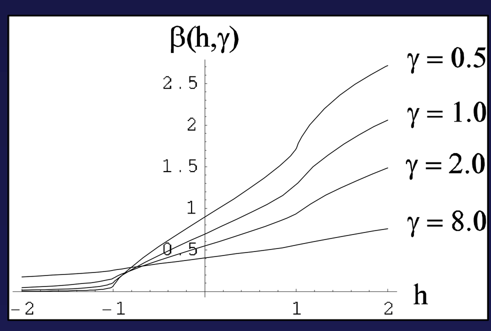
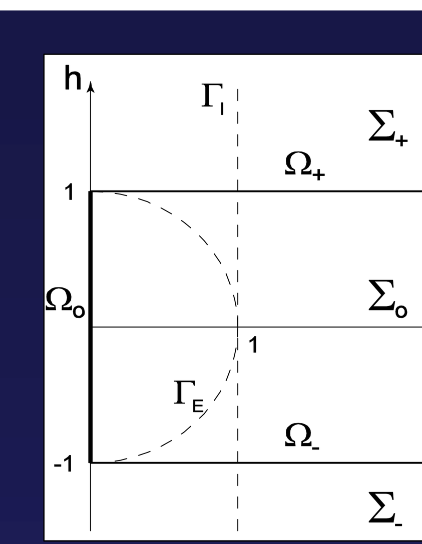


Plot of the function β(h,γ) for certain values of g (β(h,γ) is defined for γ≠0)

Emptiness Formation Probability for the Anisotropic XY Spin Chain in Transverse Magnetic Field

Fabio Franchini & Alexander Abanov



Phase Diagram of the XY Model (only γ>0 shown)

Introduction

- For 1-Dim. theories, the ground state can be found using Bethe Ansatz, but no general way is known to calculate the correlators of the theory, which stands as an open problem
- Korepin et Al. introduced a determinant represantation for correlators in terms of a generating functional: from this analysis, a special correlator known as Emptiness Formation Probability (EFP) is introduced to be the simplest correlator
- EFP it is the probability that a system doesn't present any particle in a region of a certain lenght
- In one dimensional spin models $\mathbf{H} = \sum_{i,j} J_{ij}\vec{\sigma}_i \times \vec{\sigma}_j + \mathbf{h}\sum_i \sigma^z_i$ we are interested in the Probability of Formation of a Ferromagnetic String (PFFS) of lenght n:

$$P(n) = \left\langle \prod_{i=1}^{n} \frac{1-\sigma^z_i}{2} \right\rangle = \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H} \prod_{i=1}^{n} \frac{1-\sigma^z_i}{2} \right)$$

- In the mapping to spinless fermion, PFFS becomes EFP: the EFP $P(n)$ measures the probability of formation of a string of **n** aligned spins
- Considerable efforts has been devoted to the study of this n-points correlator for the XXZ Spin Chain aiming to completely solve the model

The Critical XXZ Model

$$H = J\sum_i \left(\sigma^x_i\sigma^x_{i+1} + \sigma^y_i\sigma^y_{i+1} + \Delta\sigma^z_i\sigma^z_{i+1}\right)$$

$$\left(-1<\Delta<1,\ \nu = \frac{1}{\pi}\cos^{-1}(\Delta)\right)$$

$T \neq 0$: $\quad P(n) \underset{n\to\infty}{\sim} e^{-\gamma n}$, (γ: density of free energy)
(Korepin et al. 1993)

$T = 0$: $\quad P(n) \sim A n^{-\gamma} C^{-n^2}$, $\quad \begin{cases} C = \dfrac{\Gamma^2(1/4)}{\pi\sqrt{2\pi}} \exp\left\{ -\int_0^\infty \dfrac{\sinh^2(t\nu)e^{-t}}{\cosh(2t\nu)\sinh(t)}\dfrac{dt}{t} \right\} \\ \gamma = \dfrac{1}{12} + \dfrac{\nu^2}{3(1-\nu)} \end{cases}$
(Lukyanov et al. 2002)

- Lukyanov's result shows that P(n) is Gaussian for the critical phase, but it comes without a derivation and doesn't explain what is the physical picture
- Abanov and Korepin tackled the problem using a bosonization technique and derived the Gaussian form from first principles: they described a crossover (at finite temperature T) from a Gaussian behavior to an exponential one as **n** increases so that this crossover happens at infinity at zero temperature, but the procedure failed in providing quantitative results
- The behavior in the critical regime ($-1<\Delta<1$) is Gaussian, but what happens in the non-critical "Ising Regime" ($\Delta>1$)? Is the Gaussian behavior general for critical models?
- We study a simple model in order to understand better the meaning of the EFP

The Anisotropic XY Model

$$H = \sum_i \left[\left(\frac{1+\gamma}{2}\right)\sigma^x_i\sigma^x_{i+1} + \left(\frac{1-\gamma}{2}\right)\sigma^y_i\sigma^y_{i+1} \right] - h\sum_i \sigma^z_i$$

