]This is, in fact, a general result relating D-dimensional quantum theories with the D+1-dimensional classical ones.

For the Ising model (1.4) at vanishing magnetic field ($h = 0$)

$$H = -\sum_{j=1}^{N} \hat{S}_j^x \hat{S}_{j+1}^x \,, \tag{1.5}$$

an explicit construction of the two degenerate ground states can be achieved very easily as they are the two possible ferromagnetic states polarized along the positive and negative x-direction:

$$\begin{aligned} |GS_1\rangle &= |\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\ \ldots\rangle \\ &= \prod_{j=1}^{N} \frac{1}{\sqrt{2}}\Big(|\uparrow_j\rangle + |\downarrow_j\rangle\Big), \end{aligned} \tag{1.6}$$

$$\begin{aligned} |GS_2\rangle &= |\leftarrow\leftarrow\leftarrow\leftarrow\leftarrow\ \ldots\rangle \\ &= \prod_{j=1}^{N} \frac{1}{\sqrt{2}}\Big(|\uparrow_j\rangle - |\downarrow_j\rangle\Big), \end{aligned} \tag{1.7}$$

where $|\uparrow_j\rangle$ ($|\downarrow_j\rangle$) indicates the state with positive (negative) projection of the spin along the z-axis at the j-th lattice point. In [13] it was claimed that in the Ising model the degeneracy is killed by a non vanishing magnetic field, but we will show that that is not the case. In fact, the factorized structure (1.7) for the degenerate ground states propagates on the line $\gamma^2 + h^2 = 1$, where an explicit form of the two ground states exists [12]:

$$\begin{aligned} |GS_1\rangle &= \prod_{j=1}^{N} \left(|\uparrow_j\rangle + \tan\theta\, |\downarrow_j\rangle \right), \\ |GS_2\rangle &= \prod_{j=1}^{N} \left(|\uparrow_j\rangle - \tan\theta\, |\downarrow_j\rangle \right) \end{aligned} \tag{1.8}$$

where $\cos^2(2\theta) = (1-\gamma)/(1+\gamma)$.

In section 1.2 we provide all the details for the diagonalization of the Hamiltonian. In section 1.3 we determined the excitation spectrum, calculate the partition function of the model and discuss the phase diagram of the model. In section 1.4 we calculate the correlation functions of the model and show that they are expressible as determinant of Toeplitz matrices.

1.2 Diagonalization of the Hamiltonian

We are going to diagonalize the Hamiltonian of the ferromagnetic XY model:

$$H = -\frac{1}{2} \sum_{j=1}^{N} \left[\left(\frac{1+\gamma}{2} \right) \sigma_j^x \sigma_{j+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_j^y \sigma_{j+1}^y + h\, \sigma_j^z \right], \tag{1.9}$$

and we impose periodic boundary conditions: $\sigma_{j+N}^{\alpha} = \sigma_j^{\alpha}$.

The XY spin model defined by (1.9) has been firstly solved in [8] in the case of zero magnetic field and in [10] in the presence of a magnetic field. The fundamental correlators of the model were then calculated in the extensive work of McCoy [1].

Following the standard prescription [8], we reformulate the Hamiltonian (1.9) in terms of spinless fermions ψ_i by means of a Jordan-Wigner transformation:

$$\begin{aligned}
\sigma_j^+ &= \mathrm{e}^{\mathrm{i}\pi\sum_{l<j}\psi_l^\dagger\psi_l}\,\psi_j = \prod_{l=1}^{j-1}\left(1-2\psi_l^\dagger\psi_l\right)\psi_j, \\
\sigma_j^- &= \psi_j^\dagger\,\mathrm{e}^{-\mathrm{i}\pi\sum_{l<j}\psi_l^\dagger\psi_l} = \prod_{l=1}^{j-1}\left(1-2\psi_l^\dagger\psi_l\right)\psi_j^\dagger, \\
\sigma_j^z &= 1-2\psi_j^\dagger\psi_j,
\end{aligned} \tag{1.10}$$

where, as usual, $\sigma^\pm = (\sigma^x \pm i\sigma^y)/2$:

$$\begin{aligned}
H &= -\frac{1}{2}\sum_{j=1}^{N-1}\left(\psi_j^\dagger\psi_{j+1}+\psi_{j+1}^\dagger\psi_j+\gamma\,\psi_j^\dagger\psi_{j+1}^\dagger+\gamma\,\psi_{j+1}\psi_j\right) \\
&\quad +\frac{\mu_N^x}{2}\left(\psi_N^\dagger\psi_1+\psi_1^\dagger\psi_N+\gamma\,\psi_N^\dagger\psi_1^\dagger+\gamma\,\psi_1\psi_N\right) \\
&\quad +h\sum_{j=1}^{N}\psi_j^\dagger\psi_j-\frac{hN}{2},
\end{aligned} \tag{1.11}$$

where[2]

$$\mu_N^x \equiv \prod_{j=1}^{N}\left(1-2\psi_j^\dagger\psi_j\right) = \prod_{j=1}^{N}\sigma_j^z\ . \tag{1.12}$$

The boundary terms on the second line of (1.11) are normally discarded, since their effect is meant to be negligible in the thermodynamic limit. In fact, they are important to established the degeneracy of the model below the phase transition.

For non-vanishing γ, the Hamiltonian (1.9) does not commute with σ^z and therefore (1.11) does not conserve the number of fermions. Nonetheless, since fermions are created/destroyed in pairs, the even/oddness of their number is conserved, i.e.

$$[\mu_N^x, H] = 0. \tag{1.13}$$

This observation allow us to separate the theory into two disconnected sectors with $\mu_N^x = \pm 1$, where the plus sign characterizes configurations with an even number of particles and the minus the one with odd number:

$$H \;=\; \frac{1+\mu_N^x}{2}\,H^+ \;+\; \frac{1-\mu_N^x}{2}\,H^-\ , \tag{1.14}$$

here $\frac{1\pm\mu_N^x}{2}$ are the projector operators to the states with even/odd number of particles and $H^\pm$ have the form (1.11) with $\mu_N^x = \pm 1$.

We take care of the boundary terms in (1.11) for the two sectors by applying the appropriate boundary conditions to the spinless fermions: for $\mu_N^x = +1$ (even number of particles) we have to

[2] We choose the symbol μ_N^x to represent this operator, according to the traditional notation for the dual lattice operators for the quantum Ising Model.

impose **anti-periodic boundary conditions** on the fermions and for $\mu_N^x = -1$ (odd number of particles) we require **periodic boundary conditions**:

$$\begin{aligned} \psi_{j+N} &= -\psi_j \qquad \text{for} \qquad \mu_N^x = +1 \\ \psi_{j+N} &= \psi_j \qquad \text{for} \qquad \mu_N^x = -1. \end{aligned} \tag{1.15}$$

With these definitions, we can write both the Hamiltonians in (1.14) in the compact form:

$$H^{\pm} = -\frac{1}{2}\sum_{j=1}^{N}\left(\psi_j^\dagger\psi_{j+1} + \psi_{j+1}^\dagger\psi_j + \gamma\,\psi_j^\dagger\psi_{j+1}^\dagger + \gamma\,\psi_{j+1}\psi_j - 2h\,\psi_j^\dagger\psi_j\right) - \frac{hN}{2}\,. \tag{1.16}$$

Let us emphasize this very important point in the derivation: despite the fact that we imposed periodic boundary conditions on the original spin system, the model is mapped into **two** spinless theories $H^{\pm}$ with the same Hamiltonian, but with different boundary conditions (1.15). The degeneracy originates from the fact that the ground states of the two theories happen to have the same energy in the thermodynamic limit (for $h < 1$).

We are now going to consider consider the two sectors of the theory separately:

1.2.1 Even number of particles

For $\mu_N^x = 1$ we have anti-periodic boundary conditions. Therefore, we perform an anti-periodic Fourier transform by defining the Fourier components as[3]

$$\psi_j = \frac{e^{i\pi/4}}{N}\sum_{q=0}^{N-1} e^{i\frac{2\pi}{N}\left(q+\frac{1}{2}\right)j}\psi_q \qquad j = 1\ldots N, \tag{1.17}$$

$$\psi_q = e^{-i\pi/4}\sum_{j=1}^{N} e^{-i\frac{2\pi}{N}\left(q+\frac{1}{2}\right)j}\psi_j \qquad q = 0\ldots N-1, \tag{1.18}$$

and the Hamiltonian in Fourier space reads:

$$\begin{aligned} H^+ = &\frac{1}{N}\sum_{q=0}^{N-1}\left\{h - \cos\left[\tfrac{2\pi}{N}\left(q+\tfrac{1}{2}\right)\right]\right\}\,\psi_q^\dagger\psi_q \\ &+\frac{\gamma}{2N}\sum_{q=0}^{N-1}\sin\left[\tfrac{2\pi}{N}\left(q+\tfrac{1}{2}\right)\right]\,\left\{\,\psi_q\psi_{-q-1} + \psi_{-q-1}^\dagger\psi_q^\dagger\,\right\} - \frac{hN}{2}\,, \end{aligned} \tag{1.19}$$

which can also be written as

$$\begin{aligned} H^+ = &\frac{1}{2N}\sum_{q=0}^{N-1}\left(\psi_q^\dagger;\psi_{-q-1}\right)\begin{pmatrix} h - \cos\left[\tfrac{2\pi}{N}\left(q+\tfrac{1}{2}\right)\right] & -\gamma\,\sin\left[\tfrac{2\pi}{N}\left(q+\tfrac{1}{2}\right)\right] \\ -\gamma\,\sin\left[\tfrac{2\pi}{N}\left(q+\tfrac{1}{2}\right)\right] & \cos\left[\tfrac{2\pi}{N}\left(q+\tfrac{1}{2}\right)\right] - h \end{pmatrix}\begin{pmatrix} \psi_q \\ \psi_{-q-1}^\dagger \end{pmatrix} \\ &-\frac{hN}{2}\,. \end{aligned} \tag{1.20}$$

[3]We prefer the asymmetric version of the Fourier transform, since it will make it easier to consider the thermodynamic. In fact, from $\{\psi_j, \psi_j^\dagger\} = \delta_{j,l}$ in real space it follows that in momentum space $\{\psi_q, \psi_k^\dagger\} = N\delta_{q,k} \overset{N\to\infty}{\to} \delta(q-k)$, where the last one is the Dirac delta function in the continuum.

We diagonalize this Hamiltonian by means of a Bogoliubov transformation, which, in our notation, is nothing but a simple rotation:

$$\begin{pmatrix} \psi_q \\ \psi^\dagger_{-q-1} \end{pmatrix} = \begin{pmatrix} \cos\vartheta_q & \sin\vartheta_q \\ -\sin\vartheta_q & \cos\vartheta_q \end{pmatrix} \begin{pmatrix} \chi_q \\ \chi^\dagger_{-q-1} \end{pmatrix} , \tag{1.21}$$

or

$$\chi_q = \cos\vartheta_q\, \psi_q - \sin\vartheta_q\, \psi^\dagger_{-q-1} \qquad \chi_{-q-1} = \cos\vartheta_q\, \psi_{-q-1} + \sin\vartheta_q\, \psi^\dagger_q \tag{1.22}$$

with the rotation angle ϑ_q defined by

$$\tan(2\vartheta_q) = \frac{\gamma\ \sin\left[\frac{2\pi}{N}\left(q+\frac{1}{2}\right)\right]}{h - \cos\left[\frac{2\pi}{N}\left(q+\frac{1}{2}\right)\right]} \qquad q = 0 \ldots N-1 , \tag{1.23}$$

or equivalently

$$\mathrm{e}^{\mathrm{i}2\vartheta_q} = \frac{h - \cos\left[\frac{2\pi}{N}\left(q+\frac{1}{2}\right)\right] + \mathrm{i}\gamma\ \sin\left[\frac{2\pi}{N}\left(q+\frac{1}{2}\right)\right]}{\sqrt{\left(h - \cos\left[\frac{2\pi}{N}\left(q+\frac{1}{2}\right)\right]\right)^2 + \gamma^2\ \sin^2\left[\frac{2\pi}{N}\left(q+\frac{1}{2}\right)\right]}} . \tag{1.24}$$

In terms of the Bogoliubov quasi-particles the Hamiltonian describes free fermions

$$H^+ = \sum_{q=0}^{N-1} \varepsilon\left[\tfrac{2\pi}{N}\left(q+\tfrac{1}{2}\right)\right]\ \left\{\frac{1}{N}\ \chi^\dagger_q \chi_q - \frac{1}{2}\right\} , \tag{1.25}$$

with spectrum

$$\varepsilon(\alpha) \equiv \sqrt{(h-\cos\alpha)^2 + \gamma^2\ \sin^2\alpha}. \tag{1.26}$$

The ground state of the theory $|GS\rangle_+$ is defined by

$$\chi_q\, |GS\rangle_+ = 0 \qquad q = 0 \ldots N-1 \tag{1.27}$$

and is "empty of quasi-particles". To express this ground state in terms of physical fermions, we start from the empty state $|0\rangle_+$ defined by

$$\psi_q\, |0\rangle_+ = 0 \qquad q = 0 \ldots N-1 \tag{1.28}$$

and verify that the state

$$|GS\rangle_+ \equiv \prod_{q=0}^{[(N-1)/2]} \left(1 + \tan\vartheta_q\ \psi^\dagger_q \psi^\dagger_{-q-1}\right) |0\rangle_+ \tag{1.29}$$

satisfy (1.27), where $[x]$ is the closest integer smaller than x.

The ground state energy is

$$E_0^+ = -\frac{1}{2}\ \sum_{q=0}^{N-1} \varepsilon\left[\tfrac{2\pi}{N}\left(q+\tfrac{1}{2}\right)\right] \overset{N\to\infty}{\to} -\frac{N}{2}\int_0^{2\pi} \frac{\mathrm{d}q}{2\pi}\ \varepsilon(q) , \tag{1.30}$$

where the last expression holds in the thermodynamic limit $N \to \infty$.

Excitations on top of the ground state $|GS\rangle_+$ are obtained by applying the creation operators $\chi^\dagger_q$ to it, with the constraint that such operators always have to be applied in pairs to satisfy the condition that only an even number of particles are allowed in this sector.

1.2.2 Odd number of particles

For $\mu_N^x = -1$ we have periodic boundary conditions. We can proceed with the ordinary (still asymmetric) Fourier transform

$$\psi_j = \frac{e^{i\pi/4}}{N} \sum_{q=0}^{N-1} e^{i\frac{2\pi}{N} q\, j} \tilde{\psi}_q \qquad j = 1 \dots N, \tag{1.31}$$

$$\tilde{\psi}_q = e^{-i\pi/4} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N} q\, j} \psi_j \qquad q = 0 \dots N-1, \tag{1.32}$$

and the Hamiltonian becomes

$$\begin{aligned} H^- &= \frac{1}{N} \sum_{q=0}^{N-1} \left[h - \cos\left(\tfrac{2\pi}{N} q\right)\right] \tilde{\psi}_q^\dagger \tilde{\psi}_q \\ &\quad + \frac{\gamma}{2N} \sum_{q=0}^{N-1} \sin\left(\tfrac{2\pi}{N} q\right) \left\{\tilde{\psi}_q \tilde{\psi}_{-q} + \tilde{\psi}_{-q}^\dagger \tilde{\psi}_{-q}^\dagger\right\} - \frac{hN}{2}. \end{aligned} \tag{1.33}$$

Compared to the previous case, we notice an important difference. The $q = 0$ component of the Hamiltonian does not have superconducting terms[4]. Therefore, we will single out its contribution:

$$\begin{aligned} H^- &= \frac{h-1}{N} \tilde{\psi}_0^\dagger \tilde{\psi}_0 \\ &\quad + \frac{1}{N} \sum_{q=1}^{N-1} \left[h - \cos\left(\tfrac{2\pi}{N} q\right)\right] \tilde{\psi}_q^\dagger \tilde{\psi}_q \\ &\quad + \frac{\gamma}{2N} \sum_{q=1}^{N-1} \sin\left(\tfrac{2\pi}{N} q\right) \left\{\tilde{\psi}_q \tilde{\psi}_{-q} + \tilde{\psi}_{-q}^\dagger \tilde{\psi}_{-q}^\dagger\right\} - \frac{hN}{2}. \end{aligned} \tag{1.34}$$

The Bogoliubov transformation

$$\tilde{\chi}_q = \cos\tilde{\vartheta}_q\, \tilde{\psi}_q - \sin\tilde{\vartheta}_q\, \tilde{\psi}_{-q}^\dagger \qquad \tilde{\chi}_{-q} = \cos\tilde{\vartheta}_q\, \tilde{\psi}_{-q} + \sin\tilde{\vartheta}_q\, \tilde{\psi}_q^\dagger \tag{1.35}$$

with "rotation angle" $\tilde{\vartheta}_q$

$$\tan\left(2\tilde{\vartheta}_q\right) = \frac{\gamma\, \sin\left(\frac{2\pi}{N} q\right)}{h - \cos\left(\frac{2\pi}{N} q\right)} \qquad q = 0 \dots N-1, \tag{1.36}$$

does not transform the zero-momentum excitation:

$$\tilde{\chi}_0 = \tilde{\psi}_0. \tag{1.37}$$

The diagonalized Hamiltonian in terms of the quasi-particles is

$$H^- = (h-1)\left\{\frac{1}{N}\, \tilde{\chi}_0^\dagger \tilde{\chi}_0 - \frac{1}{2}\right\} + \sum_{q=1}^{N-1} \varepsilon\left(\tfrac{2\pi}{N} q\right) \left\{\frac{1}{N}\, \tilde{\chi}_q^\dagger \tilde{\chi}_q - \frac{1}{2}\right\}, \tag{1.38}$$

[4]For $N = 2M$ the same holds for the $q = M$ component (i.e. a π-momentum particle), but this does not contribute to the effect we are discussing here and we will not single out this contribution for now. When we discuss the anti-ferromagnetic case in the appendix, we will see that the π-momentum excitation is the important contribution to single out.

with the spectrum given in (1.26).

For $h > 1$

$$h - 1 = \varepsilon(0) > 0 \tag{1.39}$$

and we can write the Hamiltonian as

$$H^- = \sum_{q=0}^{N-1} \varepsilon\left(\tfrac{2\pi}{N} q\right) \left\{ \frac{1}{N}\, \tilde{\chi}_q^\dagger \tilde{\chi}_q - \frac{1}{2} \right\} \qquad\qquad h > 1\,, \tag{1.40}$$

but for $h < 1$

$$h - 1 = -\varepsilon(0) < 0 \tag{1.41}$$

and the zero-momentum component must be considered separately like in (1.38), since a $q = 0$ excitation lowers the energy of the system.

The state with no quasi-particle excitations $|GS'\rangle_-$ is defined by

$$\tilde{\chi}_q\, |GS'\rangle_- = 0 \qquad q = 0 \ldots N-1, \tag{1.42}$$

but this state is not allowed by the condition that this sector of the theory must contain an odd number of particles, nor is this the state with the lowest energy for $h < 1$. The ground state of the theory is in fact

$$|GS\rangle_- = \tilde{\chi}_0^\dagger\, |GS'\rangle_- = \tilde{\psi}_0^\dagger\, |GS'\rangle_- \tag{1.43}$$

and its expression in terms of physical fermions is

$$|GS\rangle_- \equiv \tilde{\psi}_0^\dagger \prod_{q=1}^{[N/2]} \left(1 + \tan\tilde{\vartheta}_q\, \tilde{\psi}_q^\dagger \tilde{\psi}_{-q}^\dagger\right) |0\rangle_- \tag{1.44}$$

where $|0\rangle_-$ is defined as the state empty of fermions:

$$\tilde{\psi}_q\, |0\rangle_- = 0 \qquad q = 0 \ldots N-1\,. \tag{1.45}$$

The energy of this ground state for $h < 1$ is

$$\begin{aligned} E_0^- &= \frac{1}{2}\,(h-1) \;- \frac{1}{2} \sum_{q=1}^{N-1} \varepsilon\left(\tfrac{2\pi}{N} q\right) \\ &= -\frac{1}{2} \sum_{q=0}^{N-1} \varepsilon\left(\tfrac{2\pi}{N} q\right) \overset{N\to\infty}{\to} -\frac{N}{2} \int_0^{2\pi} \frac{\mathrm{d}q}{2\pi}\, \varepsilon(q)\,, \end{aligned} \tag{1.46}$$

where the last expression holds in the thermodynamic limit $N \to \infty$.

We see that in the thermodynamic limit $E_0^+ = E_0^-$, i.e. the two ground states are degenerates. Let us now investigate how the gap between these two states closes as $N \to \infty$, by expanding the energy difference in powers of $1/N$:

$$\begin{aligned} E_0^+ - E_0^- &= -\frac{1}{2} \sum_{q=0}^{N-1} \left\{ \varepsilon\left[\tfrac{2\pi}{N}\left(q + \tfrac{1}{2}\right)\right] - \varepsilon\left(\tfrac{2\pi}{N} q\right) \right\} \\ &= \frac{\pi}{2N} \sum_{q=0}^{N-1} \frac{\mathrm{d}}{\mathrm{d}x} \varepsilon\left(\tfrac{2\pi}{N} q + x\right)\Big|_{x=0} + \frac{\pi^2}{2N^2} \sum_{q=0}^{N-1} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \varepsilon\left(\tfrac{2\pi}{N} q + x\right)\Big|_{x=0} + \ldots \\ &= \mathrm{O}\left(\frac{1}{N}\right) \end{aligned} \tag{1.47}$$

since

$$\sum_{q=0}^{N-1} \frac{\mathrm{d}}{\mathrm{d}x} \varepsilon\left(\tfrac{2\pi}{N} q + x\right)\Big|_{x=0} = 0. \tag{1.48}$$

This means that the gap between the two ground states closes faster than the 1-particle excitation[5] and constitutes a true thermodynamic degeneracy. Clearly, for $h > 1$ this degeneracy disappears. So, the quantum phase transition at $h = 1$ is the Ising transition between a degenerate Z_2 spontaneously broken ground state and a non-broken one.

1.3 The Phase-Diagram

We established that the ground states of the two sectors of the model (with even or odd total number of fermions/spin down) have the same energy in the thermodynamic limit. We are now going to show that every allowed energy level of the system is doubly degenerate, as long as $h < 1$.

Let us consider the energy level of the sector with an even number of particles. From (1.25) we have:

$$\begin{aligned}
|GS\rangle_+ \quad &\to \quad E_0^+ \,, \\
\chi_q^\dagger \chi_{q'}^\dagger |GS\rangle_+ \quad &\to \quad E_0^+ + \varepsilon\left(\tfrac{2\pi}{N}\, q + \tfrac{\pi}{N}\right) + \varepsilon\left(\tfrac{2\pi}{N}\, q' + \tfrac{\pi}{N}\right) \,, \qquad q \neq q' \\
&\vdots
\end{aligned} \tag{1.49}$$

For $h < 1$, from (1.38) the sector with an odd number of particles gives:

$$\begin{aligned}
|GS\rangle_- = \tilde{\chi}_0^\dagger |GS'\rangle_- \quad &\to \quad E_0^- \,, \\
\tilde{\chi}_q^\dagger |GS'\rangle_- \quad &\to \quad E_0^- + \varepsilon(0) + \varepsilon\left(\tfrac{2\pi}{N}\, q\right) \,, \qquad q \neq 0 \\
\tilde{\chi}_0^\dagger \tilde{\chi}_q^\dagger \tilde{\chi}_{q'}^\dagger |GS'\rangle_- \quad &\to \quad E_0^- + \varepsilon\left(\tfrac{2\pi}{N}\, q\right) + \varepsilon\left(\tfrac{2\pi}{N}\, q'\right) \,, \qquad q \neq q' \neq 0 \\
\tilde{\chi}_q^\dagger \tilde{\chi}_{q'}^\dagger \tilde{\chi}_{q''}^\dagger |GS'\rangle_- \quad &\to \quad E_0^- + \varepsilon(0) + \varepsilon\left(\tfrac{2\pi}{N}\, q\right) + \varepsilon\left(\tfrac{2\pi}{N}\, q'\right) + \varepsilon\left(\tfrac{2\pi}{N}\, q''\right) \,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q \neq q' \neq q'' \neq 0 \\
&\vdots
\end{aligned} \tag{1.50}$$

where $|GS'\rangle_-$ was defined in (1.43). For $h > 1$, using (1.40) this sector has no more surprises:

$$\begin{aligned}
|GS\rangle_- = \tilde{\chi}_0^\dagger |GS'\rangle_- \quad &\to \quad E_0^- + \varepsilon(0) \,, \\
\tilde{\chi}_q^\dagger |GS'\rangle_- \quad &\to \quad E_0^- + \varepsilon\left(\tfrac{2\pi}{N}\, q\right) \,, \qquad q \neq 0 \\
\tilde{\chi}_q^\dagger \tilde{\chi}_{q'}^\dagger \tilde{\chi}_{q''}^\dagger |GS'\rangle_- \quad &\to \quad E_0^- + \varepsilon\left(\tfrac{2\pi}{N}\, q\right) + \varepsilon\left(\tfrac{2\pi}{N}\, q'\right) + \varepsilon\left(\tfrac{2\pi}{N}\, q''\right) \,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q \neq q' \neq q'' \\
&\vdots
\end{aligned} \tag{1.51}$$

[5] The ground state energy E_0 is O(N), the 1-particle excitation is O(1), while this gap closes at least like O($1/N$), more probably exponentially.

In conclusion, we see that for $h < 1$ in the spectrum the energy levels always appear in pairs and in the thermodynamic limit every allowed level is doubly degenerate, with one contribution from each sector.

Now that we have determined the spectrum, the partition function of the XY model for $h < 1$ is

$$\begin{aligned}
\mathcal{Z} &= \sum \mathrm{e}^{-\beta E_i} \\
&= \frac{1}{2}\mathrm{e}^{-\beta E_0^+}\left[\prod_{q=0}^{N-1}\left(1+\mathrm{e}^{-\beta\varepsilon\left(\frac{2\pi}{N}\,q+\frac{\pi}{N}\right)}\right)+\prod_{q=0}^{N-1}\left(1-\mathrm{e}^{-\beta\varepsilon\left(\frac{2\pi}{N}\,q+\frac{\pi}{N}\right)}\right)\right] \\
&\quad +\frac{1}{2}\mathrm{e}^{-\beta E_0^-}\left[\prod_{q=0}^{N-1}\left(1+\mathrm{e}^{-\beta\varepsilon\left(\frac{2\pi}{N}\,q\right)}\right)+\prod_{q=0}^{N-1}\left(1-\mathrm{e}^{-\beta\varepsilon\left(\frac{2\pi}{N}\,q\right)}\right)\right] \\
&= 2^{N-1}\left\{\prod_{q=0}^{N-1}\cosh\left[\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}\,q+\tfrac{\pi}{N}\right)\right]+\prod_{q=0}^{N-1}\sinh\left[\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}\,q+\tfrac{\pi}{N}\right)\right]\right\} \\
&\quad +2^{N-1}\left\{\prod_{q=0}^{N-1}\cosh\left[\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}\,q\right)\right]+\prod_{q=0}^{N-1}\sinh\left[\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}\,q\right)\right]\right\}\,. \qquad (1.52)
\end{aligned}$$

Taking the thermodynamic limit, the free energy per site is:

$$\begin{aligned}
\mathcal{F} &= -\frac{1}{\beta}\lim_{N\to\infty}\frac{1}{N}\ln\mathcal{Z} \\
&= -\frac{1}{\beta}\ln 2-\frac{1}{\pi\beta}\int_0^\pi \ln\cosh\left[\frac{\beta}{2}\varepsilon(\omega)\right]\,\mathrm{d}\omega \\
&\quad -\frac{1}{\beta}\lim_{N\to\infty}\frac{1}{N}\ln\left[1+\prod_{q=0}^{N-1}\tanh\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}q\right)\right]\,, \qquad (1.53)
\end{aligned}$$

where the last term encodes the degeneracy of the model, but is clearly negligible in the thermodynamic limit.

For $h > 1$

$$\begin{aligned}
\mathcal{Z} &= 2^{N-1}\left\{\prod_{q=0}^{N-1}\cosh\left[\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}\,q+\tfrac{\pi}{N}\right)\right]+\prod_{q=0}^{N-1}\sinh\left[\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}\,q+\tfrac{\pi}{N}\right)\right]\right\} \\
&\quad +2^{N-1}\left\{\prod_{q=0}^{N-1}\cosh\left[\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}\,q\right)\right]-\prod_{q=0}^{N-1}\sinh\left[\frac{\beta}{2}\varepsilon\left(\tfrac{2\pi}{N}\,q\right)\right]\right\}\,. \qquad (1.54)
\end{aligned}$$

and the free energy per site in the thermodynamic limit is

$$\mathcal{F} = -\frac{1}{\beta}\ln 2-\frac{1}{\pi\beta}\int_0^\pi \ln\cosh\left[\frac{\beta}{2}\varepsilon(\omega)\right]\,\mathrm{d}\omega\,. \qquad (1.55)$$

Clearly, from the partition function we can derive the whole thermodynamic of the model, which is essentially that of free fermions.

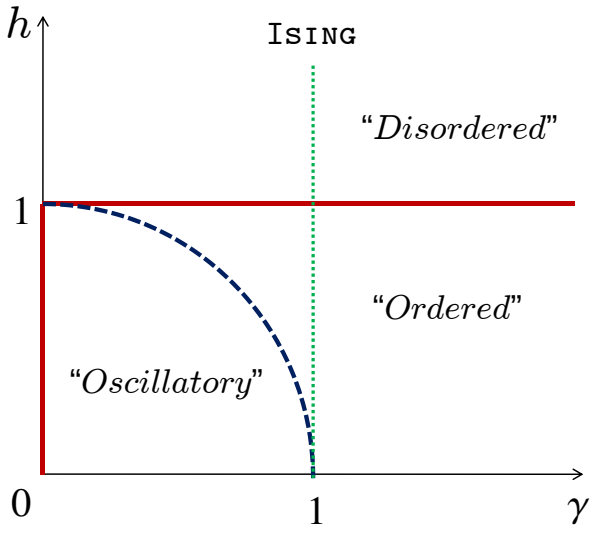

Figure 1.1: Phase diagram of the XY Model (only the part $\gamma \geq 0$ and $h \geq 0$ is shown). The theory is critical for $h = 1$ and for $\gamma = 0$ and $h < 1$ (in bold red). The line $\gamma = 1$ is the Ising Model in transverse field (dashed line). On the line $\gamma^2 + h^2 = 1$ the ground states of the theory can be factorized as a product of single spin states (blue dotted line).

However, at zero temperature, the phase diagram of the model is more interesting, with the presence of two different quantum phase transitions of different type. From the spectrum 1.26

$$\varepsilon(k) \equiv \sqrt{(h - \cos k)^2 + \gamma^2 \ \sin^2 k} \, , \tag{1.56}$$

we see that the model is gapped for most of the values of the parameters, but we have gapless excitations for $\gamma = 0$, $h < 1$ (isotropic XX model: $c = 1$ CFT) and for $h = 1$ (critical magnetic field: $c = 1/2$ CFT).

In Fig. 1.1 we draw the phase diagram of the XY model for $\gamma \geq 0$ and $h \geq 0$. It shows the critical lines $\gamma = 0$ and $h = 1$ and the line $\gamma = 1$ corresponding to the Ising model in transverse magnetic field and the line $\gamma^2 + h^2 = 1$ on which the wave function of the ground state is factorized into a product of single spin states [12].

1.4 The correlation functions

In this section we review the derivation of the fundamental correlators of the ground state $|GS\rangle_+$ for the XY model at zero temperature following McCoy and co-authors [1].

The ground state $|GS\rangle_+ = |0\rangle$ of the model in terms of the Bogoliubov quasi-particles is defined as

$$\chi_q |0\rangle = 0 \qquad \forall q \tag{1.57}$$

i.e. it is the conventional ground state of free fermions. The correlators for this theory are easily found to be

$$\langle 0 | \chi_q \chi_k^\dagger | 0 \rangle \quad = \quad \delta_{k,q}, \tag{1.58}$$

$$\langle 0|\chi_q^\dagger \chi_k|0\rangle = 0, \tag{1.59}$$
$$\langle 0|\chi_q \chi_k|0\rangle = 0, \tag{1.60}$$
$$\langle 0|\chi_q^\dagger \chi_k^\dagger|0\rangle = 0. \tag{1.61}$$

This vacuum is the ground state for the XY model, but it is not so simple when expressed in terms of physical particles. The Hamiltonian (1.19) contains superconducting-like terms, so its ground state is non-trivial. One can invert the Bogoliubov transformation (1.21)

$$\psi_q = \cos\vartheta_q \, \chi_q + \sin\vartheta_q \, \chi_{-q}^\dagger \tag{1.62}$$

to calculate the fundamental correlators in terms of physical fermions:

$$\langle 0|\psi_q^\dagger \psi_k|0\rangle = \sin^2\vartheta_q \, \delta_{k,q} = \frac{1-\cos 2\vartheta_q}{2} \, \delta_{k,q} \, , \tag{1.63}$$
$$\langle 0|\psi_q \psi_k^\dagger|0\rangle = \cos^2\vartheta_q \, \delta_{k,q} = \frac{1+\cos 2\vartheta_q}{2} \, \delta_{k,q} \, , \tag{1.64}$$
$$\langle 0|\psi_q \psi_k|0\rangle = -\cos\vartheta_q \, \sin\vartheta_q \, \delta_{-k,q} = -\frac{\sin 2\vartheta_q}{2} \, \delta_{-k,q} \, , \tag{1.65}$$
$$\langle 0|\psi_q^\dagger \psi_k^\dagger|0\rangle = \cos\vartheta_q \, \sin\vartheta_q \, \delta_{-k,q} = \frac{\sin 2\vartheta_q}{2} \, \delta_{-k,q} \, . \tag{1.66}$$

Now, the two-point fermionic correlators are easy to obtain by Fourier transform. In the thermodynamic limit they read [8, 1]

$$F_{jl} \equiv \mathrm{i}\langle 0|\psi_j \psi_l|0\rangle = -\mathrm{i}\langle 0|\psi_j^\dagger \psi_l^\dagger|0\rangle = \int_0^{2\pi} \frac{\mathrm{d}q}{2\pi} \, \frac{\sin 2\vartheta_q}{2} \mathrm{e}^{\mathrm{i}q(j-l)}, \tag{1.67}$$
$$G_{jl} \equiv \langle 0|\psi_j \psi_l^\dagger|0\rangle = \int_0^{2\pi} \frac{\mathrm{d}q}{2\pi} \, \frac{1+\cos 2\vartheta_q}{2} \mathrm{e}^{\mathrm{i}q(j-l)}. \tag{1.68}$$

These correlators will be fundamental in our calculation of the EFP in the next chapter.

To calculate the correlation functions for the original spin chain model (1.9),

$$\rho_{lm}^\nu \equiv \langle 0 \, |\sigma_l^\nu \, \sigma_m^\nu| \, 0\rangle \qquad \nu = x, y, z, \tag{1.69}$$

we need more work. We follow [8] and express these correlators in terms of spin lowering and raising operators:

$$\rho_{lm}^x = \langle 0 \, \big| \big(\sigma_l^+ + \sigma_l^-\big) \, \big(\sigma_m^+ + \sigma_m^-\big) \big| \, 0\rangle, \tag{1.70}$$
$$\rho_{lm}^y = -\langle 0 \, \big| \big(\sigma_l^+ - \sigma_l^-\big) \, \big(\sigma_m^+ - \sigma_m^-\big) \big| \, 0\rangle, \tag{1.71}$$
$$\rho_{lm}^z = \langle 0 \, \big| \big(1 - 2\sigma_l^+\sigma_l^-\big) \, \big(1 - 2\sigma_m^+\sigma_m^-\big) \big| \, 0\rangle. \tag{1.72}$$

and use (1.10) to write them using spinless fermions operators.

For instance, let us consider ρ_{lm}^x:

$$\begin{aligned} \rho_{lm}^x &= \langle 0 \, \big| \big(\sigma_l^+ + \sigma_l^-\big) \, \big(\sigma_m^+ + \sigma_m^-\big) \big| \, 0\rangle \\ &= \langle 0| \left(\psi_l^\dagger + \psi_l\right) \prod_{i=l}^{m-1} \left(1 - 2\psi_i^\dagger \psi_i\right) \left(\psi_m^\dagger + \psi_m\right) |0\rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle 0| \left(\psi_l^\dagger - \psi_l\right) \prod_{i=l+1}^{m-1} \left(1 - 2\psi_i^\dagger \psi_i\right) \left(\psi_m^\dagger + \psi_m\right) |0\rangle \\
&= \langle 0| \left(\psi_l^\dagger - \psi_l\right) \prod_{i=l+1}^{m-1} \left(\psi_i^\dagger + \psi_i\right) \left(\psi_i^\dagger - \psi_i\right) \left(\psi_m^\dagger + \psi_m\right) |0\rangle, \qquad (1.73)
\end{aligned}
$$

where we have used two identities

$$
\sigma_j^+ = \mathrm{e}^{\mathrm{i}\pi \sum_{k<j} \psi_k^\dagger \psi_k} \, \psi_j = \psi_j \, \mathrm{e}^{-\mathrm{i}\pi \sum_{k<j} \psi_k^\dagger \psi_k} \qquad (1.74)
$$

and

$$
\mathrm{e}^{\mathrm{i}\pi \psi_i^\dagger \psi_i} = 1 - 2\psi_i^\dagger \psi_i = \left(\psi_i^\dagger + \psi_i\right) \left(\psi_i^\dagger - \psi_i\right) = -\left(\psi_i^\dagger - \psi_i\right) \left(\psi_i^\dagger + \psi_i\right). \qquad (1.75)
$$

Now we define the operators

$$
\begin{aligned}
A_i &\equiv \psi_i^\dagger + \psi_i \qquad (1.76) \\
B_i &\equiv \psi_i^\dagger - \psi_i \qquad (1.77)
\end{aligned}
$$

which allow us to write the correlators (1.69) as

$$
\begin{aligned}
\rho_{lm}^x &= \langle 0| B_l A_{l+1} B_{l+1} \ldots A_{m-1} B_{m-1} A_m |0\rangle \\
\rho_{lm}^y &= (-1)^{m-1} \langle 0| A_l B_{l+1} A_{l+1} \ldots B_{m-1} A_{m-1} B_m |0\rangle \\
\rho_{lm}^z &= \langle 0| A_l B_l A_m B_m |0\rangle. \qquad (1.78)
\end{aligned}
$$

We can use Wick's Theorem to expand these expectation values in terms of two point correlation functions. By noticing that

$$
\langle 0| A_l A_m |0\rangle = \langle 0| B_l B_m |0\rangle = 0 \qquad (1.79)
$$

we write ρ_{lm}^z as

$$
\begin{aligned}
\rho_{lm}^z &= \langle 0| A_l B_l |0\rangle \langle 0| A_m B_m |0\rangle - \langle 0| A_l B_m |0\rangle \langle 0| A_m B_l |0\rangle \\
&= H^2(0) - H(m-l) H(l-m) \qquad (1.80)
\end{aligned}
$$

where

$$
H(m-l) \equiv \langle 0| B_l A_m |0\rangle = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}q}{2\pi} \, \mathrm{e}^{\mathrm{i}2\vartheta_q} \mathrm{e}^{\mathrm{i}q(m-l)}. \qquad (1.81)
$$

The other two correlators in (1.78) involve more terms. It can be shown [1, 8] that the Wick's expansion can be expressed as the determinant of a matrix with elements given by expectation values of each combination of two operators.

We can then write the spin correlators (1.78) as $m-l \times m-l$ matrix determinants:

$$
\begin{aligned}
\rho_{lm}^x &= \det |H(i-j)|_{i=l\ldots m-1}^{j=l+1\ldots m}, \qquad (1.82) \\
\rho_{lm}^y &= \det |H(i-j)|_{i=l+1\ldots m}^{j=l\ldots m-1} \qquad (1.83)
\end{aligned}
$$

with matrix elements given by (3.1).

$n \to \infty$	$\rho_x(n) \simeq$	$\rho_y(n) \simeq$	$\rho_z(n) \simeq$
"Disordered Phase" $h > 1$	$X_D \ \frac{\lambda_+^{-n}}{n^{1/2}} + \dots$	$Y_D \ \frac{\lambda_+^{-n}}{n^{3/2}} + \dots$	$\frac{1}{4} - \frac{1}{8\pi} \ \frac{\lambda_+^{-2n}}{n^2} + \dots$
"Ising Transition" $h = 1$	$C_x \ \frac{\gamma}{1+\gamma} \ \frac{1}{(\gamma n)^{1/4}} + \dots$	$C_y \ \frac{\gamma(1+\gamma)}{(\gamma n)^{9/4}} + \dots$	$m_z^2 - \frac{1}{4(\pi n)^2} + \dots$
"Ordered Phase" $\Re\sqrt{1-\gamma^2} < h < 1$	$m_x^2 \left[1 + X_O^+ \ \frac{\lambda_+^{2n}}{n^2} + \dots\right]$	$Y_{Or} \ \frac{\lambda_+^{2n}}{n^3} + \dots$	$m_z^2 - \frac{1}{8\pi} \ \frac{\lambda_+^{2n}}{n^2} + \dots$
"Factorizing Field" $h^2 = 1 - \gamma^2$	$\frac{1}{2} \ \frac{\gamma}{1+\gamma}$	0	m_z^2
"Oscillatory Phase" $h^2 < 1 - \gamma^2$	$m_x^2 \left[1 + X_O^+ \ \frac{\lambda_+^{2n}}{n^2} + X_O^- \ \frac{\lambda_-^{2n}}{n^2} + \dots\right]$	$Y_{Os} \ \frac{\lambda_+^n \lambda_-^n}{n} + \dots$	$m_z^2 - \frac{1}{4\pi} \ \frac{(\lambda_+^n + Z_{Os}\lambda_-^n)(\lambda_+^n + Z_{Os}^{-1}\lambda_-^n)}{n^2} + \dots$
"Free Fermions" $\gamma = 0, \ \lvert h\rvert < 1$	$C \ (1-h^2)^{1/4} \ \frac{1}{n^{1/2}}$	$C \ (1-h^2)^{1/4} \ \frac{1}{n^{1/2}}$	$m_z^2 - \frac{\sin^2(n \arccos h)}{\pi^2 n^2}$

Table 1.1: Asymptotic behavior of the fundamental two-point correlation functions (1.69). $C_x = \frac{1}{2A^3} e^{1/4} 2^{1/12}$, $C_y = -\frac{1}{32A^3} e^{1/4} 2^{1/12}$, and $C = \frac{1}{A^6} e^{1/2} 2^{2/3}$, where $A = 1.282\dots$ is the Glaisher's constant. The other prefactors are also known from [1] and are listed in table 1.4. The subleading correction are all suppressed by order $1/n$.

Matrices like (1.82,1.83) are very special. Their entries depend only on the difference between the row and column index, so that the same elements appear on each diagonal. Therefore they look like:

$$\rho^x_{lm} = \rho^x(n) = \begin{vmatrix} H(-1) & H(-2) & H(-3) & \dots & H(-n) \\ H(0) & H(-1) & H(-2) & \dots & H(1-n) \\ H(1) & H(0) & H(-1) & \dots & H(2-n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H(n-2) & H(n-3) & H(n-4) & \dots & H(-1) \end{vmatrix}, \tag{1.84}$$

$$\rho^y_{lm} = \rho^y(n) = \begin{vmatrix} H(1) & H(0) & H(-1) & \dots & H(2-n) \\ H(2) & H(1) & H(0) & \dots & H(3-n) \\ H(3) & H(2) & H(1) & \dots & H(4-n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H(n) & H(n-1) & H(n-2) & \dots & H(1) \end{vmatrix}, \tag{1.85}$$

where $n = m - l$.

Matrices like (1.82,1.83) are known as *"Toeplitz Matrices"* and a vast mathematical literature has been devoted to the study of the asymptotic behavior of their determinants (*"Toeplitz Determinants"*). The development of the theory of Toeplitz Determinants is tightly connected with the Ising and XY model since the seminar works by Wu, McCoy and collaborators [15],[1]. In the second paper of the series [1], these techniques were applied to calculate the fundamental correlators of the XY model. It is beyond the scope of these lecture to reproduce this derivation. The main results on the asymptotic behavior of Toeplitz determinants are summarized in appendix A and we collected the **zero-temperature** behavior of the 2-point functions $\rho^\nu(n)$ in table 1.4 and 1.4 as a function of the parameters

$$\lambda_\pm \equiv \frac{h \pm \sqrt{\gamma^2 + h^2 - 1}}{1+\gamma} \ . \tag{1.86}$$

$m_x^2 \equiv \frac{1}{4}\left[(1-\lambda_-^2)(1-\lambda_+^2)(1-\lambda_+\lambda_-)^2\right]^{1/4}$	$m_z \equiv \int_0^\pi \frac{p_1\left(e^{iq}\right)+p_2\left(e^{iq}\right)}{\sqrt{p_1\left(e^{iq}\right)p_2\left(e^{iq}\right)}}\,\frac{dq}{2\pi}$
$X_D \equiv \frac{1}{4\sqrt{\pi}}\left[\frac{(1-\lambda_-^2)}{(1-\lambda_+^{-2})}(1-\lambda_-\lambda_+)^2\right]^{1/4}$	$Y_D \equiv -\frac{1}{8\sqrt{\pi}}\left[\frac{(1-\lambda_+^{-2})^3(1-\lambda_-^2)}{(1-\lambda_-\lambda_+)^2}\right]^{1/4}\frac{1}{1-\lambda_-\lambda_+^{-1}}$
$X_O^+ \equiv \frac{1}{2\pi}\frac{\lambda_+^2}{1-\lambda_+^2}$, $X_O^- \equiv \frac{1}{2\pi}\frac{\lambda_-^2}{1-\lambda_-^2}$	$Y_{Or} \equiv -\frac{1}{8\pi}\left[\frac{(1-\lambda_-^2)}{(1-\lambda_+^2)^3(1-\lambda_-\lambda_+)^2}\right]^{1/4}\frac{1}{1-\lambda_-\lambda_+^{-1}}$
$Z_{Os} \equiv \frac{\lambda_+}{\lambda_-}\sqrt{\frac{1-\lambda_+^2}{1-\lambda_-^2}}$	$Y_{Os} \equiv \frac{[(1-\lambda_-\lambda_+^{-1})(1-\lambda_-^{-1}\lambda_+)]^{1/2}}{4\pi\left[(1-\lambda_-^2)(1-\lambda_+^2)(1-\lambda_+\lambda_-)^2\right]^{1/4}}$

Table 1.2: Dependence on $\lambda_\pm$ of the prefactors in the asymptotic behavior of table 1.4.

From these results we can derive a better interpretation of the different phases of the model. For $h > 1$ we have a "*Disordered Phase*", since there is no net magnetization along the x-direction. In this region the λ's are real and λ_- and λ_+^{-1} are inside the unit circle, with the latter with a bigger modulus. For $|h| < 1$, the model is in an "*Ordered Phase*", with a net magnetization m_x. The $\lambda_\pm$ are still real and both inside the unit circle. If $h > 0$ and $h^2 + \gamma^2 > 1$, λ_+ is closer to the circle and $\rho_x(n)$ approaches the saturation exponentially. For $h^2 + \gamma^2 < 1$ both $\lambda_\pm$ acquire an imaginary part and become complex conjugated. Thus, they both contribute to the asymptotic behavior and the correlation functions develop a periodic modulation. Hence, the name of "*oscillatory phase*". Going to negative magnetic field, the role of λ_+ and λ_- gets inverted, while crossing the line $\gamma = 0$ one should exchange $\lambda_\pm$ with $\lambda_\pm^{-1}$.

Excercises

1. **Ising self-duality**: One very interesting feature of the Ising model is its self-duality. This can be seen both in the classical two-dimensional model (Kramers-Wannier duality [16]) and in the 1-D quantum. In the latter, the statement is that the Hamiltonian (1.4) is invariant under the transformation between *order* and *disorder* spin operators:

$$\sigma_j^x = \prod_{l\le j}\mu_l^z \ , \qquad \sigma_j^z = \mu_j^x\mu_{j+1}^x \ , \tag{1.87}$$

i.e.

$$\sigma_j^x\sigma_{j+1}^x - h\sigma_j^z = \mu_{j+1}^z - h\mu_j^x\mu_{j+1}^x \ . \tag{1.88}$$

Thus relating a system with magnetic field h to one with $1/h$ and hence establishing the existence of a phase transition at the self dual point $h = 1$. What is the duality relation in terms of the Jordan-Wigner fermions:

$$\sigma_j^z = 1 - 2c_j^\dagger c_j \ , \qquad \mu_j^z = 1 - 2d_j^\dagger d_j \ . \tag{1.89}$$

Chapter 2

The Lieb-Liniger Model

2.1 Introduction

As a first example of application of the coordinate Bethe Ansatz we will use a model of bosons with contact interaction. This is the simplest example of BA and it will make the meaning of the different ingredients needed as transparent as possible [17].

