New approaches to non-equilibrium and random systems:
KPZ integrability, universality, applications and experiments

KITP – Santa Barbara

Spontaneous Ergodicity Loss in Invariant Matrix Models

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Support by:

arXiv:1412.6523

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My Claim

- Eigenvector and eigenvalue statistics <u>are linked</u>:

The U(N) symmetry matrix models are endowed with can be spontaneously broken

- Similar ideas introduced before:
  - *Moshe, Neuberger, Shapiro;* **PRL '94**
  - *Canali, Kravtsov;* **PRE '95**
  - *Bonnet, David, Eynard;* **JPA '00** …
- Peculiar SSB: thermodynamic limit also takes symmetry's rank to infinity

Outline

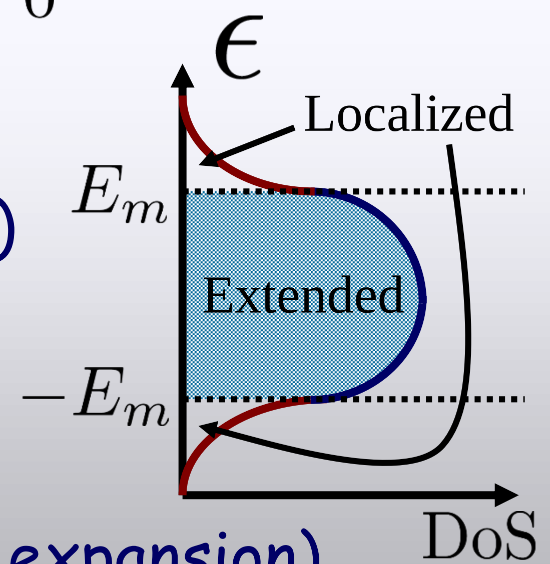
1. Physical motivation: Anderson Model
2. Spontaneous Symmetry Breaking:
   - Geometrical argument
   - Symmetry Breaking term
   - Numerical finite size detection
4. Weakly Confined Matrix Models
   - Spectral Statistics (known)
   - Energy landscape (new)
5. Conclusions & Outlook

PART 1

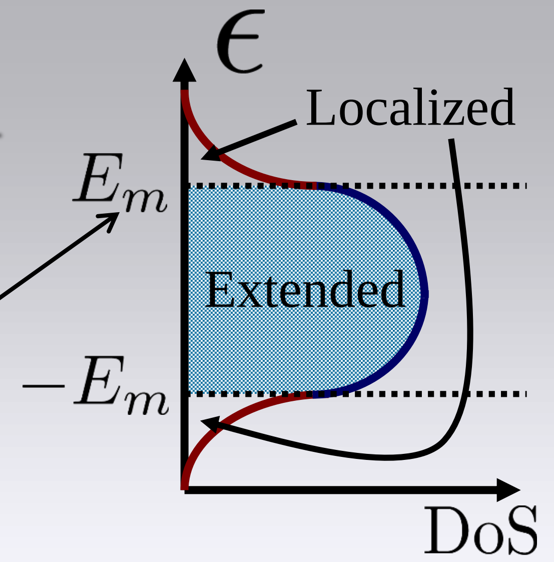
Introduction
on
Localization
due to Disorder

Disorder & Localization

- Anderson Model: $\mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[c_l^\dagger c_j + c_j^\dagger c_l \right]$ (Anderson. '58)
- Tight-binding model (nearest neighbor hopping)
- Random on-site energies: $\epsilon_j \in [-W, +W]$
- 1 (& 2) Dimensions: localized for any $W \neq 0$
- Higher D:
  - Small W : conducting (weak loc., Random Matrices)
  - $W > W_c$: insulating (localized at low energies)
- Hard problem (uncontrolled perturbation expansion)



Metal/Insulator Transition



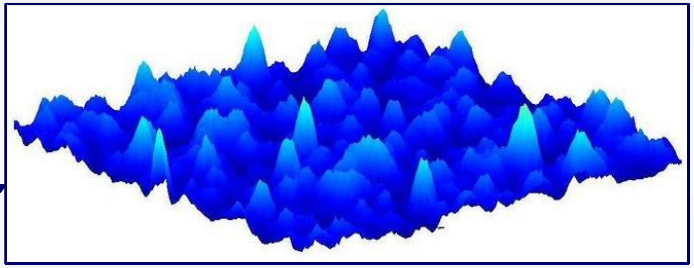
$$\mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[c_l^\dagger c_j + c_j^\dagger c_l \right]$$

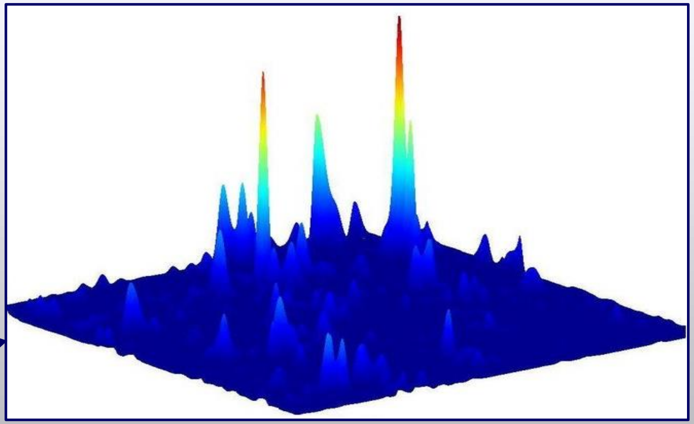
$$\epsilon_j \in [-W, +W]$$

$$W < W_c$$

$$D \geq 3$$

- Mobility Edge separates extended from localized states
- Transition as Intermediate state (multifractal)

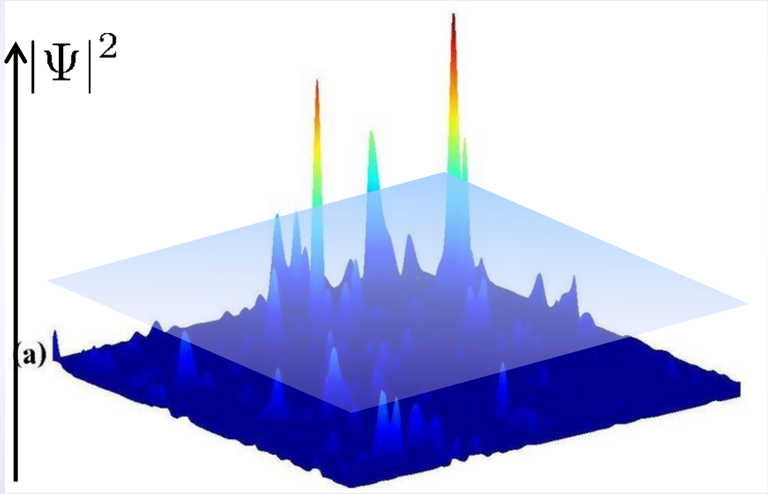


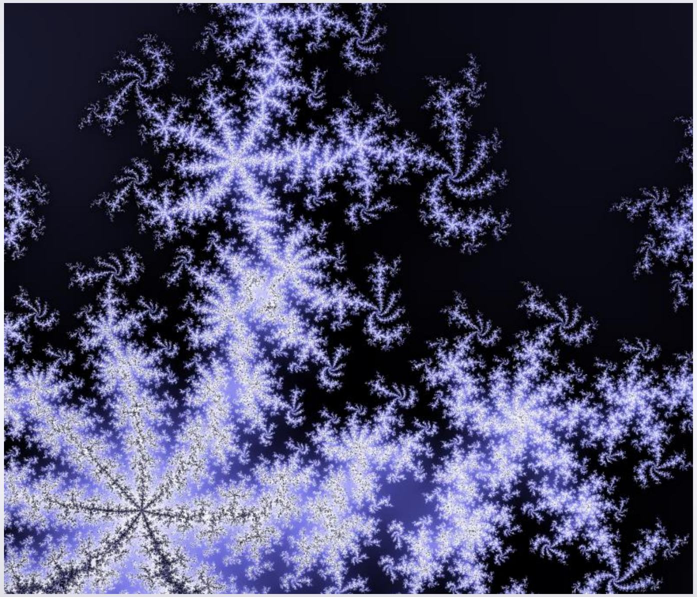


Van Tiggelen group (PRL 2009)