- A Jordan-Wigner transformation takes the spin Hamiltonian to the spinless fermions hamiltonian

$$\sigma^{\pm}_j = \frac{1}{2}\left(\sigma^x_j \pm i\sigma^y_j\right) \qquad \begin{cases} \sigma^z_j = 2\psi^\dagger_j\psi_j - 1 \\ \sigma^+_j = \psi^\dagger_j e^{i\pi\sum_{l<j}\psi^\dagger_l\psi_l} \end{cases}$$

- Switching to Fourier components we get:

$$H = \sum_q 2\left(\cos q - h\right)\psi^\dagger_q\psi_q + i\gamma\sin q\left(\psi^\dagger_q\psi^\dagger_{-q} - \psi_{-q}\psi_q\right)$$

- A Bogoliubov transformation diagonalize the Hamiltonian:

$$c_q = \cos\frac{\vartheta_q}{2}\psi_q + i\sin\frac{\vartheta_q}{2}\psi^\dagger_{-q}$$

$$H = \sum_q \varepsilon_q\left(c^\dagger_q c_q - 1/2\right)$$

$$\varepsilon_q = \sqrt{(\cos q - h)^2 + \gamma^2\sin^2 q}$$

Phase Diagram:
- 3 non-critical regions ($\Sigma_0,\Sigma_\pm$)
- 3 critical phases:
  Ω_0: Isotropic XY
  $\Omega_\pm$: Critical magnetic field

- The Emptiness Formation Probability can be written as:

$$P(n) = \left\langle \prod_{i=1}^{n} \psi_i\psi^\dagger_i \right\rangle = \mathrm{Pf}(M) = \sqrt{\det(M)}\,; \qquad M = \begin{pmatrix} \langle\psi_j\psi_k\rangle & \langle\psi_j\psi^\dagger_k\rangle \\ -\langle\psi_j\psi^\dagger_k\rangle & \langle\psi^\dagger_j\psi^\dagger_k\rangle \end{pmatrix}$$

- We can rotate the "2n × 2n" skew-symmetric matrix M:

$$M' = UMU^\dagger = \begin{pmatrix} 0 & S_n \\ -S^\dagger_n & 0 \end{pmatrix}; \qquad \det(M) = \det(M') = \left|\det(S_n)\right|^2$$

- Thus, EFP is (exactly):

$$P(n) = \left|\det(S_n)\right|$$

$$S_n = \left[\frac{1}{2}\int_{-\pi}^{\pi}\left(1 + \frac{\cos q - h + i\gamma\sin q}{\sqrt{(\cos q - h)^2 + \gamma^2\sin^2 q}}\right) e^{iq(j-k)}\frac{dq}{2\pi} \right]_{j,k=1}^{n}$$

- In this work, we evaluate the asymptotic behavior of this matrix as $n\to\infty$, in the different regions of the phase diagram of the XY Model

Toeplitz Matrices and the Generalized Fisher-Hartwig Conjecture

- Matrices like S_n are called Toeplitz, because their elements depend only on the difference of the indices and they are determined by a periodic generating function $\sigma(q)=\sigma(q+2\pi)$

$$\left(S_n\right)_{jk} = \int_{-\pi}^{\pi}\sigma(q)e^{iq(j-k)}\frac{dq}{2\pi}$$

- Analytical results are know in the literature regarding the asymptotic behavior of their determinant
- These behaviors strongly depend on the "singularities" of the generating function, so we parameterize the generating function to singled them out as:

$$\sigma(q) = \tau^i(q)\prod_{r=1}^{R} e^{-\kappa^i_r\left(\pi-(q-q_r)\,\mathrm{mod}\,2\pi\right)}\left(2-2\cos(q-q_r)\right)^{\lambda^i_r}$$

where $\tau^i(q)$ is a smooth, non-zero function with winding number 0

- The index i labels the different possible parameterizations
- The asymptotic behavior of the determinant is:

$$\det(S_n) \underset{n\to\infty}{\sim} \sum_{i\in T} E[\tau^i,\kappa^i,\lambda^i]\,n^{\Omega}e^{-b[\tau^i]n}; \quad b[\tau^i] = -\int_{-\pi}^{\pi}\mathrm{Log}\left(\tau^i(q)\right)\frac{dq}{2\pi}$$

$$T = \left\{ i;\ \sum\left((\lambda^i_r)^2 - (\kappa^i_r)^2\right) = \max_j \sum\left((\lambda^j_r)^2 - (\kappa^j_r)^2\right) = \Omega \right\}$$

- In our case the generating function is:

$$\sigma(q) = \frac{1}{2}\left(1 + \frac{\cos q - h + i\gamma\sin q}{\sqrt{(\cos q - h)^2 + \gamma^2\sin^2 q}}\right)$$

and the parameterizations (and asymptotic behaviors of the determinant) depend on the region of the phase diagram one considers