The first step is to identify the two-particle phase-shift. In a one-dimensional setting, conservation of Energy and Momentum means that when two identical particles scatter, they can only keep their momenta or exchange them. Therefore, the only effect of the interaction could be to add a phase shift to the wavefunction.

There is no consensus on a definition of what makes a model integrable, but it is known that a necessary condition is that the Yang-Baxter equations hold. It is far from the scope of these lecture to dwell into these equations and their physical meaning (entire books have been written on the subject, see for instance [18]). In essence, they mean that an integrable model does not have any true three-particle interaction. Any scattering between three particles can be decomposed in consecutive scattering between the particles in pairs and, most of all, the order in which these interactions take place is not important. Checking this latter condition is the most important test to determine whether a model can be integrable or not.

Once the effect of the interaction has been encoded in the phase shift, an ansatz wavefunction can be constructed in terms of superposition of some plane-wave modes with unknown *quasi-momenta*. Applying this wavefunction with undetermined coefficients to the Hamiltonian generates the consistency equations known as the *Bethe Equations*.

Finally, we apply periodic boundary conditions (the effect of this assumption can be relaxed by letting the size of the system go to infinity at the end of the calculation) to impose the quantization of the quantum numbers.

After all this procedure, for a system on N particles we will have a set of N coupled algebraic equations in N unknown quasi-momenta and depending on N parameters. These parameters are in fact integer numbers and they specify the quantum state of the system we are interested in. Once we decide the state, i.e these parameters, we solve the set of algebraic equations to get the quasi-momenta and this provide a complete description of the eigenfunctions of the model

with the given quantum numbers.

We will observe that these states have a fermionic nature, in that to obtain a solution the set of integer numbers cannot contain the same value twice. This is general feature of any Bethe Ansatz solution, valid for bosonic systems as well. There is a treatment of integrable systems due to the so called "Japanese group" by which any model is mapped, exactly although possibly implicitly, into a system of free fermions.

To conclude, the problem of solving the differential problem related to the Schrödinger equation is reduced by Bethe Ansatz to a much simpler system of algebraic equations. In practice, a further simplification arise by reaching the thermodynamic limit. Then, one is interested in the distribution of quasi-momenta and the set of algebraic equations can be written as an integral equation for this distribution, for which both numerical and sometimes analytical solutions can be derived. The distribution function can be used to formulate all the thermodynamical quantities of the system. Therefore, Bethe Ansatz is a very efficient tool for describing the thermodynamics of an integrable system.

The main limitation of this technique is that it provide only with an implicit knowledge of the eigenfunctions. Therefore, the calculation of correlation functions is absolutely not straightforward and, for many practical purposes, not attainable. In many instances, even a numerical approach based directly on the Bethe Ansatz solutions does not yield any improvement over a "brute force" numerical approach. In recent years, however, many developments have come from a combination of Bethe Ansatz and numerical techniques, which have improved the size limits of the systems by several order of magnitude (see, for instance, the works of Jean-Sbastien Caux and collaborators).

These improvements rely on the advancements reached in the late '80s/early '90s with the use of the *Algebraic Bethe Ansatz* approach in expressing correlation functions in terms of Fredholm determinants [19]. In the previous chapter we solved the XY model and we saw that the correlation functions can be written in terms of determinants of Toeplitz matrices. This is a particularly simple case of the general fact that in integrable models the correlators can be expressed as determinants of integral operators though the Inverse Scattering Method. The construction of these determinants is a beautiful subject and a great accomplishment of high mathematics, but too often it has been so far of very limited practical use. However, as pointed out above, recent lines of research have shown very interesting progresses and brought a renewed interest in Bethe Ansatz techniques.

2.2 Motivations

The Lieb-Liniger is a model of one-dimensional bosons with contact interaction. Thus defined by the Hamiltonian

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<l} \delta(x_j - x_l) \,, \tag{2.1}$$

where c parametrizes the strength of the interaction. Physically, this is a very realistic model for 1-D particles with short range interaction and constitute a powerful analytical tool to interpret

experimental results. However, the main limitation is that the Bethe Ansatz solution assumes translational invariance, which is spoiled by the trapping used in current experimental setup to confine the system. In fact, due to the external potential, real systems are inhomogeneous, with a density of particles that varies in space. Nonetheless, signs of integrability have been observed to persist even in realistic systems [20].

While introducing an external potential in (2.1) spoils integrability, we should mention that there is a different kind of potential that admits a parabolic confinement and is also exactly solvable, namely the Calogero-Moser model [21]:

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j<l} \frac{\lambda(\lambda-1)}{(x_j - x_l)^2} + \frac{\omega^2}{2} \sum_{j=1}^{N} x_j^2 \, . \tag{2.2}$$

This model belongs to a family of integrable systems (for an excellent review on them [22]) that includes the Calogero-Sutherland model (a periodic version of the CM model)

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{\pi^2}{L^2} \sum_{j<l} \frac{\lambda(\lambda-1)}{\sin^2 \frac{\pi}{L}(x_j - x_l)} \, , \tag{2.3}$$

and several others. This family is solved by a different kind of ansatz, called the *asymptotic Bethe ansatz*, that is not considered in this set of notes. Originally, the trapping in (2.2) was introduced to prevent the repulsive potential from pushing the particles at infinity. One would then use the trapping to keep the particle density fixed and then take the $\omega \to 0$ limit together with the thermodynamic limit. While the presence of the parabolic potential makes this model "realistic", its long-range potential is not (nobody has yet found a reasonable way to create a inverse-square interaction in a 1-D system).

Let us now come back to our Lieb-Liniger model and we write it in second quantized form

$$\mathcal{H} = \int \mathrm{d}x \left[\partial_x \Psi^\dagger(x) \partial_x \Psi(x) + c \Psi^\dagger(x) \Psi^\dagger(x) \Psi(x) \Psi(x) - \mu \Psi^\dagger(x) \Psi(x) \right] \, . \tag{2.4}$$

The Euler equation is then

$$\mathrm{i} \Psi_t = -\Psi_{xx} - \mu \Psi + 2c \Psi^\dagger \Psi \Psi \, . \tag{2.5}$$

In the small interaction regime, the bosonic field will not completely condense (it is forbidden in one-dimension, since long range order is always destroyed by fluctuations), but nonetheless, a large fraction of the particles will be in the zero momentum state. Thus we can treat these equations semiclassically and recognize in (2.5) the Gross-Pitaevskii equation. The Hamiltonian (2.4) is known as the Non-Linear Schrödinger equation (NLS) and the Lieb-Liniger model is also called the Quantum NLS. We will come back to the relation between the quantum and classical version of the NLS and their modes in the discussion of the LL model.

2.3 The two-particle problem

Let us start by considering two bosonic particles with a contact interaction:

$$\mathcal{H} = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2 \, c \, \delta(x_1 - x_2) \, , \tag{2.6}$$

where c is the strength of the interaction (attractive for $c < 0$ or repulsive for $c > 0$). An attractive interaction will create bound states and in the thermodynamic limit it will give a divergent (negative) energy for the ground state. Therefore, we will mostly be interested in the repulsive case here.

We write the generic eigenstate by dividing the $x_1 < x_2$ and $x_1 > x_2$ configurations:

$$\Psi(x_1, x_2) = f(x_1, x_2)\vartheta(x_2 - x_1) + f(x_2, x_1)\vartheta(x_1 - x_2) \; , \tag{2.7}$$

where we wrote the wavefunction explicitly as symmetric function of x_1 and x_2, and where

$$\vartheta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \tag{2.8}$$

is the Heaviside step function.

As usual with a Schrödinger problem, we assume a superposition of plane-waves for the solution, i.e.

$$\begin{aligned} f(x_1, x_2) &\equiv A(k_1, k_2)\mathrm{e}^{\mathrm{i}(k_1 x_1 + k_2 x_2)} + A(k_2, k_1)\mathrm{e}^{\mathrm{i}(k_2 x_1 + k_1 x_2)} \\ &= A_{12}\mathrm{e}^{\mathrm{i}(k_1 x_1 + k_2 x_2)} + A_{21}\mathrm{e}^{\mathrm{i}(k_2 x_1 + k_1 x_2)} \; . \end{aligned} \tag{2.9}$$

In solving the eigenvalue equation, one should remember that $\partial_x \vartheta(x) = \delta(x)$. Thus

$$\begin{aligned} \partial_{x_1}^2 \Psi(x_1, x_2) &= \partial_{x_1}^2 f(x_1, x_2)\vartheta(x_2 - x_1) + \partial_{x_1}^2 f(x_2, x_1)\vartheta(x_1 - x_2) \\ &\quad -\partial_{x_1} f(x_1, x_2)\delta(x_2 - x_1) + \partial_{x_1} f(x_2, x_1)\delta(x_1 - x_2) \; , \end{aligned} \tag{2.10}$$

where we also used the identity (valid under integral): $f(x)\partial_x \delta(x) = -\partial_x f(x)\delta(x)$.

Therefore we have:

$$\begin{aligned} \mathcal{H}\Psi &= (k_1^2 + k_2^2)\Psi \\ &\quad +2\delta(x_1 - x_2)\left[c\left(A_{12} + A_{21}\right) - \mathrm{i}\left(A_{12} - A_{21}\right)(k_1 - k_2)\right]\mathrm{e}^{\mathrm{i}(k_1 + k_2)x_1} \; . \end{aligned} \tag{2.11}$$

This eigenvalue equation is satisfied if

$$\frac{A_{12}}{A_{21}} = \frac{\mathrm{i}(k_1 - k_2) + c}{\mathrm{i}(k_1 - k_2) - c} \; . \tag{2.12}$$

It is easy to see that this factor has unit modulus and it is therefore a pure phase:

$$\frac{A_{12}}{A_{21}} = \mathrm{e}^{\mathrm{i}\tilde{\theta}(k_1 - k_2)} \tag{2.13}$$

with[1]

$$\tilde{\theta}(k) \equiv 2\arctan\frac{k}{c} + \pi \; . \tag{2.14}$$

This is the phase shift due to the contact interaction. It is a unique signature of the kind of interaction, each integrable potential is characterized by a phase shift function. We will see

[1] One has to pay attention to that the branch-cut of the logarithm coincide to the one of the arc-tangent.

that this function plays a major role in the Bethe equations. Notice that in the limit $c \to \infty$ the scattering phase becomes that of free fermions.

It is customary to define the scattering angle as an odd function of its argument that vanishes for $k = 0$. This is easy to achieve by singling out the constant π phase, signature of the fermionic nature of the interaction, as $\tilde{\theta}(k) \equiv \theta(k) + \pi$, with

$$\theta(k) \equiv 2 \arctan \frac{k}{c} . \tag{2.15}$$

Since the interaction is local and it is connected to the discontinuity of the wave-functions between sectors with different particle ordering (each ordering can be written as a two-particle permutation), there are no three-particle scattering (diffraction) and it is easy to check that the Yang-Baxter relations are trivially satisfied.

Excerices

1. Solve the Lieb-Liniger model with three particles. Write the wavefunction as a sum of exponentials with the all the different pairings between the particles positions and momenta and consider the different ordering of the particles. The coefficients in front of the exponentials can be determined taking into account the bosonic statistics and by the jump of the derivative. Check explicitly that every interaction can be decomposed into a series of two-body scattering, in particular considering what happens when $x_1 = x_2 = x_3$.

2.4 Bethe Ansatz Wavefunction

We consider now a system with N particles. The Hamiltonian is

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<l} \delta(x_j - x_l) . \tag{2.16}$$

We make the ansatz that the wave function can be written with plane-waves as

$$\Psi(x_1, \ldots, x_N; \mathcal{Q}) = \sum_{\mathcal{P}} A_{\mathcal{P}}(\mathcal{Q}) \mathrm{e}^{\mathrm{i}(k_{\mathcal{P}1}x_1 + k_{\mathcal{P}2}x_2 + \ldots k_{\mathcal{P}N}x_N)} \tag{2.17}$$

with quasi-momenta k_j to be determined. $\mathcal{P}$ is a permutation of the quasi-momenta, while $\mathcal{Q}$ is the permutation of the particle order. Since the wavefunction is bosonic, a permutation of the order should leave the wavefunction unchanged. We see by inspection that if $\mathcal{Q}$ and $\mathcal{Q}'$ are two different permutation, consistency requires that $A_{\mathcal{P}}(\mathcal{Q}') = A_{\mathcal{P}'}(\mathcal{Q})$, where $\mathcal{P}'$ is the permutation that differs from $\mathcal{P}$ in the same indices where $\mathcal{Q}'$ differs from $\mathcal{Q}$. Therefore, we can concentrate on the sector where $x_1 < x_2 < \ldots < x_N$ and solve the system in that sector. When we need to consider a different sector related to this by a permutation, we know that we can generate it by the equivalent permutation on the momenta. Thus, from now on we will drop the explicit dependence on the coordinate permutation $\mathcal{Q}$, with the understanding that we are working in the given sector and the others can be reached by symmetry.

All possible permutations can be generated by exchanging two indices at a time. This is equivalent to the two-particle scattering we consider in the previous section. Therefore

$$\frac{A_{\mathcal{P}}}{A_{\mathcal{P}'}} = \mathrm{e}^{\mathrm{i}\tilde{\theta}(k-k')} \tag{2.18}$$

where k, k' are the momenta interchanged between permutation $\mathcal{P}$ and $\mathcal{P}'$ and the scattering phase is given by (2.15). We get:

$$A_{\mathcal{P}} = C(-1)^{\mathcal{P}} \prod_{j<l} (k_{\mathcal{P}j} - k_{\mathcal{P}l} + \mathrm{i}\, c) \tag{2.19}$$

where C is a normalization constant.

Wavefunction (2.17) is an eigenfunction of the Hamiltonian operator (2.16) with eigenvalue

$$E = \sum_{j=1}^{N} k_j^2 \tag{2.20}$$

and an eigenvector of the momentum operator

$$\hat{K} = -\mathrm{i} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \tag{2.21}$$

with eigenvalue

$$K = \sum_{j=1}^{N} k_j \ . \tag{2.22}$$

2.5 Periodic Boundary Conditions

In order to quantize the system, we need to put it first in a box of finite length and will allow it to infinity at the end. We impose periodic boundary condition on the walls of this box, effectively considering the system on a circle of length L:

$$\Psi(x_1, x_2, \ldots, x_j + L, \ldots, x_N) = \Psi(x_1, x_2, \ldots, x_j, \ldots, x_N) \ , \qquad j = 1, \ldots, N \ . \tag{2.23}$$

Taking a particle around the circle means that the particle has to scatter across all other particles in the system. Through all these scattering events it will acquire a phase equal to the sum of the scattering phases associated with each scattering event plus the dynamical phase acquired through the motion (namely its momentum times L). To satisfy periodic boundary conditions, the sum of these contributions will add up to an integer multiple of 2π.

Things can be made more formal by realizing that taking a particle around the circle means moving through sectors with different particle orderings and we saw before that they can be expressed as a different permutation of the momenta. So periodic boundary conditions are satisfied if for every permutation $\mathcal{P}$ we have

$$\mathrm{e}^{\mathrm{i}k_{\mathcal{P}N}L} = \frac{A_{\tilde{\mathcal{P}}}}{A_{\mathcal{P}}} \ , \tag{2.24}$$

where $\tilde{\mathcal{P}}$ is the permutation where the last index has been moved to the beginning compared to $\mathcal{P}$ ($\tilde{\mathcal{P}}(N) = \mathcal{P}(N-1)$, $\tilde{\mathcal{P}}'(1) = \mathcal{P}(N)$, and so forth). Equivalently, this condition can be written as

$$\mathrm{e}^{\mathrm{i}k_j L} = \prod_{l\neq j} \left(\frac{k_j - k_l + \mathrm{i}\, c}{k_j - k_l - \mathrm{i}\, c} \right) , \qquad j = 1, \ldots, N . \tag{2.25}$$

taking the logarithm of these we get

$$\begin{aligned} k_j L &= 2\pi \tilde{I}_j - (N-1)\pi - 2\sum_{l=1}^{N} \arctan\left(\frac{k_j - k_l}{c} \right) \\ &= 2\pi I_j - \sum_{l=1}^{N} \theta(k_j - k_l) , \end{aligned} \tag{2.26}$$

where the $\tilde{I}_j$ are a set of integers which define the state. In the second line we introduced a new set I_j of quantum numbers to define the states: they are integers if the number of particles N is odd and half-integers if N is even. For brevity I will often refer to the I_j's as integers, since it is customary to consider systems with an odd number of particles to avoid spurious degeneracies, but the reader should remember that it is a general property of Bethe Ansatz solutions of many systems to be parametrized by integers/half-integers for odd/even number of excitations.

The (2.26) are the *Bethe equations*, a set of N coupled algebraic equations in N unknown k_j. For $c \to \infty$ we have hard-core bosons and $k_j = 2\pi I_j / L$. In general, it can be proven and it is easy to convince oneself that the ground state configuration (i.e. the state with lowest energy) is given by a set of integers symmetrically distributed around 0:

$$I_j = -\frac{N+1}{2} + j , \qquad j = 1 \ldots N . \tag{2.27}$$

From (2.22) this clearly defines a state with zero momentum. To see that it also minimizes the energy (2.20), we can start from the $c \to \infty$ solution, where this is clearly true. As we decrease the coupling constant, the k's will move, but the integers defining this state cannot change because they are quantized in integer or semi-integer values. Moreover, since it can be proven that the ground state is never degenerate, upon changing c we cannot have a level crossing and hence (2.27) always defines the state with the lowest energy. An alternative proof is based on the fact that if $I_j < I_l$, then $k_j < k_l$. Thus any state with a different set of integers will have a higher energy, compared to (2.27).

One can show [19] using the action formalism we will introduce in section 2.6 that to a given set of integers I_j's corresponds a unique solution for the k_j's (and, as a corollary, that the ground state is never degenerate). It is important to notice that setting two parameters to the same value ($I_j = I_{j'}$) would generate two identical quasi-momenta ($k_j = k_{j'}$), as can be seen by subtracting the two corresponding Bethe equations (2.26). The wavefunction (2.17) is antisymmetric in the momenta, therefore it vanishes if two of them coincide. Physically, one can think about this by considering that if two particles move with the same momentum they will never meet and interact or will always coincide and yield an infinite contribution to the energy coming from the contact interaction. The coincidence of two (or more) momenta will produce a pathology in the Bethe Ansatz scheme that has to be avoided. In summary, the quantum

numbers I_js have to be different to generate a physical state. This confers a fermionic nature to the Bethe Ansatz solution (in momentum space!), even if the underlying system is composed by bosons (in the real space) as in this case.

Finally, pleas note that since $\theta(-k) = -\theta(k)$, the momentum of the system is

$$K = \frac{2\pi}{L} \sum_j I_j \ . \tag{2.28}$$

So the momentum is quantized and does not vary as the coupling constant is varied. This is also a general result valid not only for the Lieb-Liniger model.

2.6 Action formulation of the Bethe equations

Following [26], we can introduce the action:

$$\mathcal{A} = \frac{L}{2} \sum_{j=1}^{N} k_j^2 - 2\pi \sum_{j=1}^{N} I_j k_j + \frac{1}{2} \sum_{j,l}^{N} \theta_1(k_j - k_l) \ , \tag{2.29}$$

where

$$\begin{aligned} \theta_1(k) &\equiv \int_0^k \theta(k')\mathrm{d}k' \\ &= 2k \arctan \frac{k}{c} - c \ln \left(1 + \frac{k^2}{c^2}\right) \ . \end{aligned} \tag{2.30}$$

It is easy to see that the variation of this action reproduces the Bethe equations (2.26)

$$\frac{\partial \mathcal{A}}{\partial k_j} = L k_j - 2\pi I_j + \sum_{l \neq j} \theta(k_j - k_l) = 0 \ . \tag{2.31}$$

Moreover, we can show that the extremal point, solution of the Bethe equation, is a stable solution, since the second variation

$$\mathcal{A}_{jl} \equiv \frac{\partial^2 \mathcal{A}}{\partial k_j \partial k_l} = \delta_{jl} \left[L + \sum_m \mathcal{K}(k_j, k_m) \right] - \mathcal{K}(k_j, k_l) \tag{2.32}$$

is positive definite

$$\sum_{j,l} u_j \mathcal{A}_{j,l} u_l = L \sum_j u_j^2 + \sum_{j<l} \mathcal{K}(k_j, k_l)(u_j - u_l)^2 \geq 0 \ . \tag{2.33}$$

This action formulation of the problem is not directly connected to the original Hamiltonian of the problem (eq. (2.16, for instance) and it is somehow artificial, but has several advantages and it can lead to important mathematical results. For instance it allows to prove the existence and uniqueness of the Bethe solution, it provides the easiest way to calculate the norm of the Bethe ansatz wavefunction (and to prove it's finiteness) and plays an important role in the calculation of correlation functions in the Algebraic Bethe Ansatz approach, see [19].

2.7 Thermodynamic limit

If we order the integers I_j's (and therefore the momenta k_j's) in increasing order, we can write the Bethe Equations (2.26) as

$$k_j + \frac{1}{L}\sum_{l=1}^{N}\theta(k_j - k_l) = y(k_j) \tag{2.34}$$

where we defined the *"counting function"* $y(k_j) \equiv \frac{2\pi I_j}{L}$, which is a monotonically increasing function that counts the integers as a function of the quasi-momenta (by associating its integer to each quasi-momentum). By definition,

$$y(k_j) - y(k_l) = \frac{2\pi}{L}(I_j - I_l)\ . \tag{2.35}$$

We now take the limit $N, L \to \infty$, keeping the density N/L fixed and finite. We introduce a density of quasi momenta as

$$\rho(k_j) = \lim_{N,L\to\infty} \frac{1}{L(k_{j+1} - k_j)} > 0\ . \tag{2.36}$$

Then we can replace sums with integrals over k as

$$\sum_j \to L\int \rho(k)\mathrm{d}k\ . \tag{2.37}$$

It is easy to prove that

$$y'(k_j) = \lim_{N,L\to\infty} \frac{y(k_j) - y(k_{j-1})}{k_j - k_{j-1}} = \lim_{N,L\to\infty} \frac{2\pi}{L(k_j - k_{j-1})} = 2\pi\ \rho(k_j) \tag{2.38}$$

and therefore

$$\frac{1}{2\pi}y(k) = \int^k \rho(k')\mathrm{d}k'\ , \tag{2.39}$$

establishing a direct connection between the distribution of the integers and of the quasi-momenta.

With these definitions, in the thermodynamic limit, the system of algebraic equations (2.26) can be written as an integral equation for the counting function and the momentum distribution:

$$y(k) = k + \int_{k_{min}}^{k_{max}} \theta(k - k')\rho(k')\mathrm{d}k' \tag{2.40}$$

and, by taking the derivative of this equation by k,

$$\begin{aligned}\rho(k) &= \frac{1}{2\pi} + \frac{1}{2\pi}\int_{k_{min}}^{k_{max}} \theta'(k - k')\rho(k')\mathrm{d}k' \\ &= \frac{1}{2\pi} + \frac{1}{2\pi}\int_{k_{min}}^{k_{max}} \mathcal{K}(k - k')\rho(k')\mathrm{d}k'\end{aligned} \tag{2.41}$$

where we introduced the kernel of the integral equation as the derivative of the scattering phase:

$$\mathcal{K}(k) \equiv \frac{\mathrm{d}}{\mathrm{d}k}\,\theta(k) = \frac{2c}{c^2 + k^2}\ . \tag{2.42}$$

The integral equation (2.41) with this kernel is known as the Lieb-Liniger equation [17] and it is a Fredholm type linear integral equation.

Equation (2.41) allows us to determine the distribution of the quasi-momenta. This distribution depends on the support of the kernel, in equation (2.41) the limits of integration k_{min} and k_{max}. The support is determined by the choice of the integers in the original equations (2.26). For the ground state, the limits of integration are symmetric ($k_{min} = -k_{max} = q$).

Note that for infinite repulsion ($c \to \infty$), the ground state solution of (2.41) gives the free-fermions distribution of momenta:

$$\rho(k) = \begin{cases} \frac{1}{2\pi} , & |k| \leq q , \\ 0 , & |k| > q . \end{cases} \tag{2.43}$$

A direct way to determine the limits of integration is to calculate the number of particles per unit length:

$$n \equiv N/L = \int_{-q}^{q} \rho(k) \mathrm{d}k \tag{2.44}$$

and invert this equation to calculate q in terms of N.

Finally, we can write (2.22) in the thermodynamic limit as

$$p = K/L = \int_{-q}^{q} k \, \rho(k) \mathrm{d}k = 0, \tag{2.45}$$

where we have used the fact that $\rho(-k) = \rho(k)$, and we rewrite (2.20) as

$$e \equiv E/L = \int_{-q}^{q} k^2 \, \rho(k) \mathrm{d}k. \tag{2.46}$$

Exercises

1. Solve (2.41) in the large c limit perturbatively. Start from the $c \to \infty$ solution (2.43) and write a perturbative series in powers of $1/c$

$$\rho(k) = \frac{1}{2\pi} \vartheta(q^2 - k^2) + \sum_n \frac{1}{c^n} \rho_n(k) . \tag{2.47}$$

   Then substitute it in (2.41) and solve it by equating powers of c. Can you keep q constant at each step of the perturbative series or do you have to expand it in powers of $\frac{1}{c}$ and take that in account as well?

2.8 Bound States

The construction we developed so far was based on the (implicit) assumption that the potential was repulsive ($c > 0$). For the attractive case additional solutions can be constructed, corresponding to bound states. To see this, let us consider the two-body interaction again, this time in the center of mass frame.

We rewrite (2.7,2.9) as

$$\begin{aligned}\Psi(x_1,x_2) &= \mathrm{e}^{\mathrm{i}(k_1x_1+k_2x_2)}\left[A_{12}\vartheta(x_1-x_2)+A_{21}\vartheta(x_2-x_1)\right] \\ &\quad +\mathrm{e}^{\mathrm{i}(k_2x_1+k_1x_2)}\left[A_{21}\vartheta(x_1-x_2)+A_{12}\vartheta(x_2-x_1)\right] . \end{aligned} \tag{2.48}$$

By introduction center of mass and relative coordinates and momenta as:

$$X \equiv \frac{x_1+x_2}{2} , \qquad x \equiv \frac{x_1-x_2}{2} , \tag{2.49}$$

$$K \equiv k_1+k_2 , \qquad k \equiv k_1-k_2 , \tag{2.50}$$

we can write the wavefunction (2.48) as

$$\Psi(X,x) = \mathrm{e}^{\mathrm{i}KX}\begin{cases} A_{12}\mathrm{e}^{\mathrm{i}kx}+A_{21}\mathrm{e}^{-\mathrm{i}kx}, & x>0 \\ A_{21}\mathrm{e}^{\mathrm{i}kx}+A_{12}\mathrm{e}^{-\mathrm{i}kx}, & x<0 \end{cases} . \tag{2.51}$$

The standard way to calculate the discontinuity of the wavefunction's derivative at $x=0$ is by integrating the Schrödinger equation around the discontinuity:

$$\begin{aligned}\int_{-\delta}^{\delta}\mathcal{H}\Psi(x)\mathrm{d}x &= \int_{-\delta}^{\delta}\left[-\Psi''(x)+2c\delta(x)f(x)\right]\mathrm{d}x \\ &= \Psi'(-\delta)-\Psi'(\delta)+2c\Psi(0) \\ &= E\int_{-\delta}^{\delta}\Psi(x)\mathrm{d}x = 0 . \end{aligned} \tag{2.52}$$

This condition is exactly (2.12) and can be written as

$$(A_{12}-A_{21})\,k+\mathrm{i}c\,(A_{12}+A_{21})=0 \tag{2.53}$$

We now look for solutions with complex momenta. For the solution to behave well at infinity we need $\Im K=0$. For the same reason, If $\Im k>0$, then $A_{21}=0$ and if $\Im k<0$, then $A_{12}=0$. For (2.53) to be satisfied, these conditions mean that if k has an imaginary part, its real part has to vanish and

$$A_{21}=0 \qquad \Rightarrow \qquad k=-\mathrm{i}c , \tag{2.54}$$

$$A_{12}=0 \qquad \Rightarrow \qquad k=\mathrm{i}c . \tag{2.55}$$

These solutions are compatible with the behavior at infinity only if $c<0$. Note that these conditions are equivalent to requiring that the scattering phase (2.15) diverges so that (2.12) goes to zero (or diverges) in order to kill the exponentially growing component of the wave function.

They represent a bound state of two particles with momenta

$$k_{1,2} = \frac{K\pm\mathrm{i}c}{2} , \qquad \Im K=0 , \tag{2.56}$$

with energy $E=K^2/2-c^2/2$ and total momentum K:

$$f(x_1,x_2) = \mathrm{e}^{\mathrm{i}K(x_1+x_2)/2}\mathrm{e}^{c|x_1-x_2|/2} . \tag{2.57}$$

With three particles, there are two solutions with complex momenta for $c < 0$:

$$k_1 = \alpha - \mathrm{i}c\ , \qquad k_2 = \alpha + \mathrm{i}c\ , \qquad k_3 = \beta\ , \tag{2.58}$$

$$k_1 = k_3 - \mathrm{i}c\ , \qquad k_2 = k_3 + \mathrm{i}c\ , \qquad \Im k_3 = 0\ . \tag{2.59}$$

The former is still a two-particle bound state, scattering with a third independent particle. The second is a proper three-particle bound state. n-particle bound states appear in *strings* of particles with the same real part of the momentum. A string of length n has the momenta equispaced symmetrically with respect of the real axis. This structure is quite general for may integrable models. However, while a symmetric distribution of momenta with respect to the real axis is always needed in order to have a real total momentum, the momenta do not need to lie exactly on the same line. For instance, significant deviations from this regular pattern have been observed in the XXZ model. These deviations tend to disappear in the thermodynamic limit, but the *string hypothesis* is still not proven. The repulsive Lieb-Liniger model does not create any bound state, while the attractive case creates strings of length n defined by the quasi-momenta

$$k_j = \frac{K}{n} - \mathrm{i}\,\frac{n+1-2j}{2}\,c\ , j = 1 \cdots n\ , \tag{2.60}$$

and total momentum K and energy

$$E = \frac{K^2}{n} - \frac{n(n^2-1)}{12}\,c^2\ . \tag{2.61}$$

These bound states have lower energy compared to the unbound ones. Therefore the ground state of a system of N particles is characterized by a zero total momentum string of length N, whose energy diverges negatively like N^3. Since for the thermodynamic limit to exist, the ground state energy should scale linearly with N, the attractive Lieb-Liniger is unstable in the thermodynamic limit. We will see in chapter 4 that the XXZ model also posses string solutions, but their energy (and length) is bounded from below and thus they have to be considered in its thermodynamics, enriching its physics considerably.

For a finite number of particles, the attractive Lieb-Liniger is somewhat simpler than the repulsive one, because we know the ground state solution of the Bethe equations explicitly, namely a bound state (string) of all the N particles with zero momentum. The first excitations are *one-string* $N_s = 1$ state with all N particles in a bound states with finite momentum. The *two-strings* $N_s = 2$ are made by two bound states with respectively $N - M$ and M particles, whose momentum can be determined by a system of 2 Bethe equations (for more details, see [24] or the discussion of string solutions in chapter 4. Notice that if $M = 1$ the second string collapses to a single real momentum and is not a real bound state. Multi-strings solutions can be constructed in similar ways. This "simple" structure for the spectrum of the attractive model made it possible also an explicit calculation of the basic response functions [24].

2.9 Some formalities on Integral Equations

Linear integral equations like (2.41) are a subject of a vast mathematical literature that has developed advanced ways to deal with them. Integral equations are in some sense the inverse

of differential equations.

The linear integral operator $\hat{\mathcal{K}}$ is associated with a positive kernel $\mathcal{K}(k,k')$ through the equation:

$$\left(\hat{\mathcal{K}}\rho\right)(k) \equiv \int_{-q}^{q} \mathcal{K}(k,k')\rho(k')\mathrm{d}k' \tag{2.62}$$

and equation (2.41) can be written compactly as

$$\rho - \frac{1}{2\pi}\hat{\mathcal{K}}\rho = \left(1 - \frac{1}{2\pi}\hat{\mathcal{K}}\right)\rho = \frac{1}{2\pi}\,. \tag{2.63}$$

One can then define the *resolvent* $\hat{L}$ of this operator as the operator that satisfies

$$\begin{aligned}\left(1+\hat{\mathcal{L}}\right)\,\left(1 - \frac{1}{2\pi}\hat{\mathcal{K}}\right) = 1\,, \\ \left(1+\hat{\mathcal{L}}\right)\,\hat{\mathcal{K}} = 2\pi\,\hat{\mathcal{L}}\,.\end{aligned} \tag{2.64}$$

We will need the Green function associated to a linear operator as the symmetric ($\mathcal{U}(k,k') = \mathcal{U}(k',k)$) function satisfying:

$$\mathcal{U}(k,k') - \frac{1}{2\pi}\int_{-q}^{q}\mathcal{K}(k,k'')\mathcal{U}(k',k'')\mathrm{d}k'' = \delta(k-k') \quad \Longleftrightarrow \quad \left(1 - \frac{1}{2\pi}\hat{\mathcal{K}}\right)\mathcal{U} = \mathcal{I}\,. \tag{2.65}$$

Using the Green function and the resolvent, for instance, from (2.41) we can write the density of quasi-momenta as

$$\rho(k) = \frac{1}{2\pi}\int_{-q}^{q}\mathcal{U}(k,k')\mathrm{d}k'\,, \tag{2.66}$$

$$\rho(k) - \frac{1}{2\pi} = \int_{-q}^{q}\mathcal{L}(k,k')\,\mathrm{d}k' = \frac{1}{2\pi}\int_{-q}^{q}\mathcal{K}(k,k')\,\rho(k')\,\mathrm{d}k'\,. \tag{2.67}$$

Even when the green function and the resolvent cannot be calculated analytically, often formal manipulations in terms of these operators can provide us with useful physical results. We will see some examples of this in the next section.

For the Lieb-Liniger kernel (2.42), Kac and Pollard showed that the Green function can be calculated in the singular $c \to 0$ limit (where $\mathcal{K}(k) \to \delta(k)$) and it gives

$$\mathcal{U}(x,y) \simeq \frac{1}{2\pi c}\ln\left[\frac{q^2 - xy + \sqrt{(q^2-x^2)(q^2-y^2)}}{q^2 - xy - \sqrt{(q^2-x^2)(q^2-y^2)}}\right] + \ldots \tag{2.68}$$

Exercises

1. Similarly to the exercise at the end of section 2.7, calculate the kernel's Green function in the limit $c \to \infty$ as a perturbative series in powers of $1/c$.

2.10 Elementary excitations

One of the fundamental advantage of the Bethe ansatz solution of a system is that ability to classify and understand its excitations. As we know that the Fermi liquid description breaks down in one dimension, it is of fundamental importance to identify the fundamental excitations of the system. For an integrable system, these excitations are actual stable *quasi-particles* (and this is the reason that will allow us to arrive at a complete solution of the thermodynamic of the model). In non-integrable systems these quasi-particle will not be eigenstates, but for sufficiently small perturbations from integrability, they will still be long-lived. In particular, low-energy excitations are likely to retain their nature also away from integrability. Thus, characterizing the quasi-particle content of an integrable model has the potential of teaching us much about other systems as well.

There are three kinds of elementary excitations over the ground state given by (2.27), corresponding to a symmetric distribution of momenta across the origin: we can excite one of the momenta $|k| \leq q$ and move it from its original position to one above the q-threshold; we can add a new particle with momentum $|k_p| > q$ (**Type I** excitation); and we can remove a particle and create a hole with $|k_h| \leq q$ (**Type II** excitation). Clearly, the first kind of excitation can be seen (and constructed) as a combination of the latter two and we will use this and concentrate only on type I & II excitations.

Let us start with the ground state, given by

$$\{I_j\} = \left\{ -\frac{N-1}{2}, -\frac{N-3}{2}, \ldots, \frac{N-1}{2} \right\} , \tag{2.69}$$

and consider the excitation that adds one particle (say with positive momentum), taking the number of particles from N to $N+1$. The new state is realized starting from the ground state of a system with $N+1$[2] by boosting the quantum number at the Fermi edge to a higher value:

$$\{I'_j\} = \left\{ -\frac{N}{2}, -\frac{N}{2}+1, \ldots, \frac{N}{2}+m \right\} , \tag{2.70}$$

with $m > 0$ and momentum

$$K = \frac{2\pi}{L}\, m \, . \tag{2.71}$$

This total momentum is realized through a complete rearrangement of the momenta of the particles in the system. If the original ground state configuration has momenta $\{k_1, k_2, \ldots, k_N\}$, this excited state is characterized by $\{k'_1, k'_2, \ldots, k'_N, k_p\}$, solution of a system like (2.26) but with a set of integers given by (2.70).

We see that, while the momentum of the new particle is just k_p, the momentum gained by the whole system is different and given by (2.71). The former is referred to as the *bare momentum* of the particle, in contrast with the latter, the *observed* or *dressed momentum*, due to the rearrangement of the whole system in reaction to the insertion of a new particle. This is a sign of the intrinsic sort of non-local nature of a one-dimensional system and gives a concrete

[2]Notice that adding a particle turns the integers into half-integers or viceversa.

meaning to the concept that in low-dimensional systems are intrinsically strongly interacting, regardless on the actual strength of the coupling constant.

To calculate the reaction of the system to the addition of this extra particle, let us calculate the quantity $\Delta k_j = k'_j - k_j$ by subtracting the Bethe equations for the two configurations:

$$\Delta k_j \; L = -\pi - \sum_{l=1}^{N} \left[\theta(k'_j - k'_l) - \theta(k_j - k_l)\right] - \theta(k'_j - k_p) \; . \tag{2.72}$$

(The first constant term in the right-hand-side appears because by changing the numbers of particles by one the quantum numbers shift from integers to half-integers, or viceversa.) Since Δk_j if of the order of $\mathrm{O}(L^{-1})$ (or equivalently $\mathrm{O}(N^{-1})$), the left-hand side is of order $\mathrm{O}(1)$: we can expand the right-hand side to the same order and, remembering the definition of the kernel as the derivative of the scattering phase (2.42), we obtain

$$\Delta k_j \; L = -\pi - \sum_{l=1}^{N} K(k_j - k_l) \, (\Delta k_j - \Delta k_l) - \theta(k_j - k_p) \; . \tag{2.73}$$

Collecting the terms in the following way:

$$\Delta k_j \left[1 + \frac{1}{L} \sum_{l=1}^{N} \mathcal{K}(k_j - k_l)\right] = -\frac{1}{L} \left[\pi + \theta(k_j - k_p)\right] + \frac{1}{L} \sum_{l=1}^{N} \mathcal{K}(k_j - k_l) \Delta k_l \; , \tag{2.74}$$

we can go in the thermodynamic limit and with the help of (2.41) write

$$2\pi \; \Delta k \; \rho(k) = -\frac{1}{L} \left[\pi + \theta(k - k_p)\right] + \int_{-q}^{q} \mathcal{K}(k - k') \; \Delta k' \; \rho(k') \; \mathrm{d}k' \; . \tag{2.75}$$

We introduce the *back-flow* or *shift function*

$$J(k|k_p) \equiv L \; \Delta k \; \rho(k) = \lim_{k \to k_j} \lim_{N,L \to \infty} \frac{k_j - k'_j}{k_{j+1} - k_j} \; , \tag{2.76}$$

which satisfies the integral equation above

$$J(k|k_p) - \frac{1}{2\pi} \int_{-q}^{q} \mathcal{K}(k - k') \; J(k'|k_p) \; \mathrm{d}k' = -\frac{1}{2\pi} \tilde{\theta}(k - k_p) \; , \tag{2.77}$$

where we remember the definition of the scattering phase $\tilde{\theta}(k)$ from (2.14). Using the Green function introduced in (2.65), this equation can be written as

$$J(k|k_p) = -\frac{1}{2\pi} \int_{-q}^{q} \mathcal{U}(k, k') \tilde{\theta}(k' - k_p) \; \mathrm{d}k' \; . \tag{2.78}$$

The backflow is particularly useful in calculating the changes in the macroscopical quantities under the addition of an excitation with momentum $|k_p| \geq q$, namely

$$\begin{aligned} \Delta p(k_p) &= \frac{2\pi}{L} \; m = k_p + \sum_{j=1}^{N} \left[k'_j - k_j\right] \\ &= k_p + \sum_{j=1}^{N} \Delta k_j \\ &= k_p + \int_{-q}^{q} J(k|k_p) \mathrm{d}k \; , \end{aligned} \tag{2.79}$$

where we used (2.76). Similarly, for the energy

$$\begin{aligned}\Delta e(k_p) &= k_p^2 + \sum_{j=1}^{N} \left[k_j'^2 - k_j^2\right] \\ &= k_p^2 + \sum_{j=1}^{N} \left[2k_j \Delta k_j + (\Delta k_j)^2\right] \\ &= k_p^2 + \int_{-q}^{q} 2k\ J(k|k_p)\mathrm{d}k\ , \end{aligned} \tag{2.80}$$

remembering that the term $(\Delta k_j)^2$ is suppressed like $1/N$ in comparison with the leading one.

These equations really show that this excitation has a collective nature and cannot be assigned simply at the single bosons we added. This is the difference between the bare and dressed quantities. We have added a particle with bare momentum k_p and bare energy k_p^2, but the all system rearranges itself and acquire a different momentum and energy, as per (2.79) and (2.80), i.e. the dressed quantities.

Using formal manipulations, additional identities can be reached. For instance, using (2.78) and (2.66) we can remove the backflow from (2.79)

$$\begin{aligned}\Delta p(k_p) &= k_p - \frac{1}{2\pi}\int_{-q}^{q}\mathrm{d}k \int_{-q}^{q}\mathrm{d}k' \mathcal{U}(k,k')\tilde{\theta}(k'-k_p) \\ &= k_p - \int_{-q}^{q}\rho(k)\tilde{\theta}(k-k_p)\ \mathrm{d}k\ , \end{aligned} \tag{2.81}$$

where we also used the fact that the Green function is symmetric in the exchange of variables.

These manipulations are useful to study the dispersion relation of these modes. As we mentioned, the addition of a new particle is referred to as a **Type I** excitation. Comparing its dispersion relation to that of solutions of the classical Non-Linear Schrödinger equation in the $c \to 0$ limit, one can identify these Type I excitations with the Bogoliubov excitation (i.e. a sound wave) [25].

Let us now consider a different kind of excitation, a hole, obtained by removing a particle from the Fermi sea of the ground state configuration. To take into account the shift of the quantum numbers from integers to half-integers (and viceversa) when the number of particles change by one unity, the process of creating a hole can be realized starting from the ground state of the system with $N-1$ particles by displacing a quantum number within the Fermi sea to the nearest empty space, over the Fermi point, i.e. (for a hole carrying a positive momentum $K = \frac{2\pi}{L}\ m$)

$$\{I_j''\} = \left\{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, \frac{N}{2}-m-1, \frac{N}{2}-m+1, \ldots, \frac{N}{2}\right\}\ . \tag{2.82}$$

As before we can consider the reaction of the system as the momenta of the quasi-particles change to accommodate for the absence of a particle with momentum $|k_h| < q$, corresponding to the missing quantum number. Proceeding in the same way, we can introduce the back-flow for this hole excitation, which in this case satisfies the following integral equation:

$$J(k|k_h) - \frac{1}{2\pi}\int_{-q}^{q}\mathcal{K}(k-k')\ J(k'|k_h)\ \mathrm{d}k' = \frac{1}{2\pi}\tilde{\theta}(k-k_h)\ . \tag{2.83}$$

The change in energy and momentum for the whole system are

$$\Delta p(k_h) = -k_h + \int_{-q}^{q} J(k|k_h)\mathrm{d}k \,, \tag{2.84}$$

$$\Delta e(k_h) = -k_h^2 + \int_{-q}^{q} 2k\, J(k|k_h)\mathrm{d}k \,. \tag{2.85}$$

One can prove [25] that these **Type II** excitations are not sound waves, but have a solitonic nature. In fact, they correspond to the dark solitons of the Gross-Pitaevskii equations that describe the semi-classical limit of the non-linear Schrödinger equation (i.e. the second quantized form of the Lieb-Liniger model (2.16)) in the weakly interacting limit (small c). In this semiclassical limit, the dispersion relations of the solitons and the Type II excitations agree remarkably well. As one increases the interaction strength, the semiclassical limit breaks down, but the Type II excitation's dispersion relation retains it qualitative shape. Moreover, in the $c \to \infty$ limit, it matches the dispersion relation of the solitons of the hydrodynamic description of free fermions. This justifies considering Type II excitations as a quantum sort of soliton, even beyond the semiclassical limit where this makes sense [25].

Due to the linear nature of the integral equations defining the backflow (2.77, 2.83), all excitations can be constructed from these two fundamental ones we just considered. In particular, the excitation of a particle from its ground state configuration to an excited level can be seen as the sequence of the removal of that particle from the system and reinsertion of a particle in the desired level. After each operation of this type the whole system goes through a rearrangement, that dresses the particles and the final configuration is given by the backflow defined by an integral equation like (2.77, 2.83), but with the source term given the sums of each contribution:

$$\begin{aligned} J(k|k_{p1}\dots k_{pM^+};k_{h1}\dots k_{hM^-}) &= \frac{1}{2\pi}\int_{-q}^{q} \mathcal{K}(k-k')J(k'|k_{p1}\dots k_{pM^+};k_{h1}\dots k_{hM^-})\mathrm{d}k' \\ &\quad -\frac{1}{2\pi}\sum_{j=1}^{M^+}\tilde{\theta}(k-k_{pj}) - \frac{1}{2\pi}\sum_{j=1}^{M^-}\tilde{\theta}(k-k_{hj}) \\ &= \sum_{j=1}^{M^+} J(k|k_{pj}) + \sum_{j=1}^{M^-} J(k|k_{hj}) \,. \end{aligned} \tag{2.86}$$

So far, we described the excitations using a representation in terms of density of quasi-momenta. Let us now introduce a function $\varepsilon(k)$ as the solution of the linear integral equation

$$\varepsilon(k) - \frac{1}{2\pi}\int_{-q}^{q} \mathcal{K}(k,k')\,\varepsilon(k')\,\mathrm{d}k' = k^2 - h \equiv \varepsilon_0(k) \,, \tag{2.87}$$

with the boundary condition

$$\varepsilon(q) = \varepsilon(-q) = 0 \,. \tag{2.88}$$

This is the same integral equation satisfied by the momentum density $\rho(k)$, but with a different source (the right-hand-side of (2.87)). This source term comes out naturally in a grand-canonical approach. So far, we implicitly worked in the macro-canonical approach. If we relax the

fixed number of particle condition, we introduce a chemical potential h, which is the Lagrange multiplier appearing in (2.87). The relation between the chemical potential and the number of particles is given by the boundary condition (2.88), that implicitly relates h to the support of the integral equation q.

Physically, the function $\varepsilon(k)$ defined by (2.87) can be interpreted as the dressed energy of a particle with momentum k above the ground state. From this physical point of view, condition (2.88) follows from equilibrium and shows that the theory is gapless. (2.87) will be derived as the zero-temperature limit of the free energy of the system, see later in section 2.11. Therefore the function satisfies the following properties:

$$\varepsilon'(k) > 0 \qquad \text{for} \qquad k > 0 \tag{2.89}$$

$$\varepsilon(k) = \varepsilon(-k) \ , \tag{2.90}$$

$$\varepsilon(k) < 0 \qquad \text{for} \qquad |k| < q \ , \tag{2.91}$$

$$\varepsilon(k) > 0 \qquad \text{for} \qquad |k| > q \ . \tag{2.92}$$

To support our interpretation of the function $\varepsilon(k)$, let us calculate the change in energy due, for instance, by the insertion of a new particle and the removal of another (hole). From (2.46):

$$\begin{aligned} \Delta e(k_p, k_h) &= \varepsilon_0(k_p) - \varepsilon_0(k_h) + \sum_j \left[\varepsilon_0(k'_j) - \varepsilon_0(k_j)\right] \\ &= \varepsilon_0(k_p) - \varepsilon_0(k_h) - \int_{-q}^{q} \varepsilon'_0(k') \ J(k'|k_p, k_h) \ \mathrm{d}k' \ . \end{aligned} \tag{2.93}$$

where

$$\varepsilon_0(k) \equiv k^2 - h \tag{2.94}$$

was introduced in (2.87).