Multifractality

- At each height $|\Psi|^2 = \alpha$, the wavefunction's amplitude draws a "curve" with a different fractal dimension $f(\alpha)$





- Behavior at mobility edge known in "perturbative" regimes

  $\Rightarrow$ long-standing open problem

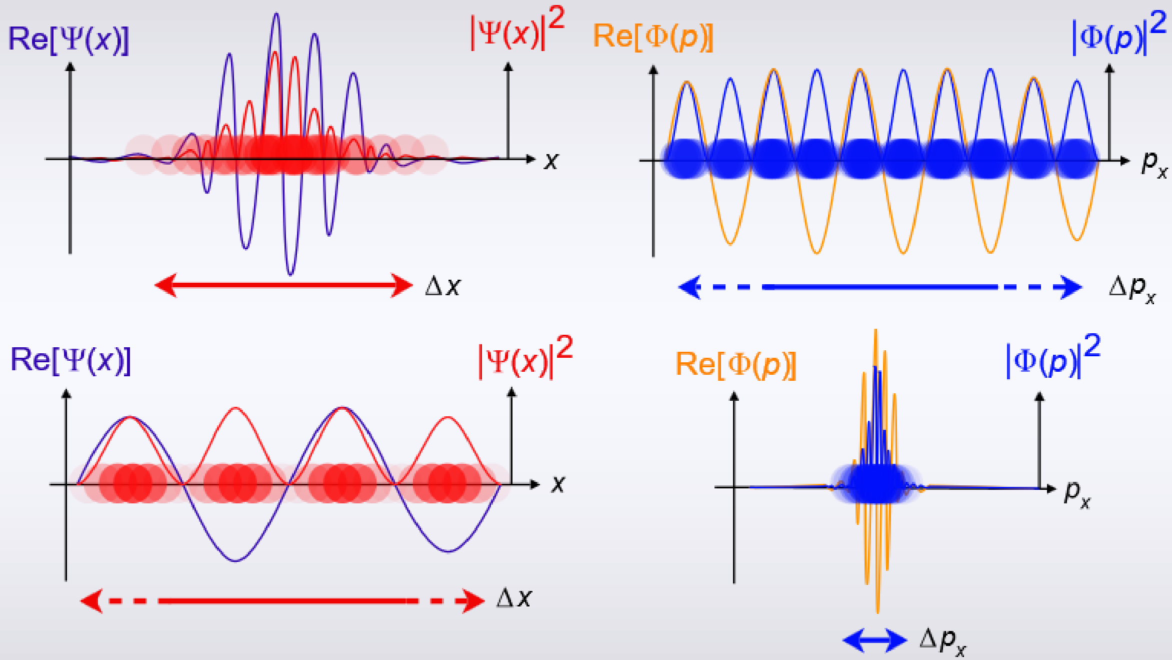
$$\boxed{\begin{array}{l} d = 2 + \epsilon \\ d \to \infty \end{array}}$$

Motivating Question for this work

- Localization/extendedness of wavefunctions is a basis-dependent property

Localized

Extended



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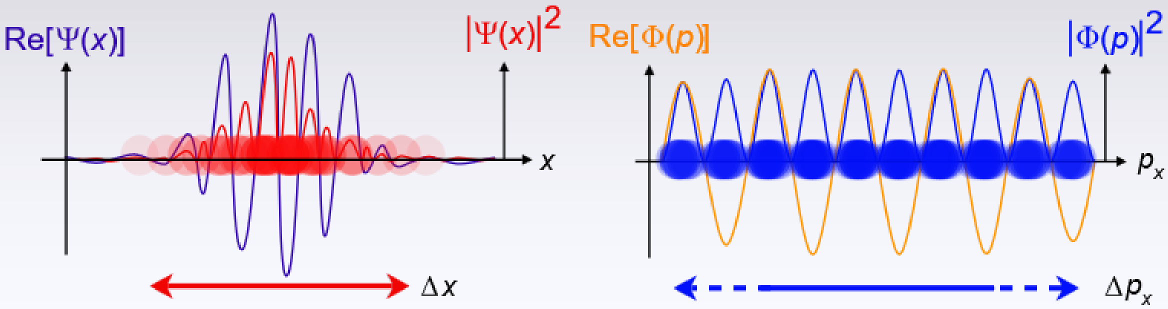
Localized

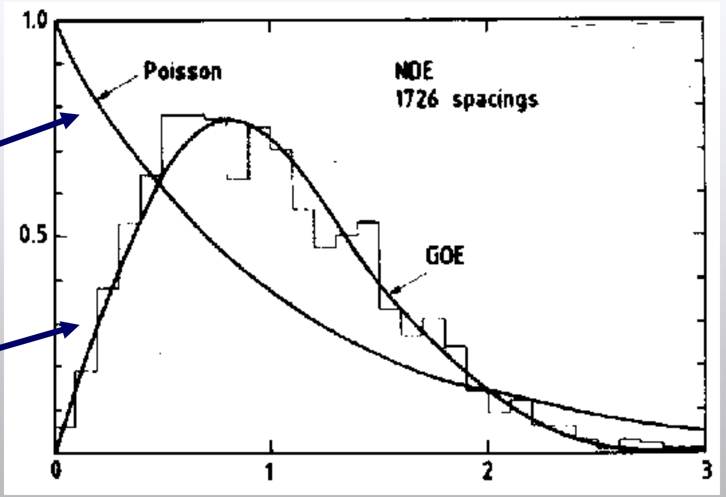


Extended

- However, level spacing statistics characterizes insulating/conducting systems

Poisson ↔ Localized

Wigner Dyson ↔ Extended



My Approach

- Spectral signature hints toward localization as basis independent property
- Random Matrix Theory ideal to test this hypothesis
- However: lack of analytical tools to study eigenstate behavior in RMT

(Allez & Bouchaud ’11-’12; Allez & Guionnet ‘13)

- Need to develop new machinery: abstract setting to study relation between eigenvalues and eigenvectors
- Seeking for fundamental structure of insulators