Non Critical Regions

Σ_- : σ(q) has no singularities
Σ_0 : σ(q) vanishes and presents a phase jump at $q = \pi$
Σ_+ : σ(q) vanishes and has phase jumps at $q = 0, \pi$

Using the FH Conjecture, the EFP is found to be:

$$P(n) \sim E(h,\gamma)e^{-\beta(h,\gamma)n}; \qquad \beta(h,\gamma) = -\int_{-\pi}^{\pi}\mathrm{Log}\left(\sigma(q)\right)\frac{dq}{2\pi}$$

For $h \geq 1$, there is Z_2 symmetry breaking and we have to use the generalized FH Conjecture to find:

$$P(n) \sim E(h,\gamma)\left[1 + (-1)^n A(h,\gamma)\right]e^{-\beta(h,\gamma)n}$$

which is in very good agreement with numerical calculations

Critical Phase: Ω_0

- For $\gamma = 0$, σ(q) is supported only for $-\cos^{-1}(h) < q < \cos^{-1}(h)$
- This case has already been studied by Shiroshi et al. (2001) and in the '70s in the context of Unitary Random Matrices
- The Fisher-Hartwig conjecture and its generalization don't apply
- We use Widom's Theorem and the behavior is Gaussian times a power law pre-factor:

$$P(n) \sim 2^{\frac{5}{24}}e^{3\zeta'(-1)}(1-h)^{-\frac{1}{8}}n^{-\frac{1}{4}}\left(\frac{1+h}{2}\right)^{n^2/2}$$

Critical Phases: $\Omega_\pm$

Ω_- : σ(q) develops a phase jump at $q = \pi$
Ω_+ : σ(q) vanishes at $q = \pi$ and presents two phase jumps at $q = 0, \pi$

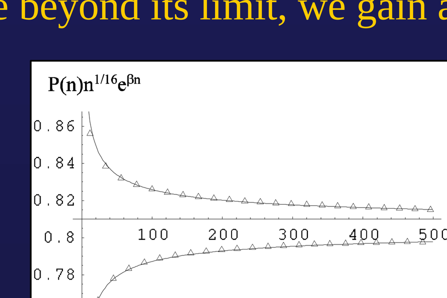
Using the FH Conjecture the result would be:

$$P(n) \sim E(\gamma)n^{-\frac{1}{16}}e^{-\beta(\pm 1,\gamma)n}$$

but, by pushing the generalized FH Conjecture beyond its limit, we gain a better agreement with the numerics:

$$\Omega_- : \quad P(n) \sim E(\gamma)n^{-\frac{1}{16}}\left[1 + A(\gamma)n^{-\frac{1}{2}}\right]e^{-\beta(-1,\gamma)n}$$

$$\Omega_+ : \quad P(n) \sim E(\gamma)n^{-\frac{1}{16}}\left[1 + (-1)^n A(\gamma)n^{-\frac{1}{2}}\right]e^{-\beta(1,\gamma)n}$$



Numerical vs. Analytical results at γ=1, h=1

Conclusions and Discussions

Region	Critical	γ, h	P(n)	Zeros of σ(q)	Phase Jumps of σ(q)
Ω_0	Yes	$\gamma = 0$, $-1<h<1$	$E\,n^{-1/4}e^{-\alpha n^2}$	$q \notin (-k_f,k_f)$	none
Σ_-	No	$h < -1$	$E\,e^{-\beta n}$	none	none
Ω_-	Yes	$h = -1$	$E\,n^{-1/16}[1+An^{-1/2}]\,e^{-\beta n}$	none	π
Σ_0	No	$-1<h<1$	$E\,e^{-\beta n}$	π	π
Ω_+	Yes	$h = 1$	$E\,n^{-1/16}[1+(-1)^n An^{-1/2}]\,e^{-\beta n}$	π	$0, \pi$
Σ_+	No	$h > 1$	$E\,[1+(-1)^n A]\,e^{-\beta n}$	$0, \pi$	$0, \pi$

- The power law contributions in $\Omega_\pm$ remind us of the scaling dimension of the square root of σ^x and σ^y
- Common feature for critical theories seems to be the presence of an universal power-law contribution (from which operators is it coming?)
- Gaussian behavior seems to be connected to the length of the "Fermi Surface"
- The bosonization argument for Gaussian behavior fails at the critical magnetization, because the EFP has non-local terms when expressed in terms of the bose field, due to the quasi-particle transformation
- This is the first physically-motivated example of application of the generalized Fisher-Hartwig Conjecture
- We suggested a way to find subleading behavior to the asymptotics using the generalized FH Conjecture

Perspective for the future

- Understand the meaning of the different behaviors and identify the signature of criticality
- Complete the phase diagram of the XXZ Model (Δ>1)
- Understand better EFP

New Methods

- Bosonization approach
- New Hydrodynamic Model for Bosonization
- Bethe Ansatz WaveFunctions (?)

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