We wish to prove that

$$\Delta e(k_p, k_h) = \varepsilon(k_p) - \varepsilon(k_h) \ . \tag{2.95}$$

To do so, a certain formal manipulations along the lines of the previous section will be useful. By taking the derivative by k of (2.87) we have

$$\left[\left(1 - \frac{1}{2\pi}\hat{\mathcal{K}}\right)\varepsilon'\right](k) = \varepsilon'_0(k) \ , \tag{2.96}$$

Acting on this with the operator $1+\hat{\mathcal{L}}$ and using the defining property of the resolvent in (2.64):

$$\varepsilon'(k) - \varepsilon'_0(k) = \int_{-q}^{q} \mathcal{L}(k, k') \ \varepsilon'_0(k') \ \mathrm{d}k' \ . \tag{2.97}$$

We can rewrite (2.77,2.83) as

$$\left[\left(1 - \frac{1}{2\pi}\hat{\mathcal{K}}\right) J\right](k) = -\frac{1}{2\pi}\int_{k_h}^{k_p} \tilde{\theta}'(k - k') \ \mathrm{d}k' = -\frac{1}{2\pi}\int_{k_h}^{k_p} \mathcal{K}(k - k') \ \mathrm{d}k' \ . \tag{2.98}$$

By acting with $1 + \hat{\mathcal{L}}$ we get

$$J(k|k_p, k_h) = -\int_{k_h}^{k_p} \mathcal{L}(k - k') \ \mathrm{d}k' \ . \tag{2.99}$$

By combining (2.97) and (2.99) with (2.93) we have

$$\begin{aligned}\Delta e(k_p,k_h) &= \varepsilon_0(k_p)+\varepsilon_0(k_h)+\int_{k_h}^{k_p}\mathrm{d}k\int_{-q}^{q}\mathcal{L}(k,k')\,\varepsilon_0'(k')\,\mathrm{d}k' \\ &= \varepsilon_0(k_p)-\varepsilon_0(k_h)+\int_{k_h}^{k_p}\left[\varepsilon'(k)-\varepsilon_0'(k)\right]\mathrm{d}k \\ &= \varepsilon(k_p)-\varepsilon(k_h)\;, \end{aligned} \tag{2.100}$$

as we set out to prove.

To conclude, we can define the one-particle excitation energy from

$$\varepsilon(k)-\frac{1}{2\pi}\int_{-q}^{q}\mathcal{K}(k,k')\,\varepsilon(k')\,\mathrm{d}k' = k^2-h \tag{2.101}$$

and its momentum from (2.81)

$$p(k) = k-\int_{-q}^{q}\tilde{\theta}(k-k')\,\rho(k')\,\mathrm{d}k'\;. \tag{2.102}$$

As we add particles with $|k_p|\geq q$ and holes with $|k_h|<q$, the total change in energy and momentum of the system is

$$\Delta e = \sum_{\text{particles}}\varepsilon(k_p)-\sum_{\text{holes}}\varepsilon(k_h)\;, \tag{2.103}$$

$$\Delta p = \sum_{\text{particles}}p(k_p)-\sum_{\text{holes}}p(k_h)\;. \tag{2.104}$$

In these equations we see once more the fermionic nature of a one-dimensional bosonic system.

Finally, let us consider the dressed scattering matrix. The scattering matrix of several particles is equal to the product of consecutive two-particle scattering matrices, each of which are just a phase factor:

$$\mathcal{S}\equiv \mathrm{e}^{\mathrm{i}\delta(k_1,k_2)}\;, \tag{2.105}$$

where the real phase can be written as

$$\delta(k_1,k_2)\equiv\phi_{12}-\phi_1\;. \tag{2.106}$$

The second factor ϕ_1 is the phase acquired by particle 1 by moving through the whole system with particle 2 removed from it, and the first factor ϕ_{12} is the same but with particle 2 in the system. Without particle 2, the bare momenta are $\{k_j\}$, but adding this particle shifts them to $\{k_j'\}$. We have

$$\phi_1 = Lk_1-\sum_{l=1}^{N}\tilde{\theta}(k_1-k_l)\;, \tag{2.107}$$

$$\phi_{12} = Lk_1-\sum_{l=1}^{N}\tilde{\theta}(k_1-k_l')-\tilde{\theta}(k_1-k_2)\;. \tag{2.108}$$

$$\tag{2.109}$$

From this we have

$$\begin{aligned}\delta(k_1,k_2) &= -\tilde{\theta}(k_1-k_2) + -\sum_{l=1}^{N}\left[\tilde{\theta}(k_1+k_l') - \tilde{\theta}(k_1-k_l)\right] \\ &= -\tilde{\theta}(k_1-k_2) + -\sum_{l=1}^{N}\mathcal{K}(k_1,k_l)\left[J(k_l|k_1,k_2) - J(k_l|k_1)\right](k_{l+1}-k_l) \\ &\to -\tilde{\theta}(k_1-k_2) + \int_{-q}^{q}\mathcal{K}(k_1,k)\; J(k|k_2)\;\mathrm{d}k = 2\pi J(k_1|k_2) \end{aligned} \tag{2.110}$$

where in the last line we took the thermodynamic limit and used (2.77). Therefore, we see that the scattering phase satisfies a linear integral equation of the same type as before:

$$\delta(k,k') - \frac{1}{2\pi}\int_{-q}^{q}\mathcal{K}(k,k'')\;\delta(k'',k')\;\mathrm{d}k'' = \delta(k-k')\;. \tag{2.111}$$

Proceeding in the same way, one can calculate the scattering phase of two holes and find it equal to the one of two particles. The scattering phase of a particle with a hole is instead

$$\mathcal{S} = \mathrm{e}^{-\mathrm{i}\delta(k_h,k_p)}\;. \tag{2.112}$$

In this section we saw that the same kind of integral equation, with the same kernel but with different source terms (and the appropriate boundary conditions) generates the various physical quantities that characterize each state, both in their bare and dressed form. As we saw, these formulations are not independent from one another and a certain dexterity with formal manipulation can be required to relate them. Therefore, to keep things as simple as possible one should choose from which to start taking into consideration the problem one is interested in. We will see that a very natural choice in terms of the dressed energy appear in the thermodynamic formulation of Bethe Ansatz.

2.11 Thermodynamics of the model: the Yang-Yang equation

We now want to describe the system at finite temperature and therefore we will study in more generality excited and mixed states.

In (2.34) we introduced the counting function $y(k)$, which interpolates between the quantum numbers of the system. This means that for a state characterized by the set $\{I_j\}$, it is constrain to satisfy

$$y(k_j) = \frac{2\pi}{L}I_j\;, \tag{2.113}$$

for the values k_j which are the solutions of the Bethe Equations.

Let us now take the view that the set of quasi-momenta k_j (solutions of the Bethe equations) are given, and thus the counting function is defined by the equation

$$y(k) = k + \frac{1}{L}\sum_{l=1}^{N}\theta\left(k-k_j\right) \tag{2.114}$$

for generic k. At $k = k_j$, by construction we will have (2.113), but we can look for all the points $\tilde{k}$ for which the counting function takes the quantized value

$$y(\tilde{k}) = \frac{2\pi}{L} n \ , \tag{2.115}$$

for some n, integer or half integer as the I_j. The set of $\tilde{k}$ are called vacancies, and correspond to all the possible solutions of the Bethe equation for a set given set of quasi-momenta k_j. The subset of $\tilde{k}$ that are the physical solutions k_j represents the *particles*. The remaining solutions are the *holes* of that state. In particular, if we consider a state generated from the ground states by removing some quantum numbers from inside the Fermi sphere, those k are the holes. In particular, the hole momenta k_j^h are associated to a set of integers (or half integers) J_j, which are not present in the set I_j specifying the state.

As we defined the density of momenta for the particles

$$\rho(k_j) = \lim_{N,L\to\infty} \frac{1}{L\left(k_{j+1} - k_j\right)} \ , \tag{2.116}$$

we can define a density of vacancies

$$\rho_t(k_j) = \lim_{N,L\to\infty} \frac{1}{L\left(\tilde{k}_{j+1} - \tilde{k}_j\right)} \tag{2.117}$$

and a hole density

$$\rho_h(k_j) = \lim_{N,L\to\infty} \frac{1}{L\left(k^h_{j+1} - k^h_j\right)} \ . \tag{2.118}$$

This means that $L\rho(k)\mathrm{d}k$ is the number of k's in the interval $\mathrm{d}k$, while $L\rho_h(k)\mathrm{d}k$ is the number of missing k's, or holes, and $L\rho_t(k)\mathrm{d}k = L\left[\rho(k) + \rho_h(k)\right]\mathrm{d}k$.

Using the same reasoning at for zero-temperature, we have

$$\begin{aligned} y'(\tilde{k}_j) &= \lim_{N,L\to\infty} \frac{y(\tilde{k}_j) - y(\tilde{k}_{j-1})}{\tilde{k}_j - \tilde{k}_{j-1}} \\ &= \lim_{N,L\to\infty} \frac{2\pi}{L(\tilde{k}_j - \tilde{k}_{j-1})} \\ &= 2\pi\rho_t(\tilde{k}_j) \ , \end{aligned} \tag{2.119}$$

and thus

$$y(k) = 2\pi \int^k \left[\rho(k') + \rho_h(k')\right] \mathrm{d}k' \ , \tag{2.120}$$

Taking the thermodynamic limit, we write (2.114) as

$$y(k) = k + \int_{-\infty}^{\infty} \theta\left(k - k'\right) \rho(k')\mathrm{d}k' \ . \tag{2.121}$$

By taking its derivative with respect to k like we did in (2.40), we get

$$\rho(k) + \rho_h(k) = \frac{1}{2\pi} + \int_{-\infty}^{\infty} \mathcal{K}(k,k')\ \rho(k')\ \mathrm{d}k' \ . \tag{2.122}$$

Note that, compared to the integral equations we dealt with so far, the introduction of the densities of holes now allows the support of the integral to extend over the whole real axis. Compared to the zero-temperature case, this integral equation in not closed, as $\rho(k)$ depends on the, yet undetermined, density of holes $\rho_h(k)$.

In the thermodynamic description of a system, we always passes from a microscopic to a macroscopic characterization of the states. For a Bethe ansatz solvable models, this amounts to switching from the quantum numbers $\{I_j\}$ to the particle and holes densities. Thus, let us start with a given set of vacancies $\{\tilde{k}_j\}$. A different choices of the quantum numbers $\{I_j\}$ of the state corresponds to a different partition of the vacancies into a set of particles and holes. However, small fluctuations over a given configuration (such as, for instance, the exchange of two neighboring vacancies of a particle with a hole) are not macroscopically distinguishable, since they differ by order $1/N$. Thus, for every infinitesimal interval, we can arbitrarily arrange the order of particles and holes. This amounts to an entropy of the macroscopical configuration that we can compute in the standard way as the number of ways we can order a given number of particles and holes between k and $k + \mathrm{d}k$, i.e.

$$\begin{aligned} \mathrm{d}\mathcal{S} &= \ln \frac{[L\left(\rho(k)+\rho_h(k)\right)\mathrm{d}k]!}{[L\rho(k)\mathrm{d}k]!\,[L\rho_h(k)\mathrm{d}k]!} \\ &\approx L\,\mathrm{d}k\left[\left(\rho(k)+\rho_h(k)\right)\ln\left(\rho(k)+\rho_h(k)\right)-\rho(k)\ln\rho(k)-\rho_h(k)\ln\rho_h(k)\right]\,, \end{aligned} \tag{2.123}$$

where in the last line we used Stirling's formula ($\ln n! \approx n\ln n - n$).

We are now interested in calculating the free energy of the model. We start from the partition function:

$$\mathcal{Z} = \frac{1}{N!}\sum_{\{I_j\}} \exp\left[-\frac{E_N}{T}\right] = \sum_{I_1<I_2<\ldots<I_N} \exp\left[-\frac{E_N}{T}\right]\,, \tag{2.124}$$

where $E_N = \sum_{j=1}^{N} k_j^2$ and the quasi-momenta k_j are the solutions of the Bethe equation with the given set of quantum numbers $\{I_j\}$. Introducing the variables $n_j = I_{j+1} - I_j$, we can rewrite (2.124) as

$$\mathcal{Z} = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_{N-1}=1}^{\infty} \mathrm{e}^{-E_N/T}\,. \tag{2.125}$$

Taking the thermodynamic limit, we notice that n_j counts how many holes are present between two consecutive particles, thus:

$$n_j = I_{j+1} - I_j \simeq \frac{\rho(k)+\rho_h(k)}{\rho(k)}\,. \tag{2.126}$$

This allows us to switch in (2.125) from sums to integrals and use the macroscopical variables ρ and ρ_h

$$\mathcal{Z} = \mathrm{const}\int \mathcal{D}\left(\frac{\rho(k)+\rho_h(k)}{\rho(k)}\right)\delta\left(\int\rho(k)\mathrm{d}k - n\right)\mathrm{e}^{\mathcal{S}-e/T}\,, \tag{2.127}$$

where $\mathcal{S}$ is the entropy (2.123) and $e = E_N/L = \int k^2\rho(k)\mathrm{d}k$ is the energy of the state. We also introduced a delta-function to enforce number of particles conservation in a macro-canonical ensemble.

By using the representation

$$\delta(x) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{hx} \mathrm{d}h \ , \tag{2.128}$$

we can write (2.127) as

$$\mathcal{Z} = \text{const} \int \mathrm{d}h \int \mathcal{D} \left(\frac{\rho(k) + \rho_h(k)}{\rho(k)} \right) \mathrm{e}^{W[\rho, \rho_h; h]} \ , \tag{2.129}$$

where

$$\begin{aligned} \mathcal{W}[\rho, \rho_h; h] \ \equiv \ & -\frac{L}{T} \int \mathrm{d}k \left\{ k^2 \rho(k) + h \left[\rho(k) - n \right] \right. \\ & \left. -T \Big[\left(\rho(k) + \rho_h(k) \right) \ln \left(\rho(k) + \rho_h(k) \right) - \rho(k) \ln \rho(k) - \rho_h(k) \ln \rho_h(k) \Big] \right\} \ . \end{aligned} \tag{2.130}$$

The Lagrange multiplier h has the physical interpretation of a chemical potential.

As $L \to \infty$, we can employ a saddle-point approximation to find the configuration that extremize the action and give the most relevant contribution to the partition function:

$$\begin{aligned} \delta\mathcal{W}[\rho, \rho_h; h] \ = \ & -\frac{L}{T} \int \mathrm{d}k \left\{ \left[k^2 - h - T \ln \left(\frac{\rho(k) + \rho_h(k)}{\rho(k)} \right) \right] \delta\rho(k) \right. \\ & \left. -T \ln \left(\frac{\rho(k) + \rho_h(k)}{\rho_h(k)} \right) \delta\rho_h(k) \right\} = 0 \ . \end{aligned} \tag{2.131}$$

Using (2.122) as

$$\delta\rho_h(k) = -\delta\rho(k) + \frac{1}{2\pi} + \int_{-\infty}^{\infty} \mathcal{K}(k, k') \ \delta\rho(k') \ \mathrm{d}k' \ , \tag{2.132}$$

we can eliminate ρ_h from (2.131) to get

$$\int \mathrm{d}k \left\{ k^2 - h - T \ln \left(\frac{\rho_h(k)}{\rho(k)} \right) - \frac{T}{2\pi} \int \mathcal{K}(k, k') \ln \left(1 + \frac{\rho(k')}{\rho_h(k')} \right) \mathrm{d}k' \right\} \delta\rho(k) = 0 \ . \tag{2.133}$$

For this condition to hold for any $\delta\rho$, we demand

$$\varepsilon(k) = k^2 - h - \frac{T}{2\pi} \int_{-\infty}^{\infty} \mathcal{K}(k, k') \ln \left(1 + \mathrm{e}^{-\varepsilon(k')/T} \right) \mathrm{d}k' \ , \tag{2.134}$$

where we defined

$$\varepsilon(k) \equiv T \ln \left(\frac{\rho_h(k)}{\rho(k)} \right) \qquad \to \qquad \frac{\rho_h(k)}{\rho(k)} = \mathrm{e}^{\varepsilon(k)/T} \ . \tag{2.135}$$

Equation (2.134) is a non-linear integral equation whose solution gives us the dressed energy per particle excitation, using which the thermodynamic quantities are at hand. Equation (2.134) is known as the Yang-Yang equation. The interpretation of the meaning of the function $\varepsilon(k)$ is supported by noting that the number of excitation over the number of available states is

$$\frac{\rho(k)}{\rho(k) + \rho_h(k)} = \frac{1}{1 + \mathrm{e}^{\varepsilon(k)/T}} \equiv \vartheta(k) \ , \tag{2.136}$$

where we see that $\vartheta(k)$ is the usual Fermi weight distribution.

By substituting this saddle-point configuration in the expression for the partition function (2.127) we find the free energy to be

$$\mathcal{F} = -T \ln \mathcal{Z} = Nh - \frac{TL}{2\pi} \int \mathrm{d}k \ln \left(1 + \mathrm{e}^{-\varepsilon(k)/T}\right) . \tag{2.137}$$

We see now why the Bethe Ansatz construction is so powerful in addressing the thermodynamic of an integrable model. Equation (2.137) is the partition function of a system of non-interacting particles with single-particle spectrum $\varepsilon(k)$. That is, once the Yang-Yang equation (2.134) has been solved (maybe numerically, or by a series expansion...) and the dressed energies have been calculated, the strongly interacting problem of the integrable theory is reduced to the partition function of a free theory with a non-trivial spectrum.

To conclude, from the knowledge of the partition function (2.137), the whole thermodynamic of the model can be calculated. The pressure is

$$P = -\left(\frac{\partial \mathcal{F}}{\partial L}\right)_T = \frac{T}{2\pi} \int \mathrm{d}k \ln \left(1 + \mathrm{e}^{-\varepsilon(k)/T}\right) . \tag{2.138}$$

(it satisfies $\mathrm{d}P = \mathcal{S}/L\mathrm{d}T + n\mathrm{d}h$) and

$$n = -\frac{\partial}{\partial h}\left(\mathcal{F} - Nh\right) , \qquad \mathcal{S} = -\frac{\partial \mathcal{F}}{\partial T} , \qquad e = \mathcal{F} + T\mathcal{S} , \ldots \tag{2.139}$$

The density of quasi momenta can be determined from the energy per particle using (2.122):

$$2\pi\rho(k)\left[1 + \mathrm{e}^{\varepsilon(k)/T}\right] = 1 + \int \mathcal{K}(k,k')\rho(k')\mathrm{d}k' . \tag{2.140}$$

The number of particle is always given by

$$n = \int \rho(k)\mathrm{d}k . \tag{2.141}$$

Before leaving this model, let us consider some limiting cases:

2.11.1 $T \to 0^+$

For $h < 0$, one can show that $n = 0$.

For $h > 0$, one can show that the function $\varepsilon(k)$ has two zeros on the real axis for

$$\varepsilon(\pm q) = 0 , \qquad h > 0 , \tag{2.142}$$

and

$$\varepsilon(k) > 0 , \qquad |k| > q , \tag{2.143}$$

$$\varepsilon(k) < 0 , \qquad |k| < q , \tag{2.144}$$

$$\varepsilon'(k) > 0 , \qquad k > 0 , \tag{2.145}$$

$$\varepsilon'(k) < 0 , \qquad k < 0 . \tag{2.146}$$

Taking the zero-temperature limit of the Yang-Yang equation (2.134) it becomes linear

$$\varepsilon(k) - \frac{1}{2\pi} \int_{-q}^{q} \mathcal{K}(k,k')\, \varepsilon(k')\mathrm{d}k' = k^2 - h . \tag{2.147}$$

Consequently,

$$\rho(k) = 0 \,, \qquad |k| > q \,, \tag{2.148}$$

$$\rho_h(k) = 0 \,, \qquad |k| < q \,, \tag{2.149}$$

$$\tag{2.150}$$

and

$$\rho(k) - \frac{1}{2\pi}\int_{-q}^{q} \mathcal{K}(k,k')\,\rho(k')\mathrm{d}k' = \frac{1}{2\pi}\,. \tag{2.151}$$

The zero-temperature pressure is

$$P = -\frac{1}{2\pi}\int_{-q}^{q} \varepsilon(k)\mathrm{d}k\,. \tag{2.152}$$

2.11.2 $c \to \infty$

In this limit the kernel vanishes, therefore

$$\varepsilon(k) = k^2 - h \tag{2.153}$$

and

$$\rho(k) = \frac{1}{2\pi}\frac{1}{1+\mathrm{e}^{(k^2-h)/T}}\,, \tag{2.154}$$

$$\mathcal{F} = Nh - \frac{T}{2\pi}\int \mathrm{d}k \ln\left(1+\mathrm{e}^{-(k^2-h)/T}\right)\,. \tag{2.155}$$

This is equivalent to free fermions.

2.11.3 $c \to 0^+$

In this limit

$$\mathcal{K}(k,k') \to \pi\delta(k-k')\,. \tag{2.156}$$

Therefore

$$\varepsilon(k) = T\ln\left[\mathrm{e}^{(k^2-h)/T} - 1\right] \tag{2.157}$$

and

$$\rho(k) = \frac{1}{2\pi}\frac{1}{\mathrm{e}^{(k^2-h)/T}-1}\,, \tag{2.158}$$

$$\rho_h(k) = \frac{1}{2\pi}\,, \tag{2.159}$$

$$\mathcal{F} = Nh - \frac{T}{2\pi}\int \mathrm{d}k \ln\left(1-\mathrm{e}^{-(k^2-h)/T}\right)\,. \tag{2.160}$$

This is coherent with what we know as free bosons.

Exercises

1. Calculate the correction to the Free Fermions result in a $1/c$ series and verify that

$$\begin{aligned}
\varepsilon(k) &= k^2 - h - \frac{2}{c}P + \mathrm{O}\left(\frac{1}{c^3}\right) , && (2.161)\\
\rho(k) &= \frac{1}{2\pi}\left(\frac{1+\frac{2}{c}\,n}{1+\mathrm{e}^{\varepsilon(k)/T}}\right) , && (2.162)\\
\rho_h(k) &= \frac{1}{2\pi}\left(\frac{1+\frac{2}{c}\,n}{1+\mathrm{e}^{-\varepsilon(k)/T}}\right) , && (2.163)\\
\rho_t(k) &= \frac{1}{2\pi}\left(1+\frac{2}{c}\,n\right) , && (2.164)\\
n &= \int_{-\infty}^{\infty}\rho(k)\mathrm{d}k\ . && (2.165)
\end{aligned}$$

Chapter 3

The Heisenberg chain

3.1 Introduction

The Heisenberg spin chain is the prototype model for the application of the Bethe ansatz approach. The basic techniques to analyze it are those introduced in the previous chapter about the Lieb-Liniger model. And we will see that the solution of the XXX chain goes through the same steps. However, the physics of this model is very rich and much more complex than before. To give a complete picture is quite complicated and out of the scope of these lectures. For the interested reader, we strongly recommend the set of notes [27][1], which is an introduction to the coordinate Bethe ansatz, mostly from a computational point of view, that gives a simple and clear overview over the complexity of the XXX chain.

The isotropic Heisenberg Chain was solved by Bethe in 1931 [29] using the intuition which will become the Bethe Ansatz technique. At the time, Bethe was very intrigued by the success of his idea and promised to go back to it to investigate it further. But he never did. In his career, Bethe contributed to virtually all fields of physics and in many of them he brought innovative ideas and concepts. His creativity was such that he never had time to get involved in the development of the Bethe Ansatz techniques and eventually lost track of the most advanced progresses in them.

Since then, the XXX model (and the XXZ chain, its generalization we will consider in the next chapter) have been the bench tools to advance the Bethe idea from one side, and to understand one-dimensional magnetism from the other. We should remark that nowadays we have numerical methods to diagonalize one-dimensional models that are more efficient than the Bethe ansatz approach. However the analytical nature of the latter and the characterization of the states in terms of a set of quantum numbers render this technique very useful for understanding what are the relevant physical processes and to develop even better approximation schemes. In particular, similarly to what we did for the Lieb-Liniger model, we will be able to give a very accurate representation of the low-energy states of the system, which are particularly important in view of taking a scaling limit.

[1] The first draft of these notes was based on [27] as well.

3.2 Definition of the model

The Hamiltonian of the Heisenberg model of spin-1/2 in one-dimension with N sites with periodic boundary conditions $\mathbf{S}_{N+j} = \mathbf{S}_j$ is given by

$$\begin{aligned} \mathcal{H} &= -J\sum_{n=1}^{N} \mathbf{S}_n \cdot \mathbf{S}_{n+1} \\ &= -J\sum_{n=1}^{N}\left[\frac{1}{2}(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+) + S_n^z S_{n+1}^z\right], \end{aligned} \tag{3.1}$$

where $S_n^{\pm} \equiv S_n^x \pm i S_n^y$ are spin flip operators. $\mathcal{H}$ acts on a Hilbert space of dimension 2^N spanned by the orthogonal basis vectors $|\sigma_1 \dots \sigma_N\rangle$, where $\sigma_n =\uparrow$ represents an up spin and $\sigma_n =\downarrow$ a down spin at site n. The spin commutation relations (with $\hbar = 1$) are

$$[S_n^z, S_{n'}^{\pm}] = \pm S_n^{\pm}\delta_{nn'}, \quad [S_n^+, S_{n'}^-] = 2S_n^z\delta_{nn'}. \tag{3.2}$$

The coupling J sets the energy scales, thus the Hamiltonian (3.1) has the same eigenstates, independently of J. However, the order of the states is reversed by changing the sign of the coupling. $J > 0$ favors ferromagnetic alignment, while $J < 0$ gives an antiferromagnet. The Bethe ansatz diagonalization gives the same result for any J, but the ground state nature (and hence the low-energy excitations) will differ greatly in the FM and AFM case.

Looking for the solution of the model, we will take advantage of the symmetries of the model. The (lattice) translational invariance will be used in constructing the eigenstates as superpositions of plane wave (same as for the Lieb-Liniger model). The Heisenberg chain also posses full $SU(2)$ rotational invariance. However, we will only need rotational symmetry about the z-axis in spin space, which we take to be the quantization axis. Thus, the z-component of the total spin $S^z \equiv \sum_{n=1}^{N} S_n^z$ is conserved: $[\mathcal{H}, S^z] = 0$. Since the magnetization is conserved, we can consider separately sectors defined by the quantum number $S^z = N/2 - r$, where r is the number of down spins. The full $SU(2)$ invariance renders the spectrum degenerate in states belonging to the same representation. These degeneracies are destroyed for the XXZ chain we will consider in the next chapter.

The block with $r = 0$ (all spins up) consists of a single vector $|0\rangle \equiv |\uparrow \dots \uparrow\rangle$. It is an eigenstate, $\mathcal{H}|0\rangle = E_0|0\rangle$, with energy $E_0 = -JN/4$.

The N basis vectors in the invariant subspace with $r = 1$ (one down spin) are labeled by the position of the flipped spin:

$$|n\rangle = S_n^-|0\rangle \qquad n = 1, \dots, N. \tag{3.3}$$

These states are clearly not eigenstates of $\mathcal{H}$, but out of them we can construct N linear combinations that take into account the translational symmetry, i.e., the invariance of $\mathcal{H}$ with respect to discrete translations by any number of lattice spacings. Translationally invariant basis vectors can be constructed from the vectors in (3.3) by writing

$$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}kn}|n\rangle;, \tag{3.4}$$

for wave numbers $k = 2\pi m/N,\ m = 0, \ldots, N-1$. (The lattice spacing has been set equal to unity.) The vectors $|\psi\rangle$ are eigenvectors of the translation operator with eigenvalues $\mathrm{e}^{\mathrm{i}k}$ and are also eigenvectors of $\mathcal{H}$ with eigenvalues

$$E - E_0 = J(1 - \cos k)\ , \tag{3.5}$$

as can be verified by inspection. The vectors (3.4) represent *magnon* excitations ($\Delta S = 1$ excitations), in which the complete spin alignment of the ferromagnetic ground state $|0\rangle$ is periodically disturbed by a spin wave with wavelength $\lambda = 2\pi/k$.

Note that the $k = 0$ state is degenerate with $|0\rangle$. It is easy to see that this state is the $S^z = \frac{N}{2} - 1$ component of the $S = \frac{N}{2}$ multiplet. Thus, its degeneracy with the fully ferromagnetic state is a consequence of the full $SU(2)$ invariance of the Heisenberg chain.

3.3 The two-body problem

The invariant subspace with $r > 1$ is not a simple superposition of magnons, as can be immediately inferred from comparing the number of basis states. For $r = 2$, for instance, we write a generic eigenstate as

$$|\psi\rangle = \sum_{1 \le n_1 < n_2 \le N} f(n_1, n_2)|n_1, n_2\rangle, \tag{3.6}$$

where $|n_1, n_2\rangle \equiv S^-_{n_1} S^-_{n_2}|F\rangle$ are the basis vectors in this subspace of dimension $N(N-1)/2$. Bethe's preliminary ansatz to determine the coefficients $f(n_1, n_2)$ is

$$f(n_1, n_2) = A\mathrm{e}^{\mathrm{i}(k_1 n_1 + k_2 n_2)} + A'\mathrm{e}^{\mathrm{i}(k_1 n_2 + k_2 n_1)}\ . \tag{3.7}$$

Setting $A = A'$ would correspond to a simple superposition of magnons, but this would be an overcomplete set of nonorthogonal and nonstationary states. Superimposed spin waves are in conflict with the requirement that the two flipped spins must be at different sites. The eigenvalue equation for (3.6) translates into $N(N-1)/2$ equations for as many coefficients $f(n_1, n_2)$:

$$\begin{aligned} 2[E - E_0]f(n_1, n_2) &= J[4f(n_1, n_2) - f(n_1 - 1, n_2) - f(n_1 + 1, n_2) \\ &\quad - f(n_1, n_2 - 1) - f(n_1, n_2 + 1)]\ , \quad \text{for } n_2 > n_1 + 1, \qquad (3.8) \\ 2[E - E_0]f(n_1, n_2) &= J[2f(n_1, n_2) - f(n_1 - 1, n_2) - f(n_1, n_2 + 1)]\ , \quad \text{for } n_2 = n_1 + 1. \qquad (3.9) \end{aligned}$$

Equations (3.8) are automatically satisfied by $f(n_1, n_2)$ in the form of a plain waves like (3.7) with arbitrary A, A', k_1, k_2 for $n_2 > n_1 + 1$ and for $n_2 = n_1 + 1$, provided the energy depends on k_1, k_2 as follows:

$$E - E_0 = J \sum_{j=1,2} (1 - \cos k_j)\ . \tag{3.10}$$

Equations (3.9), which are not automatically satisfied by the ansatz (3.7), are then equivalent to the N conditions

$$2f(n_1, n_1 + 1) = f(n_1, n_1) + f(n_1 + 1, n_1 + 1) \tag{3.11}$$

obtained by subtracting (3.9) from (3.8) for $n_2 = n_1 + 1$. The conditions (3.11) imply the following relation for the amplitude ratio:

$$\frac{A}{A'} \equiv \mathrm{e}^{\mathrm{i}\theta} = -\frac{\mathrm{e}^{\mathrm{i}(k_1+k_2)} + 1 - 2\mathrm{e}^{\mathrm{i}k_1}}{\mathrm{e}^{\mathrm{i}(k_1+k_2)} + 1 - 2\mathrm{e}^{\mathrm{i}k_2}}. \tag{3.12}$$

This requirement can be incorporated into the Bethe ansatz as extra phase factors

$$f(n_1, n_2) = \mathrm{e}^{\mathrm{i}(k_1 n_1 + k_2 n_2 + \frac{1}{2}\theta_{12})} + \mathrm{e}^{\mathrm{i}(k_1 n_2 + k_2 n_1 + \frac{1}{2}\theta_{21})}, \tag{3.13}$$

where the phase angle $\theta_{12} = -\theta_{21} \equiv \theta$ depends on the as yet undetermined k_1, k_2 via (3.12) or, in real form, via

$$2\cot\frac{\theta}{2} = \cot\frac{k_1}{2} - \cot\frac{k_2}{2}. \tag{3.14}$$

The quasi-momenta k_1, k_2 of the Bethe ansatz wave function can be determined from the requirement that the wave function (3.6) be translationally invariant: $f(n_1, n_2) = f(n_2, n_1 + N)$. This condition is satisfied by the coefficients (3.13) if

$$\mathrm{e}^{\mathrm{i}k_1 N} = \mathrm{e}^{\mathrm{i}\theta}\ , \qquad \mathrm{e}^{\mathrm{i}k_2 N} = \mathrm{e}^{-\mathrm{i}\theta}\ . \tag{3.15}$$

Equivalently, we can write (after taking their logarithm)

$$Nk_1 = 2\pi I_1 + \theta, \qquad Nk_2 = 2\pi I_2 - \theta, \tag{3.16}$$

where the $I_j \in \{0, 1, \ldots, N-1\}$ are integer quantum numbers. Note that, due to the lattice (and hence to the existence of the Brilluoin zone), the range of inequivalent quantum numbers is restricted. This was not the case in the Lieb-Liniger model.

The total momentum of this state is

$$K = k_1 + k_2 = \frac{2\pi}{N}(I_1 + I_2)\ . \tag{3.17}$$

The magnons interaction is reflected in the phase shift θ and in the deviation of the momenta k_1, k_2 from the single (free) magnon wave numbers. This is because the magnons either scatter off each other or form bound states. Note that the momenta k_1, k_2 specify the Bethe ansatz wave function (3.6), while the wave number K is the quantum number associated with the translational symmetry of $\mathcal{H}$ and exists independently of the Bethe ansatz.

The allowed (I_1, I_2) pairs are restricted to $0 \le I_1 \le I_2 \le N-1$. Switching I_1 with I_2 simply interchanges k_1 and k_2 and produces the same solution. There are $N(N+1)/2$ pairs that meet the ordering restriction, but only $N(N-1)/2$ of them yield a solution of Eqs. (3.14) and (3.16). The solutions can be determined analytically or computationally. Some of them have real k_1, k_2, and others yield complex conjugate momenta, $k_2 = k_1^*$. Note that, compared to the Bethe equations we found for the Lieb-Liniger, the scattering phase (3.14) does not depend on the difference between the momenta of the scattering particles. This means that equal Bethe quantum numbers are allowed.

If $I_1 = 0$ all solutions are real and $k_1 = 0$, $k_2 = 2\pi I_2/N$, $\theta = 0$ and $I_2 = 0, 1, \ldots, N-1$. These states are degenerate with the single magnon states and they belong to the same multiplet.

The majority of solutions are real and different from zero. They can be determined by combining (3.14), (3.16), and (3.17) into a single equation for k_1:

$$2\cot\frac{Nk_1}{2} = \cot\frac{k_1}{2} - \cot\frac{K-k_1}{2}. \tag{3.18}$$

Considering that the total momentum of the state is quantized ($K = 2\pi n/N$), we can substitute for different n in (3.18) to determine k_1 and $k_2 = K - k_1$.

There are also few state characterized by complex quasi-momenta. To find them, we write

$$k_1 \equiv \frac{K}{2} + iv, \qquad k_2 \equiv \frac{K}{2} - iv, \qquad \theta \equiv \phi + i\chi, \tag{3.19}$$

and use (3.14) and (3.16) for fixed K to obtain the relation

$$\cos\frac{K}{2}\sinh(Nv) = \sinh[(N-1)v] + \cos\phi\sinh v, \tag{3.20}$$

where $\phi = \pi(I_1 - I_2)$, and $\chi = Nv$ is inferred from the solution. Also $K = 2\pi/N(I_1 + I_2)$ and we can take $v > 0$ without loss of generality. The energy (3.10) of any complex solution can be rewritten in the form

$$E - E_0 = 2J\left(1 - \cos\frac{K}{2}\cosh v\right). \tag{3.21}$$

Complex solution exists only for $I_2 = I_1$ ($\phi = 0$) or $I_2 = I_1 + 1$ ($\phi = \pi$). Moreover, a careful analysis of (3.20) shows that only $N - 3$ choices of I_1, I_2 out of the possible $2N - 3$ actually give solutions. For more details on this, the reader is referred, for instance, to the first of [27].

For real solutions, in the large N limit the states are not too different from simple superpositions of two magnons, as the quasi-momentum of each excitation differers from the "free" quantization as

$$k_{1,2} = \frac{2\pi}{N}I_{1,2} + \mathrm{O}\left(\frac{1}{N^2}\right) . \tag{3.22}$$

Thus, the dispersion relations of these states form a continuum with boundaries

$$E - E_0 = 2J\left(1 \pm \cos\frac{K}{2}\right) . \tag{3.23}$$

For complex solutions, one can show that in the large N limit,

$$k_{1,2} = \frac{K}{2} \pm \mathrm{i}\ln\cos\frac{K}{2} , \tag{3.24}$$

and thus this states do not form a two-parameter continuum, but all lie on a single branch with dispersion

$$E - E_0 = \frac{J}{2}(1 - \cos K) . \tag{3.25}$$

This is a clear signature of the bound-state character of the complex solutions, that shows how the two magnons behave as a single entity. Substituting the parameters in the Bethe ansatz for the wavefunction, one also sees that the wavefunction amplitudes vanishes exponentially as the distance between the flipped spins grows larger, as we expect from a bound state. These states are clearly different from the others (as it is clear from their dispersion relation) and the Bethe ansatz construction provides us with a clear characterization of their different nature.

3.4 The Bethe Solution

Now that we discussed the basic features of the two-body problem, we can proceed with the construction of the eigenstates with generic r overturned spins. We generalize (3.6) and expand the eigenstates in the form

$$|\psi\rangle = \sum_{1\le n_1<\ldots<n_r\le N} f(n_1,\ldots,n_r)|n_1,\ldots,n_r\rangle. \tag{3.26}$$

The subspace has dimension $N!/[(N-r)!r!]$. The generalization of (3.13) for the coefficients in terms of r quasi-momenta k_j, and the scattering phase $\theta_{jl} = -\theta_{lj}$ for each (k_j, k_l) pair is as follows:

$$f(n_1,\ldots,n_r) = \sum_{\mathcal{P}\in S_r} \exp\left(\mathrm{i}\sum_{j=1}^{r} k_{\mathcal{P}j} n_j + \frac{\mathrm{i}}{2}\sum_{l<j}\theta_{\mathcal{P}l\mathcal{P}j}\right) . \tag{3.27}$$

The sum $\mathcal{P}\in S_r$ is over all $r!$ permutations of the labels $\{1,2,\ldots,r\}$. For $r=2$ the two permutations are the identity $(1,2)$ and the transposition $(2,1)$, which produce the two terms of (3.13). The consistency equations for the coefficients $f(n_1,\ldots,n_r)$ are extracted from the eigenvalue equation $\mathcal{H}|\psi\rangle = E|\psi\rangle$. They are a straightforward generalization of the two-particle case (3.8, 3.9). The energy eigenvalue equation becomes

$$E - E_0 = J\sum_{j=1}^{r}(1-\cos k_j) , \tag{3.28}$$

and the eigenstate condition can be written as

$$2f(n_1,\ldots,n_j,n_j+1,\ldots,n_r) = f(n_1,\ldots,n_j,n_j,\ldots,n_r) + f(n_1,\ldots,n_j+1,n_j+1,\ldots,n_r) , \tag{3.29}$$

for $j=1,\ldots,r$. These conditions relate every phase angle θ_{jl} to the (as yet undetermined) k_j in (3.26):

$$\mathrm{e}^{\mathrm{i}\theta_{jl}} = -\frac{\mathrm{e}^{\mathrm{i}(k_j+k_l)} + 1 - 2\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}(k_j+k_l)} + 1 - 2\mathrm{e}^{\mathrm{i}k_l}} , \tag{3.30}$$

which can be casted in real form as

$$2\cot\frac{\theta_{jl}}{2} = \cot\frac{k_j}{2} - \cot\frac{k_l}{2} , \quad j,l = 1,\ldots,r . \tag{3.31}$$

The translational invariance of (3.26) implies that the coefficients (3.27) satisfy the relation $f(n_1,\ldots,n_r) = f(n_2,\ldots,n_r,n_1+N)$. Consequently, we must have

$$\sum_{j=1}^{r} k_{\mathcal{P}j} n_j + \frac{1}{2}\sum_{i<j}\theta_{\mathcal{P}i,\mathcal{P}j} = \frac{1}{2}\sum_{i<j}\theta_{\mathcal{P}'i,\mathcal{P}'j} - 2\pi I_{\mathcal{P}'r} + \sum_{j=2}^{r} k_{\mathcal{P}'(j-1)} n_j + k_{\mathcal{P}'r}(n_1+N) , \tag{3.32}$$

where the relation between the permutations on the left and the right is $\mathcal{P}'(j-1) = \mathcal{P}j$, $j = 2,\ldots,r$; $\mathcal{P}'r = \mathcal{P}1$. If we take into account that all terms not involving the index $\mathcal{P}'r = \mathcal{P}1$ cancel, we are left with r additional relations between the phase angles and the momenta:

$$Nk_j = 2\pi\tilde{I}_j + \sum_{l\neq j}\theta_{jl}, \qquad j=1,\ldots,r, \tag{3.33}$$

where $\tilde{I}_j \in \{0, 1, \ldots, N-1\}$ as in (3.16). What remains to be done is to find those sets of Bethe quantum numbers $(I_1, \ldots, I_r)$ which yield (real or complex) solutions of the Bethe ansatz equations (3.31) and (3.33). Every solution represents an eigenvector (3.26) with energy (3.28) and wave number

$$K = \frac{2\pi}{N} \sum_{j=1}^{r} \tilde{I}_j \ . \tag{3.34}$$

3.5 String solutions

The scattering phase (3.30, 3.31) does not depend on the difference of the quasi-momenta of the particles. To restore this "translational invariance" it is convenient to introduce the *rapidities* λ_j to parametrize the quasimomenta:

$$\cot \frac{k_j}{2} = \lambda_j \ , \quad \text{or} \qquad k_j = \frac{1}{\mathrm{i}} \ln \frac{\lambda_j + \mathrm{i}}{\lambda_j - \mathrm{i}} = \pi - \theta_1(\lambda_j) \ , \tag{3.35}$$

where

$$\theta_n(\lambda) = 2 \arctan \frac{\lambda}{n} \ . \tag{3.36}$$

For the individual magnon, characterized by a real quasi-momentum k, we write its momentum and energy as

$$p_0(\lambda) = \frac{1}{\mathrm{i}} \ln \frac{\lambda + \mathrm{i}}{\lambda - \mathrm{i}} = k \ , \tag{3.37}$$

$$\epsilon_0(\lambda) = -J \frac{\mathrm{d}k}{\mathrm{d}\lambda} = \frac{2J}{\lambda^2 + 1} = J(1 - \cos k) \ . \tag{3.38}$$

The subscript 0 indicates that this particles correspond to a single (real) solution, i.e. a 0-type string (as we will define below).

In terms of these rapidities, the scattering phase is

$$\theta_{jl} = -\theta_2(\lambda_j - \lambda_l) + \pi \, \mathrm{sgn} \, [\Re(\lambda_j - \lambda_l)] \ , \tag{3.39}$$

where $\Re(x)$ is the real part of x and $\mathrm{sgn}\,(y) = \pm 1$ denotes the sign of y. The Bethe equations (3.33) in terms of the rapidities become

$$N\theta_1(\lambda_j) = 2\pi I_j + \sum_{l \neq j} \theta_2(\lambda_j - \lambda_l) \ , \qquad j = 1, \ldots, r \ . \tag{3.40}$$

The $\{I_j\}, j = 1, \ldots, r$ are the true Bethe quantum numbers characterizing the state. It is not easy to relate them exactly to the $\tilde{I}_j$, because of the second term in (3.39). The I_j have the same "fermionic" properties we found for the Lieb-Liniger model, i.e. we cannot have physical solutions where two of them are equal. Different sets of quantum numbers give rise to different sets of rapidities and eventually to all the eigenstates of the model, with energy and momentum

$$E = E_0 + J \sum_{j=1}^{r} \epsilon_0(\lambda_j) \ , \tag{3.41}$$

$$K = \sum_{j=1}^{r} p_0(\lambda_j) = \pi \, r - \frac{2\pi}{N} \sum_{j=1}^{r} I_j \ . \tag{3.42}$$

Some (actually, the majority of) states are characterized by complex rapidities. Standard numerical algorithms to solve the Bethe equations have problems in finding some of these complex solutions or have very low efficiency. It is thus desirable to understand ahead of time the structure of these complex rapidities to develop more suitable ways to determine them. Unfortunately, a complete understanding of this structure is still missing to date. But we have a fairly good account of what happens in the thermodynamic limit. This structure goes under the name of *string hypothesis*.

Let us look again at the $r = 2$ case (i.e. only two overturned spins). The Bethe Equations written in terms of the rapidities are:

$$\left(\frac{\lambda_1 + \mathrm{i}}{\lambda_1 - \mathrm{i}}\right)^N = \frac{\lambda_1 - \lambda_2 + 2\mathrm{i}}{\lambda_1 - \lambda_2 - 2\mathrm{i}} , \tag{3.43}$$

$$\left(\frac{\lambda_2 + \mathrm{i}}{\lambda_2 - \mathrm{i}}\right)^N = \frac{\lambda_2 - \lambda_1 + 2\mathrm{i}}{\lambda_2 - \lambda_1 - 2\mathrm{i}} . \tag{3.44}$$

As we need the total momentum of the state $k(\lambda_1) + k(\lambda_2)$ to be real, we require

$$\left(\frac{\lambda_1 + \mathrm{i}}{\lambda_1 - \mathrm{i}}\right)^N \left(\frac{\lambda_2 + \mathrm{i}}{\lambda_2 - \mathrm{i}}\right)^N = 1 . \tag{3.45}$$

We argued before that the Bethe equations admit complex solutions. In general, these have to be found numerically and sometimes they are even hard to find. However, a simple structure emerge if we take the thermodynamic limit $N \to \infty$. This structure is known as *string hypothesis*, as it is not yet proven that the string solutions we are about to describe exhaust the whole Hilbert space. However, this is commonly believed to the true, at least to the point that the states missed do not contribute significantly to the thermodynamic of the model. Thus, if $\Im\lambda_1 \neq 0$, the LHS in (3.43) will grow (or decrease) exponentially in N. Therefore, in the thermodynamic limit the LHS is strictly zero or infinity and the RHS will have to do the same. Thus, we must have

$$\lambda_1 - \lambda_2 = \pm 2\mathrm{i} , \qquad \text{i.e.} \qquad \lambda_{1,2} = \lambda \pm \mathrm{i} . \tag{3.46}$$

The energy and momentum of this state are

$$\mathrm{e}^{\mathrm{i}p_{1/2}(\lambda)} = \mathrm{e}^{\mathrm{i}p_0(\lambda+\mathrm{i})+\mathrm{i}p_0(\lambda-\mathrm{i})} = \frac{\lambda + 2\mathrm{i}}{\lambda - 2\mathrm{i}} , \tag{3.47}$$

$$\epsilon_{1/2}(\lambda) = -J\frac{\mathrm{d}p_{1/2}}{\mathrm{d}\lambda} = \epsilon_0(\lambda + \mathrm{i}) + \epsilon_0(\lambda - \mathrm{i}) = \frac{4J}{\lambda^2 + 4} , \tag{3.48}$$

which gives the dispersion relation

$$\epsilon_{1/2}(p) = \frac{1}{2}\left(1 - \cos p_{1/2}\right) . \tag{3.49}$$

Moreover, for $J > 0$, $\epsilon_{1/2}(p) < \epsilon_0(p - p') + \epsilon_0(p')$ for every $0 \leq p, p' < 2\pi$, and thus these bound states are energetically favored compare to the real solutions in the ferromagnetic regime.

For $r > 2$, more possible complex solution can appear and we describe them similarly. We can have *complexes* (or *strings*) of $2M + 1$ rapidities characterized by the same real value λ_M

and different, by equidistant, imaginary part. Here $M = 0, 1/2, 1, \ldots$ and these complexes have the structure

$$\lambda_{M;m} = \lambda_M + 2\mathrm{i}m \ , \qquad -M \leq m \leq M \ , \tag{3.50}$$

where m is integer or half-integer together with M. Counting all the complexes of length M by ν_M, for a state with a given magnetization we have

$$r = \sum_M (2M+1)\nu_M \ . \tag{3.51}$$

It is important to notice that the existence and structure of the string solution is strictly valid only in the thermodynamic limit $N \to \infty$ and as long as the number and length of the strings is not comparable to N. The fact that the complexes that we described are sufficient to exhaust the whole Hilbert space is not a proven fact and goes under the name of *string hypothesis*. For finite system sizes significant deviations from the string hypothesis are known to exist and have been observed even for very (very) large systems. When the number of strings becomes comparable with N, additional solution that cannot classified within the given structure could arise, but they are believed to be too few to contribute significantly to the thermodynamic of the model.