Part 2

Spontaneous Breaking of Rotational Symmetry in Invariant Multi-Cuts Matrix Model

Random Matrix Theory

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-W(\mathbf{M})}$$

matrix-valued action

- Take W(M) real: statistical model
- Consider M as a Hamiltonian:
  - M: Hermitian Matrix
  - Matrix entries randomly from a distribution
  - Interaction between every degree of freedom (no preconceived notion of locality)
- Common wisdom: RMT describes delocalized systems

Invariant Ensembles

- Action invariant under rotations: $\boxed{W(\mathbf{M}) = \mathrm{Tr} V(\mathbf{M})}$

- Switch to eigenvalues/eigenvectors: $\mathbf{M} = \mathbf{U}^{\dagger} \mathbf{\Lambda} \mathbf{U}$

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} = \int \mathcal{D}\mathbf{U} \int d^N \lambda \, \Delta^2 \left(\{\lambda\} \right) e^{-\sum_j V(\lambda_j)}$$

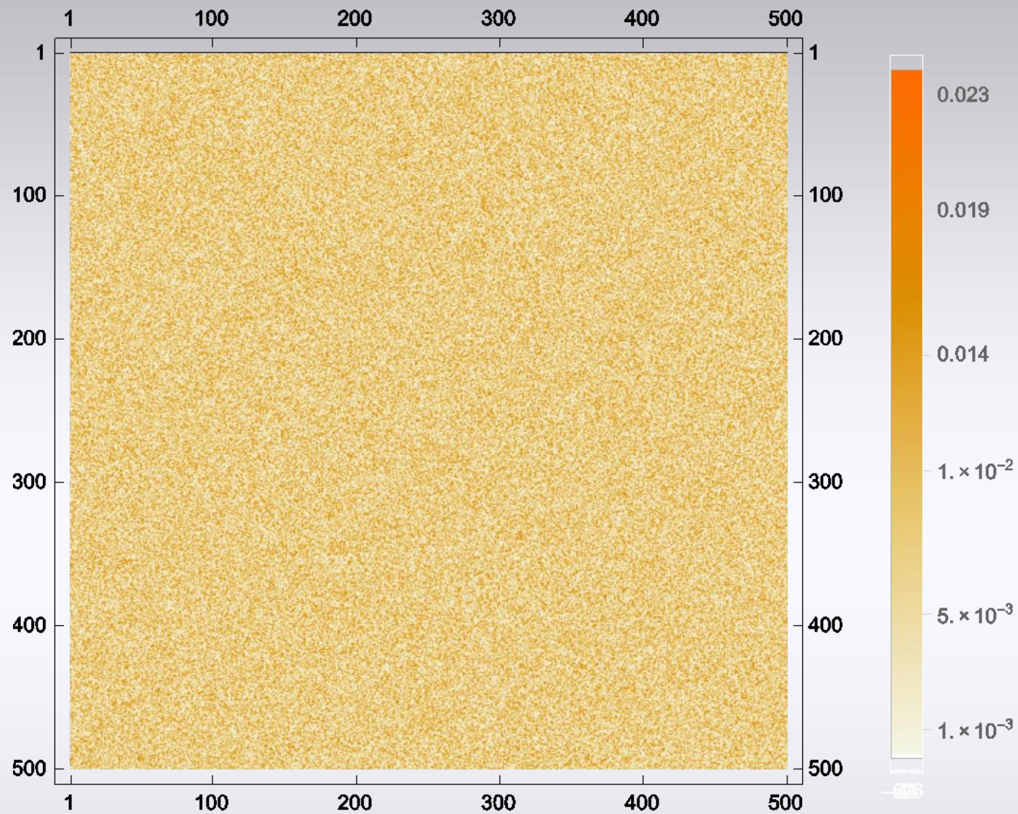
Eigenvectors uniformly distributed over the N-dimensional sphere (Hilbert space): independent from V(λ)

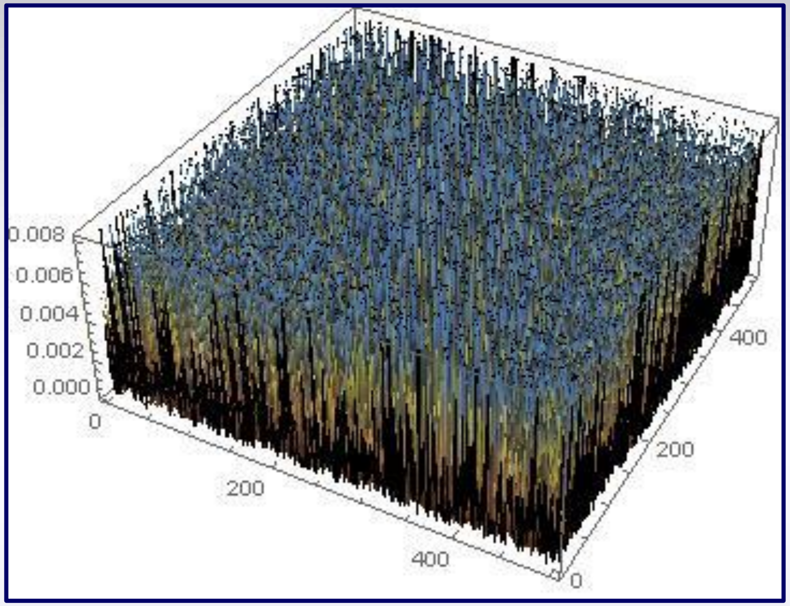
Van der Monde Determinant:

$$\Delta\left(\{\lambda\}\right) = \prod_{j>l}^{N} (\lambda_j - \lambda_l)$$

(from Jacobian)

The Haar Measure



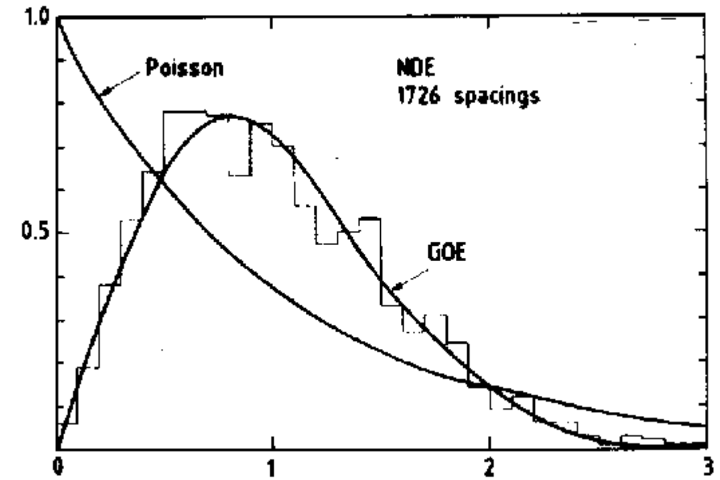


- Entries of Unitary matrix follow the Porther-Thomas Distribution: $\mathcal{P}\left(\left|\tilde{U}_{ij}\right|^2\right) = N \exp\left[-N\left|\tilde{U}_{ij}\right|^2\right]$

Wigner-Dyson Universality

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda\, \Delta^2(\{\lambda\})\, e^{-\sum_j V(\lambda_j)}$$

- Jacobian introduces interaction between eigenvalues
- Coulomb gas picture: $\mathcal{L} = -2\sum_{j>l} \ln|\lambda_j - \lambda_l| + \sum_j V(\lambda_j)$
- Eigenvalues as 1-D particles with
  - ➢ logarithmic repulsion
  - ➢ external confining potential V(λ)
- Universal level spacing distribution
  (distance between n.n. eigenvalues)



$$P(s) \propto s^\beta e^{-A(\beta)s^2}$$

- Valid for any polynomial V(λ)

Invariant Ensembles

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda\, \Delta^\beta\left(\{\lambda\}\right) e^{-\sum_j V(\lambda_j)}$$

- Wigner Dyson distribution & level repulsion:
  Jacobian introduces interaction between eigenvalues
- Extended states/conducting phases:
  uniform distribution means eigenvectors typically have all non-vanishing entries
- Eigenvalues interact through their eigenvectors:

$$\boxed{\textbf{WD} \Leftrightarrow \textbf{extended states}}$$

Non-Invariant Ensembles

- To study localization problems, introduce non-invariant random matrix ensembles (Random Banded Matrices)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\sum_{j,l} A_{jl} |M_{jl}|^2}$$

$$\langle M_{nm}^2 \rangle = A_{nn}^{-1}$$

$$\boxed{A_{nm} \propto (n-m)^{2\alpha}}$$

$\alpha > 1$ $\rightarrow$ Localized states (Poisson statistics)

(Mirlin et al. '96; …)

$\alpha = 1$ $\rightarrow$ Multi-Fractal states (Critical Statistics)

(Evers & Mirlin, '00; …)

- Limited analytical tools (SUSY, cluster expansion…)

Loophole: Spontaneous Breaking of Rotational Invariance

- Invariant models are endowed with superior (non-perturbative) analytical techniques
- Can invariant models spontaneously break rotational symmetry and realize non-trivial eigenvector statistics like non-invariant ensembles? If so,
  $\Rightarrow$ Invariant machinery for localization problems!
- Recall a ferromagnet:
  - From partition function, rotational invariance
    $\rightarrow$ no spontaneous magnetization
  - Need, e.g., a symmetry breaking term

Multi-Cut Solutions

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda\, e^{-\sum_j V(\lambda_j) + 2\sum_{j>l} \ln|\lambda_j - \lambda_l|}$$

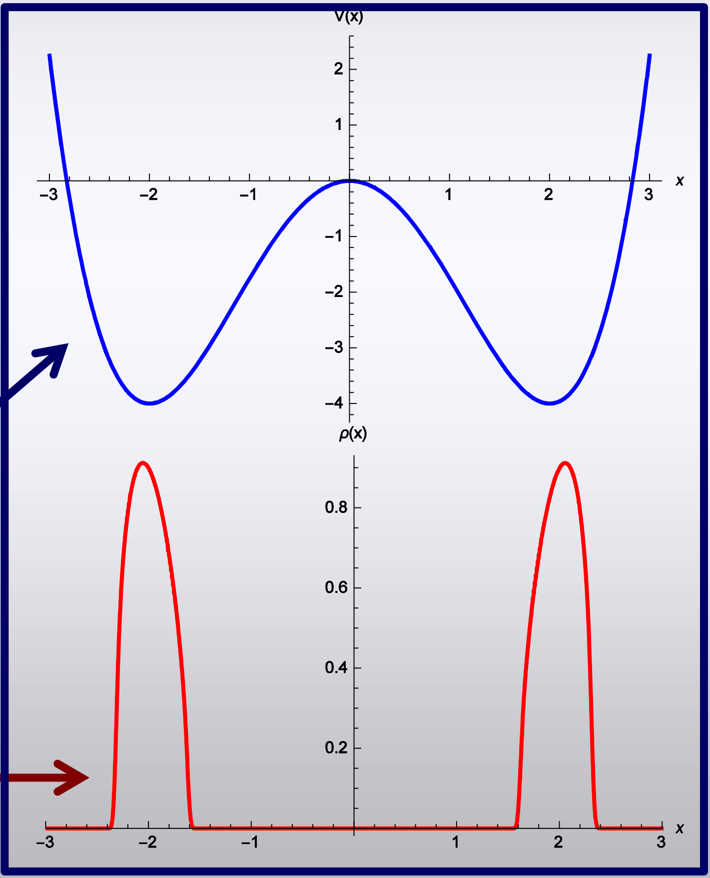
- V(x) with several, well separated, minima

  ⇒ disconnected support for eigenvalues (multi-cuts)

- For example: double well potential

$$V_{2\mathrm{W}}(x) = \frac{1}{4}x^4 - \frac{t}{2}x^2$$

(2-cuts for $t > 2$)



Level Density: $\rho(x) = \sum_j^N \delta(x - \lambda_j)$

Understanding the matrix SSB

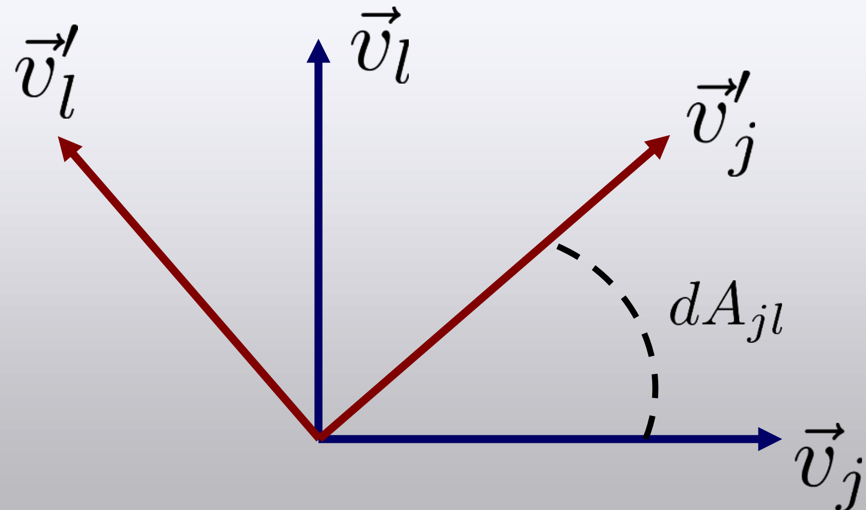
- Geometrical argument: line element

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2 \, |dA_{jl}|^2$$

$$d\mathbf{A} \equiv \mathbf{U}^\dagger d\mathbf{U}$$

$$\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$

- Angular degrees of freedom live on spheres of radii $r_{jl} = |\lambda_j - \lambda_l|$



Understanding the matrix SSB

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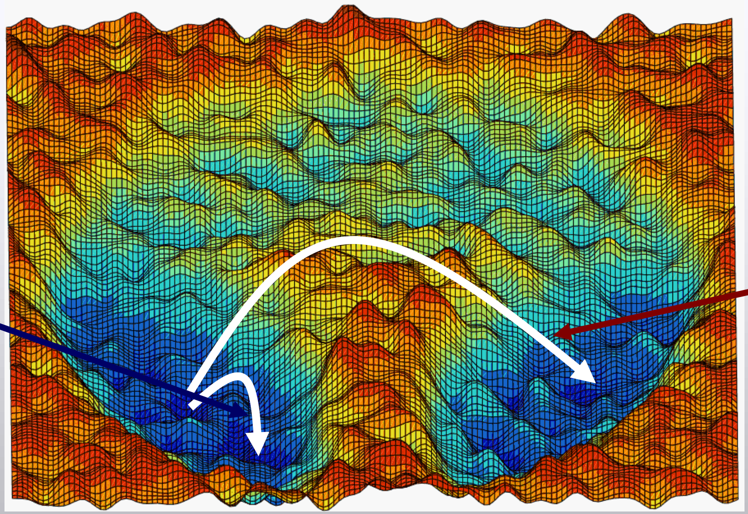
- Angular degrees of freedom live on spheres of radii $r_{jl} = |\lambda_j - \lambda_l|$