Let us therefore study a bit more carefully these string solutions. The first thing to notice is that the rapidities belonging to a complex can be grouped together and treated as a single entity. In fact, all the interactions of individual rapidities can be factorized and summed over separately in the interactions between the complexes. The energy and momentum of a M-complex is

$$p_M(\lambda_M) = \frac{1}{\mathrm{i}} \ln \frac{\lambda_M + \mathrm{i}(2M+1)}{\lambda_M - \mathrm{i}(2M+1)} \ , \tag{3.52}$$

$$\epsilon_M(\lambda_M) = \frac{2J(2M+1)}{\lambda_M^2 + (2M+1)^2} = \frac{J}{2M+1}(1 - \cos p_M) \ , \tag{3.53}$$

which is obtained by summing over all the rapidities in one complex. Due to their regular structure we have a lot of cancelations: taking them into account we can see that we can consider the scattering phase of a M-complex with a simple particle (0-complex) again by taking the product with respect with all the particles in a given complex, obtaining

$$S_{0,M}(\lambda_0 - \lambda_M) = S_{0,M}(\lambda) = \frac{\lambda + \mathrm{i}2M}{\lambda - \mathrm{i}2M} \, \frac{\lambda + \mathrm{i}2(M+1)}{\lambda - \mathrm{i}2(M+1)} \ , \tag{3.54}$$

and the scattering of two complexes of length M and M' is

$$S_{M,M'}(\lambda) = \prod_{L=|M-M'|}^{M+M'} S_{0,L}(\lambda) \ , \tag{3.55}$$

which should remind some of you of the Clebsh-Gordan coefficients. This is not surprising, since the whole structure we develop is very reminding of an $SU(2)$ algebra.

We can then describe the scattering of complexes by concentrating just on their real center $\lambda_{M,j}$, where $j = 1, \ldots, \nu_M$[2]. As we argued above, this treatment is correct as long as ν_0, i.e.

[2]To be clear in the notation used, $\lambda_{M,j}$ represent the real part of the j-th complex of length M.

the number of single-particle solutions, is dominating over all other complexes. Then we can assume the string hypothesis to be true and characterize the state simply by the $\lambda_{M,j}$.

The Bethe equation for the complexes is obtained by grouping all the rapidities belonging to the same complex and performing fist the products within respect with them to have effective consistency condition on their real centers $\lambda_{M,j}$. Doing so, we get the scattering phase we found before and the Bethe equations are:

$$\mathrm{e}^{\mathrm{i}p_M(\lambda_{M,j})N} = \prod_{M'} \prod_{\substack{j' \\ (M',j')\neq(M,j)}} S_{M,M'}\left(\lambda_{M,j} - \lambda_{M',j'}\right) \ . \tag{3.56}$$

As usual, we take the logarithm of (3.56), introduce the (half-)integers $I_{M,j}$ to take into account the branches of the logarithms and, using the familiar identity

$$\frac{1}{\mathrm{i}} \ln \frac{\lambda + \mathrm{i}n}{\lambda - \mathrm{i}n} = \pi - 2 \arctan \frac{\lambda}{n} = \pi - \theta_n(\lambda) \ , \tag{3.57}$$

we get

$$N\theta_{2M+1}(\lambda_{M,j}) = 2\pi I_{M,j} + \sum_{(M',j')\neq(M,j)} \theta_{M,M'}\left(\lambda_{M,j} - \lambda_{M',j'}\right) \ , \tag{3.58}$$

where

$$\theta_{M,M'}(\lambda) \equiv \sum_{L=|M-M'|}^{M+M'} \left[\theta_{2L}(\lambda) + \theta_{2L+2}(\lambda)\right] \ , \tag{3.59}$$

where the $L = 0$ is intended to be omitted.

Now, we shall see that, since the momenta are constraint within a Brillouin zone (due to the existence of a lattice in real space), the quantum numbers $I_{M,j}$ cannot take arbitrary values and the existence of complexes poses additional bounds on their behavior. First, we notice that, since

$$\arctan \pm\infty = \pm\frac{\pi}{2} \ , \tag{3.60}$$

for a diverging rapidity $\lambda_{M,\infty} = \infty$[3]

$$I_{M,\infty} = -\sum_{M'\neq M} \left[2\min(M,M')+1\right] \nu_{M'} - \left(2M + \frac{1}{2}\right)(\nu_M - 1) + \frac{N}{2} \ . \tag{3.61}$$

Then, since a M-complex has $2M+1$ roots, the maximum quantum number that characterizes a finite rapidity (before it joins the complexes of infinite rapidities) is

$$\begin{aligned} I_{M,\max} &= I_{M,\infty} - (2M+1) \\ &= \frac{N}{2} - \frac{1}{2} - \sum_{M'} J(M,M')\nu_{M'} \ , \end{aligned} \tag{3.62}$$

where

$$J(M,M') \equiv \begin{cases} 2\min(M,M')+1 & M \neq M' \\ 2M + \frac{1}{2} & M = M' \end{cases} \ . \tag{3.63}$$

[3]The ∞ index does not refer to $j \to \infty$, but to a rapidity outside of the physical regime.

Since all the scattering phases are odd functions of their argument, we have that

$$I_{M,\min} = -I_{M,\max} \ , \tag{3.64}$$

which means that there are

$$P_M = 2I_{M,\max} + 1 = N - 2\sum_{M'} J(M, M')\nu_{M'} \tag{3.65}$$

vacancies for an M-complex.

Note that the range of allowed values becomes narrower for complexes of any size if any string is added to the system. Using these results, one can estimate the number of states accessible within the string hypothesis and it can be shown that it scales like 2^N as one would desire, meaning that only few states are neglected in this framework and that these solutions are sufficient to accurately describe the thermodynamics of the model. More subtle issues emerge if one is interested in other aspects of the system, such as the correlation functions, or if one wants to address the finite-size corrections.

3.6 The Ferromagnetic case: $J = 1$

For a ferromagnetic coupling, the completely aligned state $|0\rangle$ can be taken as the ground state. In fact, it is degenerate with all the other members of the $S = N/2$ multiplet, which can be generated from $|0\rangle$ by adding zero-momentum magnons. The excited states can be constructed in terms of magnon excitations, keeping into considerations that bound-states corresponding to $2M + 1$ complexes have lower energies compared to states with $2M + 1$ real magnons. Thus, magnons are the natural excitations and the ground state can be considered as the magnon-vacuum.

3.7 The Anti-Ferromagnetic case: $J = -1$

The completely polarized state $|0\rangle$ that we chose as our reference state is clearly very different from an anti-ferromagnetic ground state. Moreover, as we saw in the previous section, in the ferromagnetic regime string solutions have lower energy than unbound, purely real ones, which means that they will have higher energy in the anti-ferromagnetic regime. This makes intuitive sense, since string solutions correspond to some bound states, i.e. clusters of flipped spins, which are energetically favored in the ferromagnetic regime, but not in the AFM one.

From these considerations one can see that this time the ground state configuration must be composed by single quasi-particle excitations, i.e.[4]

$$\nu_0 = \frac{N}{2} \ ; \qquad \nu_M = 0, \quad M \geq \frac{1}{2} \ , \tag{3.66}$$

in order to minimize the magnetization, since $r = \frac{N}{2}$ and $S^z = \frac{N}{2} - r = 0$. Now, from (3.65) we know that the number of vacancies for this configuration is

$$P_0 = N - 2J(0,0)\nu_0 = N - \frac{N}{2} = \frac{N}{2} \ , \tag{3.67}$$

[4]Let us assume for now that N is even, so that we will avoid spurious degeneracies that emerge for odd N.

which equals the number of roots. The quantum numbers are therefore occupying all the vacancies in the interval

$$-\frac{N}{4}+\frac{1}{2}\leq I_{0,k}\leq\frac{N}{4}-\frac{1}{2}\,, \tag{3.68}$$

and are integer (half-integer) for N/2 odd (even). Thus this singlet state is unique and will be the ground state for the anti-ferromagnetic case. It has again a Fermi sea structure like the one we observed for the Lieb-Liniger model.

To consider excited states over this ground state, we will progressively take away quasi-particles from the single state and move them into complexes, i.e. we will characterize the excited states by κ, with

$$\nu_0=\frac{N}{2}-\kappa\,. \tag{3.69}$$

Let us remark that we always take κ to be finite, which means that even in the thermodynamic limit $N\to\infty$ the number of excitations stays finite since

$$\sum_{M\geq 1/2}(2M+1)\nu_M=r-\nu_0\leq\frac{N}{2}-\nu_0=\kappa\,, \tag{3.70}$$

where we used the fact that $r\leq N/2$.

For $\kappa=1$, we cannot excite any complexes, this means that only the $r=N/2-1$ configuration is possible, which has total spin $S^z=1$. The number of vacancies in this case is

$$P_0=N-2\cdot\frac{1}{2}\left(\frac{N}{2}-1\right)=\frac{N}{2}+1 \tag{3.71}$$

which exceeds the number of roots by two. This means that we can used all the integers in the allowed range but two, and the choice of these two holes characterizes the state.

For $\kappa=2$ we can still decide to keep $\nu_M=0$ for $M\geq 1/2$ like before and have a state with magnetization $S^z=2$. The physics is similar to what we describe just before (with 4 holes this time) and we will not investigate it further. However, we also have a second possibility this time: we can have $\nu_{1/2}=1$ (and $\nu_M=0$ for $M\geq 1$), which keeps $r=N/2$ and $S^z=0$. The vacancies are

$$P_0 = N-2\left(\frac{N}{2}-2\right)\frac{1}{2}-2J\left(0,\frac{1}{2}\right)=\frac{N}{2}\,, \tag{3.72}$$

$$P_{1/2} = N-2\left(\frac{N}{2}-2\right)J\left(\frac{1}{2},0\right)-2J\left(\frac{1}{2},\frac{1}{2}\right)=4-3=1\,. \tag{3.73}$$

Once more, the number of vacancies for real roots allows for two holes, while there is no freedom for the 1/2-complex, whose state is therefore fixed.

From these examples we understand what is the general structure for general κ. We can have configurations with

$$\nu_0=\frac{N}{2}-\kappa\,,\qquad\qquad \nu_M=0\,,\qquad M\geq 1/2 \tag{3.74}$$

with $P_0=\frac{N}{2}+\kappa$ vacancies, which give rise to 2κ holes and states of total spin $S^z=\kappa$[5]. Then, we can have states with smaller magnetization (all the way to 0) and a proliferation of complexes.

[5]These excitation are holes with respect to the construction we developed, but they should be considered as particle excitations on top of the physical ground state of the AFM model.

Before we discuss these further, let us stop for while longer on the given examples and analyze them in the continuum limit obtained in the $N \to \infty$ limit. As in the Lieb-Liniger model, we can approximate the distribution of the real solutions of the Bethe equation with their continuous distribution. Let us start with the ground state, for which the quantum numbers fill the allowed interval of vacancies without holes. Let us assume that $N/2$ is odd (the even case requires just minor modifications) so that

$$I_{0,j} = j \ , \qquad\qquad j = -\frac{N}{4} + \frac{1}{2}, -\frac{N}{4} + \frac{3}{2}, \ldots, \frac{N}{4} - \frac{1}{2} \ . \tag{3.75}$$

The Bethe Equations can be written as

$$\arctan \lambda_j = \frac{\pi}{N} \, j + \frac{1}{N} \sum_k \arctan \left(\frac{\lambda_j - \lambda_k}{2} \right) \ . \tag{3.76}$$

In the $N \to \infty$ limit, the variable $x = \frac{j}{N}$ becomes continuous and limited in the range $-1/4 \leq x \leq 1/4$. The set of roots λ_j turn into a function $\lambda(x)$ and (3.76) becomes

$$\arctan \lambda(x) = \pi x + \int_{-1/4}^{1/4} \arctan \left(\frac{\lambda(x) - \lambda(y)}{2} \right) \mathrm{d}y \ . \tag{3.77}$$

As observables depend on (are best expressed in terms of) the rapidities λ_j and not on the integers $I_{0,j}$, we like to perform a change of variables and integrate over λ rather than x:

$$\sum_j f(\lambda_j) = N \int_{-1/4}^{1/4} f\left(\lambda(x)\right) \mathrm{d}x = N \int_{-\infty}^{\infty} f(\lambda) \rho_0(\lambda) \mathrm{d}\lambda \ , \tag{3.78}$$

where the change of variables $x \to \lambda(x)$ maps interval $-1/4 \leq x \leq 1/4$ into whole real line $-\infty < \lambda < \infty$ due to the monotonicity of $\lambda(x)$. More explicitly, the density $\rho(\lambda)$ is

$$\rho_0(\lambda) = \frac{\mathrm{d}x}{\mathrm{d}\lambda} = \left. \frac{1}{\lambda'(x)} \right|_{x = \lambda^{-1}(\lambda)} \ . \tag{3.79}$$

Finally, differentiating (3.77) with respect to λ we obtain an linear integral equation for the density of real roots $\rho_0(\lambda)$:

$$\rho_0(\lambda) = \frac{1}{\pi} \frac{1}{1 + \lambda^2} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2}{(\lambda - \mu)^2 + 4} \, \rho_0(\mu) \mathrm{d}\mu \ . \tag{3.80}$$

Notice that this integral equation is of the same time as the one we found for the Lieb-Liniger model and can be casted in the same form by remembering the definition of the scattering phase (4.24)

$$\rho_0(\lambda) = \frac{1}{2\pi} \theta_1'(\lambda) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{K}(\lambda - \nu) \, \rho_0(\mu) \mathrm{d}\mu \ , \tag{3.81}$$

where we introduced the kernel

$$\mathcal{K}(\lambda) \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda} \, \theta_2(\lambda) = \frac{2}{\lambda^2 + 4} \ . \tag{3.82}$$

However, since in this case the support of the density is on the whole real axis, this integral equation can be solved by Fourier transform:

$$\tilde{\rho}_0(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega\lambda}\rho_0(\lambda)\mathrm{d}\lambda \; . \tag{3.83}$$

Using

$$\frac{1}{\pi}\int \frac{n}{\lambda^2+n^2}\mathrm{e}^{-\mathrm{i}\lambda\omega}\mathrm{d}\lambda = \mathrm{e}^{-n|\omega|} \; , \tag{3.84}$$

we can turn the integral equation (3.80) into

$$\tilde{\rho}_0(\omega) \; \left(1+\mathrm{e}^{-2|\omega|}\right) = \mathrm{e}^{-|\omega|} \; , \tag{3.85}$$

which yields

$$\rho_0(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\lambda}\tilde{\rho}_0(\omega)\mathrm{d}\omega = \frac{1}{4\cosh\left(\frac{\pi\lambda}{2}\right)} \; . \tag{3.86}$$

The momentum and energy of the ground state are then given by

$$K = N\int p_0(\lambda)\rho_0(\lambda)\mathrm{d}\lambda = \frac{\pi}{2}\, N \bmod 2\pi \equiv K_{\mathrm{AFM}} \; , \tag{3.87}$$

$$E = E_0 + N\int \epsilon_0(\lambda)\rho_0(\lambda)\mathrm{d}\lambda = N\left(\frac{1}{4}-\ln 2\right) \equiv E_{\mathrm{AFM}} \; , \tag{3.88}$$

where $\rho_0(\lambda)$ and $\epsilon_0(\lambda)$ where defined in (3.52, 3.53).

Now we can turn to the state with $\nu_0 = N/2-1$ and $\nu_M = 0$ for $M \geq 1/2$. We remember that the state is characterize by two holes, that we place at j_1 and j_2:

$$I_{0,j} = j + \vartheta(j-j_1) + \vartheta(j-j_2) \; , \tag{3.89}$$

where $\vartheta(x)$ is the usual Heaviside step-function. The integral equation for the rapidity density of the real roots $\rho_{\mathrm{t}}(\lambda)$ (where t stands for triplet) is

$$\rho_{\mathrm{t}}(\lambda) = \frac{1}{\pi}\frac{1}{1+\lambda^2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{K}(\lambda-\nu)\; \rho_{\mathrm{t}}(\mu)\mathrm{d}\mu - \frac{1}{N}\left[\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)\right] \; , \tag{3.90}$$

where $\lambda_{1,2}$ are the images of $x_1 = j_1/N$ and $x_2 = j_2/N$ under the map $x \to \lambda(x)$. Since we are dealing with linear equations, we can write the solution of (3.90) as

$$\rho_{\mathrm{t}}(\lambda) = \rho_0(\lambda) + \frac{1}{N}\left[\tau(\lambda-\lambda_1)+\tau(\lambda-\lambda_2)\right] \; , \tag{3.91}$$

where $\tau(\lambda)$ solves the equation

$$\tau(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{K}(\lambda-\mu)\; \tau(\mu)\mathrm{d}\mu + \delta(\lambda) = 0 \; , \tag{3.92}$$

whose solution, in Fourier space, reads:

$$\tilde{\tau}(\omega) = \frac{1}{1+\mathrm{e}^{-2|\omega|}} \; . \tag{3.93}$$

Its real space form is a bit convoluted, but we can evaluate its contribution to the momentum and energy by working in the ω space ($\tilde{\theta}_0(\omega) = 2\pi \frac{\mathrm{e}^{-|\omega|}}{\mathrm{i}\omega}$):

$$\begin{aligned}
\int \theta_0(\lambda)\tau(\lambda-\lambda')\mathrm{d}\lambda &= \frac{1}{2\pi}\int \tilde{p}_0(\omega)\tilde{\tau}(-\omega)\mathrm{e}^{\mathrm{i}\omega\lambda'} \\
&= \int \frac{\mathrm{e}^{-|\omega|}}{1+\mathrm{e}^{-2|\omega|}}\,\frac{\mathrm{e}^{\mathrm{i}\omega\lambda'}}{\mathrm{i}\omega}\,\mathrm{d}\omega \\
&= \int \tilde{\rho}_0(\omega)\left(\int^{\lambda'} \mathrm{e}^{\mathrm{i}\omega\lambda'}\mathrm{d}\lambda\right)\mathrm{d}\omega \\
&= \frac{\pi}{2}\int^{\lambda'} \rho_0(\lambda) = \arctan\left[\sinh\frac{\pi\lambda'}{2}\right] , \qquad (3.94) \\
\int \epsilon_0(\lambda)\tau(\lambda-\lambda')\mathrm{d}\lambda &= -\int p_0'(\lambda)\tau(\lambda-\lambda')\mathrm{d}\lambda \\
&= -\frac{\mathrm{i}}{2\pi}\int \omega\ \tilde{p}_0(\omega)\tilde{\tau}(-\omega)\mathrm{e}^{\mathrm{i}\omega\lambda'} \\
&= -\int \tilde{\rho}_0(\omega)\ \mathrm{e}^{\mathrm{i}\omega\lambda'}\mathrm{d}\omega \\
&= -\frac{\pi}{2}\,\frac{1}{\cosh\frac{\pi\lambda}{2}} . \qquad (3.95)
\end{aligned}$$

Hence the total momentum and energy of this state with density of rapidities given by (3.91) is

$$\begin{aligned}
K &= N\int p_0(\lambda)\ \rho_{\mathrm{t}}(\lambda)\mathrm{d}\lambda \\
&= K_{\mathrm{AFM}} + k(\lambda_1) + k(\lambda_2) , \qquad (3.96) \\
E &= N\int \epsilon_0(\lambda)\ \rho_{\mathrm{t}}(\lambda)\mathrm{d}\lambda \\
&= E_{\mathrm{AFM}} + \varepsilon(\lambda_1) + \varepsilon(\lambda_2) , \qquad (3.97)
\end{aligned}$$

where

$$k(\lambda) \equiv \frac{\pi}{2} - \arctan\sinh\frac{\pi\lambda}{2} , \qquad \varepsilon(\lambda) \equiv -\frac{\pi}{2\cosh\frac{\pi\lambda}{2}} . \qquad (3.98)$$

Thus, this is a state with two-particle excitations (spinons) and (3.98) are their dressed energy and momentum. Combining the two, we see that these excitations are characterized each by the dispersion relation

$$\varepsilon(k) = -\frac{\pi}{2}\sin k , \qquad -\frac{\pi}{2} \le k \le \frac{\pi}{2} . \qquad (3.99)$$

We see therefore that each hole in the quantum numbers generates a quasi-particle excitation, which is called a *spinon*, i.e. an excitation with spin-1/2. Spinons only exist as collective excitations (since flipping a spin-1/2 creates a spin-1 excitation) and they are an example of *fractionalization* that happens in one-dimension. Individual spinons cannot be excited in a chain with an even number of sites (while they can be present with an odd number of sites, due to the degeneracy of a state with $r = \frac{N-1}{2}$ and $r = \frac{N+1}{2}$). However, the dispersion relation of a spin-1 excitation made by two spinons shows it composite nature in that it makes a band and not a single line (as it is for a simple magnon).

Let us now look back into a state we considered before, i.e. the state with $\nu_0 = N/2 - 2$, $\nu_{1/2} = 1$, and $\nu_M = 0$ for $M \geqslant 1$. For the density of real roots $\rho_{\rm s}(\lambda)$ (s is for singlet) we get the integral equation

$$\begin{aligned} \rho_{\rm s}(\lambda) &= \frac{1}{\pi}\frac{1}{1+\lambda^2} - \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{K}(\lambda-\mu)\,\rho_{\rm s}(\mu)\mathrm{d}\mu \\ &\quad -\frac{1}{N}\left[\delta(\lambda-\lambda_1)+\delta(\lambda-\lambda_2)+\frac{1}{\pi}\,\theta'_{0,1/2}(\lambda-\lambda_{1/2})\right] , \end{aligned} \tag{3.100}$$

where $\lambda_{1,2}$ stand for the holes and the last term in the RHS is the contribution from the interacting with the complex of type 1/2 with rapidity $\lambda_{1/2}$, solution of the Bethe equation (see 3.58):

$$\begin{aligned} 2\arctan\frac{\lambda_{1/2}}{2} &= \frac{1}{N}\sum_j \theta_{1/2,0}\left(\lambda_{1/2}-\lambda_{0,j}\right) \\ &= \int_{-\infty}^{\infty}\theta_{1/2,0}\left(\lambda_{1/2}-\lambda\right)\rho_{\rm s}(\lambda)\mathrm{d}\lambda , \end{aligned} \tag{3.101}$$

where

$$\theta_{1/2,0}\left(\lambda\right) = 2\arctan\lambda + 2\arctan\frac{\lambda}{3} \tag{3.102}$$

was given in (3.59) and in the second line we took the continuous limit for $N \to \infty$. As we explained before, the quantum number $I_{1/2,1} = 0$ in (3.101), since its allowed range is limited to just one point.

The solution of (3.100) is

$$\rho_{\rm s}(\lambda) = \rho_0(\lambda) + \frac{1}{N}\left[\tau(\lambda-\lambda_1)+\tau(\lambda-\lambda_2)+\sigma(\lambda-\lambda_{1/2})\right] \quad , \tag{3.103}$$

where $\tau(\lambda)$ is given by (3.92) and $\sigma(\lambda)$ is the solution of

$$\sigma(\lambda) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{K}(\lambda-\mu)\,\sigma(\mu)\mathrm{d}\mu + \frac{1}{\pi}\,\theta'_{0,1/2}(\lambda) = 0 , \tag{3.104}$$

which, in Fourier space, reads

$$\tilde{\sigma}(\omega)\left(1+\mathrm{e}^{-2|\omega|}\right) = -\left(\mathrm{e}^{-|\omega|}+\mathrm{e}^{-3|\omega|}\right) , \tag{3.105}$$

i.e.

$$\tilde{\sigma}(\omega) = -\mathrm{e}^{-|\omega|} . \tag{3.106}$$

To evaluate $\lambda_{1/2}$ we can rewrite (3.101) as

$$\begin{aligned} 2\arctan\frac{\lambda_{1/2}}{2} &= \int_{-\infty}^{\infty}\theta_{1/2,0}\left(\lambda_{1/2}-\lambda\right)\rho_0(\lambda)\mathrm{d}\lambda = \\ &\quad +\frac{1}{N}\int_{-\infty}^{\infty}\theta_{1/2,0}\left(\lambda_{1/2}-\lambda\right)\left[\tau(\lambda-\lambda_1)+\tau(\lambda-\lambda_2)+\sigma(\lambda-\lambda_{1/2})\right]\mathrm{d}\lambda . \end{aligned} \tag{3.107}$$

The last term in the RHS due to σ vanishes due to the oddness of the integrand. Moreover, we have

$$\begin{aligned}
\int \theta_{1/2,0}\left(\lambda_{1/2}-\lambda\right)\rho_0(\lambda)\mathrm{d}\lambda &= \frac{1}{2\pi}\int \tilde{\theta}_{1/2,0}(\omega)\;\tilde{\rho}_0(\omega)\;\mathrm{e}^{\mathrm{i}\omega\lambda_{1/2}}\mathrm{d}\omega \\
&= \int\left[\frac{\mathrm{e}^{-|\omega|}}{\mathrm{i}\omega}+\frac{\mathrm{e}^{-3|\omega|}}{\mathrm{i}\omega}\right]\frac{\mathrm{e}^{-|\omega|}}{1+\mathrm{e}^{-2|\omega|}}\;\mathrm{e}^{\mathrm{i}\omega\lambda_{1/2}}\mathrm{d}\omega \\
&= \int\frac{\mathrm{e}^{-2|\omega|}}{\mathrm{i}\omega}\;\mathrm{e}^{\mathrm{i}\omega\lambda_{1/2}}\mathrm{d}\omega \\
&= \int^{\lambda_{1/2}}\mathrm{d}\lambda'\int\mathrm{e}^{-2|\omega|}\;\mathrm{e}^{\mathrm{i}\omega\lambda'}\mathrm{d}\omega \\
&= \int^{\lambda_{1/2}}\frac{4}{\lambda'^2+4}\,\mathrm{d}\lambda \\
&= 2\arctan\frac{\lambda_{1/2}}{2}\;. \qquad (3.108)
\end{aligned}$$

This means that (3.107) reduces to

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty}\theta_{1/2,0}\left(\lambda_{1/2}-\lambda\right)\left[\tau(\lambda-\lambda_1)+\tau(\lambda-\lambda_2)\right]\mathrm{d}\lambda \\
&= \frac{1}{2\pi}\int\tilde{\theta}_{1/2,0}(\omega)\;\tilde{\tau}(\omega)\;\mathrm{e}^{\mathrm{i}\omega\lambda_{1/2}}\left(\mathrm{e}^{-\mathrm{i}\omega\lambda_1}+\mathrm{e}^{-\mathrm{i}\omega\lambda_2}\right)\mathrm{d}\omega \\
&= \int\frac{\mathrm{e}^{-|\omega|}}{\mathrm{i}\omega}\left[\mathrm{e}^{\mathrm{i}\omega(\lambda_{1/2}-\lambda_1)}+\mathrm{e}^{\mathrm{i}\omega(\lambda_{1/2}-\lambda_2)}\right]\mathrm{d}\omega \\
&= \arctan(\lambda_{1/2}-\lambda_1)+\arctan(\lambda_{1/2}-\lambda_2)\;, \qquad (3.109)
\end{aligned}$$

i.e.

$$\lambda_{1/2}=\frac{\lambda_1+\lambda_2}{2}\;. \qquad (3.110)$$

Thus, the rapidity of the type 1/2 complex is fixed and determined by the rapidities of the holes.

Moreover, if we evaluate the momentum for this state we find:

$$\begin{aligned}
K &= N\int p_0(\lambda)\;\rho_{\mathrm{s}}(\lambda)\mathrm{d}\lambda+p_{1/2}\left(\lambda_{1/2}\right) \\
&= K_{\mathrm{AFM}}+k(\lambda_1)+k(\lambda_2)+\int p_0(\lambda)\;\sigma\left(\lambda-\lambda_{1/2}\right)+2\arctan\frac{\lambda_{1/2}}{2} \\
&= K_{\mathrm{AFM}}+k(\lambda_1)+k(\lambda_2)\;, \qquad (3.111)
\end{aligned}$$

since, using (3.106),

$$\begin{aligned}
\int p_0(\lambda)\;\sigma\left(\lambda-\lambda_{1/2}\right) &= \frac{1}{2\pi}\int\tilde{p}_0(\omega)\;\tilde{\sigma}(-\omega)\;\mathrm{e}^{\mathrm{i}\omega\lambda_{1/2}}\mathrm{d}\omega \\
&= -\int\frac{\mathrm{e}^{-2|\omega|}}{\mathrm{i}\omega}\;\mathrm{e}^{\mathrm{i}\omega\lambda_{1/2}}\mathrm{d}\omega \\
&= -2\arctan\frac{\lambda_{1/2}}{2}\;. \qquad (3.112)
\end{aligned}$$

Similarly, for the energy

$$\begin{aligned} E &= E_{\rm AFM} + N \int \epsilon_0(\lambda)\, \rho_{\rm s} {\rm d}\lambda + \epsilon_{1/2}\left(\lambda_{1/2}\right) \\ &= E_{\rm AFM} + \varepsilon(\lambda_1) + \varepsilon(\lambda_2)\ , \end{aligned} \tag{3.113}$$

which can be easily derived from the previous result remembering that $\epsilon_M(\lambda) = -\frac{\rm d}{{\rm d}\lambda} k(\lambda)$. Hence, we see that the contributions from the string cancel exactly and this state has exactly the same momentum, energy (and dispersion relation) as the one without complexes that we calculated before (3.96, 3.97, 3.99). In particular, the two observable excitations in both cases obey (3.98).

Thus, we saw that these two families of states with two holes in the distribution of purely real roots have the same energy and momentum (when the same holes are taken in the two cases) and they only differ in their total spin, being $S^z = 1$ in the first case and $S^z = 0$ in the latter. One notices that, since applying the operator $S^+ \equiv \sum_{n=1}^{N} S_j^+$ to any of these states kills them, these are highest-weight states. This supports the interpretation of each hole excitation as a spin-1/2 excitation (spinon). In the first case we described the combination of two excitations into a triplet (in its highest-weight state $S^z = 1$), while in the second we got a singlet ($S^z = 0$).

For general κ the same picture holds: the states with $\nu_0 = N/2 - \kappa$ and $\nu_M = 0$ for $M \geq 1/2$ are 2κ-particle states in the highest-weight state of spin $S^z = k$. All other states with the same κ are states of lower magnetization, entering into multiplets with a number of particles non exceeding 2κ. In all these case, the contribution of M-complexes to the energy and momentum identically vanishes and so the energy/momentum depend only on the number of particles, i.e. on the holes in the purely real solutions. These multiplets are exactly degenerate only at the Heisenberg point (in zero external magnetic field) and will get split in the general XXZ model.

It is known that only spin-1 excitations are observed in the Heisenberg chain, but we see that these excitations are not pure *magnons*, but a combination of an even number (since the number of particles is 2κ) of spin-1/2 excitations (*spinons* or *spin waves*) with dispersion relation (3.99). Note that the dispersion relation for each spinon is defined only on half of the Brillouin zone, while the dispersion for the integer spin collective excitation is defined for $-\pi \leq k \leq \pi$.

One should note that all the states we described so far are highest-weight state. To lower the magnetization in a multiplet by one we place an extra rapidity at infinity, which corresponds to an excitation with zero-momentum. This corresponds to adding a quantum number at $I_{M,\infty}$ (remember that by adding one particle, one has to shift all the quantum numbers by 1/2), which leaves the existing rapidities unaffected. Additional complexes at infinity generate all the members in a given multiplet.

We have sketched how all excitations can be constructed in terms of spinon excitations over the anti-ferromagnetic ground state $|{\rm AFM}\rangle$, which can be interpreted as a spinon-vacuum. This point of view is particularly suitable to describe the anti-ferromagnetic case. We saw that in the ferromagnetic one, instead, the ground state is a magnon-vacuum and excitations are the magnons. While the whole Hilbert space can be described perfectly in either pictures, the two are somewhat dual to one another and it is not straightforward to see how a state constructed in terms of magnons can be also be generated using spinons.

3.8 Interaction with a magnetic field

In the presence of a magnetic field h, the Hamiltonian (3.1) must be supplemented by a Zeeman energy:

$$\mathcal{H} = -J\sum_{n=1}^{N}\mathbf{S}_n\cdot\mathbf{S}_{n+1} - h\sum_{n=1}^{N}S_n^z\,. \tag{3.114}$$

As the Hamiltonian commutes with the total magnetization, the magnetic field does not affect the eigenstates and only alters the eigenenergies. For the ferromagnetic case $J>0$, the ground state remains the fully polarized one and the magnetic field only splits the energy of the element in each multiplets.

In the anti-ferromagnetic case $J<0$, the two parts of $\mathcal{H}$ are in competition. Spin alignment in the positive z-direction is energetically favored by the Zeeman term, but any aligned nearest-neighbor pair costs exchange energy. The $2S+1$ components (with $|S^z|\le S$) of any S-multiplet fan out symmetrically about the original level position and depend linearly on h.

The largest downward energy shift in each multiplet is experienced by the state with $S^z=S$, and that shift is proportional to S. The state $|\mathrm{AFM}\rangle$, which has $S=0$, does not move at all, whereas the state $|0\rangle$ with $S=N/2$ descends more rapidly than any other state. Even though $|0\rangle$ starts out at the top of the spectrum, it is certain to become the ground state in a sufficiently strong field. The saturation field h_S marks the value of h where $|0\rangle$ overtakes its closest competitor in the race of levels down the energy axis.

The pattern in which levels with increasing S^z become the ground state of $\mathcal{H}$ as h increases depends on their relative starting position along the energy axis. From the zero-field energies of this set of states, we will now determine the magnetization $m_z\equiv S^z/N$ of the ground state as a function of h.

The lowest energy state of a sector with $S^z=N/2-r\ge 0$ is characterized by $\nu_0=r$ and $\nu_M=0$ for $M>0$ and the Bethe quantum numbers for the 0-complexes of this state are

$$I_{0,j}=\frac{1}{2}\left(S^z-1+2j-\frac{N}{2}\right),\quad j=1,\ldots,r\,. \tag{3.115}$$

At $h\neq 0$, all of these levels experience a downward shift of magnitude hS^z. All spacings between adjacent levels shrink by the same amount h. The first level crossing occurs between the state $|\mathrm{AFM}\rangle$ with $S^z=0$ and the state with $S^z=1$, which thereby becomes the new ground state. Next, this state is overtaken and replaced as the ground state by the state with $S^z=2$ and so forth. The last of exactly $N/2$ replacements involves the state with $S^z=N/2-1$ and the state $|0\rangle$ with $S^z=N/2$. Their energy difference in zero field is $2J$ independent of N. Consequently, the saturation field is simply $h_S=2J$.

The magnetization m_z grows in $N/2$ steps of width $1/N$ between $h=0$ and $h=h_S$. In the thermodynamic limit, the energy per site of the lowest level with given S^z becomes the internal energy density at zero temperature,

$$u(m_z)=\lim_{N\to\infty}\frac{E(S^z)-E_0}{JN}=\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N\left(\frac{1}{2}-m_z\right)}\epsilon_0(\lambda_j)\,. \tag{3.116}$$

From (3.116) we obtain, via the thermodynamic relations,

$$h = \frac{\mathrm{d}u}{\mathrm{d}m_z}, \quad \chi_{zz} = \frac{\mathrm{d}m_z}{\mathrm{d}h} = \left(\frac{\mathrm{d}^2 u}{\mathrm{d}m_z^2}\right)^{-1} \tag{3.117}$$

In a finite system, where $m_z = S^z/N$ varies in steps of size $1/N$, Eqs. (3.117) are replaced by

$$h(m_z) = E(S^z) - E(S^z - 1), \tag{3.118}$$

$$\chi_{zz}(m_z) = \frac{1/N}{E(E^z + 1) - 2E(S^z) + E(S^z - 1)}. \tag{3.119}$$

Using these relations, one can also relate the susceptibility to the Fermi velocity of low-energy excitations:

$$\chi_{zz}(m_z) = \frac{2}{\pi v_F} \, . \tag{3.120}$$

One can show that $\chi_{zz}(h)$ has a nonzero value at $h = 0$, grows monotonically with h, and finally diverges at the saturation field $h = h_S$. The initial value can be shown to be ($v_F = 2J\pi$)

$$\chi_{zz}(0) = \frac{1}{J\pi^2} \, . \tag{3.121}$$

This is a non-trivial result of the Bethe ansatz analysis, which is impossible to obtain with any perturbative approach, due to the logarithmic singularity which produces an infinite curvature in $m_z(h)$ at $h = 0$. The divergence of $\chi_{zz}(h)$ at h_S is of the type

$$\chi_{zz}(h) \stackrel{h \to h_S}{\longrightarrow} \frac{1}{2\pi} \frac{1}{\sqrt{J(h_S - h)}}. \tag{3.122}$$

The characteristic upwardly bent magnetization curve with infinite slope at the saturation field is a quantum effect unreproducible by any simple and meaningful classical model system. The Hamiltonian (3.114), reinterpreted as the energy function for coupled three-component vectors, predicts a function $m_z(h)$ which increases linearly from zero all the way to the saturation field.

Chapter 4

The XXZ Model

4.1 Introduction

We consider the model defined by the Hamiltonian:

$$\mathcal{H} = -J \sum_{j=1}^{N} \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta \, S_j^z S_{j+1}^z \right] - 2h \sum_{j=1}^{N} S_j^z, \tag{4.1}$$

and we impose periodic boundary conditions: $S_{j+N}^{\alpha} = S_j^{\alpha}$. Here, $S_j^{\alpha} = \frac{1}{2}\sigma_j^{\alpha}$, where σ_j^{α} are familiar the Pauli matrices. For $\Delta = 1$ we recover the Heisenberg chain we discussed in the previous chapter.

In (4.1), J sets the energy scale and only its sign is really important: for $J > 0$ the *ferromagnetic* order is preferred along the $x-y$ plane, while when $J < 0$ we have an *antiferromagnet* in the plane. The parameter Δ sets the strength of the uniaxial anisotropy along the z direction competing with the planar $x-y$ term and distinguishes a planar regime ($|\Delta| < 1$) from the axial regime ($|\Delta| > 1$). Clearly, for $|\Delta| > 1$ we have a *ferromagnet* along the z direction for $J\Delta > 0$ and an *antiferromagnet* when $J\Delta < 0$. In the literature, sometimes the phases are named after the planar ordering and other times after the axial one and one should pay attention in order to avoid ambiguities. Clearly, there are some symmetries that relates these different phases. For instance a rotation around the z axis of every-other spin, followed by the transformation $\Delta \to -\Delta$ maps the $J > 0$ ferromagnetic case into the $J < 0$ antiferromagnet and viceversa.

The $J < 0$ antiferromagnetic coupling is the most relevant for physical realization of the model (ferromagnets are more rare) and the case $|\Delta| < 1$ is the most interesting from the theoretical perspective, as it corresponds to a paramagnet, i.e. a critical (gapless) phase for sufficiently low magnetic field. For $\Delta = 0$ the model reduces to the isotropic XY model (also known as the XX model), corresponding to free fermions on a lattice. The point $\Delta = 1/2$ also turns out to be very interesting, as it develops additional symmetries. For $\Delta \to \pm\infty$ the model tends to a (anti)ferromagnetic classical Ising model.

An interesting exercise is to apply the Jordan-Wigner transformation we introduced in Chapter 1 to (4.1): one would find that the x and y coupling generates a standard nearest neighbor hopping for the fermions, the magnetic field sets the chemical potential and the z interact-

ing corresponds to a nearest neighbor interaction which corresponds to a lattice version of the contact interaction (that we considered in Chapter 2 for bosons in the Lieb-Liniger model).

4.2 Physical preview

Much of the complications in discussing the physics of the XXZ chain can be avoided by a smart choice of the reference "ground state" over which one consider excitations to constructs the different eigenstates. In different parts of the phase-diagram, different choices are more "natural" and thus low-energy state are characterized by different quasi-particles. A very good account of this variety can be found in [28] and we already saw an aspect of this at the isotropic point, when we argued that for $J > 0$ we should describe excitations as magnons, while for $J < 0$ spinons are more suitable.

For instance, for $J\Delta \to \infty$, the completely ferromagnetic state $|0\rangle \equiv \prod_{j=1}^{N} |\uparrow_j\rangle$ becomes a natural ground state. Low-energy excitations can be constructed in terms of magnons (i.e. single overturned spins), but also *domains* of consecutive flipped spins, since their interaction cost only lies at the boundary between different domains and thus longer domains do not cost more. These excitation have spin $S = 1$. For $J\Delta \to -\infty$, there are two degenerate ground states, the Neel states $|N_1\rangle \equiv |\uparrow\downarrow\uparrow\downarrow\uparrow \ldots\rangle$ and $|N_2\rangle \equiv |\downarrow\uparrow\downarrow\uparrow\downarrow \ldots\rangle$. Now the low-energy excitations are *domain walls*, i.e. regions where the one type of Neel order changes into the other, thus creating two consecutive ferromagnetically aligned spins. To create one of these domain walls one needs to overturn a macroscopically large number of spins (that is, the number of lattice sites between two consecutive domain walls), but, again, their interaction costs only lies at the boundary and this their energy does not depend on the number of flipped spins. This means that these are collective excitations. It is also clear that each domain wall carries spin $S = 1/2$ and they are thus spinons.

In the previous section we discussed in details the physical content of the Heisenberg model, i.e. what happens at $\Delta = 1$ and we also found that the low-energy excitations are magnons or spinons, depending on the sign of J.

Another "simple" point is the free case $\Delta = 0$, where the model can be exactly mapped into free lattice fermions. Thus, the ground state is best characterized in momentum space as a Fermi sea and excitations are easily constructed as particles and holes over this sea.

The intermediate cases interpolate between these extremal configurations. It is important to remark that the Bethe ansatz construction is valid everywhere and generates the full spectrum of the model. However, these initial consideration can be helpful to extract their physical meaning.

4.3 Bethe Ansatz Approach

To begin the study of the XXZ Hamiltonian (4.1), we observe that it does conserve the magnetization along the z-axis (unlike the XY model):

$$S^z \equiv \sum_j S_j^z \,. \tag{4.2}$$

The maximum magnetization $S^z = N/2$ is given by the configuration where all spins point up. Since a rotation along the x-axis followed by the transformation $h \to -h$ leaves the Hamiltonian unchanged, we can consider only the case $0 \leq S^z < N/2$. Moreover, because $[\mathcal{H}, S^z] = 0$ we can divide the Hilbert space in sectors with a given magnetization and consider them separately.

Let us start with the *absolute ground state* $|0\rangle \equiv |\uparrow\uparrow\uparrow \ldots\rangle$ (i.e. a state with no excitations, but not necessarily with the lowest energy) with all spins up ($S^z = N/2$) as the usual reference state. This is the *highest weight state* defined by

$$S_j^+ |0\rangle = 0 \,, \qquad j = 1, \ldots, N \,. \tag{4.3}$$

Acting on it with the Hamiltonian (4.1) gives

$$\mathcal{H}|0\rangle = E_0 \, |0\rangle \,, \qquad E_0 = -\left(\frac{J}{4}\,\Delta + h\right) N \,. \tag{4.4}$$

This is the only state in the sector with this magnetization.

If we flip one spin, we have N possible states in this sector with magnetization $S^z = \frac{N}{2} - 1$, corresponding to all the sites where the spin can be flipped. As we flip more spins, the dimension of the Fock space of states with a given magnetization increases very quickly (exponentially). We can write the generic state with r spin-flips as

$$\Psi = \sum_{\{n_l\}} f\,(n_1, n_2, \ldots, n_r)\, |n_1, n_2, \ldots, n_r\rangle \,, \tag{4.5}$$

where the sum is over all the choices of r lattice sites out of N and

$$|n_1, n_2, \ldots, n_r\rangle \equiv S_{n_1}^- S_{n_2}^- \ldots S_{n_r}^- |0\rangle \tag{4.6}$$

is the state with r spins flipped at the lattice sites $\{n_l\}$. We order the coordinates such that $1 \leq n_1 < n_2 < \ldots < n_r \leq N$. This state has magnetization $S^z = \frac{N}{2} - r$. For the above-mentioned symmetry, we can take $r \leq N/2$ and we notice that only for *even* N we can have a $SU(2)$ invariant state, i.e. $S^z = 0$, while for odd N the magnetization is a half integer.

Instead of determining the two-body scattering phase by considering a system of just two particles interacting like we did in the previous chapters, let us apply the Hamiltonian (4.1) directly to the state (4.5) and see that a series of conditions involving the successive permutations of two particles arise naturally, proving that every interaction can be decomposed in single two-particle processes.

The eigenvalue equation for the Hamiltonian (4.1) using the wave function (4.5) is

$$\begin{aligned}
&(\mathcal{H} - E)\,\Psi = \\
&-\frac{J}{2} \sum_{j=1}^{r} \left(1 - \delta_{n_j+1, n_{j+1}}\right) \Big[f\,(n_1, \ldots, n_j + 1, n_{j+1}, \ldots, n_r) + \\
&\qquad\qquad + f\,(n_1, \ldots, n_j, n_{j+1} - 1, \ldots, n_r) \Big] \\
&+ \left[E_0 - E + (J\Delta + 2h)r - J\Delta \sum_{j=1}^{r} \delta_{n_j+1, n_{j+1}} \right] f\,(n_1, n_2, \ldots, n_r) = 0 \,,
\end{aligned} \tag{4.7}$$

where we dropped writing the spin part of the wave-function, as it is assumed to be paired in an obvious way to the coordinate part. Notice that acting with the Hamiltonian on the state, leaves a diagonal part and a series of terms involving only two-particle (nearest neighbor) interaction.

Now, we make an ansatz for the coordinate wave function, the **Bethe Ansatz**, i.e. we assume that it is written as a superposition of plane-waves[1] in the following way:

$$\begin{aligned} f\left(n_1, n_2, \ldots, n_r\right) &\equiv \sum_{\mathcal{P}}^{r!} A[\mathcal{P}] \mathrm{e}^{\mathrm{i}\sum_{j=1}^{r} k_{\mathcal{P}j} n_j} \\ &= \sum_{\mathcal{P}}^{r!} \exp\left[\mathrm{i}\sum_{j=1}^{r} k_{\mathcal{P}j} n_j + \frac{\mathrm{i}}{2}\sum_{j<l}^{r} \tilde{\Theta}(k_{\mathcal{P}j}, k_{\mathcal{P}l})\right] , \end{aligned} \tag{4.8}$$

where $\mathcal{P}$ is a permutations of $1, 2, \ldots, r$:

$$\mathcal{P} = \begin{pmatrix} 1, & 2, & \ldots, & r \\ \mathcal{P}1, & \mathcal{P}2, & \ldots, & \mathcal{P}r \end{pmatrix} , \tag{4.9}$$

$k_1, k_2, \ldots, k_r$ are called quasi-momenta and will be determined later and in the second line we highlighted the fact that the relative amplitudes have unit modulus and can in fact be written as a phase (the *scattering phase*).