Small r_{jl}

Small ds

Small $d\lambda_j$

if overshoot



http://www.math.nus.edu.sg/~matrw/string/fig/rough_ener.gif

Large r_{jl}

Large ds

Large $d\lambda_j$

if overshoot

Understanding the matrix SSB

- Geometrical argument: line element

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2 \, |dA_{jl}|^2$$

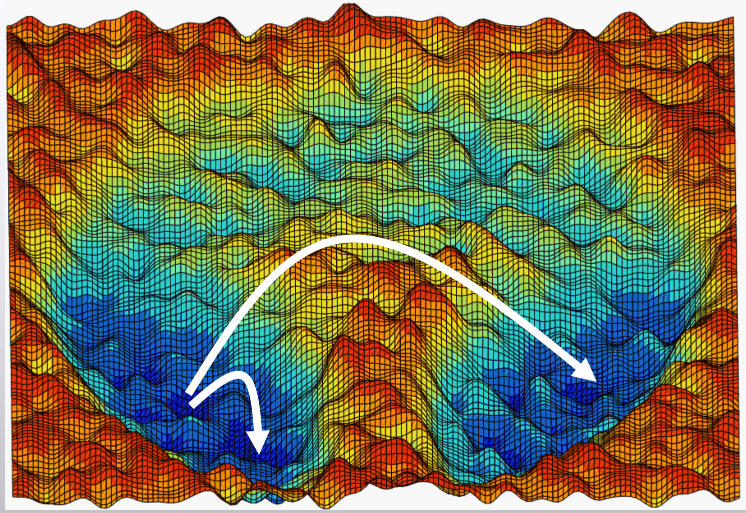
$$d\mathbf{A} \equiv \mathbf{U}^\dagger d\mathbf{U}$$

$$\mathbf{M} = \mathbf{U}^\dagger \boldsymbol{\Lambda} \mathbf{U}$$

- For large $r_{jl} = |\lambda_j - \lambda_l|$, rotations generate large ds

  $\Rightarrow$ move to far point in configuration space

- Entropic (fine tuning) origin of SSB (same as level repulsion)



Understanding the matrix SSB

- Geometrical argument: line element

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2\,|dA_{jl}|^2$$

  - Two lengths scales:
    - Eigenvalues spacing: $\mathcal{O}\left(\frac{1}{N}\right)$
    - Support of distribution: $\mathcal{O}\,(1)$

- Multi-cut solutions:

> Eigenvectors of eigenvalues
>
> in different cuts cannot mix

Generating a Random Matrix

- Gaussian Models: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}\mathbf{M}^2} = \int \prod dM_{jl} e^{-\sum_{jl} M_{jl}^2}$

  $\rightarrow$ each matrix entries sampled independently

- One-Cut Models: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}V(\mathbf{M})} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}\sum_k g_k \mathbf{M}^k}$

  $\rightarrow$ entries correlated: generated as perturbation of Gaussian case in a Metropolis scheme

- Multi-Cut Solutions: Gaussian case unstable

  $\rightarrow$ start from initial seed and evolve it to equilibrium

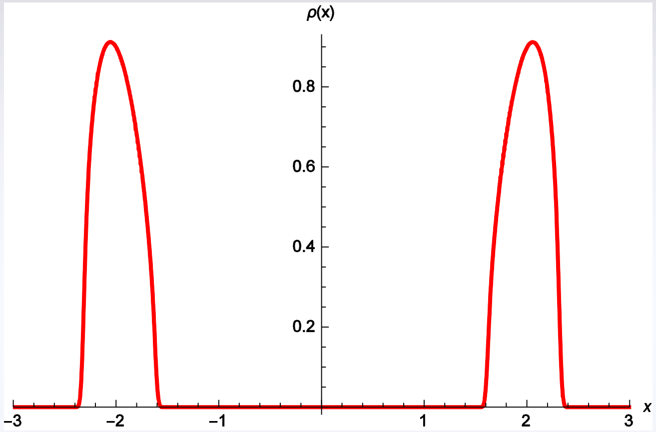
  $\rightarrow$ SSB: final configuration has memory of eigenvectors of initial seed

Multi-Cuts SSB

- Level repulsion resolves degeneracy:

  $\Rightarrow$ each of the n cuts contains m_j eigenvalues

- Gap between cuts breaks rotational invariance: $\boxed{U(N) \overset{N\to\infty}{\to} \prod_{j=1}^{n} U(m_j)}$



Double well

$$U(N) \overset{N\to\infty}{\to} U(N/2) \times U(N/2)$$

(assume N even)

- Three Arguments:
  - ★ Brownian motion;
  - ★ Numerical finite size analysis;
  - ★ Symmetry Breaking Term

F.F. arXiv:1412.6523

Symmetry Breaking Term

- To detect SSB introduce symmetry breaking term
- Most natural one is $\mathrm{Tr}\left([\mathbf{M},\mathbf{S}]\right)^2$, but too hard to handle

$\mathbf{S}$: given Hermitian Matrix
Favors alignment of eigenvectors

- We introduce: $W(J) = \ln \int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}$

J : source strength

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

Absolute value can be removed by sorting eigenvalues in increasing order

Symmetry Breaking: Double Well

$$W(J) = \ln \int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}$$

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

- Double well: $U(N) \overset{N\to\infty}{\to} U(N/2) \times U(N/2)$

  (assume N even)

- Take $\mathbf{S}$ with 2 sets of N/2-degenerate eigenvalues: $t = \pm 1$ to induce correct symmetry breaking

- Use (regularized) Itzykson-Zuber formula: (Itzykson & Zuber, '80)

$$\int d\mathbf{U} e^{JN\mathrm{Tr}\mathbf{M}\,\mathbf{S}} \propto \frac{1}{\Delta(\{\lambda\})} \sum_{\{\alpha\}\cup\{\alpha'\}=\{\lambda\}}' e^{-JN\sum_j(\alpha_j - \alpha'_j)} \Delta(\{\alpha\})\Delta(\{\alpha'\})$$

Sum over ways to partition eigenvalues of M according to degeneracies of S

Symmetry Breaking Term

$$\boxed{W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\, \mathbf{T} - \mathbf{M}\, \mathbf{S})|}}$$

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

- Calculate (dis-)order parameter:
  - $\frac{d}{dJ} \lim_{N\to\infty} W(J)\Big|_{J=0} = 0$ ⟶ Symmetry is Broken!
  - Finite N: $\frac{dW(J)}{dJ}\Big|_{J=0} = \langle \mathrm{Tr}\,(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S}) \rangle \neq 0$

Eigenvectors misaligned

Symmetry Breaking Term

$$\boxed{W(J) = \ln \int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}}$$