The Bethe wave function (4.8) has total momentum

$$K = \left(\sum_{j=1}^{r} k_j\right) \bmod (2\pi) , \tag{4.10}$$

(where we took into account that on a lattice momentum is defined only within a periodicity) and is an eigenfunction of (4.7) with eigenvalue

$$\begin{aligned} E &= E_0 + \sum_{l=1}^{r} \left[J(\Delta - \cos k_l) + 2h\right] \\ &= E_0 + (J\Delta + 2h)r - \sum_{l=1}^{r} \cos k_l \end{aligned} \tag{4.11}$$

if

$$\begin{aligned} &A[\mathcal{P}]\left(\mathrm{e}^{\mathrm{i}k_{\mathcal{P}j}} + \mathrm{e}^{-\mathrm{i}k_{\mathcal{P}(j+1)}} - 2\Delta\right)\mathrm{e}^{\mathrm{i}k_{\mathcal{P}(j+1)}} + \\ &+A[\mathcal{P}(j, j+1)]\left(\mathrm{e}^{\mathrm{i}k_{\mathcal{P}(j+1)}} + \mathrm{e}^{-\mathrm{i}k_{\mathcal{P}j}} - 2\Delta\right)\mathrm{e}^{\mathrm{i}k_{\mathcal{P}j}} = 0 , \end{aligned} \tag{4.12}$$

i.e. for

$$A[\mathcal{P}] = (-1)^{\mathcal{P}} \prod_{j<l} \left(\mathrm{e}^{\mathrm{i}(k_{\mathcal{P}j} + k_{\mathcal{P}l})} + 1 - 2\Delta\mathrm{e}^{\mathrm{i}k_{\mathcal{P}j}}\right) \tag{4.13}$$

or, alternatively, by fixing the scattering phases as

$$\mathrm{e}^{\mathrm{i}\tilde{\Theta}(k,k')} = -\frac{\mathrm{e}^{\mathrm{i}(k+k')} + 1 - 2\Delta\mathrm{e}^{\mathrm{i}k}}{\mathrm{e}^{\mathrm{i}(k+k')} + 1 - 2\Delta\mathrm{e}^{\mathrm{i}k'}} \tag{4.14}$$

[1]Please note that the assumptions of using plane-waves as a basis is not restrictive, as later we will find a change of variable that will give a more appropriate basis for the expansion.

or

$$\begin{aligned}\Theta(k,k') &\equiv \tilde{\Theta}(k,k') - \pi \\ &= 2\arctan\frac{\Delta\sin\frac{1}{2}(k-k')}{\cos\frac{1}{2}(k+k') - \Delta\cos\frac{1}{2}(k-k')} \,. \end{aligned} \tag{4.15}$$

By applying periodic boundary conditions, we get the following quantization equations:

$$\begin{aligned}\mathrm{e}^{\mathrm{i}k_j N} &= \prod_{j\neq l}\mathrm{e}^{\mathrm{i}\tilde{\Theta}(k_j,k_l)} \\ &= (-1)^{r-1}\prod_{j\neq l}\frac{\mathrm{e}^{\mathrm{i}(k_j+k_l)}+1-2\Delta\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}(k_j+k_l)}+1-2\Delta\mathrm{e}^{\mathrm{i}k_l}} \,, \qquad j=1,\ldots,r \,. \end{aligned} \tag{4.16}$$

By taking its logarithm we get the Bethe equations

$$k_j N = 2\pi\tilde{I}_j - \sum_{l=1}^{r}\Theta(k_j,k_l) \,, \tag{4.17}$$

where the $\{\tilde{I}_j\}$ are the integer/half integers quantum numbers defining the state.

The two-body scattering phase (4.15) has the unpleasant property of not being translational invariant for shifts of the momenta and this makes it harder to show the factorizations of the scattering matrix. It is then convenient to reparametrize the quasi-momenta by a change of variable $k_j \to \lambda_j$ to introduce proper *rapidities*:

$$\mathrm{e}^{\mathrm{i}k_j} = \frac{\sin\frac{\phi}{2}(\lambda_j+\mathrm{i})}{\sin\frac{\phi}{2}(\lambda_j-\mathrm{i})} \,, \tag{4.18}$$

where ϕ is a (yet-undetermined) parameter. The new variables λ_j (the *rapidities*) arise naturally because the transfer and scattering matrices for the model have simple combination rules in terms of them and we see that they will simplify our analysis in the same way. They also introduce a different basis for the Bethe Ansatz expansion (4.8), which turns out to be more natural than plane-waves for the model at hand. In terms of these new variables, one of the terms of the products on the right-hand side of (4.16) becomes

$$\begin{aligned}&\frac{\mathrm{e}^{\mathrm{i}(k_j+k_l)}+1-2\Delta\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}(k_j+k_l)}+1-2\Delta\mathrm{e}^{\mathrm{i}k_l}} = \\ &\frac{\cos\left[\frac{\phi}{2}(\lambda_j+\lambda_l)\right](\cosh\phi-\Delta)+\left\{\Delta\cos\left[\frac{\phi}{2}(\lambda_j-\lambda_l+2\mathrm{i})\right]-\cos\left[\frac{\phi}{2}(\lambda_j-\lambda_l)\right]\right\}}{\cos\left[\frac{\phi}{2}(\lambda_j+\lambda_l)\right](\cosh\phi-\Delta)+\left\{\Delta\cos\left[\frac{\phi}{2}(\lambda_l-\lambda_j+2\mathrm{i})\right]-\cos\left[\frac{\phi}{2}(\lambda_j-\lambda_l)\right]\right\}} \,. \end{aligned} \tag{4.19}$$

To remove the unwanted terms we set

$$\cosh\phi \equiv \Delta \,. \tag{4.20}$$

With a bit more algebra we then get

$$\frac{\mathrm{e}^{\mathrm{i}(k_j+k_l)}+1-2\Delta\mathrm{e}^{\mathrm{i}k_j}}{\mathrm{e}^{\mathrm{i}(k_j+k_l)}+1-2\Delta\mathrm{e}^{\mathrm{i}k_l}} = \frac{\sin\left[\frac{\phi}{2}(\lambda_j-\lambda_l+2\mathrm{i})\right]}{\sin\left[\frac{\phi}{2}(\lambda_j-\lambda_l-2\mathrm{i})\right]} \,. \tag{4.21}$$

An equivalent representation for (4.18) is

$$\cot\frac{k_j}{2} = \coth\frac{\phi}{2}\ \tan\left(\frac{\phi\lambda_j}{2}\right)\ , \tag{4.22}$$

which allows to write Bethe equations (4.17) directly as:

$$N\theta_1(\lambda_j) = 2\pi I_j + \sum_{l\neq j}^{r}\theta_2(\lambda_j-\lambda_l)\,,\quad j=1\dots r\ , \tag{4.23}$$

with

$$\theta_n(\lambda) = 2\arctan\left[\coth\left(\frac{n\phi}{2}\right)\ \tan\left(\frac{\phi\lambda}{2}\right)\right]\ . \tag{4.24}$$

In terms of the rapidities, the energy and momentum are given by:

$$E \;=\; E_0 + 2hr + \sum_{j=1}^{r}\epsilon(\lambda_j)\ , \tag{4.25}$$

$$K \;=\; 2\sum_{j=1}^{r}\cot^{-1}\frac{\tan\left(\phi\lambda_j/2\right)}{\tanh\left(\phi/2\right)}\ , \tag{4.26}$$

with

$$\epsilon(\lambda) = \frac{J\sinh^2\phi}{\cosh\phi-\cos(\phi\lambda)} \tag{4.27}$$

the quasi-particle energy. It should be clear that the phase $\theta_1(\lambda_j)$ is actually the original quasi-momentum k_j. This means that the momentum of the quasi-particle can be written in terms of the rapidities as

$$p(\lambda_j)\equiv\theta_1(\lambda_j) = k_j\ . \tag{4.28}$$

Moreover, notice that we have[2]

$$\frac{\epsilon(\lambda)}{J} = -\frac{\sinh\phi}{\phi}\ \frac{\mathrm{d}}{\mathrm{d}\lambda}p(\lambda)\ \ \Big(=\Delta-\cos k\Big)\ , \tag{4.29}$$

supporting our interpretation of λ as a rapidity.

Note also that, while for $\Delta>1$ ϕ stays real, it becomes imaginary for $\Delta<1$. The isotropic point $\Delta=1$ ($\phi=0$) can be found by continuity. In fact, taking the limit $\Delta\to1$ we get

$$\cot\frac{k_j}{2} = \lambda_j\ ,\quad \text{or}\qquad \mathrm{e}^{\mathrm{i}k_j} = \frac{\lambda_j+\mathrm{i}}{\lambda_j-\mathrm{i}} \tag{4.30}$$

and

$$\theta_n(\lambda) = 2\arctan\frac{\lambda}{n}\ . \tag{4.31}$$

Similarly, around the point $\Delta=-1$, the parameter ϕ goes from purely imaginary to complex and the $\Delta=-1$ is defined by continuity. This is the main advantage of the parametrization (4.18, 4.22), since it spans the whole range of Δ with the only caution of tracking down the analytical structure. Notice that (4.18) is not the only transformation that gives translationally invariant scattering phases and other choices can preferred for some purposes. However, other parametrizations differs mostly for shifting and/or rescaling of the rapidities.

To be more explicit in these analytical properties, it is useful at this point to separate the different cases $\Delta>1$, $\Delta<-1$ and $|\Delta|\leq1$.

[2]The minus sign arises as a consequence of the different branch cut between the cotangent defying k_j in (4.22) and the tangent used for the phase in (4.24).

4.3.1 Axial Case: $\Delta > 1$

In this case the angle $\phi > 0$ is real and positive and the Bethe equations (4.16) can be written as

$$\left(\frac{\sin\left[\frac{\phi}{2}(\lambda_j + \mathrm{i})\right]}{\sin\left[\frac{\phi}{2}(\lambda_j - \mathrm{i})\right]}\right)^N = \prod_{l\neq j} \frac{\sin\left[\frac{\phi}{2}\left(\lambda_j - \lambda_l + 2\mathrm{i}\right)\right]}{\sin\left[\frac{\phi}{2}\left(\lambda_j - \lambda_l - 2\mathrm{i}\right)\right]} \tag{4.32}$$

4.3.2 Axial Case: $\Delta < -1$

To work with real and positive angle $\phi > 0$, it is more convenient to define

$$\phi = \cosh^{-1}(-\Delta)\ , \tag{4.33}$$

i.e. we change from $\phi \to \phi + \mathrm{i}\pi$. The Bethe equations (4.16) are as exactly as before

$$\left(\frac{\sin\left[\frac{\phi}{2}(\lambda_j + \mathrm{i})\right]}{\sin\left[\frac{\phi}{2}(\lambda_j - \mathrm{i})\right]}\right)^N = \prod_{l\neq j} \frac{\sin\left[\frac{\phi}{2}\left(\lambda_j - \lambda_l + 2\mathrm{i}\right)\right]}{\sin\left[\frac{\phi}{2}\left(\lambda_j - \lambda_l - 2\mathrm{i}\right)\right]} \tag{4.34}$$

4.3.3 Paramagnetic/Planar Case: $|\Delta| \leq 1$

In this case the angle ϕ is purely imaginary. Therefore, we prefer to use the real angle *gamma* defined by:

$$\gamma = \arccos(-\Delta)\ , \qquad\qquad 0 < \gamma < \pi\ . \tag{4.35}$$

With these notations the Bethe equations (4.16) are

$$\left(\frac{\sinh\left[\frac{\gamma}{2}(\lambda_j + \mathrm{i})\right]}{\sinh\left[\frac{\gamma}{2}(\lambda_j - \mathrm{i})\right]}\right)^N = \prod_{l\neq j} \frac{\sinh\left[\frac{\gamma}{2}\left(\lambda_j - \lambda_l + 2\mathrm{i}\right)\right]}{\sinh\left[\frac{\gamma}{2}\left(\lambda_j - \lambda_l - 2\mathrm{i}\right)\right]}\ . \tag{4.36}$$

4.4 String solutions

In chapter 3 we studied the isotropic point $\Delta = 1$ and we discussed how string solutions can be constructed and classified, and how their interactions can be factored out into interactions between different complexes. We will not repeat the same construction for here for $\Delta \neq 1$ since most of it carries out in the same way. Also, most of the qualitative features survive, except that many states that were degenerate get split.

One can repeat the analysis in chapter 3 to see the differences: most formulae remain valid after the substitution of rational functions with trigonometric one like:

$$\frac{\lambda + \mathrm{i}a}{\lambda - \mathrm{i}a} \to \frac{\sin\left[\phi(\lambda + \mathrm{i}a)\right]}{\sin\left[\phi(\lambda - \mathrm{i}a)\right]}\ . \tag{4.37}$$

The energy and momentum were given in (4.25, 4.26, 4.27). For a M-type complex

$$\lambda_{M,j} = \lambda_M + \mathrm{i}(M - j)\ , \qquad j = 0, \ldots, 2M\ , \tag{4.38}$$

we still have cancelations as in the trigonometric case and using trigonometric identities we get

$$\begin{aligned} \epsilon_M(\lambda_M) &= \frac{J\sinh\phi\sinh\left[(2M+1)\phi\right]}{\cosh\left[(2M+1)\phi\right]-\cos(\phi\lambda_M)}\ , \\ p_M(\lambda_M) &= 2\cot^{-1}\frac{\tan\left(\frac{\phi\lambda}{2}\right)}{\tanh\left[\left(M+\frac{1}{2}\right)\phi\right]}\ . \end{aligned} \tag{4.39}$$

We can use the last one to solve for the rapidity as a function of P_M and insert it in the previous expression to find the dispersion relation

$$\epsilon_M(p_M) = J\sinh\phi\frac{\cosh\left[(2M+1)\phi\right]-\cos p_M}{\sinh\left[(2M+1)\phi\right]}\ . \tag{4.40}$$

For a state where all the r flipped spins belong to a single complex of size $r = 2M+1$, we have

$$E = E_0 + 2(2M+1)h + \epsilon_M(p_M)\ , \tag{4.41}$$

For large Δ (or ϕ), this energy tends to

$$E \sim E_0 + 2(2M+1)h + J\Delta\ . \tag{4.42}$$

This state corresponds to a bound state of $r = 2M+1$ consecutive spin flipped down in a sea of up spins. For smaller Δ, the lowest energy state for such string solution is the one with zero-momentum. Thus it has

$$E = -\frac{J\Delta}{4}\ N - (N-2r)\,h + J\sinh\phi\ \tanh\frac{r\phi}{2}\ . \tag{4.43}$$

While the rapidities of a string are equally spaced along the imaginary axis, with steps that do not depend on ϕ, the corresponding quasi-momenta have imaginary part that grow further apart as Δ increases. In the $\Delta \to \infty$ limit, the quasi-momenta acquire an infinite imaginary component that creates infinitely tight bound state, corresponding to the domains we discussed in the introduction. This phenomenon is sometimes referred to as *stretched strings*.

4.4.1 The planar (paramagnetic) regime: $-1 < \Delta < 1$

As we just saw that for larger Δ the strings become stretched and the bound states tighter, as $\Delta \to 0$ the complexes progressively shrink and the bound states dissolves into elementary excitations.

The paramagnetic regime is arguably the most interesting phase of the XXZ model, mostly because it is an extended gapless phase, but also because the structures that we observed at the isotropic point progressively crumble as one approaches the point $\Delta = 0$. At this point we recover the isotropic XY (or XX model), i.e. one of the critical lines of the XY model. As we saw in chapter 1, on this line the Bogoliubov angle vanishes and the Bogoliubov quasi-particles coincide with the physical fermions. Therefore the model is such of free fermions on a lattice. To the best of my knowledge, a detail description of how the Bethe ansatz construction evolves into these free fermions as $\Delta \to 0$ has not been worked out, but we will try to sketch the qualitative picture.

The first thing to notice is that the scattering phase (4.14) has a special behavior if one of the particles scattering has quasi-momentum k_0 such that $\Delta = \cos k_0$:

$$e^{i\tilde{\Theta}(k,\pm k_0)} = e^{\mp 2ik_0} \equiv -e^{\mp i\theta_0} . \tag{4.44}$$

Thus, the scattering phase is independent from the other momentum k and equal to $\theta_0 \equiv \pi - 2k_0$. This mean that a particle with quasi-momentum k_0 factorizes out in the Bethe equations and its only effect is to introduce an overall phaseshift for the whole system, as if a flux was threading the system imposing and Aharonov-Bohm phase. This translate in writing the Bethe equations as

$$e^{ik_j L} \prod_{l\neq j} e^{i\theta(k_j,k_l)} = e^{i\Phi} , \tag{4.45}$$

where in this case the phase is $\Phi = \pm\theta_0$.

Moreover, this critical k_0 also corresponds to a *threshold* state, i.e. a type-1/2 complex that has just coalesced into two real momenta. To see this, let us consider once more the two-body scattering phase in its original form (4.14). Following the treatment of section 2.8, let us introduce the center of mass coordinate and the total momentum of the two-body state $K = k_1 + k_2$ and the relative momentum $k = k_1 - k_2 = \pm i\kappa$. Using these variables, the scattering phase is

$$e^{i\tilde{\Theta}(K,k)} = -\frac{\cos(K/2) - \Delta e^{ik/2}}{\cos(K/2) - \Delta e^{-ik/2}} . \tag{4.46}$$

The condition for the bound state to be normalizable is that either $e^{i\tilde{\Theta}} = 0$ or $e^{i\tilde{\Theta}} = \infty$ (see 2.55), i.e.

$$\cos(K/2)\, e^{\kappa/2} = \Delta , \qquad \kappa > 0 . \tag{4.47}$$

This implies that for $-1 \leq \Delta \leq 1$, the total momentum of the complex of size 2 has to satisfy $\cos(K/2) \leq \Delta$ and the decay factor is $e^{\kappa/2} = \frac{\Delta}{\cos(K/2)}$. On threshold we have $\cos(K_0/2) = \Delta$, i.e. $K_0 = \pm 2k_0$.

Therefore, we have determined that k_0 is the branch point of the scattering phase. As $\Delta \to 0$, string solutions move in the complex plane coming closer to the real axis and they coalesce to the real axis on the points $\pm k_0$ before splitting into two distinct real solutions. We notice that exceptionally we can have more than one particle with the same quasi momentum, if this is $\pm k_0$ and we can interpret these particles as remnants of a string solution. The rapidities of such solutions are at infinity ($k(\lambda = \pm\infty) = \pm k_0$) and they correspond to the maximum allowed integer I_∞. One also notices that these excitation do not contribute to the energy, since $\epsilon(\pm k_0) = \Delta - \cos(\pm k_0) = 0$ and they only contribute to the total momentum (i.e. the flux). In the previous section we mentioned that at the isotropic point these threshold rapidities can be used to generate the non-highest weight state.

For $|\Delta| > 1$ we can reason in the same way, but in this case the branch points occur for complex values of the quasi-momentum and therefore it is much harder to observe a state that would contain such excitations.

If we consider longer strings, we see that the total momentum of a type M complex is bounded by the configuration where all quasi-momenta are $\pm k_0$, i.e. $P_M^{\max/\min} = \pm(2M +$

$1)k_0 \bmod 2\pi$. This means that for $\Delta = \cos\frac{\pi}{2M+1}$, the range of allowed momenta has shrunk to zero and the allowed string solutions have already completely collapsed on the real axis. As $\Delta \to 0$, complexes with large M ($2M+1 > \pi/\arccos\Delta$) will have become unstable and dissolved into real solutions and shorter and shorter complexes can be stable closer to $\Delta = 0$, until they completely disappear and only real solutions are supported at the XX point.

This can also be seen from the dispersion relation. In the paramagnetic regime, (4.40) becomes

$$\epsilon_M(p_M) = J\sin\gamma\frac{\cos\left[(2M+1)\gamma\right]-\cos p_M}{\sin\left[(2M+1)\gamma\right]}\ , \tag{4.48}$$

where $\Delta = -\cos\gamma$. We see that the region of stability of the bound states is bounded by the points for which $\epsilon_M(p_M^{\max/\min}) = 0$ and these are given by $P_M^{\max/\min} = \pm(2M+1)k_0 \bmod 2\pi$, since we observed that excitations with $\pm k_0$ carry no energy and these type-M complexes are allowed as long as $\gamma > \gamma_M = \pi - \pi/(2M+1)$.

4.5 Interaction with a magnetic field

Until now, in the Bethe ansatz solution we always assume that there was no external magnetic field. This implied that in the paramagnetic regime the ground state had zero magnetization. This situation, in the particle language, is called *half filling*, since it corresponds to the maximum number of excitations, i.e. to half of the spins in the system turned.

We remind that the ansatz wavefunction can be written as

$$f(n_1,n_2,\ldots,n_r) = \sum_P (-1)^P \left[\prod_{j<l}\sinh\frac{\gamma}{2}\left(\lambda_{Pj}-\lambda_{Pl}+2\mathrm{i}\right)\right]\left[\prod_{n=1}^{r}\left(\frac{\sinh\frac{\gamma}{2}\left(\lambda_{Pn}+\mathrm{i}\right)}{\sinh\frac{\gamma}{2}\left(\lambda_{Pn}-\mathrm{i}\right)}\right)^{n_j}\right]\ , \tag{4.49}$$

where $\gamma = \arccos(-\Delta)$.

Remembering that in this regime

$$k(\lambda) = 2\arctan\left[\frac{\tanh\frac{\gamma\lambda}{2}}{\tan\frac{\gamma}{2}}\right] = \theta_1(\lambda) \equiv \theta(\lambda|\gamma)\ , \tag{4.50}$$

and taking the logarithm of (4.36) we get the Bethe equation in logarithmic form

$$k(\lambda_j)N = \theta_1(\lambda_j)N = 2\pi I_j + \sum_{l=1}^{r}\theta_2(\lambda_j-\lambda_l)\ , \tag{4.51}$$

where

$$\theta_2(\lambda) = 2\arctan\left[\frac{\tanh(\frac{\gamma\lambda}{2})}{\tan\gamma}\right] \equiv \theta(\lambda|2\gamma)\ , \tag{4.52}$$

and the I_j's are, as usual, integers or half-integers specifying the state for r odd or even respectively.

In the thermodynamic limit, we introduce the density of momenta $\tilde\rho(k)$ and by using the counting function

$$y(k) \equiv \int^k \tilde\rho(k')\mathrm{d}k'\ , \tag{4.53}$$

from (4.51) we have

$$k = 2\pi y(k) + \int_{-q}^{q} \theta(k-k')\rho(k')\mathrm{d}k' \ . \tag{4.54}$$

We are more interested in the density of rapidities, which we introduce by

$$\rho(\lambda)\mathrm{d}\lambda \equiv \tilde{\rho}(k)\mathrm{d}k \ , \tag{4.55}$$

i.e.

$$\rho(\lambda) = k'(\lambda)\rho(\lambda) \ . \tag{4.56}$$

We rewrite (4.54) as

$$k(\lambda) = 2\pi y(\lambda) + \int_{-\Lambda}^{\Lambda} \theta(\lambda-\lambda')\rho(\lambda')\mathrm{d}\lambda' \ , \tag{4.57}$$

where now

$$y(\lambda) \equiv \int^{\lambda} \rho(\lambda')\mathrm{d}\lambda' \ . \tag{4.58}$$

By taking the derivative of (4.57) with respect to λ we get

$$\rho(\lambda) + \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} \mathcal{K}(\lambda-\mu)\rho(\mu)\mathrm{d}\mu = \frac{1}{2\pi}k'(\lambda) \ , \tag{4.59}$$

where

$$\mathcal{K}(\lambda) \equiv \frac{\mathrm{d}\theta_2}{\mathrm{d}\lambda} = \frac{\sin^2(\gamma)}{\cosh(\gamma\lambda)-\cos\gamma} \ . \tag{4.60}$$

Eq. (4.59) is very similar to the one we derived for the XXX model (3.81), but clearly with a different kernel. However, the main difference is that in (3.81) we noticed that $\Lambda = \infty$, while the presence of a magnetic field has the effect of shrinking the support of the density function.

The magnetization and energy density are given by

$$s^z \equiv \frac{S^z}{N} = \frac{1}{2} - m \ , \tag{4.61}$$

$$m \equiv \int_{-\Lambda}^{\Lambda} \rho(\lambda)\mathrm{d}\lambda \ , \tag{4.62}$$

$$e \equiv \frac{\tilde{E}}{N} = -\sin\gamma\int_{-\Lambda}^{\Lambda} k'(\lambda)\rho(\lambda)\mathrm{d}\lambda - hm = \int_{-\Lambda}^{\Lambda} \epsilon_0(\lambda)\rho(\lambda)\mathrm{d}\lambda \ , \tag{4.63}$$

where the bare energy is now

$$\epsilon_0(\lambda) \equiv -\sin\gamma k'(\gamma) - h \ . \tag{4.64}$$

The momentum is clearly

$$p \equiv \frac{K}{N} = 2\pi\sum_j I_j = \int_{-\Lambda}^{\Lambda} k(\lambda)\rho(\lambda)\mathrm{d}\lambda \ . \tag{4.65}$$

From this point on, to study the thermodynamic of the system we can proceed like in the previous chapter. We can introduce the dressed energy per particle $\varepsilon(\lambda)$ as the solution of the linear integral equation

$$\varepsilon(\lambda) - \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} \mathcal{K}(\lambda,\mu)\varepsilon(\mu)\mathrm{d}\mu = \varepsilon_0(\lambda) \ , \tag{4.66}$$

with the condition $\varepsilon(\pm\Lambda) = 0$. And we can prove, like we did for the Lieb-Liniger model, that

$$e = \int_{-\Lambda}^{\Lambda} \varepsilon_0(\lambda)\rho(\lambda)\mathrm{d}\lambda = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} \varepsilon(\lambda)k'(\lambda)\mathrm{d}\lambda\ . \tag{4.67}$$

Equation (4.66) has solution only if $h < h_c = 1 - \Delta$ as for $h \to h_c, \Lambda \to 0$ and the ground state becomes ferromagnetic. So, we see that a sufficiently large magnetic field destroys the paramagnetic phase and turn the system ferromagnetic. On the other side, it can be proven that

$$h \to 0 \qquad \Rightarrow \qquad \Lambda \to \infty \tag{4.68}$$

which brings back to the situation analyzed for the XXX model, where the integral equation can be solved by Fourier transform, giving, for instance

$$\rho(\lambda) = \left\{2(\pi - \gamma)\cosh\left[\frac{\pi\lambda}{\pi - \gamma}\right]\right\}^{-1}\ . \tag{4.69}$$

Chapter 5

Two-Dimensional Classical Integrable Systems

5.1 Overview

The Algebraic Bethe Ansatz (ABA) method is essentially a second quantization of the coordinate one. It uses the algebraic structure of the Transfer Matrix formalism to express the wavefunctions in terms of creation operators acting on a reference state (known as *pseudo-vacuum*). One of its advantage is the fact that it connects the Bethe Ansatz solution with some of the most powerful techniques and concepts in the general theory of integrable system, like the Inverse Scattering Method, Lax representation and Transfer matrix approaches.

To help understanding the ABA construction and to provide some physical intuition on it, in this chapter we will use the general fact that any 1-D quantum model is equivalent to a classical 2-D system and introduce the main ingredients in a two-dimensional settings. For simplicity, we will focus on lattice model, say made by M horizontal sites and N vertical ones, with periodic boundary conditions in both directions.

A key role will be played by the transfer matrix $\mathbf{T}$. This is an operator that propagates a given configuration from one horizontal line to the next, i.e. it gives the weight in the partition function of a state with a given configuration on the N sites of a horizontal line and another given configuration on the N sites of the next line. If one knows the transfer matrix for a given model, the partition function can be found by taking the trace of M products of the same transfer matrix $\mathbf{T}$:

$$\mathcal{Z} = \mathrm{Tr}\, \mathbf{T}^M \,, \tag{5.1}$$

where the products connects the different rows and the final trace allows to close the top and bottom line together with periodic boundary conditions. It is then clear that we are primarily interested in the eigenvalues Λ_j of the transfer matrix, and in particular on its highest one Λ_1, since

$$\mathcal{Z} = \Lambda_1^M \left[1 + \left(\frac{\Lambda_2}{\Lambda_1} \right)^M + \dots \right] \,, \tag{5.2}$$

where the terms in the brackets converge to 1 in the thermodynamic limit $M \to \infty$.

The main idea is that, instead of diagonalizing a single transfer matrix, we will try do diagonalize a whole family of transfer matrices at the same time. We identify each member within a family by a parameter λ (usually referred to as the *spectral parameter*), i.e. $\mathbf{T}(\lambda)$, and we try to diagonalize each $\mathbf{T}(\lambda)$ simultaneously for every λ. This procedure might seem too ambitious at first, since one turns the hard problem of solving a system into the seemingly harder problem of solving a bunch of them. In fact, this technique brings out a deep structure due to the integrability. In particular, we will find that all transfer matrices within a family commute with one another and therefore they share the same eigenvectors. So, in spirit, for each λ we can look the "easy" eigenvectors of $\mathbf{T}(\lambda)$ knowing that they are eigenvectors of all other matrices. Then the rich algebraic structure will allow us to track down the eigenvalues of each vector for every value of the spectral parameters.

In order to uncover this rich structure, we need to consider, in addition to the normal Transfer matrix, also the *monodromy matrix* $\mathcal{T}(\lambda)$, which is the operator that propagates an open horizontal line to the next, i.e. without imposing periodic boundary conditions. Thus, the monodromy matrix $\mathcal{T}$ posses an additional degree of freedom, compared to $\mathbf{T}$, corresponding to the state at the beginning and end of the line. The tracing of this degree of freedom is equivalent to requiring the in and out state to coincide (i.e. imposing periodic boundary conditions) and thus reproduced the transfer matrix.

If this boundary states belong to a Hilbert space of dimension k, the monodromy matrix will be a $k \times k$ dimensional matrix (where each matrix element will be an operator acting on the real N-dimensional space where the transfer matrix acts). These generalized commutation relations are one example of a Yang-Baxter equation, which in this case can be written as

$$\mathcal{T}_1(\lambda)\, \mathcal{T}_2(\mu)\, \mathcal{R}_{12}(\lambda,\mu) = \mathcal{R}_{12}(\lambda,\mu)\, \mathcal{T}_2(\mu)\, \mathcal{T}_1(\lambda)\ , \tag{5.3}$$

where the subscript 1 and 2 highlights that the monodromy matrices do not act on the same physical system (i.e. the same line of the same two-dimensional system), but on two different spaces, which are then connected by the so-called R-matrix $\mathcal{R}_{12}(\lambda,\mu)$, also known as the *intertwiner*. By taking the trace over the auxiliary space, we recover the original transfer matrix $\mathbf{T}(\lambda) = \operatorname{tr}\mathcal{T}(\lambda)$, and taking the trace of (5.3) we immediately see that

$$[\mathbf{T}(\lambda), \mathbf{T}(\mu)] = 0\ . \tag{5.4}$$

Therefore, as we claimed before, transfer matrices with different spectral parameters commute and thus share the same eigenvectors. This fact will be crucial for the Algebraic Bethe Ansatz construction that will be the subject of the next chapter.

In this chapter we will develop in detail what we have just sketched, taking as an example an important 2-dimensional classical model, known as the *ice-type* model, or six-vertex model. The choice of this model is clearly not accidental, as we will show that it is related to the XXZ spin chain (studied in the previous chapter using the coordinate Bethe Ansatz). In particular, the transfer matrix of the six-vertex model is the generating function for all the conserved charges of the Heisenberg chain, including its Hamiltonian. This chapter is meant to ease the reader into a clearer physical intuition of the ABA construction and its formalism. However, the algebraic

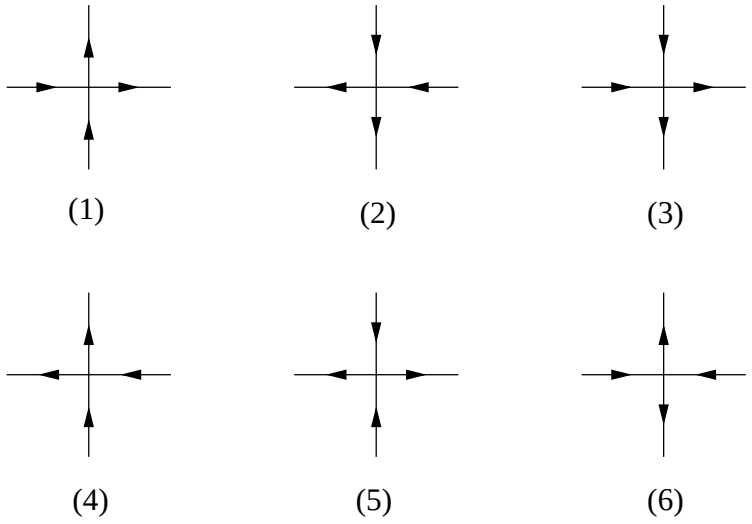

Figure 5.1: The configurations allowed by the ice-rule, representing the six possible vertexes.

Bethe Ansatz construction will be postponed till the next chapter and it does not require the knowledge of its two-dimensional analog.

5.2 Ice-type models

This model was originally introduced as a description of two dimensional ice. When water freezes, each oxygen atom is surrounded by four hydrogen ions. However, the hydrogen will be closer to one of its neighboring oxygen than to the other, but always in a way such that each oxygen has two hydrogens closer to it and two further away. This is known as the *ice rule*.

This system is therefore modeled as a square lattice where each bond between two vertices has an arrow pointing to either vertex, indicating to which of the two the hydrogen ion is closer to. What is important to us is that each bond has a degree of freedom that can assume two values, which we can represent as $+$ and $-$, 0 and 1, (or with classical spin-1/2), etc. To satisfy the ice rule we must have always two arrows pointing toward a vertex and two arrows point away: this constraint limits the number of possible vertex configurations to only 6, which are given in Figure 5.1 and labeled from 1 to 6. This is the reason for which this model is also known as the *six-vertex model*. There are other models similar to this and we should mention the 8-vertex model, where the ice rule is broken by adding two additional vertices, one with all the four arrows pointing toward the vertex and another with all arrows pointing away. The 8-vertex model is connected with the XYZ spin chain and also to the XY model. It is also called an elliptic model, since the parametrization of the coupling is achieve through elliptic functions (i.e. analytic functions that are periodic both in the real and imaginary direction). We will see that for the 6-vertex model this parametrization is done using just periodic function (and it is therefore referred to as the trigonometric case), which reduces to rational functions at the isotropic point ($\Delta = 1$).

The 6-vertex model is defined by the Boltzmann weights assigned to each vertex:

$$w_j = \mathrm{e}^{-\beta\epsilon_j} \ , \qquad j = 1, \ldots, 6 \ , \tag{5.5}$$

where $\beta = 1/k_B T$ is the usual inverse temperature scaled by the Boltzmann's constant. The partition function is given by the sum over all possible configurations of arrows on bonds, weighted by the above expressions, i.e.

$$\mathcal{Z} = \sum_{\{n_l\}} \exp\left[-\beta \left(n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 + n_4\epsilon_4 + n_5\epsilon_5 + n_6\epsilon_6\right)\right] \tag{5.6}$$

where $\{n_l\}$ are the number of vertices of type l in the system.

A first simplification arises because imposing periodic boundary condition forces the number of vertices of type 5 and 6 to be equal, since they act as sinks (source) for horizontal (vertical) arrows and vice-versa. This means that only the combination $n_5(\epsilon_5+\epsilon_6)$ appears in the partition function and we can chose $\epsilon_5 = \epsilon_6 = \epsilon_c$ with no loss of generality (since a different choice is unobservable).

If we further assume that the system is invariant under the simultaneous reversal of all arrows, then $n_1 = n_2$ and $n_3 = n_4$ and the partition function can be written as

$$\mathcal{Z} = \sum_{\{n_l\}} \exp\left\{-\beta \left[n_1 \left(\epsilon_1 + \epsilon_2\right) + n_3 \left(\epsilon_3 + \epsilon_4\right) + 2\ n_5\ \epsilon_c\right]\right\} \ , \tag{5.7}$$

and for the same reasoning as before we can choose

$$\epsilon_1 = \epsilon_2 \equiv \epsilon_a \ , \qquad \epsilon_3 = \epsilon_4 \equiv \epsilon_b \ . \tag{5.8}$$

The condition (5.8), i.e. the invariance under arrow reversal, is known as the *zero-field condition*. In fact, if we add a field E_y (E_y) in the vertical (horizontal) direction that couples to the arrows in each bond giving each up/down-pointing arrow an extra energy $\pm E_y$ and each right/left-pointing arrow the extra energy $\pm E_x$ we can break the degeneracy of the energies:

$$\begin{aligned} \epsilon'_1 &= \epsilon_a - E_x - E_y \ , & \epsilon'_2 &= \epsilon_a + E_x + E_y \ , \\ \epsilon'_3 &= \epsilon_b - E_x + E_y \ , & \epsilon'_4 &= \epsilon_b + E_x - E_y \ , \end{aligned} \tag{5.9}$$

and generates more vertices of one type or another. In the rest of this chapter we will always assume the zero-field condition $E_x = E_y = 0$, but it is possible to consider the more general case. For instance, a finite E_y corresponds to a finite external magnetic field in the XXZ model.

If our lattice has M rows and N columns, we can write the partition function as a sum of contributions from each of the M rows

$$\mathcal{Z} = \sum_{r=1}^{M} \sum_{\{m_l^r\}} a^{m_1+m_2}\ b^{m_3+m_4}\ c^{m_5+m_6} \ , \tag{5.10}$$

where $\{m_l^r\}$ is the number of vertices of type l in row r and we introduce the parameters a, b, c to identify the weights in the zero-field case:

$$a \equiv w_1 = w_2 \ , \qquad b \equiv w_3 = w_4 \ , \qquad c \equiv w_5 = w_6 \, . \tag{5.11}$$

We can rewrite the contribution from each row taking into account the configuration of arrows below and above it. If we denote by

$$\{\gamma^r\} = \{\gamma_1^r, \gamma_2^r, \ldots, \gamma_N^r\} \tag{5.12}$$

the configuration of arrows immediately below row r (since each arrow can assume two values –up or down– for each row we have 2^N possible configurations spanned by $\{\gamma^r\}$) we can write the partition function in terms of the row-to-row transfer matrix $\mathbf{T}_{\gamma^j}^{\gamma^{j+1}}$

$$\begin{aligned} \mathcal{Z} &= \sum_{\gamma^1}\sum_{\gamma^2}\ldots\sum_{\gamma^M} \mathbf{T}_{\gamma^1}^{\gamma^2}\mathbf{T}_{\gamma^2}^{\gamma^3}\ldots\mathbf{T}_{\gamma^{M-1}}^{\gamma^M}\mathbf{T}_{\gamma^M}^{\gamma^1} \\ &= \operatorname{tr}\mathbf{T}^M \,, \end{aligned} \tag{5.13}$$

where $\mathbf{T}$ is a $2^N \times 2^N$ matrix with elements

$$\mathbf{T}_{\gamma}^{\gamma'} = \sum a^{m_1+m_2}\, b^{m_3+m_4}\, c^{m_5+m_6} \,, \tag{5.14}$$

where the sum is over all configuration of vertices compatible with the vertical configuration given by γ and γ', i.e. a sum over all possible configurations of horizontal arrows on the N bonds of the row.

As a side note, we should remark that the number of up (down) arrows is conserved from one row to another (as a consequence of the toroidal boundary condition and of the fact that on each row we must have the same number of sources and sinks, i.e. vertices of type 5 and 6). This means that the transfer matrix has a block diagonal structure with blocks describing all the configurations with n up arrows entering and n up arrows exiting the row, $n = 0, \ldots, N$. This structure is equivalent to what we observed within the Bethe ansatz, that is that there is no particle productions and that states with n particles scatter and evolve only into states with the same number of particles n, so that the scattering matrix has the same block diagonal structure as the transfer matrix of the six-vertex model.

5.3 The Transfer Matrix and the Yang-Baxter equations

Let us now study the transfer matrix in more detail. Let us consider a configuration where the arrows below the row are given by the configuration $\gamma = \{\gamma_1, \ldots, \gamma_N\}$ and the ones above are $\gamma' = \{\gamma_1', \ldots, \gamma_N'\}$. We denote an up arrow by $\gamma_j = +1$ or $\gamma_j' = +1$ and a down arrow by $\gamma_j = -1$ and $\gamma_j' = -1$. We also denote the arrow on the horizontal bonds as $\alpha = \{\alpha_1, \ldots, \alpha_N\}$, with the convention that $\alpha_j = +1$ corresponds to a right-pointing arrow and a $\alpha_j = -1$ to a left-pointing one. With these notations in mind, we will refer to α_j, γ_j, γ_j' and so on as *spin variables* with spin up/down depending if they have value $+1/-1$.

We can write the transfer matrix as

$$\mathbf{T}_{\gamma}^{\gamma'} = \sum_{\alpha_1}\ldots\sum_{\alpha_N} \mathcal{L}_{\alpha_1\gamma_1}^{\alpha_2\gamma_1'}\mathcal{L}_{\alpha_2\gamma_2}^{\alpha_3\gamma_2'}\ldots\mathcal{L}_{\alpha_N\gamma_N}^{\alpha_1\gamma_N'} \,, \tag{5.15}$$

where $\mathcal{L}_{\alpha\ \gamma}^{\alpha'\gamma'}$ is a 4×4 matrix with entries given by the Boltzmann weights of the vertex configurations, i.e.

$$\mathcal{L}_{++}^{++} = \mathcal{L}_{--}^{--} = a\,, \tag{5.16}$$

$$\mathcal{L}_{+-}^{+-} = \mathcal{L}_{-+}^{-+} = b\,, \tag{5.17}$$

$$\mathcal{L}_{+-}^{-+} = \mathcal{L}_{-+}^{+-} = c\,, \tag{5.18}$$

with all other elements being zero due to the ice rule. More explicitly, this L-matrix can be written as

$$\mathcal{L} = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & a \end{pmatrix}\,. \tag{5.19}$$

As we mentioned in the introduction, our strategy at this point is not to attempt to diagonalize directly the transfer matrix, but instead to look under which conditions two transfer matrices with different parameters commute. To this end, let us introduce a second transfer matrix $\mathbf{T}'$, defined as in (5.14), but with Boltzmann weights a', b', c'. Then

$$\begin{aligned} \left(\mathbf{T}\mathbf{T}'\right)_{\gamma}^{\gamma'} &= \sum_{\{\gamma''\}} \mathbf{T}_{\gamma}^{\gamma''}\ \mathbf{T}_{\gamma''}^{\gamma'} \\ &= \sum_{\alpha_1,\ldots,\alpha_N}\ \sum_{\beta_1,\ldots,\beta_N}\ \prod_{j=1}^{N} \mathcal{S}_{\alpha_j\ \beta_j\ |\gamma_j}^{\alpha_{j+1}\beta_{j+1}|\gamma_j'}\,, \end{aligned} \tag{5.20}$$

where

$$\mathcal{S}_{\alpha\ \beta\ |\gamma}^{\alpha'\beta'|\gamma'} \equiv \sum_{\gamma''} \mathcal{L}_{\alpha\ \gamma}^{\alpha'\gamma''}\,\mathcal{L}'^{\beta'\gamma'}_{\beta\ \gamma''} \tag{5.21}$$

is the double-row transfer matrix (i.e. the operator that propagates across two rows, with different weights for each row) and with the understanding that $\alpha_{N+1} = \alpha_1$ and $\beta_{N+1} = \beta_1$. The operator $\mathcal{S}$ is a 8×8 matrix. If we keep the two vertical indices as fixed, we can write it as a 4×4 matrix as

$$\mathcal{S}_{\alpha\ \beta}^{\alpha'\beta'}(\gamma,\gamma') \equiv \mathcal{S}_{\alpha\ \beta\ |\gamma}^{\alpha'\beta'|\gamma'}\,, \tag{5.22}$$

and (5.20) as

$$\left(\mathbf{T}\mathbf{T}'\right)_{\gamma}^{\gamma'} = \operatorname{tr}\mathcal{S}(\gamma_1,\gamma_1')\mathcal{S}(\gamma_2,\gamma_2')\ldots\mathcal{S}(\gamma_N,\gamma_N')\,. \tag{5.23}$$

We can also consider to invert the order of the two rows, but keeping the external legs fixed and write the resulting double-row transfer matrix as

$$\left(\mathbf{T}'\mathbf{T}\right)_{\gamma}^{\gamma'} = \operatorname{tr}\mathcal{S}'(\gamma_1,\gamma_1')\mathcal{S}'(\gamma_2,\gamma_2')\ldots\mathcal{S}'(\gamma_N,\gamma_N')\,, \tag{5.24}$$

where

$$\mathcal{S}'^{\alpha'\beta'|\gamma'}_{\alpha\ \beta\ |\gamma} \equiv \sum_{\gamma''} \mathcal{L}'^{\alpha'\gamma''}_{\alpha\ \gamma}\,\mathcal{L}^{\beta'\gamma'}_{\beta\ \gamma''}\,. \tag{5.25}$$

We want to find under which conditions these expressions commute, i.e.

$$\mathbf{T}\mathbf{T}' = \mathbf{T}'\mathbf{T}\,. \tag{5.26}$$

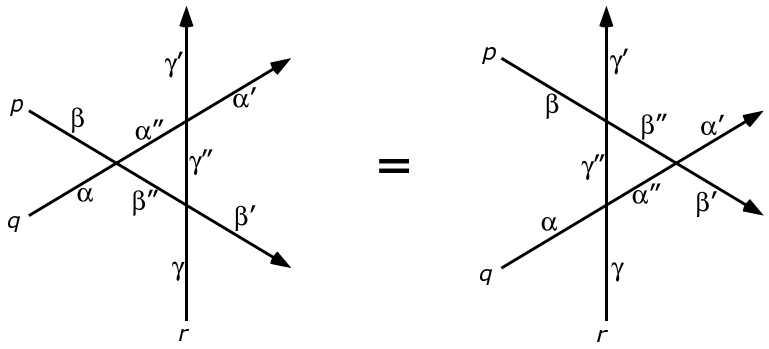

Figure 5.2: Diagrammatic representation of the Yang-Baxter equation (5.28).

This will surely be true if there exists a 4×4 non-singular matrix $\mathcal{R}$ such that

$$\mathcal{S}(\gamma, \gamma') = \mathcal{R}\, \mathcal{S}'(\gamma, \gamma') \mathcal{R}^{-1} \,, \tag{5.27}$$

where we remind that $\mathcal{S}$ is also a 4 matrix and $\gamma = \pm 1$ and $\gamma' = \pm 1$ are taken as parameters. If (5.27) is satisfied, then plugging it into (5.23) and using the cyclic property of the trace we get (5.24) as we set to achieve.

We can write (5.27) explicitly as

$$\sum_{\alpha'', \beta'', \gamma''} \mathcal{L}^{\alpha'' \gamma''}_{\alpha \gamma}\; \mathcal{L}'^{\beta'' \gamma'}_{\beta \gamma''}\; \mathcal{R}^{\alpha' \beta'}_{\alpha'' \beta''} = \sum_{\alpha'', \beta'', \gamma''} \mathcal{R}^{\alpha'' \beta''}_{\alpha \beta}\; \mathcal{L}'^{\alpha' \gamma''}_{\alpha'' \gamma}\; \mathcal{L}^{\beta' \gamma'}_{\beta'' \gamma''} \,. \tag{5.28}$$

This is the *Yang-Baxter* equation for the L-matrices and it can be understood more clearly in its diagrammatic form, fig. 5.2. We see that the R-matrix acts as an intertwiner for the two L-matrices since it connects the "horizontal" spins, but it does not act on the "vertical" ones.

At this point we make an ansatz, i.e. we assume that the R-matrix has the same structure as an L-matrix, i.e. that we can write it as in (5.19), but with different weights, namely a'', b'' and c'': $\mathcal{R} = \mathcal{L}''$. This is not to say that the R-matrix can be identified with an L-matrix (since they act on different spaces as operators), but only to assume that the ice-rules apply to $\mathcal{R}$ as well.

Then we can look for solutions of (5.28) by writing it as a system of 64 equations (coming from equating each component of the resulting matrix multiplication, or corresponding to all possible combination of the external spin variables). We take a, b, c as given and we look for which choices of a', b', c' and a'', b'', c'' (5.28) is satisfied. Notice that, since all equations are homogeneous, they do not fix the normalization of the matrices $\mathcal{L}'$ $\mathcal{R}$, so the parameters can be rescaled by a constant without violating (5.28), so only 4 out of the six parameters are meaningful to solve the Yang-Baxter equation.

Of course there is one trivial solution:

$$\mathcal{L}' \propto \mathcal{L} \,, \qquad \text{and} \qquad \mathcal{R}^{\alpha' \beta'}_{\alpha, \beta} = \delta_{\alpha \alpha'} \delta_{\beta \beta'} \,, \tag{5.29}$$

but this amounts to say that the transfer matrix commutes with simple multiples of itself and it is not interesting. To look for non-trivial solution, we notice that the ice-rule severely restricts

the number of non-zero components of the L and R-matrices (see 5.19). In fact, $\mathcal{L}_{\alpha\gamma}^{\alpha'\gamma'} = 0$ unless $\alpha + \gamma = \alpha' + \gamma'$. This means that both sides of (5.28) are identically zero only if $\alpha + \beta + \gamma \neq \alpha' + \beta' + \gamma'$ and this leaves only 20 non-trivial equations out of the 64.

Moreover, the zero-field condition implies that negating all the spin variables leaves the Boltzmann weights unchanged, so these 20 equations occur in 10 identical pairs. Finally, the symmetric structure of (5.28) under the reversal of spin pairs can be shown to lead to just these three inequivalent equations:

$$\begin{aligned} ac'a'' &= bc'b'' + ca'c'' \,, \\ ab'c'' &= ba'c'' + cc'b'' \,, \\ cb'a'' &= ca'b'' + bc'c'' \,. \end{aligned} \tag{5.30}$$

This is quite a miracle that is completely due to (and responsible for) the integrability of the model.