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  - $\frac{d}{dJ} \lim_{N\to\infty} W(J)\Big|_{J=0} = 0$
  - Finite N: $\frac{dW(J)}{dJ}\Big|_{J=0} \neq 0$

Instantons:
- Pairs of eigenvalues tunneling between cuts
- Restore broken symmetries

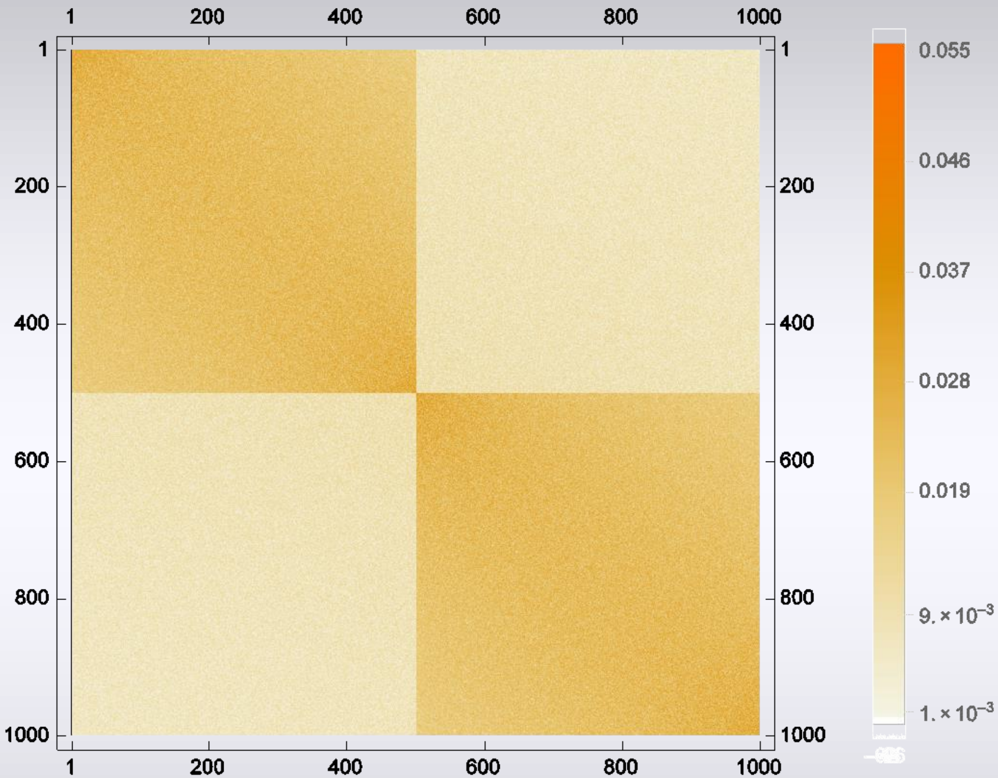
$$\int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|} \propto \mathcal{Z}_0 + \mathcal{Z}_1 \left(e^{-2JN|\lambda_j - \lambda'_l|} \right) + \ldots$$

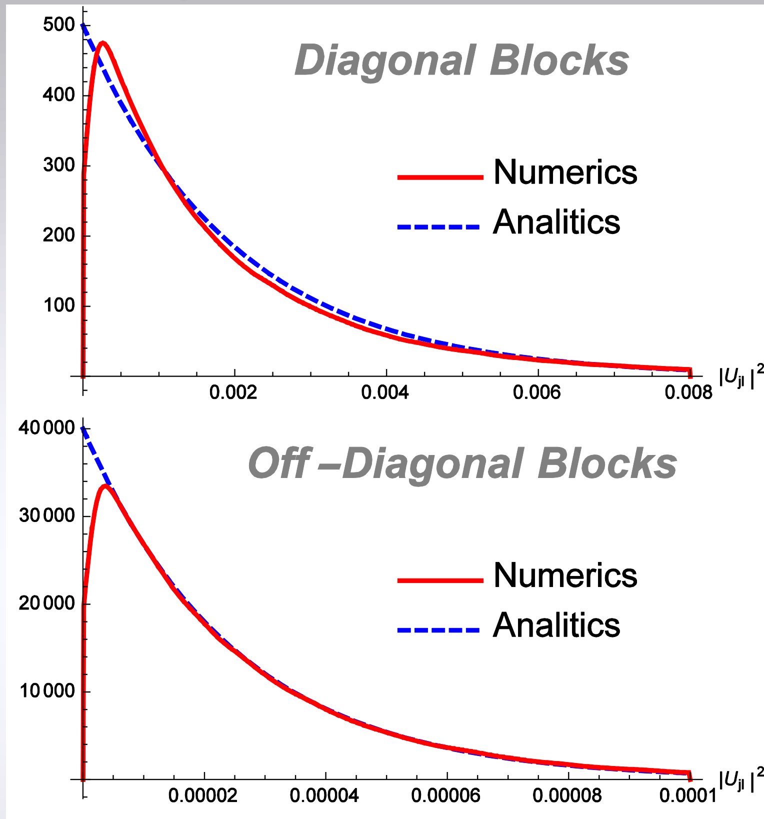
Finite Size Analysis

- Without preferred, reference basis; localization means rigidity of eigenvectors under perturbations
- Take double well matrix model: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N \mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$
- Generate a representative matrix: $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$
- Apply perturbation $\mathbf{\Delta} \mathrm{M}$ (sparse Gaussian Matrix)
- Find eigenvectors of perturbed matrix: $\mathbf{M} + \mathbf{\Delta M} = \mathbf{U}'^\dagger \mathbf{\Lambda}' \mathbf{U}'$
- Consider eigenvectors of perturbed matrix in original eigenvector basis (rotation due to perturbation): $\tilde{\mathbf{U}} = \mathbf{U}' \mathbf{U}^\dagger$

Finite Size Analysis

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N\mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$$



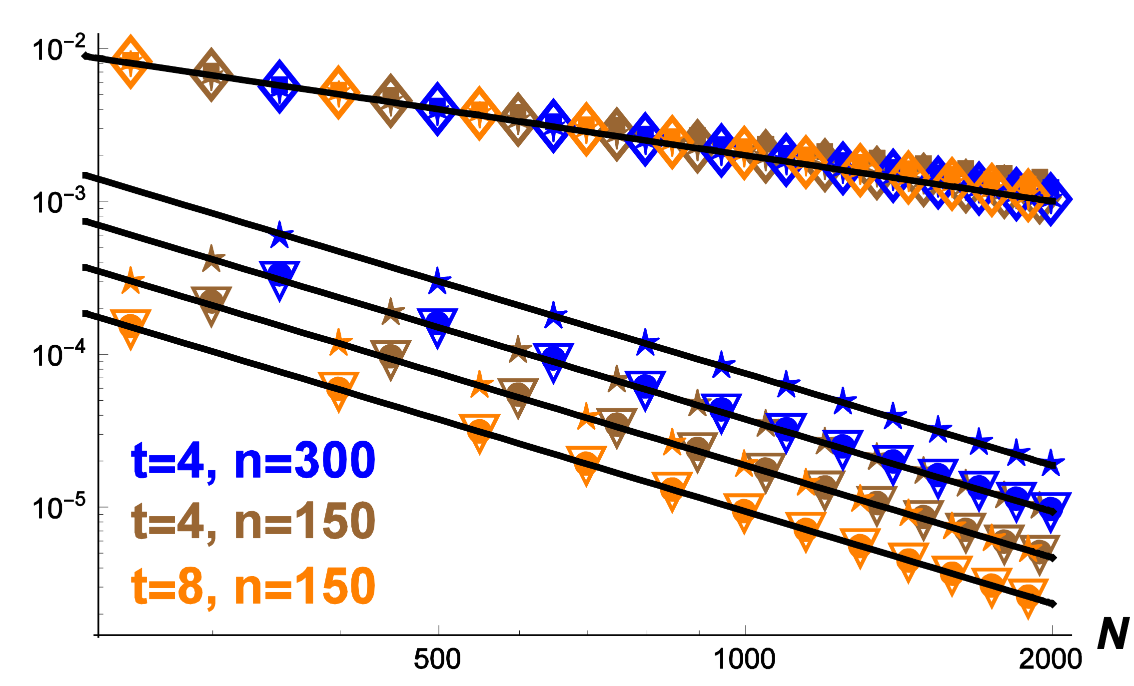


t=4, N=1000, sparse matrix with n=200 non zero elements, drawn from Gaussian with zero mean and variance N)

$$\mathcal{P}\left(\left|\tilde{U}_{ij}\right|^2\right) = \chi \exp\left[-\chi \left|\tilde{U}_{ij}\right|^2\right]$$

$$\chi_{\mathrm{D}} = \frac{N}{2} \qquad \chi_{\mathrm{OD}} = \frac{2tN^2}{n}$$

Finite Size Analysis



$$O_{jl} = \sum_m^N \left|\tilde{U}_{mj}\right|^2 \left|\tilde{U}_{ml}\right|^2$$

Overlaps between eigenstates

$$\langle |O_{jl}| \rangle_{\mathrm{D}} = \langle \left|\tilde{U}_{jl}\right| \rangle_{\mathrm{D}} = \langle \left|\Delta\tilde{U}_{jl}\right| \rangle_{\mathrm{D}} = \frac{1}{\chi_{\mathrm{D}}} = \frac{2}{N}$$

$$\langle |O_{jl}| \rangle_{\mathrm{OD}} = 2\langle \left|\tilde{U}_{jl}\right| \rangle_{\mathrm{OD}} = 2\langle \left|\Delta\tilde{U}_{jl}\right| \rangle_{\mathrm{OD}} = \frac{2}{\chi_{\mathrm{OD}}} = \frac{n}{tN^2}$$

Off-diagonal blocks suppressed as 1/N compared to diagonal ones

Onset of localizations!

Multi-Cuts SSB: Conclusions

- Gap in the eigenvalue distribution
  → Deviation from WD universality
  → Spontaneous breaking of rotational symmetry
  → Eigenvectors localized in patch of Hilbert space spanned by the other eigenvectors in the same cut
- Broken symmetries restored by instantons
- Not "localization" in usual meaning, but loss of ergodicity
- Proof that eigenvectors of invariant matrix models encode non-trivial information!
- Relevant for MBL?

Part 3

Weakly Confined Matrix Models & The Metal/Insulator Transition

Weakly Confined Invariant Models

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} \;,\; V(\lambda) \overset{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa} \ln^2 |\lambda|$$

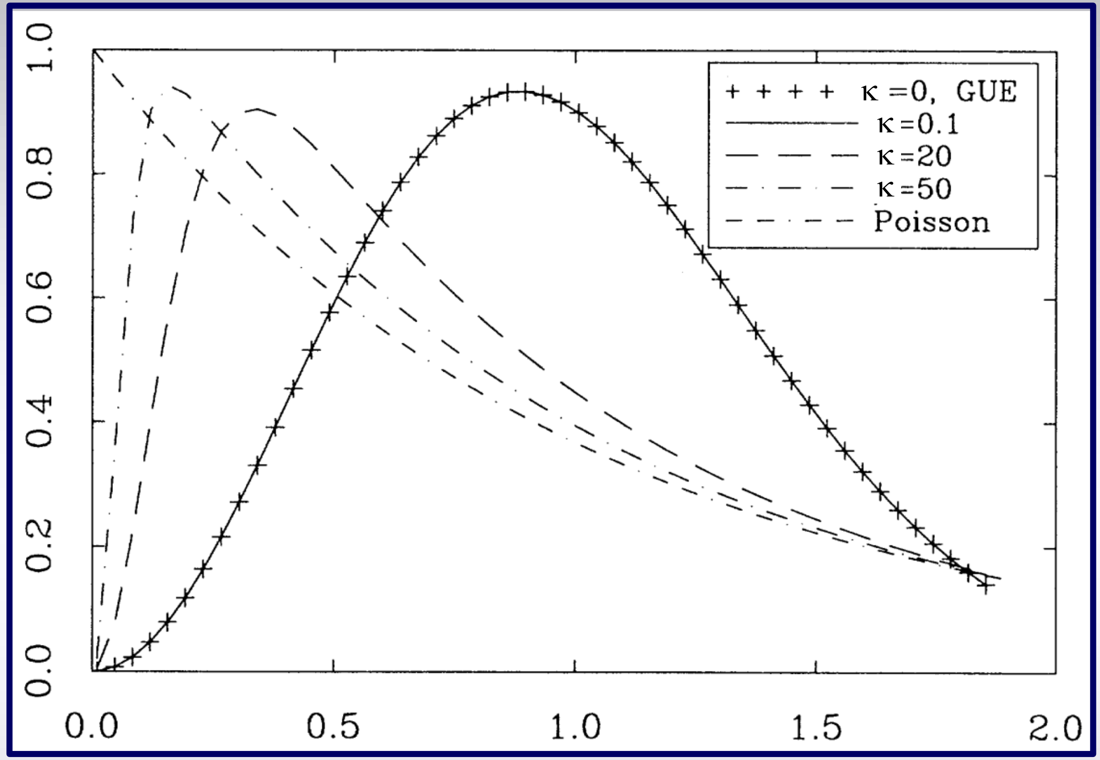
- Family of "Muttalib" ensemble (Muttalib et al. '93)
- Soft confinement sets them apart from usual polynomial potentials
  → WD universality does not apply
  → Indeterminate moment problem
- Solvable through orthogonal polynomials: q-deformed Hermite/Laguerre Polynomials
  (Muttalib et al. '93; Tierz'04)
- Arise in localization limit of Chern-Simons/ABJM: $\kappa \propto \frac{i}{g_s}$
  (Marino '02; Kapustin et a. '10; …)

Weakly Confined Matrix Models & their applica-tions

Weakly Confined Matrix Models

$$V(\lambda) \overset{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa} \ln^2 |\lambda|$$

- Intermediate level spacing statistics
- Same eigenvalue correlations as Critical Random Banded Matrices



(Muttalib et al. ‘93)

- Critical level statistics signals fractal eigenstates?
- Critical Spontaneous Breaking of U(N) Invariance?