First, let us eliminate a'', b'', c'' from (5.30): this leaves the single equation

$$\frac{a^2 + b^2 - c^2}{ab} = \frac{a'^2 + b'^2 - c'^2}{a'b'} \,. \tag{5.31}$$

This means that we can associate to each L-matrix a quantity

$$\Delta \equiv \frac{a^2 + b^2 - c^2}{2ab} \tag{5.32}$$

which has to remain invariant for each member of a family in order for the transfer matrices to commute. In other words, $\mathbf{T}$ and $\mathbf{T}'$ can have different values of a, b, c, but they would still commute as long as $\Delta = \Delta'$.

It is convenient to look for a parametrization of a, b, c that will identically satisfy (5.31). The easiest choice is clearly

$$a = a \,, \qquad b = ax \,, \qquad c = a\sqrt{1 - 2\Delta x + x^2} \,, \tag{5.33}$$

where we recognized that an overall multiplicative constant can be factorized out. The problem with this parametrization is that c is not an entire function of x and Δ, due to the branch of the square root (an entire function does not have branch point or branch cuts). This problem can be solved by introducing a new parameter t defined as

$$t = \sqrt{\frac{x - x_1}{x - x_2}} \,, \tag{5.34}$$

where $x_{1,2}$ are the zeros of c, i.e.

$$\Delta = \frac{1}{2}\,(x_1 + x_2) \,, \qquad \text{and} \qquad x_2 = \frac{1}{x_1} \,. \tag{5.35}$$

Solving (5.34) for x we get

$$x = \frac{x_1 - t^2 x_2}{1 - t^2} \,, \tag{5.36}$$

so that

$$\sqrt{1-2\Delta x+x^2} = (x_1 - x_2)\frac{t}{1-t^2} \tag{5.37}$$

and we get

$$a = a\ , \qquad b = \frac{a}{x_1}\,\frac{x_1^2 - t^2}{1-t^2}\ , \qquad c = \frac{a}{x_1}\,(x_1^2-1)\,\frac{t}{1-t^2}\ . \tag{5.38}$$

This is a entire parametrization of the L-matrix where keeping x_1 fixed assures that (5.31) is satisfied while a and t can be varied at will.

However, it is customary to make an additional change of variable from t, x_1 to λ, ϕ to achieve a parametrization in terms of hyperbolic function. Different equivalent choices can be made, we will use:

$$x_1 \equiv \mathrm{e}^{\phi}\ , \qquad t \equiv \mathrm{e}^{\lambda+\phi}\ , \qquad \Delta = \cosh\phi\ . \tag{5.39}$$

This gives

$$a = a\ , \qquad b = a\,\frac{\sinh\lambda}{\sinh(\lambda+\phi)}\ , \qquad c = a\,\frac{\sinh\phi}{\sinh(\lambda+\phi)}\ . \tag{5.40}$$

Other standard parametrization can be found by shifting or multiplying λ. We also have freedom in the overall normalization factor a. A natural one is to to eliminate the denominators:

$$a = \rho\ \sinh(\lambda+\phi)\ , \qquad b = \rho\ \sinh\lambda\ , \qquad c = \rho\ \sinh\phi\ . \tag{5.41}$$

In conclusions, we see that if the parameters of the L-matrices are chosen according to (5.41) with different ρ and λ, but the same ϕ, then (5.31) is satisfied and the corresponding transfer matrices will commute. Since ρ is an uninfluential normalization constant, the transfer matrices belonging to a commuting family will be denoted as $\mathbf{T}(\lambda)$, where λ is usually called the *spectral parameter*. The dependence of the transfer matrix on $\phi = \cosh^{-1}\Delta$ is normally assumed and not explicitly written and thus all transfer matrices in the following are taken with the same ϕ.

As the R-matrix has been chosen of the same form as the L-matrix, it will also have a parametrization like (5.41) with the same ϕ as the L-matrices (this can be seen by eliminating the prime variables from (5.30) to get that $\Delta = \Delta''$).

Thus, two of the three equations in (5.30) have given $\Delta = \Delta' = \Delta''$. Substituting our parametrization (5.41) for the unprimed, primed and double-primed variables in (5.30) we see that the last equations gives

$$\lambda'' = \lambda' - \lambda\ . \tag{5.42}$$

To conclude, we have proven that for a given $\mathcal{L}(\lambda)$, we can construct a whole family of matrices $\mathcal{L}(\lambda')$ that satisfy the Yang-Baxter equation (5.28) with $\mathcal{R}(\lambda-\lambda')$. In matrix form

$$\mathcal{L}_n(\lambda)\ \mathcal{L}_n(\lambda')\ \mathcal{R}(\lambda'-\lambda) = \mathcal{R}(\lambda'-\lambda)\mathcal{L}_n(\lambda')\ \mathcal{L}_n(\lambda)\ , \tag{5.43}$$

where the n indicates on which sites of the row does the L-operator acts.

Summing over all the configuration on a given row j corresponds to taking the products of the L-matrices at different sites: this defines the *monodromy matrix* $\mathcal{T}_j(\lambda)$:

$$\mathcal{T}_j(\lambda) \equiv \mathcal{L}_N(\lambda)\ \mathcal{L}_{N-1}(\lambda)\ \ldots\ \mathcal{L}_1(\lambda)\ . \tag{5.44}$$

This is an $2^{N+1} \times 2^{N+1}$ matrix that depends on the N spin variable above and below the line and on the first and last horizontal spin. If we consider two such monodromy matrices, acting on different rows and with different couplings, i.e. different spectral parameter, by using the Yang-Baxter equation (5.28) for the L-matrices, we can shift the intertwiner R-matrix from one end to the other of the chain to get

$$\mathcal{T}_j(\lambda)\, \mathcal{T}_l(\lambda')\, \mathcal{R}_{jl}(\lambda' - \lambda) = \mathcal{R}_{jl}(\lambda' - \lambda)\, \mathcal{T}_l(\lambda')\, \mathcal{T}_j(\lambda)\ , \tag{5.45}$$

which is the Yang-Baxter for the monodromy matrix (here we explicitly write the index j and l to remind us of the different spaces where these operators act and the R-matrix is intended to act only on the space of the horizontal spins). Taking the trace over the horizontal spins in (5.45) corresponds to closing the chain with periodic boundary conditions: since $\operatorname{tr} \mathcal{T}_j(\lambda) = \mathbf{T}_j(\lambda)$, using the periodicity of the trace we find

$$\left[\mathbf{T}_j(\lambda), \mathbf{T}_l(\lambda')\right] = 0\ . \tag{5.46}$$

Let us remark that the proper Yang-Baxter equation is a condition on the R-matrix alone. To see this, let us consider the product of three monodromy matrices and notice that by applying (5.45) in different ways (order) we can get a different result:

$$\begin{aligned}
\mathcal{T}_j(\lambda)\, \mathcal{T}_l(\mu)\, \mathcal{T}_k(\nu) &= \mathcal{R}_{jl}^{-1}(\lambda - \mu)\, \mathcal{R}_{jk}^{-1}(\lambda - \nu)\, \mathcal{R}_{lk}^{-1}(\mu - \nu) \\
&\quad \times \mathcal{T}_k(\nu)\, \mathcal{T}_l(\mu)\, \mathcal{T}_j(\lambda) \\
&\quad \times \mathcal{R}_{lk}(\mu - \nu)\, \mathcal{R}_{jk}(\lambda - \nu)\, \mathcal{R}_{jl}(\lambda - \mu) \\
&= \mathcal{R}_{lk}^{-1}(\mu - \nu)\, \mathcal{R}_{jk}^{-1}(\lambda - \nu)\, \mathcal{R}_{jl}^{-1}(\lambda - \mu) \\
&\quad \times \mathcal{T}_k(\nu)\, \mathcal{T}_l(\mu)\, \mathcal{T}_j(\lambda) \\
&\quad \times \mathcal{R}_{jl}(\lambda - \mu)\, \mathcal{R}_{jk}(\lambda - \nu)\, \mathcal{R}_{lk}(\mu - \nu)\ .
\end{aligned} \tag{5.47}$$

Thus, in order to preserve associativity, we must have

$$\mathcal{R}_{lk}(\mu - \nu)\, \mathcal{R}_{jk}(\lambda - \nu)\, \mathcal{R}_{jl}(\lambda - \mu) = \mathcal{R}_{jl}(\lambda - \mu)\, \mathcal{R}_{jk}(\lambda - \nu)\, \mathcal{R}_{lk}(\mu - \nu) \tag{5.48}$$

which is the Yang-Baxter equation for the R-matrices. This is the fundamental equation defining an integrable model. It defines an algebra and finding solutions to (5.48) is in a sense equivalent to finding representations for the group. Every time a solution is identified for (5.48) in some k-dimensional space, one can construct the corresponding L-matrices and monodromy matrices that satisfy (5.28, 5.45) and eventually identify the model one has just solved. In the case of the six-vertex model, we have found a trigonometric solution of (5.48) in terms of a $2^2 \times 2^2$ matrix[1].

The main advantage of having proven that transfer matrices at different spectral parameters commute, is that we can now interpret the transfer matrix as a generator for the conserved charges of the theory (which are in infinite number, since the model is integrable). In practice,

[1] Note that there are solution of the Yang-Baxter equation were the dimensions of the L and R-matrix are not the same.

it is more convenient to consider the logarithm of the transfer matrix as the generating function of the integrals of motion, since in this way they turn out to be local operators with simple physical interpretation. In fact, we can expand the generating function around $\lambda = 0$

$$\ln \mathbf{T}(\lambda) = \sum_{n=0}^{\infty} J_n \lambda^n \ . \tag{5.49}$$

Plugging this into (5.46) we see that

$$[J_n, J_m] = 0 \ , \tag{5.50}$$

so that the coefficients of the expansions can be interpreted as conserved densities in involution with one another.

Let us look at these conserved quantities. If we set $\lambda = 0$, we see that

$$\mathcal{L}_{\alpha\gamma}^{\alpha'\gamma'}(\lambda = 0) = \rho \ \sinh\phi \ \delta_{\alpha\gamma'} \ \delta_{\alpha'\gamma} \ . \tag{5.51}$$

This means that the L-operator transfers the in-horizontal spin to the out-vertical state and the in-vertical spin to the out-horizontal one. Successive application of this L-operator, progressively shifts the in-vertical state in one column to the out-vertical spin in the next column. Taking the final trace over the first and last horizontal spin closes the chain and effectively shifts the last vertical spin. Thus the net effect of the transfer matrix is that of a shift by one lattice sit, i.e.

$$\mathbf{T}(0) = \rho^N \ \sinh^N \phi \ \mathrm{e}^{\mathrm{i}\hat{P}} \ , \tag{5.52}$$

where $\hat{P}$ is the lattice momentum operator. Similarly, one can take the first logarithmic derivative of the transfer matrix at $\lambda = 0$ and show that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathbf{T}(\lambda)\Big|_{\lambda=0} = \frac{1}{2\sinh\phi} \sum_{j=1}^{N} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh\phi \left(1 + \sigma_j^z \sigma_{j+1}^z\right)\right] \ , \tag{5.53}$$

where σ_j^α are Pauli matrices, which emerge as matrix representations of Cronecker delta's. Thus, the logarithm of $\mathbf{T}$ at $\lambda = 0$ is proportional to the lattice momentum and its first logarithmic derivative gives an operator that is proportional to the Hamiltonian of the XXZ model (plus a constant). This shows the connection between the six-vertex model and the Heisenberg chain and implies that all higher logarithmic derivatives of the transfer matrix are in convolution with the Hamiltonian and are therefore integral of motions as asserted.

Finally, let us mention that it is possible to construct an additional operator $\mathbf{Q}(\lambda)$, called the Q-matrix, that allows for an easy derivation of the Bethe equation and of the spectrum of the transfer matrix. This construction becomes important in the algebraic version of the thermodynamical Bethe Ansatz (which we will not cover).

The Q-matrix is defined as an operator that commutes with the transfer matrix

$$\left[\mathbf{T}(\lambda), \mathbf{Q}(\lambda')\right] = \left[\mathbf{Q}(\lambda), \mathbf{Q}(\lambda')\right] = 0 \tag{5.54}$$

and satisfies the following equation

$$\mathbf{T}(\lambda)\mathbf{Q}(\lambda) = \mathbf{Q}(\lambda)\mathbf{T}(\lambda) = \sigma(\lambda - \phi) \ \mathbf{Q}(\lambda + 2\phi) + \sigma(\lambda + \phi) \ \mathbf{Q}(\lambda - 2\phi) \ , \tag{5.55}$$

with

$$\sigma(\lambda) \equiv [\rho \sinh \lambda]^N \ . \tag{5.56}$$

It can be proven [31] that a Q-operator satisfying (5.54, 5.55) exists.

When we introduced the six-vertex model at the beginning, we argued that the periodic boundary conditions imply that the number of vertical spin up and down is conserved from one row to another (and hence vertices of type 5 and 6 appear in the same numbers in each row). This means that the transfer matrix (and the partition function) of the six-vertex model have a block-diagonal structure, where each block corresponds and connect only configurations with a given number of spin down, say r. Since the Q-matrix commutes with the transfer matrix, it shares the same eigenvectors and the same block-diagonal structure. Thus, we can diagonalize $\mathbf{T}(\lambda)$ and $\mathbf{Q}(\lambda)$ separately in each $r \times r$ block.

For a given λ, in a given block, the eigenvalues $Q(\lambda)$ of $\mathbf{Q}(\lambda)$ can be parameterized in terms of r quantities λ_j in the following way

$$Q(\lambda) = C \prod_{j=1}^{r} \sinh(\lambda - \lambda_j) \ , \tag{5.57}$$

with some constant C. From (5.54), it can be proved that the same parametrization, with the same λ_j, is valid for all values of the spectral parameters.

Since $\mathbf{T}$ and Q commute, they can be simultaneously diagonalized in each $r \times r$ block and the TQ-relation (5.55) can be written as a set of scalar equations

$$\Lambda(\lambda) Q(\lambda) = \sigma(\lambda - \phi)\, Q(\lambda + 2\phi) + \sigma(\lambda + \phi)\, Q(\lambda - 2\phi) \ . \tag{5.58}$$

From (5.57) we see that $Q(\lambda)$ has r zeros located at $\lambda = \lambda_l$ (i.e. there are r values of λ at which the Q-operator has vanishing determinant): evaluating (5.58) at such zeros we get

$$\sigma(\lambda_j - \phi)\, Q(\lambda_j + 2\phi) + \sigma(\lambda_j + \phi)\, Q(\lambda_j - 2\phi) = 0 \ , \tag{5.59}$$

i.e.

$$\left(\frac{\sinh(\lambda_j + \phi)}{\sinh(\lambda_j - \phi)} \right)^N = - \prod_{l=1}^{r} \frac{\sinh(\lambda_j - \lambda_l + 2\phi)}{\sinh(\lambda_j - \lambda_l - 2\phi)} \ , \qquad j = 1, \ldots, r \ , \tag{5.60}$$

which we recognize as the Bethe equations for the XXZ model and that specify the parameters λ_j in (5.57). So, in this construction, the Bethe equations arise as consistency equations for the TQ-relation to be valid. Having found the eigenvalues of $\mathbf{Q}(\lambda)$, we can substitute them into (5.58) to find the spectrum of the transfer matrix

$$\Lambda(\lambda) = \rho^N \left[\sinh^N(\lambda - \phi) \prod_{l=1}^{r} \frac{\sinh(\lambda - \lambda_j + 2\phi)}{\sinh(\lambda - \lambda_j)} + \sinh^N(\lambda + \phi) \prod_{l=1}^{r} \frac{\sinh(\lambda - \lambda_j - 2\phi)}{\sinh(\lambda - \lambda_j)} \right] . \tag{5.61}$$

Thus, we accomplished what we set out to do, i.e. to determine the spectrum of the transfer matrices, from which we can access all information contained in the partition function. To this end it was fundamental to extend the original problem of diagonalizing a single system to a

whole family of commuting ones, since this gave us the freedom of choosing the most suitable λ for each eigenvalue, namely the one that vanishes the determinant of $\mathbf{Q}(\lambda)$, see (5.59).

The TQ-construction has thus shown to be very helpful. Its limitation is that it does not give us direct access to the eigenvectors of the system. When we derived the Yang-Yang equation for the thermodynamics of the Lieb-Liniger model we saw that one take a similar point of view, focusing directly on the energy eigenvalues, instead of the eigenfunction. The operator generalization of the Yang-Yang equation takes advantage of the TQ-relations to develop the *Thermodynamic Bethe Ansatz*, but this subject will not be addressed in these notes. The Algebraic Bethe Ansatz is a different way to use the transfer matrix that starts from its eigenstates construction to characterize the system. This is the subject of the next chapter.

Chapter 6

Algebraic Bethe Ansatz

6.1 Overview

The Algebraic Bethe Ansatz (ABA) method is essentially a second quantization of the coordinate one. It uses the Yang-Baxter algebra of the Transfer Matrix to generate the wavefunctions by applying certain operators (which can be interpreted as quasi-particle creation operators) to a reference state (known as *pseudo-vacuum*). The Bethe equations then emerge as consistency conditions for these states to be eigenvectors of the transfer matrix. The Algebraic Bethe Ansatz construction is one of the results in a long effort to understand the relation between seemingly different kind of integrable systems. In fact, it is the quantum version of the Inverse Scattering Method (ISM): a construction that, through the Lax representation of classical integrable non-linear differential equations, has allowed a deeper understanding of these system and, even most notably, the systematic construction of their soliton solutions. From another angle, the ABA is grounded on the relation between two-dimensional classical integrable statistical physics systems and 1-D quantum one. The connection between the transfer matrix $\mathbf{T}$ of the six-vertex model and the Hamiltonian of the XXZ chain that was shown at the end of the last chapter is one such example that means that the two models share the same eigenvectors.

All the techniques developed to solve non-trivial integrable models pass through the enlargement of the physical Hilbert space with the introduction of some auxiliary space. This is done to the goal of "decoupling" the interaction so that the physical degrees of freedom do not interact among themselves, but only with the auxiliary space. This simplifies the problem to the point of allowing the exact solution. Then, one only needs to trace over the auxiliary degrees of freedom to project the solution to the physical space.

One can think of this additional space as describing a new degree of freedom, a sort of a probe, that propagates inside the system (something like an unobservable gauge field that encodes the interaction between otherwise free particles in a gauge theory). The transfer matrix that takes into account the action on this auxiliary space is called the *monodromy matrix* $\mathcal{T}_a(\lambda)$ and tracing over the auxiliary space reproduced the original transfer matrix $\mathbf{T}(\lambda) = \mathrm{tr}_a \mathcal{T}_a(\lambda)$. The spectral parameter λ belongs to the probe and identifies a continuous degree of freedom it has. The auxiliary space allows to introduce generalized commutation relations

for the monodromy matrices at different spectral parameters. If the auxiliary space admits a κ-dimensional representation, the monodromy matrix will be a $\kappa \times \kappa$ dimensional matrix (while each matrix element will be an operator acting on the physical N-dimensional space where the transfer matrix acts). These generalized commutation relations are one example of a Yang-Baxter equation, which in this case can be written as

$$\mathcal{T}_a(\lambda)\, \mathcal{T}_b(\mu)\, \mathcal{R}_{a,b}(\lambda,\mu) = \mathcal{R}_{a,b}(\lambda,\mu)\, \mathcal{T}_b(\mu)\, \mathcal{T}_a(\lambda)\ , \tag{6.1}$$

where the subscript a and b highlights that the monodromy matrices act on two different auxiliary spaces, that are connected by the so-called R-matrix $\mathcal{R}_{12}(\lambda,\mu)$, also known as the *intertwiner*.

Thus, all these constructions depend on the existence of an operator, the R-matrix, satisfying the Yang-Baxter equation (YBE). Then, this R-matrix is used together with the L-operator and the monodromy matrix to satisfy a similar YBE, which is a sort of generalized commutation relation[1]. The main limitation of these approaches is that in general it is not possible to immediately write down the R-matrix for a specific system. In practice, one looks for solutions of the YBE and, every time a R-matrix is found, the Inverse Scattering Method machinery is used to identify the model at hand. Once the R-matrix is given, this identification is straightforward and direct, like calculating the derivative of a function. The inverse problem of finding the primitive of a function, however, is quite complicated and in general we do not have a systematic way to do so (even if we have a number of tricks and techniques to help), but we rely on our experience in taking derivatives. The situation is very similar to the problem of finding the R-matrix of a given model.

The ABA method supposes that the R-matrix and the YBE for a given 1-D quantum system are known. Then, the plain-wave expansion at the heart of the coordinate approach is replaced by generic "modes", which are taken as operators (i.e. they are second quantized) and applied to a reference state, similarly to what one does to move from quantum mechanics to quantum field theory. The fact that the states are now represented in terms of operators, simplifies its manipulation (in comparison with having to deal with an explicit form for the wavefunction as in the Coordinate Bethe Ansatz approach) and highlights its algebraic structures and properties. These features allow the formalism to express in a relatively compact way the correlation functions of the system. The compactness of the formalism hides a growing computational complexity when one wants to extract the behavior of the correlation functions for points further and further apart. In this course, due to the lack of time and the growing complexity of the subject, we will not show how to construct correlation functions using the algebraic approach. The interested reader can find the foundations of this construction in the book by Korepin et al. [19] and look in the current literature for the recent progresses in applying this formal techniques to numerics to significantly reduce the computational difficulty and length (see the works of J.S. Caux's group, for instance).

[1]The YBE satisfied by the L-operator can be thought of as an algebra, whose structure factors are given by the R-matrix. This algebra also has an adjoint representation, same as in traditional Lie-algebras, which is the YBE satisfied by the R-matrix with itself.

6.2 Preliminaries

Before we proceed with the algebraic construction, let us look once more at the two-body interaction from the scattering matrix point of view. Let us write the two-body wavefunction as

$$\begin{aligned}\Psi(x_1,x_2) &= \sum_{\mathcal{P}} \Psi(\mathcal{Q}|\mathcal{P}) \mathrm{e}^{\mathrm{i}\sum_j x_{\mathcal{Q}j} k_{\mathcal{P}j}} && (6.2)\\ &= \begin{cases} \Psi(1,2|1,2)\mathrm{e}^{\mathrm{i}(x_1k_1+x_2k_2)} + \Psi(1,2|2,1)\mathrm{e}^{\mathrm{i}(x_1k_2+x_2k_1)} , & x_1<x_2 \\ \Psi(2,1|1,2)\mathrm{e}^{\mathrm{i}(x_2k_1+x_1k_2)} + \Psi(2,1|2,1)\mathrm{e}^{\mathrm{i}(x_2k_2+x_1k_1)} , & x_1>x_2 \end{cases} \\ &= \mathrm{e}^{\mathrm{i}XK} \begin{cases} \Psi(1,2|1,2)\mathrm{e}^{\mathrm{i}xk} + \Psi(1,2|2,1)\mathrm{e}^{-\mathrm{i}xk} , & x<0 \\ \Psi(2,1|1,2)\mathrm{e}^{-\mathrm{i}xk} + \Psi(2,1|2,1)\mathrm{e}^{\mathrm{i}xk} , & x>0 \end{cases} , && (6.3)\end{aligned}$$

where we used center-of-mass coordinates

$$X \equiv \frac{x_1+x_2}{2} , \qquad x \equiv \frac{x_1-x_2}{2} , \tag{6.4}$$

$$K \equiv k_1+k_2 , \qquad k \equiv k_1-k_2 . \tag{6.5}$$

We explicitly wrote the dependence of the amplitudes $\Psi(\mathcal{Q}|\mathcal{P})$ on the order of particles (given by the $\mathcal{Q}$-permutation) and of the pairing with the different momenta (given by the $\mathcal{P}$-permutation).

Let us now imagine a scattering experiment. We send in a beam from the left and we measure a reflected component on the left with amplitude $R(k)$ and a transmitted one to the right with amplitude $T(k)$:

$$\Psi(1,2|1,2) = R(k)\Psi(1,2|2,1) + R(k)\Psi(2,1|2,1) . \tag{6.6}$$

Similarly, if we start with an incident ray from the right we have

$$\Psi(2,1|1,2) = R(k)\Psi(2,1|2,1) + T(k)\Psi(1,2|2,1) . \tag{6.7}$$

We can cast these equations in matrix form, in several ways. We can write

$$\Psi^r(\mathcal{P}') = \begin{pmatrix} \Psi(1,2|1,2) \\ \Psi(2,1|1,2) \end{pmatrix} = \begin{pmatrix} R(k) & T(k) \\ R(k) & T(k) \end{pmatrix} \begin{pmatrix} \Psi(1,2|2,1) \\ \Psi(2,1|2,1) \end{pmatrix} = \mathbf{S}^r(k)\Psi(\mathcal{P}) , \tag{6.8}$$

Where the identities of the particles are uncorrelated with the momenta. This representation is called *reflection-diagonal*. An alternative choice is the *transmission-diagonal representation*

$$\Psi^t(\mathcal{P}') = \begin{pmatrix} \Psi(2,1|2,1) \\ \Psi(1,2|1,2) \end{pmatrix} = \begin{pmatrix} T(k) & R(k) \\ R(k) & T(k) \end{pmatrix} \begin{pmatrix} \Psi(1,2|2,1) \\ \Psi(2,1|2,1) \end{pmatrix} = \mathbf{S}^t(k)\Psi(\mathcal{P}) , \tag{6.9}$$

where we identify each particle with the momentum it carries. Other representations are possible, but we will not use them.

The reflection and transmission-diagonal representations are related by a matrix $\mathbf{\Pi} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ that exchanges the particles:

$$\Psi^r(\mathcal{P}) = \mathbf{\Pi}\Psi^t(\mathcal{P}) , \qquad \mathbf{S}^r(k) = \mathbf{\Pi}\mathbf{S}^t(k) . \tag{6.10}$$

Using $\mathbf{\Pi}^2 = 1$, we can in fact write

$$\mathbf{S}^r(k) = T(k) + \mathbf{\Pi}\, T(k)\ , \qquad \mathbf{S}^t(k) = T(k) + \mathbf{\Pi}\, R(k)\ . \tag{6.11}$$

If the particle have bosonic/fermionic statistics, we have $\Pi = \pm 1$, thus

$$S^r(k) = R(k) \pm T(k)\ , \qquad S^t(k) = T(k) \pm R(k) = \pm S^r(k)\ . \tag{6.12}$$

The transmission and reflection coefficients are uniquely determined by the statistic of the particles and their scattering phase $S^r(k) = -\mathrm{e}^{-\mathrm{i}\theta(k)}$. Comparing with section (see 2.3), we see that for a bosonic problem (for which every coordinate permutation $\mathcal{Q}$ reproduces the same wavefunction) $r(k)$ and $t(k)$ are related to the two-body scattering phase $\theta(k)$ by

$$T(k) = \frac{1 - \mathrm{e}^{-\mathrm{i}\theta(k)}}{2}\ , \qquad R(k) = -\frac{1 + \mathrm{e}^{-\mathrm{i}\theta(k)}}{2}\ . \tag{6.13}$$

In a lattice system, the two-body problem has also to be supplemented with the information about the presence of a particle on the lattice site. Thus, the scattering matrix becomes a 4×4 matrix. Let us discuss this problem in its spin formulation (i.e. instead of discussing particle on a lattice, let us perform a Jordan-Wigner transformation and take spin on a chain).

We shall take the XXZ Hamiltonian (4.1) as an example of spin interaction. We write the two body-interaction as a matrix connecting the 4 possible states $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$. Aligned spins only scatter on the same state. Antialigned spins can be reflected on the same state or transmitted by exchanging identity. Thus, the scattering matrix can be written as

$$\mathbf{S}^r = \begin{pmatrix} \Theta & 0 & 0 & 0 \\ 0 & R & T & 0 \\ 0 & T & R & 0 \\ 0 & 0 & 0 & \Theta \end{pmatrix} = \Theta(k) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & t & 0 \\ 0 & t & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Theta(k)\, \mathbf{s}^r(k)\ , \tag{6.14}$$

where we normalized by the amplitude for the aligned scattering and introduces the reduced reflection and transmission amplitudes $r \equiv R/\Theta$, $t \equiv T/\Theta$. Here we work in the reflection-diagonal representation, but we could also write it in the transmission-diagonal by using the 4×4 exchange operator

$$\mathbf{\Pi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ , \tag{6.15}$$

as

$$\mathbf{S}^t = \mathbf{\Pi}\mathbf{S}^r = \Theta(k) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & r & 0 \\ 0 & r & t & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ . \tag{6.16}$$

Notice that these matrices have a natural representation as a product of 2×2 Pauli matrices

$$\mathbf{s}^t(k) = \frac{1}{2}\Big[1 + \sigma_z\sigma'_z + t(k)\left(1 - \sigma_z\sigma'_z\right) + r(k)\left(\sigma_x\sigma'_x + \sigma_y\sigma'_y\right)\Big]\ . \tag{6.17}$$

We also have

$$\mathbf{\Pi} = \frac{1 + \vec{\sigma} \cdot \vec{\sigma}'}{2} \,. \tag{6.18}$$

Comparing with the results of chapter 4, for the XXZ chain we have (using rapidities instead of the momentum)

$$t(\lambda) = \frac{\sin \frac{\phi}{2} (\lambda + \mathrm{i})}{\sin \frac{\phi}{2} (\lambda - \mathrm{i})} \,, \qquad\qquad r(\lambda) = \frac{\sin \phi}{\sin \frac{\phi}{2} (\lambda - \mathrm{i})} \,, \tag{6.19}$$

where $\phi = \cosh^{-1} \Delta$.

6.3 Construction of the Transfer Matrix

In chapter 5 we studied the classical six-vertex model and sketched its transfer matrix solution. This was done with the introduction of certain operators and their relations in a clear physical context. We saw the natural emergence of the Yang-Baxter equation and how this implies that transfer matrices at different spectral parameters commute and can be diagonalized simultaneously. This fact can be used to generate all the conserved quantities of the model as the coefficients of a power expansion of the transfer matrix (which then becomes a generating function of the integral of motions). At the end of the chapter, we commented that one of this conserved charges is the Hamiltonian of the XXZ model.

In this chapter we are going to show how this construction can be used to generate the eigenfunctions and eigenvalues of the Hamiltonian, in a different way compared to chapter 4. However, this will be done without referring to the previous chapter, since all operators will be introduced here anew to better show the general structure of the construction and to indicate its generalization to other one-dimensional, quantum, Bethe Ansatz solvable models. Nonetheless, a certain familiarity with the concepts of last chapter can help the reader in following the various steps with some physical intuition and without getting lost in the mathematical details or wondering about the origin of certain ansatz.

Let us start with some mathematical preliminaries. The Hilbert space $\mathcal{H}$ of the XXZ is taken to be the direct sum of the Hilbert space at each site $\mathcal{H}_j$: $\mathcal{H} = \oplus_{j=1}^{N} \mathcal{H}_j$. For the XXZ spin chain, the space at each site can be represented by a two-dimensional vector, corresponding to having the spin-1/2 up or down. We will also need an additional vector space V as an auxiliary space: this is in general κ-dimensional and for the XXZ model $\kappa = 2$ (in fact, $\mathcal{H}_j$ and V are isomorphic in this case).

We introduce the operator $\mathcal{R}_{a,b}(\lambda) : V_a \times V_b \to V_a \times V_b$, which we can represent as a $\kappa^2 \times \kappa^2$ matrix and we require it to satisfy the Yang-Baxter equation

$$\mathcal{R}_{1,2}(\lambda - \mu)\, \mathcal{R}_{1,3}(\lambda - \nu)\, \mathcal{R}_{2,3}(\mu - \nu) = \mathcal{R}_{2,3}(\mu - \nu)\, \mathcal{R}_{1,3}(\lambda - \nu)\, \mathcal{R}_{1,2}(\lambda - \mu) \,, \tag{6.20}$$

where $1, 2, 3$ indicates the three different copies of V where the R-matrix acts (in couple).

Different solutions to this equation can exist. In principle, for every κ, one should take the most general matrix and look for solutions. It can be proven that the first non-trivial solutions of the Yang-Baxter equations appear for $\kappa \geq 4$. Already for $\kappa = 4$, the Yang-Baxter is equivalent

to a system of 64 equations in 16 unknown and classifying all solutions is a demanding task. In practice, one make as ansatz on the form of the R-matrix and checks whether this ansatz could solve the Yang-Baxter and under which condition.

Thus, guided by the solution of the six-vertex model, we make an ansatz and we look for solutions of the form

$$\mathcal{R}_{a,b}(\lambda,\mu) = f(\lambda,\mu)\frac{1+\tau_a^z\tau_b^z}{2} \;+\; g(\lambda,\mu)\;\frac{1-\tau_a^z\tau_b^z}{2} \;+\; \tau_a^+\tau_b^- + \tau_a^-\tau_b^+ \;, \tag{6.21}$$

where τ_a^α are the Pauli matrices acting on V_a. Note that This is the form of the R-operator we constructed in section 5.3. Thus, without repeating the derivation, we know that it satisfies (6.20) if

$$f(\lambda,\mu) = \frac{\sinh(\lambda-\mu+\phi)}{\sinh(\lambda-\mu)} \;, \qquad\qquad g(\lambda,\mu) = \frac{\sinh\phi}{\sinh(\lambda-\mu)} \;, \tag{6.22}$$

where the parameter ϕ is kept fixed and common to every matrix satisfying (6.20).

We introduce a second operator $\mathcal{L}_{j,a}(\lambda) : \mathcal{H}_j \times V_a \to \mathcal{H}_j \times V_a$, which for the XXZ model is also a 4×4 matrix. We require it to satisfy a Yang-Baxter equation with the above R-matrix as:

$$\mathcal{L}_{j,a}(\lambda)\;\mathcal{L}_{j,b}(\lambda')\;\mathcal{R}_{a,b}(\lambda-\lambda') = \mathcal{R}_{a,b}(\lambda-\lambda')\;\mathcal{L}_{j,b}(\lambda')\;\mathcal{L}_{j,a}(\lambda)\;. \tag{6.23}$$

Since the R-matrix satisfies (6.20), we know that (6.23) has a solution, which can be found in the same for as the R-matrix. However, it is more convenient to write the L-operator in the transmission-diagonal representation (this is accomplished simply by applying the exchange operator (6.15) to the reflection-diagonal representation we have been using so far):

$$\mathcal{L}_{j,a}(\lambda) = \frac{1+\sigma_j^z\tau_a^z}{2} \;+\; t(\lambda)\;\frac{1-\sigma_j^z\tau_a^z}{2} \;+\; r(\lambda)\;\left(\sigma_j^+\tau_a^- + \sigma_j^-\tau_a^+\right) \;, \tag{6.24}$$

where σ_j^α are Pauli matrices acting on the chain at the site j and

$$t(\lambda) \equiv \frac{1}{f(\lambda)} = \frac{\sinh\lambda}{\sinh(\lambda+\phi)} \;, \qquad\qquad r(\lambda) \equiv \frac{g(\lambda)}{f(\lambda)} = \frac{\sinh\phi}{\sinh(\lambda+\phi)} \;. \tag{6.25}$$

Note that, compared to (6.92), here the rapidities are rescaled $\lambda \to \mathrm{i}\frac{\phi}{2}\,(\lambda+\mathrm{i})$ to reproduce the same parametrization we used to solve the six-vertex model.

The form of the R and L-operator can be compared with the two-body scattering matrix in the previous section and thus $t(\lambda)$ and $r(\lambda)$ can be interpreted as the *transmission reflection coefficient*, respectively. The transmission-diagonal representation is convenient because at $\lambda = 0$, $t(0) = 0$ and $r(0) = 1$ and thus the L-operator reduces to the permutation operator that simply exchanges the two particle:

$$\mathcal{L}_{j,a}(0) = \frac{1+\sigma_j^z\tau_a^z}{2} \;+\; \left(\sigma_j^+\tau_a^- + \sigma_j^-\tau_a^+\right) = \frac{1}{2}\,(\mathcal{I}_j\otimes\mathcal{I}_a + \overrightarrow{\sigma}_j\otimes\overrightarrow{\tau}_a) = \Pi_{j,a}\;. \tag{6.26}$$

Hence we can interpret the L-operator as the scattering matrix of the auxiliary spin scattering off the physical spin of the chain. It is convenient to consider the L-operator as a $\kappa\times\kappa$

matrix (in the auxiliary space) with matrix elements given by operators in the physical Hilbert space

$$\begin{aligned}\mathcal{L}_{j,a} &= \begin{pmatrix} \frac{1+t(\lambda)}{2} + \frac{1-t(\lambda)}{2}\,\sigma_j^z & r(\lambda)\sigma_j^- \\ r(\lambda)\sigma_j^+ & \frac{1+t(\lambda)}{2} - \frac{1-t(\lambda)}{2}\,\sigma_j^z \end{pmatrix} \\ &= \frac{1}{\sinh(\lambda+\phi)} \begin{pmatrix} \sinh\left(\lambda + \frac{1+\sigma_j^z}{2}\,\phi\right) & \sigma_j^- \sinh\phi \\ \sigma_j^+ \sinh\phi & \sinh\left(\lambda + \frac{1-\sigma_j^z}{2}\,\phi\right) \end{pmatrix} ,\end{aligned} \tag{6.27}$$

where in the second line we specialized the result for the XXZ model.

We can construct the *transition matrix* of the auxiliary spin as it moves across several sites:

$$\mathcal{T}_a(n,m|\lambda) \equiv \mathcal{L}_{n,a}(\lambda)\,\mathcal{L}_{n-1,a}(\lambda)\cdots\mathcal{L}_{m,a}(\lambda)\;, \qquad n \geq m\;. \tag{6.28}$$

Here, the product between L-operators is the standard matrix product in a space, as they act on the same auxiliary space. Thus the transition matrix remains a $\kappa \times \kappa$ matrix in V, but each matrix element is an operator in the Hilbert space of the physical sites of the chain involved. The transition matrix across the entire chain is called the *monodromy matrix* (or *winding matrix*)

$$\mathcal{T}_a(\lambda) \equiv \mathcal{T}_a(N,1|\lambda) = \mathcal{L}_{N,a}(\lambda)\,\mathcal{L}_{N-1,a}(\lambda)\cdots\mathcal{L}_{1,a}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} , \tag{6.29}$$

where A, B, C, D are $2^N \times 2^N$ matrices acting on $\mathcal{H}$.

Taking the trace over the auxiliary space of (6.29) we obtain the *Transfer Matrix*

$$\mathbf{T}(\lambda) \equiv \mathrm{tr}_{\,a}\mathcal{T}_a(\lambda) = A(\lambda) + D(\lambda)\;. \tag{6.30}$$

The names of this operators come from their role in solving the respective classical two-dimensional models, but have no particular meaning in the context of Bethe Ansatz solutions. The reader who skipped the previous chapter should not read to much into these names, since they are due to historical reasons.

Notice that the transfer matrix (6.30) only acts on the physical Hilbert space $\mathcal{H}$ and does not depend of the auxiliary space V. The exact form of the A, B, C, D operators is complicated, but it will not be really needed. The ABA construction is based solely on their commutation relations, as they will be read off from the Yang-Baxter equation. Even if we presented an explicit derivation of these matrices, they should be treated as operators acting on a Hilbert space and defined by their algebraic structure, more than by their specific representation in matrix form.

Thus, what we have done is to consider the scattering of a probe spin (the auxiliary vector space V) propagating through the whole system scattering on each physical spin according to the L-operator. The monodromy matrix encodes this scattering through the entire chain and can be interpreted as in (6.17, 6.27)

$$\mathcal{T}_a(\lambda) = \frac{1}{2}\left[A(\lambda)+D(\lambda)\right]\mathcal{I}_a + \frac{1}{2}\left[A(\lambda)-D(\lambda)\right]\tau_a^z + B(\lambda)\tau_a^+ + C(\lambda)\tau_a^-\;. \tag{6.31}$$

The operators A, B, C, D encode the effect of the interaction with this ghost spin on the physical system and we see that we can think of C and B as some spin raising/lower operator,

respectively. Taking the trace in (6.30) amounts to closing the system at infinity (with periodic boundary conditions) and requiring that the probe emerges from the interaction in the same state as it entered.

Let us consider two monodromy matrices, acting on different auxiliary space a and b and with different spectral parameters λ and λ'. Repeated use of (6.23) shows that

$$\mathcal{T}_a(\lambda)\,\mathcal{T}_b(\lambda')\,\mathcal{R}_{a,b}(\lambda-\lambda') = \mathcal{R}_{a,b}(\lambda-\lambda')\,\mathcal{T}_b(\lambda')\,\mathcal{T}_a(\lambda)\;, \tag{6.32}$$

or

$$\mathcal{R}^{-1}_{a,b}(\lambda-\lambda')\,\mathcal{T}_a(\lambda)\,\mathcal{T}_b(\lambda')\,\mathcal{R}_{a,b}(\lambda-\lambda') = \mathcal{T}_b(\lambda')\,\mathcal{T}_a(\lambda)\;. \tag{6.33}$$

This is the Yang-Baxter equation for the winding matrix and we will see later that, from an ABA point of view, can be considered as a generalized commutation relation. Physically, it means that it is equivalent to let two probes scatter through the physical chain, or to let these probe scatter on one another first, then propagate through the system and finally scatter again.

Taking the trace of (6.33) and using the cyclic property of the trace we get

$$\big[\mathbf{T}(\lambda),\mathbf{T}(\lambda')\big] = 0\;. \tag{6.34}$$

The Yang-Baxter equation (6.32, 6.33) means that the entanglement process due to the probe propagation that generates the monodromy matrix can be factorized at the border and, by taking the trace over the ghost variables, the two chains can be disentangled. So, the transfer matrices generated by two ghost particles propagating with different parameters commute.

This is the fundamental result of this whole construction, since it implies that the transfer matrix is the generating function of all conserved quantities. In practice, it is most convenient to consider the logarithm of the transfer matrix, since this way one can make the integral of motions local. To see this, let us introduce an operator defined as

$$\mathbf{J}_{\{c\}} \equiv \sum_n \sum_j c_{n,j}\;\; \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\;\ln\mathbf{T}(\lambda)\Big|_{\lambda=\lambda_j}\;, \tag{6.35}$$

for certain λ_j, where $c_{n,j}$ are some coefficients. It is clear that, using (6.34) we have

$$\big[\mathbf{J}_{\{c\}},\mathbf{T}(\lambda)\big] = 0 \qquad \text{and} \qquad \big[\mathbf{J}_{\{c\}},\mathbf{J}_{\{c'\}}\big] = 0\;. \tag{6.36}$$

Expressions (6.35) are known as *trace identities* and define integral of motions in involutions, which can be used to define the state of the integrable system.

Let us consider, for instance, $\mathbf{T}(0)$. From (6.26) we know that the monodromy matrix is composed by product of exchange operators. That is, the probe enters the system and exchanges its state with the first spin, then proceed to the next lattice site and exchanges it state, effectively leaving that spin with the state of the previous one and so on. After taking the trace and closing the chain, the net effect has been to shift every spin by a lattice site. In mathematical form, since

$$\mathcal{L}_{j,a}(0) = \Pi_{j,a}\;, \tag{6.37}$$

one uses the identity for the permutation

$$\Pi_{j,a}\hat{X}_a = \hat{X}_j\Pi_{j,a} \ , \tag{6.38}$$

where $\hat{X}_a$ is some operator acting on the vector space a. Thus we have

$$\Pi_{j,a}\Pi_{l,a} = \Pi_{j,l}\Pi_{j,a} = \Pi_{l,a}\Pi_{j,l} \tag{6.39}$$

and

$$\Pi_{j,l} = \Pi_{l,j} \ . \tag{6.40}$$

Then

$$\begin{aligned} \mathcal{T}_a(0) &= \Pi_{N,a}\Pi_{N-1,a}\dots\Pi_{1,a} \\ &= \Pi_{1,2}\Pi_{2,3}\dots\Pi_{N-1,N}\Pi_{N,a} \ . \end{aligned} \tag{6.41}$$

Since $\mathrm{tr}_a\Pi_{j,a} = \mathcal{I}_j$, we get

$$\begin{aligned} \mathbf{T}(0) &= \Pi_{1,2}\Pi_{2,3}\dots\Pi_{N-1,N} \\ &= \exp \mathrm{i}\hat{P} \ , \end{aligned} \tag{6.42}$$

where $\hat{P}$ is the lattice momentum operator. Thus

$$\hat{P} = -\mathrm{i}\ln\mathbf{T}(0) \ . \tag{6.43}$$

Next, let us look at the first logarithmic derivative of the transfer matrix. First, using the properties of permutation operators

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{T}_a(\lambda)\Big|_{\lambda=0} &= \sum_{j=1}^{N}\Pi_{N,a}\dots\Pi_{j+1,a}\mathcal{L}'_{j,a}(0)\Pi_{j-1,a}\dots\Pi_{1,a} \\ &= \sum_{j=1}^{N}\Pi_{N,a}\dots\mathcal{L}'_{j,j+1}(0)\Pi_{j+1,a}\Pi_{j-1,a}\dots\Pi_{1,a} \\ &= \sum_{j=1}^{N}\mathcal{L}'_{j,j+1}(0)\Pi_{1,2}\Pi_{2,3}\dots\Pi_{j-1,j+1}\dots\Pi_{N-1,N}\Pi_{N,a} \ . \end{aligned} \tag{6.44}$$

Taking the trace over the auxiliary space we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbf{T}_a(\lambda)\Big|_{\lambda=0} = \sum_{j=1}^{N}\mathcal{L}'_{j,j+1}(0)\Pi_{1,2}\Pi_{2,3}\dots\Pi_{j-1,j+1}\dots\Pi_{N-1,N} \ . \tag{6.45}$$

Finally, multiplying by the inverse shift operator most of the permutation operators cancel out and we are left with

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\ \ln\mathbf{T}(\lambda)\Big|_{\lambda=0} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbf{T}_a(\lambda)\Big|_{\lambda=0}\mathbf{T}^{-1}(0) \\ &= \sum_{j=1}^{N}\mathcal{L}'_{j,j+1}(0)\Pi_{j,j+1} \\ &= \sum_{j=1}^{N}\frac{\mathrm{d}}{\mathrm{d}\lambda}\ \ln\mathcal{L}_{j,j+1}(\lambda)\Big|_{\lambda=0} \ . \end{aligned} \tag{6.46}$$

Now, we notice that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{L}_{j,j+1}(\lambda)\Big|_{\lambda=0} = \frac{1}{\sinh\phi}\left[\frac{1-\sigma_j^z\sigma_{j+1}^z}{2} - \cosh\phi\left(\sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+\right)\right] \tag{6.47}$$

and

$$\mathcal{L}'_{j,j+1}(0)\Pi_{j,j+1} = \frac{1}{\sinh\phi}\left[-\cosh\phi\,\frac{1-\sigma_j^z\sigma_{j+1}^z}{2} + \sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+\right]. \tag{6.48}$$

The last identity is most easily derived in matrix form

$$\begin{aligned}\mathcal{L}'_{j,j+1}(0)\Pi_{j,j+1} &= \frac{1}{\sinh\phi}\begin{pmatrix}0&0&0&0\\0&1&-\cosh\phi&0\\0&-\cosh\phi&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\\ &= \frac{1}{\sinh\phi}\begin{pmatrix}0&0&0&0\\0&-\cosh\phi&1&0\\0&1&-\cosh\phi&0\\0&0&0&0\end{pmatrix}.\end{aligned} \tag{6.49}$$

We can conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\ \ln\mathbf{T}(\lambda)\Big|_{\lambda=0} = \frac{1}{\sinh\phi}\sum_{j=1}^{N}\left[\cosh\phi\,\frac{\sigma_j^z\sigma_{j+1}^z-1}{2} + \sigma_j^+\sigma_{j+1}^- + \sigma_j^-\sigma_{j+1}^+\right], \tag{6.50}$$

where we recognize the form of the XXZ Hamiltonian (4.1), with no external field. In particular, if we set $\cosh\phi = \Delta$ we have

$$\hat{H}_{XXZ} = \frac{\sqrt{\Delta^2-1}}{2}\ \frac{\mathrm{d}}{\mathrm{d}\lambda}\ \ln\mathbf{T}(\lambda)\Big|_{\lambda=0} + \frac{1}{4}\,N\,\Delta\,. \tag{6.51}$$

Thus, we proved that the Hamiltonian of the XXZ model is among the conserved quantities in convolution generated by the transfer matrix.

This is a generic feature and it all starts with the identification of an R-matrix that solves the Yang-Baxter equation (6.20). Once a solution is found, it specifies an L-operator and thus a monodromy matrix and a transfer matrix. At this point, one uses the trace identities to find out the Hamiltonian of the model that has just been solved. In general, there is no way to identify the model directly from the R-matrix and in practice one looks for solutions of (6.20) and classifies the kind of Hamiltonian that can be generated for later use. This is why this construction is called the (quantum) inverse scattering method. In our case, we made an educated ansatz for the R-matrix in (6.21), since we knew that it would generate the XXZ chain.

6.4 The ABA solution

Comparing the construction so far with what we have done in the coordinate Bethe Ansatz approach, we see that we have again factorized the interaction into a series of two-body scattering

(but this time by introducing an auxiliary particle in the system) and that we quantized it by closing the system at infinity with periodic boundary condition (in the ABA approach, by taking the trace of the auxiliary state).

To determine the eigenfunctions of the model, we look for eigenvectors of the transfer matrix and we construct them by injecting ghost particles from the auxiliary space V and constructing the eigenvalue equation using the Yang-Baxter equation. These conditions will turn out to be the same Bethe equation we have found in the coordinate approach. But in the ABA way, we will have a better characterization of the eigenstate.

In the previous sections, we have determined that transfer matrix is the generator of a series of conserved quantities in convolution, among which we found the Hamiltonian of the XXZ model. This means that the transfer matrix and the Hamiltonian can be simultaneously diagonalized, since they share the same eigenvectors. Thus, to find the eigenstates of the XXZ chain, we can look for the eigenvectors of $\mathbf{T}(\lambda)$, knowing that these will be eigenstates for every λ:

$$\mathbf{T}(\lambda)|\Psi\rangle = [A(\lambda) + D(\lambda)]\,|\Psi\rangle = \Lambda(\lambda)|\Psi\rangle\ , \tag{6.52}$$

where we have used (6.30).

The construction of the solutions, starts with the identification of a reference state $|0\rangle$, which we call *pseudo-vacuum*. This is a "trivial" eigenstate of the system which can be recognized by inspection. In general, it is specified by the requirements

$$A(\lambda)|0\rangle = \tilde{a}(\lambda)|0\rangle\ , \qquad D(\lambda)|0\rangle = \tilde{d}(\lambda)|0\rangle\ , \qquad C(\lambda)|0\rangle = 0\ , \tag{6.53}$$

since in this way we will see that $|0\rangle$ can be interpreted as the highest state of a $SU(2)$ representation.

To find this pseudo-vacuum state, we look at (6.27) and notice that

$$\mathcal{L}_{j,a}(\lambda)|\uparrow_j\rangle = \begin{pmatrix} 1 & 0 \\ 0 & t(\lambda) \end{pmatrix} |\uparrow_j\rangle + \begin{pmatrix} 0 & r(\lambda) \\ 0 & 0 \end{pmatrix} |\downarrow_j\rangle\ , \tag{6.54}$$

where $|\uparrow_j\rangle$ ($|\downarrow_j\rangle$) denotes the state with a spin up (down) at the j-th lattice site. Thus, $|\uparrow\rangle$ makes the L-operator upper-diagonal, hence the state

$$|0\rangle = \prod_{j=1}^{N} |\uparrow_j\rangle \tag{6.55}$$

makes the monodromy matrix (6.29) upper-diagonal as well and satisfies the conditions (6.53) with

$$\tilde{a}(\lambda) = 1\ , \qquad \tilde{d}(\lambda) = t^N(\lambda)\ . \tag{6.56}$$

Notice that these eigenvalues depend only on the form of the L-operator.

Comparing (6.17, 6.27) and (6.31) we see that if we "inject" in the system a ghost particle (of the auxiliary space) with spin-down and we extract it in a spin-up state, this means that a spin of the chain has made the opposite flip. This action is performed by the operator

$$\langle\uparrow_a|\mathcal{T}_a(\lambda_j)|\downarrow_a\rangle = B(\lambda_j)\ , \tag{6.57}$$

which can be interpreted as a spin-flip operator that creates an excitation over the pseudo-vacuum $|0\rangle$ with rapidity λ_j.

Thus, we look for eigenstates of the transfer matrix of the form

$$|\Psi\rangle = \prod_{j=1}^{m} B(\lambda_j)|0\rangle \tag{6.58}$$

and we look under which condition (6.52) is satisfied.

The reason to use the spin-flip operator $B(\lambda_j)$ instead of the local one σ_j^- is that for B we can use the algebra of the Yang-Baxter equation (6.33) as defining a sort of generalized commutation relations for the operators A, B, C, D. These can be worked out by writing explicitly the matrix multiplications in (6.33). A complete list can be found for instance in [19]. In the following we will only need these

$$B(\lambda')B(\lambda) = B(\lambda)B(\lambda') , \tag{6.59}$$

$$A(\lambda)B(\lambda') = f(\lambda',\lambda)\ B(\lambda')A(\lambda) + g(\lambda,\lambda')\ B(\lambda)A(\lambda') , \tag{6.60}$$

$$D(\lambda)B(\lambda') = f(\lambda,\lambda')\ B(\lambda')D(\lambda) + g(\lambda',\lambda)\ B(\lambda)D(\lambda') . \tag{6.61}$$

The first equation is important to establish that in (6.58) the order in which we multiply the B's does not matter, as we expect from the physical meaning of the Yang-Baxter, i.e. that the order of the ghost interaction does not matter. Notice that the coefficients of these generalized commutation relations depend only on the R-matrix.

To check whether (6.52) is satisfied, we will need to progressively commute the A and D through the B's. This is physically equivalent to scattering the ghost particle with the excitations created by the B's through the whole system, giving rise to transmissions and reflections of the ghost. Note that if we evaluate the transfer matrix at one of the spectral parameters λ_j, this means that one of the R-operators will become the permutation operator and the ghost particle will simply exchange its identity with one of the physical spins.

In general, for the state (6.58) to be an eigenstate it means that after each reflection, the ghost particle propagates in a way that the cumulative effect of all these processes interferes destructively and only the ghost that transmitted through the whole system keeping its degrees of freedom reaches the end of the system. For instance, the first term on the right-hand side of (6.60) is a sort of transmission, since the operators keep their degrees of freedom, while the second is a reflection, as rapidities are exchanged. We have

$$A(\lambda)\prod_{j=1}^{m} B(\lambda_j) = \prod_{j=1}^{m} f(\lambda_j,\lambda)B(\lambda_j)\ A(\lambda) - B(\lambda)\sum_{l=1}^{m} g(\lambda_l,\lambda)\prod_{\substack{j=1\\ j\neq l}}^{m} f(\lambda_j,\lambda_l)B(\lambda_j)\ A(\lambda_l) . \tag{6.62}$$

The first term come from the transmission part, the second from the reflection one, but requires a bit of explanation. Let us single out one of the B-operator and write

$$A(\lambda)B(\lambda_l)\prod_{\substack{j=1\\ j\neq l}}^{m} B(\lambda_j) = f(\lambda_l,\lambda)B(\lambda_l)\ A(\lambda) - g(\lambda_l,\lambda)\ B(\lambda)A(\lambda_l)\prod_{\substack{j=1\\ j\neq l}}^{m} B(\lambda_j) . \tag{6.63}$$

Now we should commute the A-operator with the other B's in the second term. However, any additional reflection term would look like the the first term in (6.63) and since we know that the order in which we take the B's does not matter we conclude that we can neglect all additional reflection terms and arrive at (6.62).

Reasoning in a similar way we get

$$D(\lambda)\prod_{j=1}^{m}B(\lambda_j)=\prod_{j=1}^{m}f(\lambda,\lambda_j)B(\lambda_j)\;D(\lambda)-\sum_{l=1}^{m}g(\lambda,\lambda_l)\;B(\lambda)\prod_{\substack{j=1\\ j\neq l}}^{m}f(\lambda_l,\lambda_j)B(\lambda_j)\;D(\lambda_l)\;. \quad (6.64)$$

Collecting these results, we have

$$\begin{aligned}\mathbf{T}(\lambda)|\Psi\rangle &= [A(\lambda)+D(\lambda)]\prod_{j=1}^{m}B(\lambda_j)|0\rangle\\ &= \left[\prod_{j=1}^{m}f(\lambda_j,\lambda)\tilde{a}(\lambda)+\prod_{j=1}^{m}f(\lambda,\lambda_j)\tilde{d}(\lambda)\right]|\Psi\rangle\\ &\quad-\sum_{l=1}^{m}g(\lambda_l,\lambda)\;\tilde{a}(\lambda_l)\prod_{\substack{j=1\\ j\neq l}}^{m}f(\lambda_j,\lambda_l)\;B(\lambda_j)B(\lambda)|0\rangle\\ &\quad-\sum_{l=1}^{m}g(\lambda,\lambda_l)\;\tilde{d}(\lambda_l)\prod_{\substack{j=1\\ j\neq l}}^{m}f(\lambda_l,\lambda_j)B(\lambda_j)B(\lambda)|0\rangle\;. \end{aligned} \quad (6.65)$$

For $|\Psi\rangle$ to be an eigenvector of the transfer matrix (6.52) with eigenvalue

$$\Lambda(\lambda)=\prod_{j=1}^{m}f(\lambda_j,\lambda)\tilde{a}(\lambda)\;+\prod_{j=1}^{m}f(\lambda,\lambda_j)\tilde{d}(\lambda) \quad (6.66)$$

we need to impose the vanishing of the off-diagonal terms, i.e.

$$g(\lambda_l,\lambda)\tilde{a}(\lambda_l)\prod_{\substack{j=1\\ j\neq l}}^{m}f(\lambda_j,\lambda_l)+g(\lambda,\lambda_l)\tilde{d}(\lambda_l)\prod_{\substack{j=1\\ j\neq l}}^{m}f(\lambda_l,\lambda_j)=0\;,\qquad l=1,\ldots,m\;. \quad (6.67)$$

From (6.22) we see that $g(\lambda,\mu)=-g(\mu,\lambda)$. Hence, we can remove from (6.67) the dependence on the spectral parameter λ of the ghost particle (remember that we proved that the transfer matrices commute for different λ's and therefore the eigenstate conditions cannot depend on it) and get

$$\frac{\tilde{d}(\lambda_j)}{\tilde{a}(\lambda_j)}=\prod_{\substack{l=1\\ j\neq j}}^{m}\frac{f(\lambda_l-\lambda_j)}{f(\lambda_j-\lambda_l)}\;,\qquad j=1,\ldots,m\;. \quad (6.68)$$

Substituting $\tilde{d}(\lambda)=t^N(\lambda)$, $\tilde{a}(\lambda)=1$ and $f(\lambda)=1/t(\lambda)$ from (6.22,6.25) we have

$$\left[\frac{\sinh\lambda_j}{\sinh(\lambda_j+\phi)}\right]^N=\prod_{\substack{l=1\\ j\neq j}}^{m}\frac{\sinh(\lambda_l-\lambda_j+\phi)}{\sinh(\lambda_j-\lambda_l+\phi)}\;,\qquad j=1,\ldots,m\;, \quad (6.69)$$

which are the Bethe equations for the XXZ model (see chapter 4). One can take them into the traditional form by shifting the rapidities $\lambda_j \to \lambda_j - \phi/2$ or $\lambda_j \to \mathrm{i}\frac{\phi}{2}(\lambda + \mathrm{i})$.

Thus, we have proven that the state (6.58) is an eigenstate of the transfer matrix if the rapidities of the excitations satisfy the Bethe equations (6.69). Notice that both the parameters of the L and the R-operator appear in this construction of the Bethe equations. Now, on one side we could proceed like we did in chapter 4 to study the spectrum and the thermodynamic of the model, by taking the logarithm of these equation and introducing the distribution function and the integral equation they satisfy. But the main advantage of this construction of the eigenstates is that now we have a better control of them, since the eigenfunctions are not cumbersome superposition of plane-wave solutions like in the coordinate approach, but are generated as creator operators applied to a reference vacuum. We have seen that the Yang-Baxter equation provides us with the algebra that these operator satisfy and we argued that the B operator can be interpreted as creator operators. Similarly, the C operators are destruction operators for the excitation (or creation ones for the bra states). It is also clear that this algebra is similar to the familiar $SU(2)$ algebra in a complicated representation in which the pseudo-vacuum $|0\rangle$ is the highest weight state annihilated by every $C(\lambda)$ and that the A and B operators are the Casimirs. Hence, we have a complete representation for every state and every operator of the theory and in principle we can use it to calculate the correlation functions. In practice, this is still a hard task, since the algebra is quite rich and the complexity of commuting a large number of operators for many-particles correlators becomes overwhelming. Nonetheless, correlation function in system with few excitations can be easily tackled in the ABA formalism and recently extremely promising progresses have been achieved in identifying the algebraic processes that mostly contribute to a given response function, allowing to achieve even asymptotic results. This has allowed to efficiently applying the ABA construction to the calculation of correlation functions for the physically interesting problems.

6.5 The Lax Representation

The L-operator is the central object of the so-called *Lax representation* of integrable systems. This approach is not central for the understanding of the Algebraic Bethe Ansatz, but it is worth mentioning it to recognize these techniques in relations with other aspects of integrability. However, this section can be safely skipped by the non-interested reader.

The main idea of the Lax method is to map a non-linear problem into a linear one, by enlarging the space of solutions with the introduction of an auxiliary field. In its original form, it was applied to the solution of classical integrable non-linear differential equations. One interprets the differential equation as the dynamical equation of a Hamiltonian system and, once more, the auxiliary space is used to established the commutation of the transfer matrices, from which the Hamiltonian is recovered using the trace identities. The explicit construction can be quite cumbersome and non-intuitive (and we will not give it here: a short but clear introduction, with explicit examples can be found, for instance, in [32, 19]), but in the rest of this section we will briefly outline the formal manipulations that highlight the similarity with

the approach developed in the rest of the chapter.

The Lax representation of the *Inverse Scattering Method* consists in representing a non-linear differential equation through a pair of $\kappa \times \kappa$ operators $\mathcal{U}(x|\lambda)$ and $\mathcal{V}(x|\lambda)$, satisfying the following zero-curvature condition

$$[\partial_t - \mathcal{U}(x|\lambda), \partial_x + \mathcal{V}(x|\lambda)] = 0 \ , \tag{6.70}$$

for every λ at each point x. The matrix $\mathcal{V}(x|\lambda)$ is called the potential and $\mathcal{U}(x|\lambda)$ is the time evolution for a an unknown vector function $\Phi(x,t)$, valued in the κ-dimensional auxiliary space. The condition (6.70) ensures the consistency of the evolution equations

$$\begin{aligned} \partial_t \Phi(x,t) &= \mathcal{U}(x|\lambda)\Phi(x,t) \ , \\ \partial_x \Phi(x,t) &= -\mathcal{V}(x|\lambda)\Phi(x,t) \ , \end{aligned} \tag{6.71}$$

which are easy to integrate.

For lattice models, with lattice spacing δ, the evolution equations read

$$\begin{aligned} \partial_t \Phi(j,t) &= \mathcal{U}(j|\lambda)\Phi(j,t) \ , \\ \Phi(j+1,t) &= \mathcal{L}(j|\lambda)\Phi(j,t) \ . \end{aligned} \tag{6.72}$$

The coordinate of the j-th lattice site is $x_j = j\delta$ and we have

$$\mathcal{L}(j|\lambda) = \mathcal{I} - \mathcal{V}(x_j|\lambda) + \mathrm{O}(\delta^2) \ , \tag{6.73}$$

where $\mathcal{I}$ is the $\kappa \times \kappa$ unit matrix.

The transition matrix $\mathcal{T}(x,y|\lambda)$ in the continuous case gives the evolution of the auxiliary field Φ from point y to $x \geq y$ satisfies the equation

$$[\partial_x + \mathcal{V}(x|\lambda)]\, \mathcal{T}(x,y|\lambda) = 0 \ , \tag{6.74}$$

with the boundary condition

$$\mathcal{T}(y,y|\lambda) = \mathcal{I} \ . \tag{6.75}$$

This has formal solution

$$\mathcal{T}(x,y|\lambda) = \mathrm{P} \exp\left\{ -\int_x^y \mathcal{V}(z|\lambda)\mathrm{d}z \right\} \ , \tag{6.76}$$

where P is the path ordering of non-commuting factor. (6.76) is the continuous analog of (6.28) and clearly guarantees that

$$\mathcal{T}(x,z|\lambda)\ \mathcal{T}(z,y|\lambda) = \mathcal{T}(x,y|\lambda) \ , \qquad x \geq z \geq y \ . \tag{6.77}$$

After this, one can show that the monodromy matrix defined as the transition matrix for the whole system satisfies a Yang-Baxter equation and therefore the trace over the auxiliary space ensures that the transfer matrices at different spectral parameters commute. Using the trace identities to construct the Hamiltonian among the various conserved charges. one identifies the non-linear system at hand, which has just been solved by the inverse scattering method.

As an example, let us consider the classical non-linear Schrödinger equation. We chose $\kappa = 2$ and we leave to the reader to check that the following Lax-pair satisfy the zero-curvature condition (6.70)

$$\mathcal{V}(x|\lambda) = \mathrm{i}\frac{\lambda}{2}\,\sigma^z + \mathbf{\Omega}(x)\;, \tag{6.78}$$

$$\mathcal{U}(x|\lambda) = \mathrm{i}\frac{\lambda^2}{2}\,\sigma^z + \lambda\mathbf{\Omega}(x) + \mathrm{i}\sigma^z\left(\partial_x\mathbf{\Omega} + c\mathbf{\Psi}^*\mathbf{\Psi}\right)\;, \tag{6.79}$$

where

$$\mathbf{\Omega}(x) \equiv \mathrm{i}\sqrt{c}\left[\mathbf{\Psi}^*(x)\sigma^+ - \mathbf{\Psi}(x)\sigma^-\right]\;, \tag{6.80}$$

σ^α are the usual Pauli matrices, c is a constant and $\mathbf{\Psi}(x)$, $\mathbf{\Psi}^*(x)$ are complex fields with canonical Poisson brackets

$$\{\mathbf{\Psi}(x), \mathbf{\Psi}^*(y)\} = \mathrm{i}\delta(x-y)\;. \tag{6.81}$$

The corresponding L-operator on the infinitesimal lattice is

$$\mathcal{L}(j|\lambda) = \begin{pmatrix} 1 - \mathrm{i}\frac{\lambda}{2}\,\delta & -\mathrm{i}\sqrt{c}\mathbf{\Psi}_j^*\delta \\ \mathrm{i}\sqrt{c}\mathbf{\Psi}_j\delta & 1 + \mathrm{i}\frac{\lambda}{2}\,\delta \end{pmatrix} + \mathrm{O}\delta^2\;, \tag{6.82}$$

where

$$\mathbf{\Psi}_j = \frac{1}{\delta}\int_{x_{j-1}}^{x_j}\mathbf{\Psi}(x)\mathrm{d}x\;, \qquad\qquad \{\mathbf{\Psi}_j, \mathbf{\Psi}_l^*\} = \frac{\mathrm{i}}{\delta}\delta_{j,l}\;. \tag{6.83}$$

One can show [19] that the trace identities applied to the transfer matrix generated by this Lax operator give the following conserved charges

$$\mathbf{J}_0 = \int \mathbf{\Psi}^*\mathbf{\Psi}\mathrm{d}x\;, \tag{6.84}$$

$$\mathbf{J}_1 = \int \mathbf{\Psi}^*\partial_x\mathbf{\Psi}\mathrm{d}x\;, \tag{6.85}$$

$$\mathbf{J}_2 = \int \left[\partial_x\mathbf{\Psi}^*\partial_x\mathbf{\Psi} + c\mathbf{\Psi}^*\mathbf{\Psi}^*\mathbf{\Psi}\mathbf{\Psi}\right]\mathrm{d}x\;, \tag{6.86}$$

$$\mathbf{J}_3 = \int \left[\mathbf{\Psi}^*\partial_x^3\mathbf{\Psi} - \frac{3}{2}\,c\mathbf{\Psi}^*\mathbf{\Psi}^*\partial_x\left(\mathbf{\Psi}\mathbf{\Psi}\right)\right]\mathrm{d}x\;, \tag{6.87}$$

$$\ldots$$

We recognize that $\mathbf{J}_0$ is the number of particles in the system, $\mathbf{J}_1$ the total momentum and $\mathbf{J}_2$ is the Hamiltonian of the non-linear Schrödinger model, that generates the Gross-Pitaevskii equation

$$\mathrm{i}\partial_t\mathbf{\Psi} = -\partial_x^2\mathbf{\Psi} + 2c\mathbf{\Psi}^*\mathbf{\Psi}\mathbf{\Psi}\;. \tag{6.88}$$

6.6 The simplest R-matrix: Lieb-Liniger & Heisenberg chain

In section 6.3 we used the solution of the six-vertex model we developed in chapter 5 to construct the solution of the Yang-Baxter corresponding to the XXZ chain. The R-operator is a 4×4

with the form

$$\mathcal{R}_{a,b}(\lambda,\mu) = \begin{pmatrix} f(\lambda,\mu) & 0 & 0 & 0 \\ 0 & g(\lambda,\mu) & 1 & 0 \\ 0 & 1 & g(\lambda,\mu) & 0 \\ 0 & 0 & 0 & f(\lambda,\mu) \end{pmatrix} , \tag{6.89}$$

with

$$f_{XXZ}(\lambda,\mu) = \frac{\sinh(\lambda-\mu+\phi)}{\sinh(\lambda-\mu)} , \qquad g_{XXZ}(\lambda,\mu) = \frac{\sinh\phi}{\sinh(\lambda-\mu)} . \tag{6.90}$$

The L-operator was also chosen as a 4×4 matrix and is

$$\mathcal{L}_{j,b}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t(\lambda) & r(\lambda) & 0 \\ 0 & r(\lambda) & t(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{6.91}$$

with

$$t_{XXZ}(\lambda) = \frac{\sinh\lambda}{\sinh(\lambda+\phi)} , \qquad r_{XXZ}(\lambda) = \frac{\sinh\phi}{\sinh(\lambda+\phi)} . \tag{6.92}$$

The Heisenberg chain is recovered in the limit $\phi\to 0$ ($\Delta\to 1$). To take this limit, we first rescale the rapidity as $\lambda\to\phi\lambda/c$ and then take the $\phi\to 0$ limit of (6.90,6.92) to get

$$f_{XXX}(\lambda,\mu) = 1+\frac{c}{\lambda-\mu} , \qquad g_{XXX}(\lambda,\mu) = \frac{c}{\lambda-\mu} , \tag{6.93}$$

$$t_{XXX}(\lambda) = \frac{\lambda}{\lambda+c} , \qquad r_{XXX}(\lambda) = \frac{c}{\lambda+c} . \tag{6.94}$$

This gives the following R and L-matrices

$$\mathcal{R}^{XXX}_{a,b}(\lambda,\mu) = \frac{c}{\lambda-\mu}\mathcal{I}_{a,b} + \Pi_{a,b} , \tag{6.95}$$

$$\mathcal{L}^{XXX}_{j,a}(\lambda) = \frac{1}{\lambda+c}\left[\lambda\,\mathcal{I}_{j,a} + c\,\Pi_{j,a}\right] = \frac{1}{\lambda+c}\begin{pmatrix} \lambda + c\,\frac{1+\sigma^z_j}{2} & c\,\sigma^-_j \\ c\,\sigma^+_j & \lambda + c\,\frac{1-\sigma^z_j}{2} \end{pmatrix} . \tag{6.96}$$

$\mathcal{R}^{XXX}$ is the simplest solution of a Yang-Baxter. The Lieb-Liniger model can also be generated from the same R-operator, but with a 2×2 L-matrix (since the physical Hilbert space has dimension 1). Introducing an infinitesimal lattice of lattice space δ

$$\mathcal{L}^{\text{Lieb}-\text{Liniger}}_{j,a} = \begin{pmatrix} 1-\mathrm{i}\frac{\lambda}{2}\,\delta & -\mathrm{i}\sqrt{c}\mathbf{\Psi}^\dagger_j\delta \\ \mathrm{i}\sqrt{c}\mathbf{\Psi}_j\delta & 1+\mathrm{i}\frac{\lambda}{2}\,\delta \end{pmatrix} + \mathrm{O}\delta^2 , \tag{6.97}$$

where $\mathbf{\Psi}_j, \mathbf{\Psi}^\dagger_j$ are quantum operator with commutation relations $\left[\mathbf{\Psi}_j, \mathbf{\Psi}^\dagger_l\right] = \frac{1}{\delta}\,\delta_{j,l}$. This is the quantum analog of the L-operator of the classical non-linear Schrödinger equation (6.82).

With these ingredients and the general construction presented before, one can immediately derive the algebraic solutions of these three models.

6.7 A glimpse into Quantum Groups

As we mentioned, the XXX R-matrix is the simplest solution of the Yang-Baxter equation (6.20). In fact, for every matrix dimension κ, a solution of the form (6.95) with the proper exchange operator always exists.

For $\kappa = 2$, we can use (6.26) to write the exchange operator using Pauli matrices and the identity. Thus, the Yang-Baxter equation is solved by (6.95) because of the underlying $sl(2)$ algebra, i.e. the $SU(2)$ group plus the identity. From this point of view, the algebra induced by the Yang-Baxter is nothing else but the $sl(2)$ algebra.

In this final section of this chapter, we would like to show that the $\Delta \neq 1$ solutions of (6.20) can be generated as a deformation of the $sl(2)$ group underlying: in the literature, this deformation is known as a *quantum group*.

Let us begin by introducing a complex parameter q and define the q-deformation of a number x:

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \,. \tag{6.98}$$

Notice that the $q \to 1$ limit is well-defined and $\lim_{q\to 1}[x]_q = x$. If we parametrize q as $q \equiv \mathrm{e}^{\phi}$, we can write two equivalent representation for the q-deformation (6.98)

$$[x]_q = \frac{\sinh(\phi x)}{\sinh\phi} = \prod_{n=-\infty}^{\infty} \frac{x + \pi n \phi^{-1}}{1 + \pi n \phi^{-1}} \,, \tag{6.99}$$

where the first expression is better suited to study the $\phi \to 0$ limit of the second, while the second representation has a physical interpretation as the change of the complex plane to a strip via a multiplicative averaging.

It is easy to see that q-deforming the Heisenberg model generates the XXZ chain, with $\Delta = \cosh\phi$. This can be seen, for instance, by looking at the L-operator (we can set $c = 1$ in (6.96)) :

$$\begin{aligned}
\mathcal{L}^{\mathrm{XXZ}}_{j,a}(\lambda) &= \left[\mathcal{L}^{\mathrm{XXX}}_{j,a}(\lambda)\right]_q \\
&= \prod_{n=-\infty}^{\infty} \mathcal{L}^{\mathrm{XXX}}_{j,a}\left(\lambda + \pi n \phi^{-1}\right) \\
&= \frac{1}{\sinh\phi} \begin{pmatrix} \sinh\left(\lambda + \phi\,\frac{1+\sigma^z_j}{2}\right) & \sinh\phi\sigma^-_j \\ \sinh\phi\sigma^+_j & \sinh\left(\lambda + \phi\,\frac{1-\sigma^z_j}{2}\right) \end{pmatrix} \,,
\end{aligned} \tag{6.100}$$

or

$$\mathcal{L}^{\mathrm{XXZ}}_{j,a}(\lambda) = \begin{pmatrix} x\, q^{\frac{1+\sigma^z_j}{2}} - x^{-1}\, q^{-\frac{1+\sigma^z_j}{2}} & (q - q^{-1})\sigma^-_j \\ (q - q^{-1})\sigma^+_j & x\, q^{\frac{1-\sigma^z_j}{2}} - x^{-1}\, q^{-\frac{1-\sigma^z_j}{2}} \end{pmatrix} \,, \tag{6.101}$$

with $x \equiv \mathrm{e}^{\lambda}$. This L-operator satisfies a modified Yang-Baxter equation

$$\mathcal{L}_{j,a}(x)\, \mathcal{L}_{j,b}(y)\, \mathcal{R}_{a,b}(x/y) = \mathcal{R}_{a,b}(x/y)\, \mathcal{L}_{j,b}(y)\, \mathcal{L}_{j,a}(x) \,, \tag{6.102}$$

with an R-matrix defined as

$$\mathcal{R} \equiv \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{pmatrix} , \tag{6.103}$$

where

$$a \equiv qx - q^{-1}x^{-1} , \qquad b \equiv x - x^{-1} , \qquad c \equiv q - q^{-1} . \tag{6.104}$$

As for the Heisenberg model the Yang-Baxter is satisfied because of the $SU(2)$ algebra of the Pauli matrices we are using to write the L and R-operator, this Yang-Baxter for the XXZ chain induces a deformed $sl(2)$ algebra. This is the q-deformation (**??**) of the traditional algebra:

$$q^{\sigma^z}\sigma^{\pm} = q^{\pm}\sigma^{\pm}q^{\sigma^z} , \tag{6.105}$$

$$\left[\sigma^+, \sigma^-\right] = \frac{\left(q^{\sigma^z}\right)^2 - \left(q^{\sigma^z}\right)^{-2}}{q - q^{-1}} . \tag{6.106}$$

These equations define the q-deformation of $sl(2)$ algebra, known as $sl_q(2)$, and this is one of the simplest examples of a quantum group. Using this algebra, one can still think of the solution of the Yang-Baxter as a q-deformed exchange operator.

These deformations of known solutions of the Yang-Baxter can be used to generate new solutions. For the interested reader, we suggest to start the study of this complex subject from [33].

In closing, let us just explain why are these groups named *quantum*. This is not because they are related to some particular quantum process, but just because in the limit $\phi \to 0$ they reproduces the "classical" group, just like quantum mechanics reduces to the classical in the $\hbar \to 0$ limit.

Appendix A

Asymptotic behavior of Toeplitz Determinants

As we showed in Chapter 1, the asymptotic behavior of the correlation function in the XY model can be calculated from the asymptotic behavior of the determinant of the corresponding Toeplitz matrix. These determinants can be extracted from known theorems and conjectures in the theory of Toeplitz matrices. These types of calculations have been done first in [8, 1] for spin-spin correlation functions. It is well known that the asymptotic behavior of the determinant of a Toeplitz matrix as the size of the matrix goes to infinity strongly depends upon the zeros and singularities of the generating function of the matrix.

A very good report on the subject has been recently compiled by T. Ehrhardt [7]. Here we want to recapitulate what is known about the determinant

$$D_n[\sigma] = \det(\mathbf{S_n}) = \det|s(j-k)|_{j,k=0}^{n} \tag{A.1}$$

of a $n+1 \times n+1$ Toeplitz matrix

$$\mathbf{S_n} = \begin{pmatrix} s(0) & s(-1) & s(-2) & \dots & s(-n) \\ s(1) & s(0) & s(-1) & \dots & s(1-N) \\ s(2) & s(1) & s(0) & \dots & s(2-N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s(n) & s(n-1) & s(n-2) & \dots & s(0) \end{pmatrix} \tag{A.2}$$

with entries generated by a function $\sigma(q)$:

$$s(l) \equiv \int_{-\pi}^{\pi} \sigma(q) \mathrm{e}^{-\mathrm{i}lq} \frac{\mathrm{d}q}{2\pi}\,, \tag{A.3}$$

where the generating function $\sigma(q)$ is a periodic (complex) function, i.e. $\sigma(q) = \sigma(2\pi + q)$.

In this work we dealt only with generating functions with zero winding number

$$\mathrm{Ind}\,\sigma(q) \equiv \int_{-\pi}^{\pi} \frac{\mathrm{d}q}{2\pi} \frac{\mathrm{d}}{\mathrm{d}q} \log\sigma(q) = 0 \tag{A.4}$$

and this brief review will be limited to this condition. This was not the case in the study of Barouch et al. [1], where the generating function had non-zero winding number in some regions of the phase sdiagram.

A.1 The Strong Szegö Theorem

If $\sigma(q)$ is sufficiently smooth, non-zero and satisfies $\mathrm{Ind}\,\sigma(q) = 0$ (i.e., the winding number is 0), we can apply what is known as the *Strong Szegö Limit Theorem* ([34], [35]), which states that the determinant has a simple exponential asymptotic form

$$D_n[\sigma] \sim E[\sigma]G[\sigma]^n \qquad n \to \infty, \tag{A.5}$$

where $G[\sigma]$ and $E[\sigma]$ are defined by

$$G[\sigma] \equiv \exp \hat{\sigma}_0, \qquad E[\sigma] \equiv \exp \sum_{k=1}^{\infty} k\hat{\sigma}_k\hat{\sigma}_{-k} \tag{A.6}$$

and $\hat{\sigma}_k$ are the Fourier coefficients of the expansion of the logarithm of $\sigma(q)$:

$$\log \sigma(q) \equiv \sum_{k=-\infty}^{\infty} \hat{\sigma}_k \mathrm{e}^{\mathrm{i}kq}. \tag{A.7}$$

A.2 The Fisher-Hartwig Conjecture

Over the years, the Szegö Theorem has been extended to consider broader classes of generating functions by relaxing the continuity conditions which define a "smooth function", but it remained limited to never-vanishing functions. Therefore, some extensions have been proposed to the Szegö Theorem in order to relax this latter hypothesis. When the generating function has only pointwise singularities (or zeros), there exists a conjecture known as the Fisher-Hartwig Conjecture (FH) [36]. [1]

When $\sigma(q)$ has R singularities at $q = \theta_r$ $(r = 1..R)$, we decompose it as follows:

$$\sigma(q) = \tau(q) \prod_{r=1}^{R} \mathrm{e}^{\mathrm{i}\kappa_r[(q-\theta_r) \bmod 2\pi - \pi]} \left(2 - 2\cos(q-\theta_r)\right)^{\lambda_r} \tag{A.8}$$

so that $\tau(q)$ is a smooth function satisfying the conditions stated in the previous section. Then according to FH the asymptotic formula for the determinant takes the form

$$D_n[\sigma] \sim E\left[\tau, \{\kappa_a\}, \{\lambda_a\}, \{\theta_a\}\right] \; n^{\sum_r \left(\lambda_r^2 - \kappa_r^2\right)} G[\tau]^n \qquad n \to \infty, \tag{A.9}$$

where the constant prefactor is conjectured to be

$$\begin{aligned} E\left[\tau, \{\kappa_a\}, \{\lambda_a\}, \{\theta_a\}\right] \equiv \quad & E[\tau] \quad \prod_{r=1}^{R} \tau_- \left(\mathrm{e}^{\mathrm{i}\theta_r}\right)^{-\kappa_r - \lambda_r} \tau_+ \left(\mathrm{e}^{-\mathrm{i}\theta_r}\right)^{\kappa_r - \lambda_r} \\ & \times \quad \prod_{1 \le r \ne s \le R} \left(1 - \mathrm{e}^{\mathrm{i}(\theta_s - \theta_r)}\right)^{(\kappa_r + \lambda_r)(\kappa_s - \lambda_s)} \\ & \times \quad \prod_{r=1}^{R} \frac{\mathrm{G}(1 + \kappa_r + \lambda_r)\mathrm{G}(1 - \kappa_r + \lambda_r)}{\mathrm{G}(1 + 2\lambda_r)}. \end{aligned} \tag{A.10}$$

[1] This conjecture is still not completely proven. For details and status of the conjecture see Ref. [39].

$E[\tau]$ and $G[\tau]$ are defined as in (A.6) and $\tau_\pm$ are defined by decomposition

$$\tau(q) = \tau_- \left(\mathrm{e}^{\mathrm{i}q}\right) G[\tau] \tau_+ \left(\mathrm{e}^{-\mathrm{i}q}\right), \tag{A.11}$$

so that τ_+ (τ_-) are analytic and non-zero inside (outside) the unit circle on which τ is defined and satisfy the boundary conditions $\tau_+(0) = \tau_-(\infty) = 1$. G is the *Barnes G-function*, an analytic entire function defined as

$$\mathrm{G}(z+1) \equiv (2\pi)^{z/2} \mathrm{e}^{-[z+(\gamma_E+1)z^2]/2} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^k \mathrm{e}^{-z+\frac{z^2}{2n}}, \tag{A.12}$$

where $\gamma_E \sim 0.57721\ldots$ is the Euler-Mascheroni Constant.

This conjecture is actually proven for some ranges of parameters κ_r and λ_r or fully for the case of a single singularity ($R = 1$), see [37, 38].

In many simple cases it is possible to find the factorization of τ into the product of τ_+ and τ_- by inspection. More complicated examples like the ones presented in this work require a special technique to obtain this factorization, which is known as the *Wiener-Hopf decomposition*:

$$\begin{aligned} \log \tau_+(w) &= \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\log \tau(z)}{z-w} \qquad & |w| < 1, \\ \log \tau_-(w) &= -\oint \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\log \tau(z)}{z-w} \qquad & |w| > 1, \end{aligned} \tag{A.13}$$

where the integral is taken over the unit circle.

In light of these formulas, it is useful to present the parametrization (A.8) in a form which makes the analytical structure more apparent. Changing the variable dependence from q to $z \equiv \mathrm{e}^{\mathrm{i}q}$, we write

$$\sigma(z) = \tau(z) \prod_{r=1}^{R} \left(1-\frac{z}{z_r}\right)^{\lambda_r+\kappa_r} \left(1-\frac{z_r}{z}\right)^{\lambda_r-\kappa_r}, \tag{A.14}$$

where $z_r \equiv \mathrm{e}^{\mathrm{i}\theta_r}$.

A.3 The Generalized Fisher-Hartwig Conjecture

Despite the considerable success of the Fisher-Hartwig Conjecture, few examples have been reported in the mathematical literature that do not fit this result. These examples share the characteristics that inequivalent representations of the form (A.8) exist for the generating function $\sigma(q)$. Although no theorem has been proven concerning these cases, a generalization of the Fisher-Hartwig Conjecture (gFH) has been suggested by Basor and Tracy [39] that has no counter-examples yet.

If more than one parametrization of the kind (A.8) exists, we write them all as

$$\sigma(q) = \tau^i(q) \prod_{r=1}^{R} \mathrm{e}^{\mathrm{i}\kappa_r^i [(q-\theta_r) \bmod 2\pi - \pi]} \left(2-2\cos(q-\theta_r)\right)^{\lambda_r^i}, \tag{A.15}$$

where the index i labels different parametrizations (for $R > 1$ there can be only a countable number of different parametrizations of this kind). Then the asymptotic formula for the determinant is

$$D_n[\sigma] \sim \sum_{i\in\Upsilon} E\left[\tau^i, \{\kappa_a^i\}, \{\lambda_a^i\}, \{\theta_a\}\right]\ n^{\Omega(i)} G[\tau^i]^n \qquad n \to \infty, \tag{A.16}$$

where

$$\Omega(i) \equiv \sum_{r=1}^{R} \left(\left(\lambda_r^i\right)^2 - \left(\kappa_r^i\right)^2 \right), \tag{A.17}$$

$$\Upsilon = \left\{ i \parallel \mathrm{Re}\left[\Omega(i)\right] = \max_j \mathrm{Re}\left[\Omega(j)\right] \right\}. \tag{A.18}$$

The generalization essentially gives the asymptotics of the Toeplitz determinant as a sum of (FH) asymptotics calculated separately for different leading (see Eq. (A.18)) representations (A.15).

A.4 Widom's Theorem

If $\sigma(q)$ is supported only in the interval $\alpha \le q \le 2\pi - \alpha$ as in our model for $\gamma = 0$, singularities are no longer pointwise and one should apply Widom's Theorem [40]. It states that the asymptotic behavior of the determinant in this case is

$$D_n[\sigma] \sim 2^{1/12} \mathrm{e}^{3\zeta'(-1)} \left(\sin\frac{\alpha}{2}\right)^{-1/4} E[\rho]^2 n^{-1/4} G[\rho]^n \left(\cos\frac{\alpha}{2}\right)^{n^2}, \tag{A.19}$$

where E and G are defined in (A.6) and

$$\rho(q) = \sigma\left(2\cos^{-1}\left[\cos\frac{\alpha}{2}\cos q\right]\right) \tag{A.20}$$

with the convention $0 \le \cos^{-1} x \le \pi$.

For the case considered in Section **??**, the generating function is constant, $E[\rho] = G[\rho] = 1$, and (A.19) simplifies considerably giving

$$D_n[\sigma] \sim 2^{1/12} \mathrm{e}^{3\zeta'(-1)} \left(\sin\frac{\alpha}{2}\right)^{-1/4} n^{-1/4} \left(\cos\frac{\alpha}{2}\right)^{n^2}. \tag{A.21}$$

Appendix B

Application of BA to field theories

B.1 Introduction

While the Bethe Ansatz is quite efficient in providing us with the thermodynamics of a system, the calculation of correlation functions is very involved and computationally very demanding. In $1+1$ dimensions, however, there are other advanced analytical methods to access correlation functions, especially in the low-energy, long-distance limit: the *bosonization* approach and the *Conformal Field Theory* CFT) description. Physically, they are the mathematical formulation of the fact that one-dimensional systems do not conform to the paradigm of a *Fermi liquid*, and therefore the long-living quasi-particle excitations are not close to being free fermions, but have novel features altogether. This is something that we noticed in the analysis of the previous chapters, as we saw that one-dimensional (integrable, but not only) systems respond collectively to any perturbation.

For simple systems, a new universality class emerges in 1-D and it is known as *Luttinger Liquid*. This amounts to the observation that low-energy degrees of freedom have a sound wave (phononic) nature, which allows to describe the system using just a free bosonic field. From a CFT point of view, the Luttinger liquid is a $c = 1$ theory, and thus more information is needed to uniquely identify its operatorial content. More complicated systems can have fractional central charges or $c > 1$, and thus they are not Luttinger liquids, but can still be described efficiently within a CFT approach, possibly supplemented by a Kaz-Moody algebra.

In this chapter we wish to introduce how Bethe Ansatz can be used in conjunction with these effective field theory approaches. In fact, these effective descriptions need to be supplemented with the right parameters, and the BA solution is one of the few (the only?) analytical ways to calculate them independently. This is a great advantage, since normally one has to fit a couple of correlation functions from numerical calculations or from experiments to determine these parameters and only at that point one can really check the predictions from the field theory. This latter part could be a problem, as the number of independent measurements one can perform on a sample is often very limited.

After the introduction of the main ideas, the material will become very mathematical. We have tried to reduce the number of formulae as much as possible, but the topic does not help

in this respect. The bosonization approach is well explained in [41, 42]. For the application of Bethe Ansatz to the scaling limit we mostly follow [19]. We invite the interest reader to the original sources for better explanations and more details.

B.2 Bosonization

Bosonization is a way to describe the dynamics of a critical (i.e. gapless) system in term of its collective behavior through a collective bosonic field. This is possible in one-dimension because the system is very much constraint by its dimensionality: as one particle is excited, all other particles have to rearrange to accommodate it, because there is no way for a particle to go around another without interacting. This kind of phenomenon is familiar to us already from our analysis of the excitations of integrable models from Bethe Ansatz.

Therefore, the description of the system in terms of its density of particle can be efficiently use to capture the whole dynamics, provided that the density field

$$\rho(x) = \sum_j \delta(x - x_j) \tag{B.1}$$

(where x_j is the position of the j-th particle) can be approximated with a smooth function. This amount to a hydrodynamic description for the system, where the field conjugated to the density is the velocity $v(x)$.

$$[\rho(x), v(y)] = -\mathrm{i}\delta'(x - y) \ . \tag{B.2}$$

A general structure for the evolution equations for such a system gives

$$\dot{\rho} - \partial_x (\rho v) = 0 \ , \tag{B.3}$$

$$\dot{v} - v\partial_x v + \partial_x F(\rho) = 0 \ , \tag{B.4}$$

where the first is the continuity equation and the second is the proper dynamical Euler equation.

These equations are non linear and very difficult to treat at the quantum level (there is no clear small-coupling expansion valid for all times). But they can be linearized around a classical solution and a linear hydrodynamics gives essentially a wave equation. This is to say that elementary (universal) excitations of a one-dimensional system are phonons.

Thus, under these general considerations we expect to be able to describe a 1-D system with a bosonic operator and a quadratic Hamiltonian. This description is called *bosonization*, and it should be remarked that even bosonic theories can be bosonized, since this just means to give a linear-hydrodynamics formulation.

Let us describe how to bosonize a free fermionic theory, with microscopic Hamiltonian

$$\mathcal{H} = -\frac{1}{2m} \Psi^\dagger(x)\partial_x^2\Psi(x) = \frac{k^2}{2m} \Psi^\dagger(k)\Psi(k) \ , \tag{B.5}$$

where $\partial_x \equiv \partial/\partial x$ and the last expression shows the spectrum of the Hamiltonian in the Fourier space representation.

The first step is to separate particles in left and right movers and introduce separate fields for each type. The fundamental identity one uses can be written as ($Right = +, Left = -$)

$$\psi_{\pm}(x) \equiv \frac{1}{\sqrt{2\pi}} \, : \mathrm{e}^{\mp \mathrm{i}\sqrt{4\pi}\phi_{\pm}(x)} : , \tag{B.6}$$

where $\phi_{\pm}(x)$ are collective bosonic fields and $: \mathcal{O} : \equiv \mathcal{O} - \langle 0|\mathcal{O}|0\rangle$ stands for the *normal ordering*. Note that the exponential mapping is periodic: the $\sqrt{4\pi}$ factor determines the periodicity of the fields $\phi_{\pm}$ (also called the *compactification radius*) and for free fermions is equal for both chiral fields. The choice of $\sqrt{4\pi}$ is convenient to ensure that the anti-commutation of the fermionic fields translate into canonical commutation relations for the bosonic ones. We will see at the end of the section that in bosonizing interacting theories we take different compactification radii for the two chiral fields, in a way that preserves the commutation relation (and also the quantization of the conformal spins for the vertex operators).

While the identity (B.6) between a fermion and a boson holds in generality in one-dimension, the prescription for the normal ordering depends on the theory (and its ground state). This prescription is pivotal for the bosonization construction to give meaningful results (and avoid spurious divergences) and it is not available in generality. However, if we concentrate only on low-energy excitations, we can linearize the spectrum and hence derive a clear and simple normal ordering rule.

Thus, we write the free Hamiltonian (B.5) as

$$\mathcal{H} = -\frac{1}{2m} \sum_{r=\pm} \psi_r^{\dagger} \left(\partial_x + \mathrm{i} r k_F\right)^2 \psi_r \tag{B.7}$$

and expand around the Fermi points as

$$\mathcal{H} \simeq -\frac{k_F^2}{2m} \sum_{r=\pm} \psi_r^{\dagger}\psi_r - \mathrm{i}\frac{k_F}{m} \sum_{r=\pm} r\psi_{\pm}^{\dagger}\partial_x\psi_{\pm} + \dots \tag{B.8}$$

where the first term is interpreted as a chemical potential (which can be absorbed in a re-definition of the ground state energy), while the second term shows a linear spectrum for the excitations around the Fermi points $\pm k_F$. Moreover, we defined left- and right-moving fields $\psi_{\pm}$ as the fields obtained expanding around the left/right Fermi Point:

$$\begin{aligned} \Psi(x) &= : \mathrm{e}^{\mathrm{i}k_F x}\psi_+(x) + \mathrm{e}^{-\mathrm{i}k_F x}\psi_-(x) : \\ &= \sum_{r=\pm} \mathrm{e}^{\mathrm{i}rk_F x} : \mathrm{e}^{\mathrm{i}\sqrt{4\pi}\phi_r(x)} : \ . \end{aligned} \tag{B.9}$$

We can use the mapping (B.6) to calculate various bilinears in the fermions in terms of the bosonic field. For instance, one can consider a quantity like

$$\begin{aligned} : \psi_{\pm}^{\dagger}(x)\psi_{\pm}(x+\epsilon) : &= \psi_{\pm}^{\dagger}(x)\psi_{\pm}(x+\epsilon) - \langle\psi_{\pm}^{\dagger}(x)\psi_{\pm}(x+\epsilon)\rangle \\ &= \frac{1}{2\pi}\left[: \mathrm{e}^{\mp\mathrm{i}\sqrt{4\pi}(\phi_{\pm}(x+\epsilon)-\phi_{\pm}(x))} : -1\right] \mathrm{e}^{4\pi\langle\phi_{\pm}(x)\phi_{\pm}(x+\epsilon)\rangle} = \\ &= \pm\frac{1}{2\mathrm{i}\pi\epsilon}\left[\mathrm{e}^{\mp\mathrm{i}\sqrt{4\pi}\left(\phi_{L,R}(x+\epsilon)-\phi_{L,R}(x)\right)} - 1\right] \end{aligned} \tag{B.10}$$

where we used the identity

$$: \mathrm{e}^{A} : : \mathrm{e}^{B} :=: \mathrm{e}^{A+B} : \ \mathrm{e}^{\langle AB - \frac{A^2+B^2}{2} \rangle} \tag{B.11}$$

and the fact that

$$\langle \phi_\pm(0)\phi_\pm(x) - \phi_\pm^2(0) \rangle = \lim_{\alpha \to 0} \frac{1}{4\pi} \ln \frac{\alpha}{\alpha \pm \mathrm{i}x} \ . \tag{B.12}$$

Here, α is a regulator that mimics a finite bandwidth and prevents the momentum from becoming too large (thus limiting the bandwidth to $\Lambda \sim 1/\alpha$).

The prescription to calculate bilinear like (B.10) is known as *point splitting* and it takes into account that the square of a field is coordinate space is not-defined and has to be regularized by discretizing the space. In practice, we saw in the second line of (B.10) that the normal ordering amounts to subtract $1/\epsilon$ from the exponential, corresponding to the ground state contribution. Thus, from one side α in (B.12) captures the low-energy approximation, from the other ϵ in (B.10) is related to the underlying lattice of the microscopic theory.

We can expand (B.10) in powers of ϵ

$$\begin{aligned} : \psi_\pm^\dagger(x)\psi_\pm(x+\epsilon) : \ &= \ \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \psi_\pm^\dagger(x) \partial_x^n \psi_\pm(x) \\ &= \ \pm \frac{1}{2\mathrm{i}\pi\epsilon} \left[\mathrm{e}^{\mp \mathrm{i}\sqrt{4\pi} \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \partial_x^n \phi_\pm(x)} - 1 \right] \ , \end{aligned} \tag{B.13}$$

which give the generating function of the chiral fermionic currents

$$j_n^\pm(x) \equiv \psi_\pm^\dagger(x) \partial_x^n \psi_\pm(x) \tag{B.14}$$

in terms of the bosonic field $\phi_\pm$.

By matching powers of ϵ in (B.13) we can write down these expressions. The density of fermion is

$$\rho_\pm = j_0^\pm = \psi_\pm^\dagger(x)\psi_\pm(x) = -\frac{1}{\sqrt{\pi}} \partial_x \phi_\pm(x), \tag{B.15}$$

the current density is

$$j_1^\pm = \psi_\pm^\dagger(x) \partial_x \psi_\pm(x) = \pm \mathrm{i} \left(\partial_x \phi_\pm(x) \right)^2 - \frac{1}{\sqrt{4\pi}} \partial_x^2 \phi_\pm \ . \tag{B.16}$$

The third term in the expansion can be identify with the original quadratic Hamiltonian for the left/right movers

$$\begin{aligned} j_2^\pm \ &= \ -2m \, \mathcal{H}_\pm = \psi_\pm^\dagger(x) \partial_x^2 \psi_\pm(x) \\ &= \ \frac{4}{3}\sqrt{\pi} \left(\partial_x \phi_\pm(x) \right)^3 \pm \mathrm{i} \left(\partial_x \phi_\pm \right) \left(\partial_x^2 \phi_\pm \right) - \frac{1}{3\sqrt{\pi}} \partial_x^3 \phi_\pm \ . \end{aligned} \tag{B.17}$$

While the first line is a well defined Hamiltonian operator for fermions, the second line amounts to a cubic potential, which is not bounded and thus cannot sustain a stable vacuum. Hence, it can enter in the Hamiltonian only as a perturbative interaction term.

In fact, we can neglect the last terms in (B.17) as boundary terms since they are total derivatives and thus write the bosonized version of (B.5) as

$$\mathcal{H} \propto (\partial_x \phi_+)^3 + (\partial_x \phi_-)^3 \,. \tag{B.18}$$

This is cubic theory and therefore cannot be quantized, since the spectrum for the bosonic field has no lower bound and the ground state of the theory is unstable and has an infinite energy. This is the reason for which it does not make sense to directly bosonize a non-linear theory.

If one were, instead, to consider the linearized version of the free fermionic theory (B.8), using the expressions found above, the bosonized Hamiltonian would read

$$\begin{aligned} \mathcal{H} &\sim -\mathrm{i}\frac{k_F}{m}\left[j_1^+ - j_1^-\right] + \dots \\ &= \frac{k_F}{m}\left[(\partial_x\phi_+)^2 + (\partial_x\phi_-)^2\right] + \dots \end{aligned} \tag{B.19}$$

One can then include the additional terms like (B.17) neglected in (B.8) in a perturbative way and treat them as small correction.

Out of the two chiral fields we can define a bosonic field and its dual

$$\phi(x) \equiv \phi_+(x) + \phi_-(x) \,, \qquad\qquad \theta(x) \equiv \phi_+(x) - \phi_-(x) \,. \tag{B.20}$$

Using (B.9) and the fermionic commutation relation, one can prove that these bosonic fields satisfy the commutation relation

$$[\phi(x), \theta(y)] = \mathrm{i}\vartheta(y-x) \,, \tag{B.21}$$

where $\vartheta(x)$ is the usual Heaviside step function. By differentiating we have

$$[\phi(x), \partial_y\theta(y)] = [\theta(x), \partial_y\phi(y)] = \mathrm{i}\delta(x-y) \,, \tag{B.22}$$

which means that we can identify the derivative of the dual field $\theta(x)$ with conjugate of $\phi(x)$ (or viceversa):

$$\Pi(x) \equiv \frac{1}{v_0}\partial_t\phi(x) = \partial_x\theta(x) \,, \tag{B.23}$$

where $v_0 \equiv k_F/m$ is the sound velocity of the free system.

Thus, the linearized free fermionic theory is mapped into a free bosonic theory

$$\mathcal{H} = \frac{v_0}{2}\int \left[(\Pi(x))^2 + (\partial_x\phi(x))^2\right] \mathrm{d}x \,. \tag{B.24}$$

Physically, the bosonic field is the *displacement field* and one should notice the similarity between the bosonization identity (B.6) and the Jordan-Wigner transformation (1.10). In fact, $\phi(x)$ counts the number of particles to the left of x and its derivative gives the particle density, see (B.16). In particular we have

$$\begin{aligned} \rho(x) &= \Psi^\dagger(x)\Psi(x) \\ &= \rho_0 + \psi_+^\dagger(x)\psi_+(x) + \psi_-^\dagger(x)\psi_-(x) + \mathrm{e}^{-\mathrm{i}2k_Fx}\psi_+^\dagger(x)\psi_-(x) + \mathrm{e}^{\mathrm{i}2k_Fx}\psi_-^\dagger(x)\psi_+(x) \\ &= \rho_0 - \frac{1}{\sqrt{\pi}}\partial_x\phi(x) + \frac{1}{\pi}\cos\left[\sqrt{4\pi}\phi(x) - 2k_Fx\right] \,, \end{aligned} \tag{B.25}$$

so that we identify the bosonic field with a density wave (ρ_0 is the constant, background, density of particles).

Thus, we have shown that low-energy excitations of the free fermions Hamiltonian (B.5) can be described in terms of a simple quadratic boson, corresponding to a quantum sound wave. It can be shown that any interaction term one can add to the free model can be bosonized using the mapping (B.6) and the point-splitting prescription and written in terms of the bosonic field $\phi(x)$ and its dual $\theta(x)$. The great advantage of the bosonization procedure is that this additional terms can be considered easily with a Renormalization Group (RG) analysis and divided between relevant interactions that open a gap and irrelevant perturbations that do not. The latter can be shown to contribute to (B.24) only with additional quadratic terms and thus only changes the coefficients in front of the two terms in (B.24). The renormalization of these coefficients can thus be written as

$$\mathcal{H} = \frac{u}{\pi} \int \left[K \left(\Pi(x)\right)^2 + \frac{1}{K} \left(\nabla\phi(x)\right)^2 \right] \mathrm{d}x \ , \tag{B.26}$$

where u has the dimension of a velocity and can be interpreted as the Fermi velocity of the interacting system and K is a dimensionless parameter that is related to the compactification radius of the theory, or to the exclusion statistic area occupied by a particle in phase-space. Interactions which open a gap generate sine or cosine terms in the field. A single term of this kind gives a "simple" Sine-Gordon theory, more can make the resulting theory difficult to analyze, but it can often be shown that only one of them is relevant in the RG sense.

To recap, the low-energy excitations of any one-dimensional gapless (fermionic) system can be mapped using the bosonization procedure to a bosonic Gaussian theory (B.26), where all effect of the interactions are captured by only two parameters: u and K. Notice that the *Luttinger parameter* K can be removed from the Hamiltonian (B.26) by a simple rescaling of the fields

$$\phi(x) \to \frac{1}{\sqrt{K}} \, \phi(x) \ , \qquad\qquad \theta(x) \to \sqrt{K} \, \theta(x) \ . \tag{B.27}$$

This corresponds to a redefinition of the compactification radius of the chiral fields. Using (B.20)

$$\phi_\pm = \frac{1}{2\sqrt{K}} [\phi(x) \pm K \, \theta(x)] \ . \tag{B.28}$$

In general, $K = 1$ corresponds to free fermions. $K > 1$ encodes attractive fermions and $0 < K < 1$ repulsive fermions. Free bosons are not stable in one-dimension and they would correspond to $K \to \infty$. Thus, any finite K corresponds to repulsive bosons all the way to the $K = 1$ limit of perfectly repulsive bosons (the so-called Tonks-Girardeau limit, i.e. $c \to infty$ of the Lieb-Liniger model).

One of the fundamental advantage of having mapped an interacting system to a Gaussian bosonic theory like (B.26) is that the correlation functions are easily obtainable. For instance, see (B.12) and (B.28), we have

$$\langle [\phi(x,t) - \phi(0,0)]^2 \rangle \ = \ \lim_{\alpha \to 0} \frac{K}{2\pi} \ln \frac{x^2 + (ut+\alpha)^2}{\alpha^2} \ , \tag{B.29}$$

$$\langle [\theta(x,t) - \theta(0,0)]^2 \rangle \ = \ \lim_{\alpha \to 0} \frac{1}{2\pi K} \ln \frac{x^2 + (ut+\alpha)^2}{\alpha^2} \ . \tag{B.30}$$

The principal operators of the theory are vertex operators of the form

$$V(\beta, z) \equiv \mathrm{e}^{\mathrm{i}\beta\phi_+(z=x+\mathrm{i}ut)} , \qquad \bar{V}(\bar{\beta}, \bar{z}) \equiv \mathrm{e}^{\mathrm{i}\bar{\beta}\phi_-(\bar{z}=x-\mathrm{i}ut)} . \tag{B.31}$$

Correlation functions of vertex operators can be calculated using the power of a Gaussian theory:

$$\left\langle \mathrm{e}^{\mathrm{i}\sum_j \left[\beta_j\phi_+(z_j)+\bar{\beta}_j\phi_-(\bar{z}_j)\right]} \right\rangle = \mathrm{e}^{\frac{1}{2}\left\langle \left[\sum_j \beta_j\phi_+(z_j)+\bar{\beta}_j\phi_-(\bar{z}_j)\right]^2 \right\rangle} , \tag{B.32}$$

which is non-zero only if $\sum_j \beta_j = \sum_j \bar{\beta}_j = 0$.

In general, these correlation functions decay like power-law $\langle \mathcal{O} \rangle \sim r^{-2d}$, with a characteristic exponent d. If $d < 2$ the corresponding operator is *relevant* in an RG sense; if $d > 2$ it is *irrelevant*, while $d = 2$ corresponds to the marginal case.

Using these formulae and (B.6) it is easy to calculate the asymptotic behavior of physical correlators. For instance

$$\langle \rho(x,t)\rho(0,0) \rangle \simeq \frac{K^2}{2\pi^2} \frac{1}{(x^2+u^2t^2)^2} + B\frac{\cos 2k_F x}{(x^2+u^2t^2)^{2K}} + \dots . \tag{B.33}$$

Finally, let us mentioned that the bosonization construction is very general and applicable to any one-dimensional critical system. Even if we showed the construction explicitly only for a microscopic fermionic theory, it can be generalized to any model. The approximation to linear spectrum (low-energy modes) is pivotal to ensure that the resulting theory is just quadratic. To bosonize a spin system, one can first perform a Jordan-Wigner transformation to map it into a fermionic theory and then bosonize these fermions (note that a spin chain at half filling -i.e. zero magnetization- has $k_F = \pi/2$, which corresponds to having a smooth and a staggered component in the spin density, see (B.25)). It is also possible to bosonize a bosonic theory, in that the mapping does not have to do with the statistic of the particle, but with the fact that fundamental excitations are collective, as we noticed while studying integrable models.

With systems with additional degrees of freedom, like the Hubbard model or various spin ladders, one can bosonized each of degree of freedom and study their interaction (and competition) in the collective description. However, these systems often acquire additional symmetries for which graded CFTs can provide a more powerful description.

B.3 Bosonization parameters from Bethe Ansatz

As we saw, the bosonization construction leaves only two free parameters (see B.26) that completely characterize the universal feature of the system. These parameters can then be fitted with experiments or by some other independent approach, since for most system their microscopical derivation is not possible. This is where Bethe Ansatz can help.

In fact, one can compare thermodynamic quantities calculated using the Luttinger liquid description and derived with Bethe Ansatz. Since bosonization has only two parameters, we need just two relations, and the choice of which to use is just due to convenience. In this section we are following [41] and this is mostly a numerical approach. More accurate analytical methods can be employed, but we do not have enough space/time to discuss them. However, we will give a glimpse of them in the next sections.

For instance, one easily available quantity is the *compressibility*

$$\kappa \equiv -\frac{1}{L}\frac{\partial L}{\partial \mathcal{P}} = \frac{1}{\rho^2}\frac{\partial \rho}{\partial h} \,, \tag{B.34}$$

where $\mathcal{P}$ is the pressure, ρ is the particle density and h is the chemical potential. We will use a rescaled compressibility

$$\tilde{\kappa} \equiv \frac{\partial \rho}{\partial h} \,. \tag{B.35}$$

The presence of a chemical potential adds to the Hamiltonian a term of the form

$$\delta H_h = h \int \mathrm{d}x \; \rho(x) = \frac{h}{\pi} \int \mathrm{d}x \; \nabla\phi(x) \,, \tag{B.36}$$

which can be absorbed by a shift in the field

$$\phi(x) \to \phi(x) - h\frac{K}{u}x \,. \tag{B.37}$$

The compressibility is thus calculated to be

$$\tilde{\kappa} = \frac{1}{\pi}\frac{\mathrm{d}\langle\nabla\phi(x)\rangle}{\mathrm{d}h} = \frac{K}{u\,\pi} \,. \tag{B.38}$$

From Bethe Ansatz, the compressibility is just

$$\kappa^{-1} = \frac{N^2}{L}\frac{\mathrm{d}^2 E}{\mathrm{d}N^2} \tag{B.39}$$

$$\tilde{\kappa}^{-1} = L\frac{\mathrm{d}^2 E}{\mathrm{d}N^2} \simeq L\left(\frac{E(N+2)+E(N-2)-2E(N)}{4}\right) \,. \tag{B.40}$$

Notice that the calculation of this quantity requires the knowledge of the energy of the system for system size differing by 2: this is very important, since it avoid some spurious effect that might arise for step of size 1.

Another easily accessible quantity is the *charge stiffness*. Since we placed the system on a ring (by imposing periodic boundary conditions), we can imagine to thread the ring with a magnetic field, which will induce a phase shift in the wavefunction around one circle. The minimal coupling of the vector potential with the field induce a shift

$$\delta H_A = \frac{\Phi}{L\pi} \int \mathrm{d}x \; \mathrm{d}\tau \; \partial_\tau \phi \,, \qquad \phi(\tau) \to \phi(\tau) - \frac{\Phi}{L}\, Ku\,\tau \,, \tag{B.41}$$

where Φ is the flux enclosed in the ring. By varying the flux, we generate a current

$$J = \frac{1}{\pi}\langle\partial_\tau\phi\rangle = \frac{1}{L}\sum_k \frac{\mathrm{d}\epsilon_k}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}\Phi}\sum_k \epsilon_k = \frac{\mathrm{d}E}{\mathrm{d}\Phi} \,. \tag{B.42}$$

The charge stiffness is defined as

$$\mathcal{D} \equiv \pi L \left.\frac{\mathrm{d}J}{\mathrm{d}\Phi}\right|_{\Phi=0} \tag{B.43}$$

and we see that

$$\mathcal{D} = uK = \pi L \left.\frac{\mathrm{d}^2 E}{\mathrm{d}\Phi^2}\right|_{\Phi=0} \,. \tag{B.44}$$

It is easy to implement the presence of a flux and the resulting phase shift in the Bethe Ansatz solution. This is equivalent to impose *twisted boundary conditions* and it amounts to modify the Bethe equations:

$$Lk_j = 2\pi I_j + \Phi - \sum_l \theta(k_j - k_l) \; . \tag{B.45}$$

B.4 Sound velocity

The calculation of the sound velocity and Luttinger parameter sketched in the previous section does not take full advantage of the power of Bethe ansatz and require the solution of the set of N Bethe equations for different **finite** N's and different boundary conditions. A third option which (in principle) is purely analytical, is to evaluate the Fermi velocity $u = v_F$ directly from the Bethe equations in the thermodynamic limit. Microscopically, it is defined as the derivative of the dressed energy by the dressed momentum at the Fermi point:

$$u = \left.\frac{\partial\varepsilon(\lambda)}{\partial p(\lambda)}\right|_{\lambda=q} = \left(\frac{\partial\varepsilon(\lambda)}{\partial\lambda}\right) \Big/ \left(\frac{\partial p(\lambda)}{\partial\lambda}\right) \; . \tag{B.46}$$

Comparing

$$\varepsilon(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} \mathcal{K}(\lambda,\mu)\; \varepsilon(\mu)\; \mathrm{d}\mu \;\; = \;\; \lambda^2 - h \; , \tag{B.47}$$

$$\rho(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} \mathcal{K}(\lambda,\mu)\; \rho(\mu)\; \mathrm{d}\mu \;\; = \;\; \frac{1}{2\pi} \; , \tag{B.48}$$

$$p(\lambda) - \int_{-q}^{q} \theta(\lambda-\mu)\; \rho(\mu)\; \mathrm{d}\mu \;\; = \;\; k(\lambda) \; , \tag{B.49}$$

we have

$$\frac{\partial\varepsilon(\lambda)}{\partial h} = -2\pi\rho(\lambda) \tag{B.50}$$

and

$$\frac{\partial p(\lambda)}{\partial\lambda} = 2\pi\rho(\lambda) \; . \tag{B.51}$$

Using the latter we immediately get

$$u = \left.\frac{1}{2\pi\rho(\lambda)}\frac{\partial\varepsilon(\lambda)}{\partial\lambda}\right|_{\lambda=q} \; . \tag{B.52}$$

There is also another definition of the sound velocity, a macroscopic one derived from thermodynamics:

$$v_F = \frac{\partial\mathcal{P}}{\partial n} = 2\left(\frac{\partial\mathcal{P}}{\partial h}\right) \Big/ \left(\frac{\partial n}{\partial h}\right) \; , \tag{B.53}$$

where for h is the chemical potential, n is the particle density and for the pressure we have

$$\frac{\partial\mathcal{P}}{\partial h} = -\frac{1}{2\pi}\int_{-q}^{q} \frac{\partial\varepsilon(\lambda)}{\partial h}\mathrm{d}\lambda = n \; . \tag{B.54}$$

One can prove manipulating the BA integral equations that these two definitions agree and can be alternatively used according to convenience. More on this respect can be found in [19].

The Luttinger parameter (and all the scaling dimension of the relevant correlation functions) can also be calculated by analytical manipulations of the integral equations of the Bethe solution. This approach is based on the finite size scaling for large system sizes. Before we sketch the basics of this idea, we need some additional notions of conformal field theory. These will not be developed here, so we will just set some notations and refer the reader to one of the many sources to learn CFT.

B.5 Basics of Conformal Field Theory

The bosonization approach has a deep connection to CFT, but we will now consider the latter independently.

CFT is based on the idea that at a critical point there are no relevant length scales and the theory is invariant under any rescaling. The group responsible for this invariance is the conformal group. In a two-dimensional space, this symmetry is enhanced and becomes powerful enough to constrain the theory and the correlation function in a significant way.

Conformal Field theory, being two-dimensional, is best represented in terms of complex variables

$$z = \mathrm{i}\, x + u\, t\ , \qquad \bar{z} = -\mathrm{i}\, x + u\, t\ , \tag{B.55}$$

where u is the sound (light) velocity. CFT assume Lorentz invariance and thus massless excitations all move with the same velocity U. CFT is also a chiral theory, therefore the left and right moving sector tend to be independent from one another.

The quantum generators of the conformal transformations are called *Virasoro operators* $L_n, \bar{L}_n$ and satisfy the algebra (independent for holomorphic and antiholomorphic sector)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\, n(n^2-1)\delta_{n+m,0}\ , \tag{B.56}$$

$$\left[\bar{L}_n, \bar{L}_m\right] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}\, n(n^2-1)\delta_{n+m,0}\ , \tag{B.57}$$

where $c, \bar{c}$ is the *central charge*, or *conformal anomaly*. The $L_n, \bar{L}_n$ are nothing but the coefficients in a Laurent expansion of the stress tensor in powers of $z, \bar{z}$:

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}\ , \qquad \bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}}\ . \tag{B.58}$$

Under a conformal transformation

$$z = z(w)\ , \qquad \bar{z} = \bar{z}(\bar{w})\ , \tag{B.59}$$

a primary field $\phi(z, \bar{z})$ transforms as

$$\phi(w, \bar{w}) = \left(\frac{\partial z}{\partial w}\right)^{\Delta^+} \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{\Delta^-} \phi\left(z\left(w\right), \bar{z}\left(\bar{w}\right)\right)\ , \tag{B.60}$$

where the *conformal dimensions* $\Delta^{\pm}$ characterize the field and specify the two-point correlation function

$$\langle \phi(z_1, \bar{z}_1)\phi(z_2, \bar{z}_2)\rangle = (z_1 - z_2)^{-2\Delta^+}(\bar{z}_1 - \bar{z}_2)^{-2\Delta^-}\ . \tag{B.61}$$

To consider a space with periodic boundary condition in the space direction, we map the complex plane into a cylinder with the mapping

$$z = \mathrm{e}^{2\pi w/L} , \qquad w = \mathrm{i}\, x + u\, t , \quad 0 \leq x < L . \tag{B.62}$$

The leading part of the correlation function becomes

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle_L \sim \mathrm{e}^{-\frac{2\pi\Delta^+}{L}[\mathrm{i}(x_1-x_2)+u(t_1-t_2)]}\mathrm{e}^{-\frac{2\pi\Delta^-}{L}[-\mathrm{i}(x_1-x_2)+u(t_1-t_2)]} . \tag{B.63}$$

Comparing this with a spectral decomposition

$$\langle \phi(w_1, \bar{w}_1)\phi(w_2, \bar{w}_2)\rangle_L = \sum_Q |\langle 0|\phi(0,0)|Q\rangle|^2\, \mathrm{e}^{-(t_1-t_2)(E_Q-E_0)-\mathrm{i}(x_1-x_2)(P_Q-P_0)} \tag{B.64}$$

where E_0, P_0 are the energy and momentum of the ground state, while E_Q, P_Q are the energy and momentum of the intermediate state Q's, which constitute a complete set.

The leading term of this expansion should give

$$E_Q - E_0 = \frac{2\pi u}{L}(\Delta^+ + \Delta^-) \tag{B.65}$$

$$P_Q - P_0 = \frac{2\pi}{L}(\Delta^+ - \Delta^-) \tag{B.66}$$

Comparing the energy and momentum of the different states as obtained with Bethe Ansatz with (B.65, B.66) one can identify the scaling dimensions of the different operators in the theory. Knowing the scaling dimensions of all the two-point functions and the central charge identify the theory uniquely.

An easy thermodynamic way to determine the central charge is to study the finite size effects. So far, we always considered the system in the thermodynamic limit ($L, N \to \infty$, $N/L = \rho = \mathrm{const}$). If the size of the system is large, but finite, there are going to be corrections. From CFT it can be proven, for instance, that the total energy scales like

$$E \simeq L\, e - c\frac{\pi}{6L}\, v_F + \mathrm{O}(L^{-2}) , \tag{B.67}$$

providing us with a direct way to evaluate c.

B.6 Finite size analysis of the Lieb-Liniger model

The content of a QFT is specified by the *central charge* and the *scaling dimensions* of the primary fields. We saw how to approach the scaling dimensions, let us now introduce the central charge and show how to extract it from the Bethe Ansatz solution.

As the Hamiltonian is the generator of the evolution of the theory, we write it in terms of modes

$$H(z) = \sum_{n=-\infty}^{\infty} L_n\, z^n . \tag{B.68}$$

The operators L_n are the generators for all the transformation of the Conformal Group. They satisfy the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m} . \tag{B.69}$$

c is the central charge of the theory and it is a quantum anomaly, in the sense that classical fields have $c = 0$. This anomaly arises from the fact that a CFT is a chiral theory with a relativistic spectrum. Below the Fermi point, the dirac sea extends indefinitely to negative energies and its divergent presence is the source of the anomaly.

Here we want to sketch how to use the Bethe ansatz machinery to determine the correct QFT describing the Lieb-Liniger model studied in chapter 2. We focus on this model for definiteness (also being the simpler), but we shall keep the notation as general as possible to allow the application of this method to other models with minimal corrections.

To evaluate the finite-size corrections from Bethe Ansatz, we will need the Euler-Maclaurin formula:

$$\sum_{j=1}^{N} f(x_j) = \int_a^b f(x)\mathrm{d}x + \frac{f}{2}\Big|_a^b - \frac{b_2}{2}\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_a^b + \dots \,, \tag{B.70}$$

where $b_2 = \frac{1}{6}$ is the second Bernoulli, $x_1 = a$ and $x_N = b$. This formula gives us the corrections due to approximating a series with an integral.

The energy of the ground state is given by

$$e = \sum_{j=1}^{N} k_j^2 \,, \tag{B.71}$$

where the k_j are the solution of the Bethe equation with the quantum numbers I_j symmetrically distributed around 0. As $N \to \infty$, the distance between consecutive k's is of the order of $1/L$. We define a function $\lambda(x)$ as

$$\lambda\left(\frac{I_j}{L}\right) = k_j \,. \tag{B.72}$$

Using (B.70) we have:

$$e = \int_{-N/2L}^{N/2L} \varepsilon_0(\lambda(x))\mathrm{d}x - \frac{1}{24L^2}\frac{\partial\varepsilon_0}{\partial x}\Big|_{n=-N/2L}^{n=N/2L} + \dots \tag{B.73}$$

where $\varepsilon(x) = x^2 - h$.

Similar corrections give

$$\begin{aligned} &\rho_L(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} \mathcal{K}(\lambda,\mu)\rho_L(\mu)\mathrm{d}\mu \\ &= \frac{1}{2\pi}\left\{k_0'(\lambda) + \frac{1}{24L^2\rho(q)}\left[\mathcal{K}'(\lambda,q) - \mathcal{K}'(\lambda,-q)\right]\right\} \,. \end{aligned} \tag{B.74}$$

Combining these equations we have

$$E = L\int_{-q}^{q} \varepsilon_0(\lambda)\rho_L(\lambda)\mathrm{d}\lambda - \frac{\pi}{6L}\frac{\varepsilon_0'(q)}{2\pi\rho(q)} + \dots \tag{B.75}$$

where, to zeroth order

$$q \equiv \lambda_N + \frac{1}{2L\rho(\lambda_N)} \,, \tag{B.76}$$

and therefore

$$E = L\int_{-q}^{q} \varepsilon_0(\lambda)\rho(\lambda)\mathrm{d}\lambda + \frac{1}{48\pi L\rho(q)}\int_{-q}^{q} \varepsilon(\lambda)\left[\mathcal{K}'(\lambda,q) - \mathcal{K}'(\lambda,-q)\right]\mathrm{d}\lambda - \frac{1}{6L}\frac{\varepsilon_0'(q)}{2\pi\rho(q)} + \dots \tag{B.77}$$

Using the definition of the dressed energy function in (2.87) we have

$$\varepsilon'(q) = \varepsilon'_0(q) + \frac{1}{2\pi}\int_{-q}^{q} \mathcal{K}'(q,\lambda)\varepsilon(\lambda)\mathrm{d}\lambda \tag{B.78}$$

and therefore

$$E = L\int_{-q}^{q} \varepsilon_0(\lambda)\rho(\lambda)\mathrm{d}\lambda - \frac{\pi}{6L}\, v_F + \dots \tag{B.79}$$

where the sound velocity was defined as

$$v_F = \left.\frac{\partial\varepsilon}{\partial k}\right|_{\lambda=q} = \frac{1}{2\pi\rho(q)}\left.\frac{\partial\varepsilon}{\partial\lambda}\right|_{\lambda=q} \tag{B.80}$$

as per (B.52). Thus, we have found that $c = 1$, which is what we would naively expect.

B.7 Conformal dimensions from finite size

To evaluate the conformal dimensions of the primary fields, we use (B.65,B.66) and we need the momentum and energy gap of the lowest excitations of the theory. There are three physical processes which generates these excitations

1. The particles at the Fermi points $\pm q$ can be boosted: the quantum numbers I_1 and I_N are changed by a finite amount N^- (at $-q$: $I_1 \to I_1 - N^-$) or N^+ (at q: $I_N \to I_N + N^+$).

2. A number of particles ΔN can be added (or subtracted) to (from) the system and place (removed) around the Fermi points.

3. Some particles (let say d) can backscatter, i.e. transfer from one fermi point to the other. This process is equivalent to shifting all quantum numbers $\{I_j\}$ by d, i.e. to a state with $\{I_j + d\}$.

It can be calculated using Bethe ansatz techniques (see [19]) that the energy and momentum of these elementary processes are

$$\delta E = \frac{2\pi v_F}{L}\left[\left(\frac{\Delta N}{2\mathcal{Z}}\right)^2 + (\mathcal{Z}d)^2 + N^+ + N^-\right], \tag{B.81}$$

$$\delta P = 2k_F\, d + \frac{2\pi}{L}\left(N^+ - N^- + \Delta N\, d\right), \tag{B.82}$$

where the Fermi momentum k_F is

$$k_F \equiv \pi n = \pi\frac{N}{L}, \tag{B.83}$$

and $\mathcal{Z} = Z(q) = Z(-q)$ is the value of the *dressed charge* function $Z(\lambda)$ at the Fermi boundary. This function is defined as the solution of the integral equation

$$Z(\lambda) - \frac{1}{2\pi}\int_{-q}^{q} \mathcal{K}(\lambda,\nu)Z(\nu)\mathrm{d}\nu = 1\,. \tag{B.84}$$

For the Lieb-Liniger model, the dressed charge is simply

$$Z(k) = 2\pi\rho(k)\,, \qquad \mathrm{Lieb-Liniger\ \ model}\,. \tag{B.85}$$

but is also satisfies

$$Z(\lambda) = -\frac{\partial \varepsilon(\lambda)}{\partial h} \ . \tag{B.86}$$

Using these results in (B.65, B.66) we can find the conformal dimensions as

$$\Delta^{\pm} = N^{\pm} + \frac{1}{2}\left(\frac{\Delta N}{2\mathcal{Z}} \pm \mathcal{Z}\ d\right)^2 \ . \tag{B.87}$$

In the conformal language, $N^{\pm}$ describes the level of the descendants and ΔN is a characteristic of the local field $\phi(x,t)$.

For instance, to obtain the field correlator $\langle \Psi(x,t)\Psi^{\dagger}(0,0)\rangle$ of the Lieb-Liniger, one sets $\Delta N = 1$. To obtain the leading term, we further set $N^{\pm} = d = 0$:

$$\langle \Psi(x,t)\Psi^{\dagger}(0,0)\rangle \simeq A|x + \mathrm{i}vt|^{-1/(2\mathcal{Z}^2)} \ . \tag{B.88}$$

Higher terms are obtained in a series (B.64)

$$\langle \Psi(x,t)\Psi^{\dagger}(0,0)\rangle = \sum_{d,N^{\pm}} \frac{A(d,N^{\pm})\mathrm{e}^{-2\mathrm{i}dk_F x}}{(x-\mathrm{i}vt)^{2\Delta^+}(x+\mathrm{i}vt)^{2\Delta^-}} \ , \tag{B.89}$$

where $\Delta^{\pm}$ are given by (B.87), with $\Delta N = 1$ and d and $N^{\pm}$ integers.

Similarly, for the density correlators we set $\Delta N = 0$ and we have

$$\langle \rho(x,t)\rho(0,0)\rangle = \langle \rho(0,0)\rangle^2 + \frac{A}{(x+\mathrm{i}vt)^2} + \frac{A}{(x-\mathrm{i}vt)^2} + A_3 \frac{\cos 2k_F x}{|x+\mathrm{i}vt|^{2\mathcal{Z}^2}} \ , \tag{B.90}$$

where the first term corresponds to $d = 0, N^+ = 0, N^- = 1$, the second to $d = 0, N^+ = 1, N^- = 0$, and the third to $d = \pm 1, N^{\pm} = 0$.

Note that these series are consistent with the Luttinger liquid universality and with the result of bosonization. In particular, from the asymptotic behavior of the density correlators one can extract the Luttinger parameter $K = \mathcal{Z}^2$, see (B.33). Thus, we have used the Bethe Ansatz to determine the two parameters of Bosonization, i.e. the sound velocity and the K.

In this chapter we have given a quick overview of the methods used to extract field theory parameters from Bethe Ansatz. While we think that the main ideas are fairly straightforward, the derivations of these parameters are quite long and cumbersome. In an effort to keep the ratio physical intuition/techinicalities as high as possible, we might have lacked of rigor and preciseness. The interested reader is recommended to find in [19] all the details and explanations we have skipped.

B.8 Bosonization of the XXZ model

The scaling theory of the XXZ model corresponds to a sine-Gordon theory [42]. To show this, let us write the spin model using spin-less fermions, using the familiar Jordan-Wigner transformation:

$$S^z_j =: \psi^{\dagger}_j \psi_j := \psi^{\dagger}_j \psi_j - \frac{1}{2} \ . \tag{B.91}$$

We expand the fermionic field around the fermi points, in terms of the chiral fields:

$$\psi_j \to \sqrt{a}\left[(-\mathrm{i})^j \phi_+(x) + \mathrm{i}^j \phi_-(x)\right] \ , \tag{B.92}$$

where a is the lattice spacing, $x = aj$ and we took the system at half filling ($k_F = \pi/2$), i.e. at zero magnetization.

Then, the spin density can be written in terms of chiral fields and it decomposes into the sum of a smooth and oscillating (staggered) component:

$$\begin{aligned} S^z_j &\to a S^z(x) \ , & \text{(B.93)}\\ S^z(x) &= \rho(x) + (-1)^j M(x) \ , & \text{(B.94)}\\ \rho(x) &= :\phi^\dagger_+(x)\phi_+(x): + :\phi^\dagger_-(x)\phi_-(x): \ , & \text{(B.95)}\\ M(x) &= :\phi^\dagger_+(x)\phi_-(x): + :\phi^\dagger_-(x)\phi_+(x): \ . & \text{(B.96)} \end{aligned}$$

In the Hamiltonian

$$\mathcal{H} = \sum_j \left[\frac{1}{2}\left(S^+_j S^-_{j+1} + S^-_j S^+_{j+1}\right) + \Delta S^z_j S^z_{j+1}\right] \ , \tag{B.97}$$

the first two terms are just the kinetic part of a free theory and in the linear spectrum approximation give

$$\mathcal{H}_0 = -\mathrm{i} v_F \int \mathrm{d}x \left[\phi^\dagger_+ \partial_x \phi_+ - \phi^\dagger_- \partial_x \phi_-\right] \ . \tag{B.98}$$

The interaction terms is

$$\mathcal{H}_{int} = v_f \Delta \int \mathrm{d}x \left[: \rho(x)\rho(x+a) : - M(x)M(x+a)\right] \ , \tag{B.99}$$

and $\mathcal{H} \simeq \mathcal{H}_0 + \mathcal{H}_{int}$.

The bosonization of the kinetic term clearly gives (B.24), i.e.

$$\mathcal{H}_0 = \frac{v_F}{2} \int \left[(\Pi(x))^2 + (\partial_x \phi(x))^2\right] \mathrm{d}x \ . \tag{B.100}$$

For the interaction terms we have

$$\begin{aligned} \rho(x) &= \frac{1}{\sqrt{\pi}} \partial_x \phi(x) \ , & \text{(B.101)}\\ M(x) &\simeq -\frac{1}{\pi a} : \sin \sqrt{4\pi}\phi(x) : \ , & \text{(B.102)}\\ \lim_{a\to 0} M(x)M(x+a) &= -\frac{1}{(\pi a)^2} \cos \sqrt{16\pi}\phi(x) - \frac{1}{\pi} \left(\partial_x \phi\right)^2 + \text{const} \ . & \text{(B.103)} \end{aligned}$$

The cosine term originates from the sa-called Umklapp processes $\phi^\dagger_+(x)\phi^\dagger_+(x+a)\phi_-(x+a)\phi_-(x)+$ *h.c.* where two particles are removed from one Fermi point and added at the other. This scattering event corresponds to a transfer of momentum $4k_f$ and it is possible only when the Fermi point is such to allow the lattice to recoil and absorb this excess momentum, as it happens for $k_F = \pi/2$.

Finally, putting these terms together, the continuous version of the XXZ Hamiltonian reads

$$\mathcal{H} = \int \mathrm{d}x \left\{ \frac{v_F}{2} \left[\Pi^2 + \left(1 + 4\frac{\Delta}{\phi} \right) (\partial_x \phi)^2 \right] + \frac{v_F \Delta}{(\pi a)^2} \ :\cos\sqrt{16\pi}\phi: \right\} . \tag{B.104}$$

One can normalize the fields to transfer the interaction contributions into the effective compactification radius of the bosons. Studying the conformal dimension of the cosine terms, it is easy to see that it is irrelevant for $|\Delta| < 1$. At $\Delta = 1$ a chiral symmetry gets broken by the Umklapp term and the cosine term turns relevant and opens a gap. This is consistent with what we know about the XXZ model. In the paramagnetic phase, the cosine terms can be thrown away and the effective Hamiltonian is (B.26).

The Fermi velocity can be calculated in the standard way using the Bethe Ansatz formulae and it gives

$$v_F = \frac{\pi \sin\mu}{2\mu} , \tag{B.105}$$

where $\Delta = \cos\mu$. The Luttinger parameter can be extracted directly from the fractional charge and is

$$K = \frac{\pi}{2(\pi - \mu)} . \tag{B.106}$$

This can be compared with the naive (perturbative) result one can derive from (B.104) at small Δ

$$K \simeq 1 - \frac{2\Delta}{\pi} + \mathrm{O}(\Delta^2) . \tag{B.107}$$

Notice that, since $\Delta = 0$ corresponds to free fermions ($K = 1$), $\Delta > 0$ gives attracting fermions ($K < 1$) and $\Delta < 0$ repulsive ($K > 1$). The Heisenberg chain ($\Delta = 0$) has

$$K = \frac{1}{2} , \qquad v_F = \frac{\pi}{2} . \tag{B.108}$$

Finally, one can show that the correct bosonization expression of the spin operators are [42]

$$S^z(x) = \sqrt{\frac{K}{2\pi}} \partial_x \phi(x) - \lambda_z (-1)^j \sin\sqrt{4\pi K}\phi(x) , \tag{B.109}$$

$$S^{\pm}(x) = \lambda_x (-1)^j \mathrm{e}^{\pm \mathrm{i}\sqrt{\frac{\pi}{K}}\theta(x)} . \tag{B.110}$$

Bibliography

[1] E. Barouch, B.M. McCoy, and M. Dresden, Phys. Rev. **A 2**, 1075 (1970).
Statistical Mechanics of the XY Model. I.
E. Barouch, and B.M. McCoy, Phys. Rev. **A 3**, 786 (1971).
Statistical Mechanics of the XY Model. II. Spin-Correlation Functions.
E. Barouch, and B.M. McCoy, Phys. Rev. **A 3**, 2137 (1971).
Statistical Mechanics of the XY Model. III.
B.M. McCoy, E. Barouch, and D.B. Abraham, Phys. Rev. **A 4**, 2331 (1971).
Statistical Mechanics of the XY Model. IV. Time-Dependent Spin-Correlation Functions.

[2] M. Shiroishi, M. Takahashi, and Y. Nishiyama, J. Phys. Soc. Jap. **70**, 3535 (2001). *Emptiness Formation Probability for the One-Dimensional Isotropic XY Model.*

[3] A.G. Abanov, and F. Franchini, Phis. Lett. **A 316**, 342 (2003).
Emptiness formation probability for the anisotropic XY spin chain in a magnetic field.
F. Franchini, and A.G. Abanov, J. Phys. **A 38**, 5069 (2005).
Asymptotics of Toeplitz Determinants and the Emptiness Formation Probability for the XY Spin Chain.

[4] B.-Q. Jin, and V.E. Korepin, J. Stat Phys. **116**, 79-95 (2004).
Entanglement, Toeplitz determinants and Fisher-Hartwig conjecture.
A.R. Its, B.-Q. Jin, and V.E. Korepin, J. Phys. **A 38**, 2975 (2005).
Entanglement in XY Spin Chain.
I. Peschel, J. Stat. Mech.: Theor. Exp. P12005 (2004).
On the entanglement entropy for a XY spin chain.
A.R. Its, B.-Q. Jin, and V.E. Korepin, arXiv:quant-ph/0606178.
Entropy of XY Spin Chain and Block Toeplitz Determinants.

[5] F. Franchini, A.R. Its, B.-Q. Jin, V.E. Korepin, arXiv:quant-ph/0609098 (submitted to JPA).
Ellipses of Constant Entropy in the XY Spin Chain.

[6] K.L. Mehta, *Matrix Theory - Selected Topics and Useful Results.*, Les Editions de Physique, Les Ulis Cedex, France, 1977.

[7] T. Ehrhardt, Operator Th: Advances and App. **124**, 217-241 (2001).
A status report on the asymptotic behavior of Toeplitz determinants with Fisher-Hartwig singularities.

[8] E. Lieb, T. Schultz, and D. Mattis, Ann. of Phys. **16**, 407-466 (1961).
Two Soluble Models of an Antiferromagnetic Chain.

[9] S. Katsura, Phys. Rev. **127**, 1508-1518 (1962).
Statistical Mechanic of the Anisotropic Linear Heisenberg Model.
Errata: Phys. Rev. **129**, 2835 (1963).

[10] Th. Niemeijer, Physica **36**, 377-419 (1967).
Some exact calculations on a chain of spins 1/2.
Niemeijer, Physica **39**, 313-326 (1968).
Some exact calculations on a chain of spins 1/2. *II.*

[11] T.D. Schultz, D.C. Mattis, and E.H. Lieb, Rev. Mod. Phys. **36**, 856 (1964).
Two-Dimensional Ising Model as a Soluble Problem of Many Fermions.

[12] J. Kurmann, H. Thomas, and G. Müller, Physica **A 112**, 235 (1982).
Antiferromagnetic long-range order in the anisotropic quantum spin chain.
G. Müller, and R.E. Shrock, Phys. Rev. **B 32**, 5845 (1985).
Implications of direct-product ground states in the one-dimensional quantum XYZ and XY spin chains.

[13] P. Pfeuty, Ann. Phys. **57**, 79-90 (1970).
The One-Dimensional Ising Model with a Transverse Field.

[14] C. Domb, Phil. Mag. Suppl. **9**, 151 (1960).

[15] T.T. Wu, Phys. Rev. **149**, 380 (1966).
Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. I.
B.M. McCoy, and T.T. Wu, Phys. Rev. **155**, 438 (1967).
Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. II.
H. Cheng, and T.T. Wu, Phys. Rev. **164**, 719 (1967).
it Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. III.
B.M. McCoy, and T.T. Wu, Phys. Rev. **162**, 436 (1967).
Theory of Toeplitz Determinants and the Spin Correlations of the Two-Dimensional Ising Model. IV.

[16] H.A. Kramers, and G.H. Wannier, Phys. Rev. **60**, 252 (1941).
Statistics of the two-dimensional ferromagnet.

[17] E.H. Lieb, and W. Liniger, Phys. Rev. **130**, 16005 (1963).
E.H. Lieb, Phys. Rev. **130**, 1616 (1963).

[18] M. Jimbo, *Yang-Baxter Equation in Integrable Systems.*, World Scientific Publishing, 1990.

[19] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, 1997.

[20] T. Kinoshita, T. Wenger, and D.S. Weiss, Nature **440**, 900 (2006).
A quantum Newton's cradle.

[21] F. Calogero, J. Math. Phys. **10**, 2191 (1969).
Solution of a three-body problem in one dimension.
F. Calogero, J. Math. Phys. **12**, 419 (1971). "Erratum", ibidem **37**, 3646 (1996).
Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials.
J. Moser, Adv. Math. **16**, 197 (1975).
Three integrable Hamiltonian systems connected with isospectral deformations.

[22] B. Sutherland, *Beautiful Models - 70 Years of Exactly Solved Quantum Many-Body Problems*, World Scientific.

[23] M. Kac, and H. Pollard, Can. J. Math. **2**, 375 (1950).
The distribution of the maximum of partial sums of independent random variables.
V. Hutson, Proc. Cambridge Phil. Soc. **59**, 211 (1963).
The circular plate condenser at small separations.

[24] P. Calabrese, and J.-S. Caux, Phys. Rev. Lett. **98**, 150403 (2007).
Correlation functions of the one-dimensional attractive Bose gas.
P. Calabrese, and J.-S. Caux, J. Stat. Mech. P08032 (2007).
Dynamics of the attractive 1D Bose gas: analytical treatment from integrability.

[25] M. Ishikawa, and H. Takayama, J. Phys. Soc. Japan **49**, 1242 (1980).
Solitons in a One-Dimensional Bose System with the Repulsive Delta-Function Interaction.

[26] C.N. Yang, and C.P. Yang, J. Math. Phys. **10**, 1115 (1969).
Thermodynamics of a one-dimensional system of bosons with repulsive delta-function interactions.

[27] M. Karbach, and G. Müller, Computers in Physics **11**, 36 (1997). (arXiv:cond-mat/9809162)
Introduction to the Bethe ansatz I.
M. Karbach, K. Hu, and G. Muller, Computers in Physics **12**, 565 (1998). (arXiv:cond-mat/9809163)
Introduction to the Bethe ansatz II.
M. Karbach, K. Hu, and G. Muller, arXiv:cond-mat/0008018.
Introduction to the Bethe ansatz III.

[28] P. Lu, G. Muller, and M. Karbach, arXiv:0909.2728.
Quasiparticles in the XXZ model.

[29] H.A. Bethe, Z. Physik **71**, 205 (1931).

[30] R. Orbach, Phys. Rev. **112**, 309 (1958).
L.R. Walker, Phys. Rev. **116**, 1089 (1959).
C.N. Yang, and C.P. Yang, Phys. Rev **147**, 303 (1966).
C.N. Yang, and C.P. Yang, Phys. Rev **150**, 321 (1966).
C.N. Yang, and C.P. Yang, Phys. Rev **150**, 327 (1966).
C.N. Yang, and C.P. Yang, Phys. Rev **151**, 258 (1966).

[31] R.J. Baxter, *"Exactly Solved Models in Statistical Mechanics*, Dover Publications.

[32] M. L. Mehta, *"Random Matrices"*, Academic Press.

[33] C. Gmez, M. Ruiz-Altaba, G. Sierra, *"Quantum Groups in Two-Dimensional Physics"*, Cambridge University Press, 2005.
S. Majid, *"Foundations of Quantum Group Theory"*, Cambridge University Press, 2000.

[34] I.I. Hirschman, Jr., Amer. J. Math. **88**, 577 (1966).
"The Strong Szegö Limit Theorem for Toeplitz Determinants".

[35] B.M. McCoy, and T.T. Wu, *The Two Dimensional Ising Model.*, Harvard University Press, Cambridge, USA, 1973.

[36] M.E. Fisher and R.E. Hartwig, Adv. Chem. Phys. **15**, 333 (1968).
"Toeplitz determinants, some applications, theorems and conjectures".

[37] H. Widom, Amer. J. Math. **95**, 333 (1973).
"Toeplitz Determinants with Singular Generating Functions".

[38] T. Ehrhardt and B. Silbermann, J. Funct. Anal. **148**, 229-256 (1997).
"Toeplitz Determinants with One Fisher-Hartwig Singularity".

[39] E.L. Basor, and K.E. Morrison, Lin. Alg. App. **202**, 129 (1994).
"The Fisher-Hartwig Conjecture and Toeplitz Eigenvalues".
E.L. Basor, and C.A. Tracy, Phys. **A 177**, 167 (1991).
"The Fisher-Hartwig conjecture and generalizations".

[40] H. Widom, Ind. Univ. Math. J. **21**, 277 (1971).
"The Strong Szegö Limit Theorem for Circular Arcs".

[41] T. Giamarchi, *"Quantum Physics in One Dimension"*, Oxford University Press (2004).

[42] A.O. Gogolin, A.A. Nersesyan, A.M. Tsvelik, *"Bosonization and Strongly Correlated Systems"*, Cambridge University Press (1998).