(Canali, Kravtsov, ‘95)

WCMM and Anderson Transition

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} \, , \; V(\lambda) \overset{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa} \ln^2 |\lambda|$$

- Same spectral signatures as c-RBM :

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\sum_{j,l} A_{jl} \, |M_{jl}|^2}, \; A_{nm} = 1 + \frac{(n-m)^2}{B^2}$$

- C-RBM toy model for the Anderson Transition: reproduce multifractal spectrum (analytical for $d = 2 + \epsilon$)

$$B \sim \frac{1}{\kappa} \sim d, \mathrm{Connectivity}$$

- Conjecture: SSB of WCMM to calculate analytically multifractal spectrum of Anderson MIT

New: WCMM Energy Landscape

- Take exactly log-normal ensemble (positive eigenvalues)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int_{\lambda>0} d^N\lambda\, \Delta(\{\lambda\})\, e^{-\frac{1}{2\kappa}\sum_j \ln^2 \lambda_j}$$

- Exponential mapping: $\boxed{\lambda_j = e^{\kappa x_j}}$

$$\mathcal{Z} \propto \int d^N x_j \prod_{n<m} \left(e^{\kappa x_n} - e^{\kappa x_m}\right)^2\; e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2x_l\right]}$$

- Each term of the Van der Monde shifts the equilibrium of the parabolic potential: different effective potential felt by each eigenvalue for each term for the VdM

F.F. arXiv:1503.03341

New: Energy Landscape

- Partition function has a large number of saddle points!

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int_{\lambda>0} d^N\lambda \, \Delta(\{\lambda\}) \, e^{-\frac{1}{2\kappa}\sum_j \ln^2 \lambda_j}$$

$$\boxed{q = e^{-\kappa}}$$

$$\propto e^{\frac{\kappa}{6}N(4N^2-1)} \, (2\pi\kappa)^{N/2} \, N! \prod_{n=1}^{N-1} (1-q^n)^{N-n}$$

- Inclusion of all saddle contributions correctly reproduces orthogonal polynomials result (Tierz ‘04)

F.F. arXiv:1503.03341

New: Energy Landscape

- Partition function has a large number of saddle points!

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int_{\lambda>0} d^N\lambda \, \Delta(\{\lambda\}) \, e^{-\frac{1}{2\kappa}\sum_j \ln^2 \lambda_j}$$

$$\boxed{q = e^{-\kappa}}$$

$$\propto e^{\frac{\kappa}{6}N(4N^2-1)} (2\pi\kappa)^{N/2} N! \prod_{n=1}^{N-1} (1-q^n)^{N-n}$$

- Each term of the expansion of the product
  - ➢ Corresponds to a different saddle (equilibrium conf.)
  - ➢ Has the same leading energy (differ in powers of q)
  - ➢ q^j fugacity of the instantons
  - ➢ Saddles in 1-to-1 correspondence with ways of breaking U(N) in its components!

F.F. arXiv:1503.03341

Landscape Interpretation

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int_{\lambda>0} d^N\lambda\, \Delta(\{\lambda\})\, e^{-\frac{1}{2\kappa}\sum_j \ln^2 \lambda_j}$$

$$\boxed{q = e^{-\kappa}}$$

$$\propto e^{\frac{\kappa}{6}N(4N^2-1)}\,(2\pi\kappa)^{N/2}\, N! \prod_{n=1}^{N-1} (1-q^n)^{N-n}$$

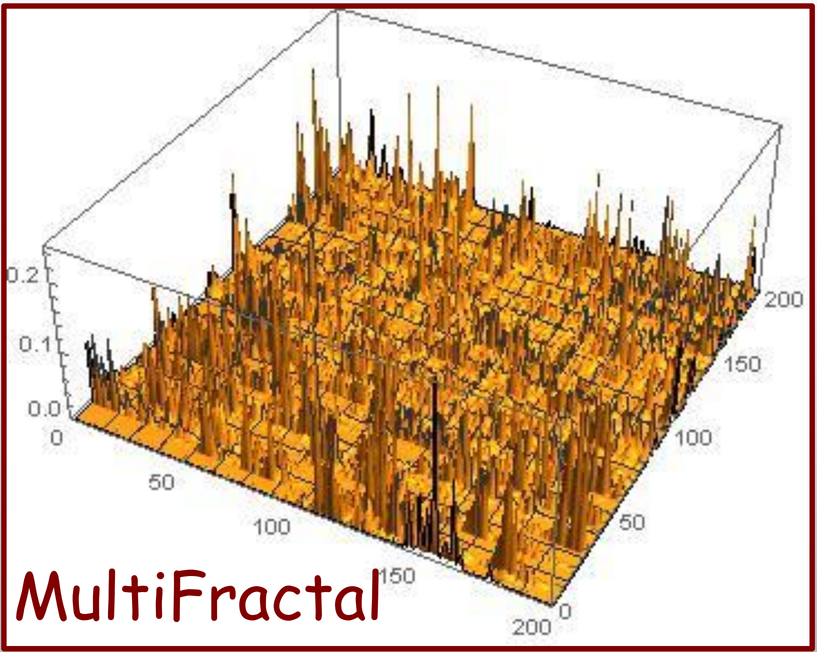
- Limit $\kappa \to \infty$ selects one saddle (Bogomolny et al. '97)
  corresponds to breaking of $U(N)$ into $U(1)^N$ (Pato, '00)
- At finite κ, instantons connect to other saddles by progressively restoring the broken symmetries
- Critical eigenvalue statistics from complex landscape

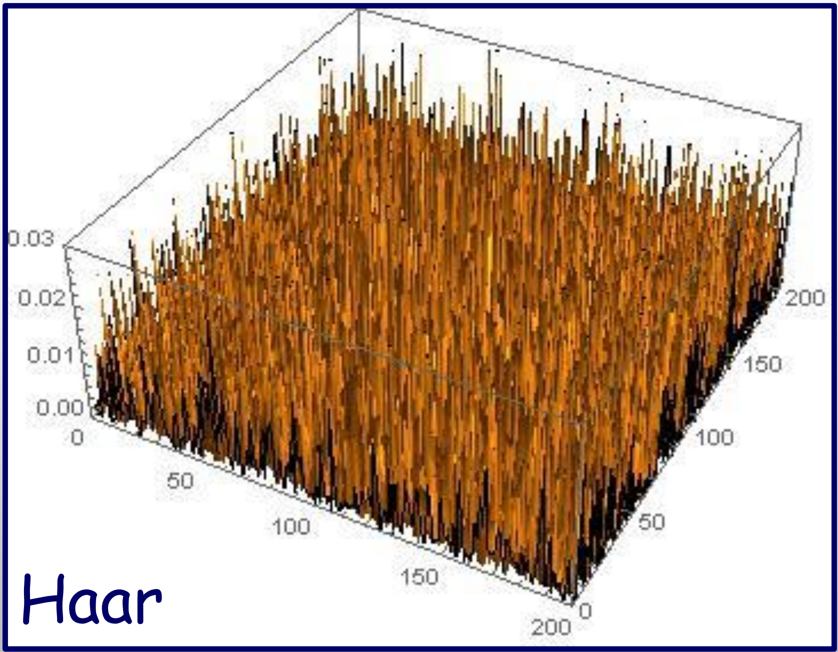
F.F. arXiv:1503.03341

Multi-fractal Spectrum

- Conjecturing form of $\mathbf{U} = e^{i\mathbf{A}}$ from landscape structure
- Inverse Participation Ratios of $\mathbf{U}$ scale with fractional powers of N (unfortunately, wrong ones…)

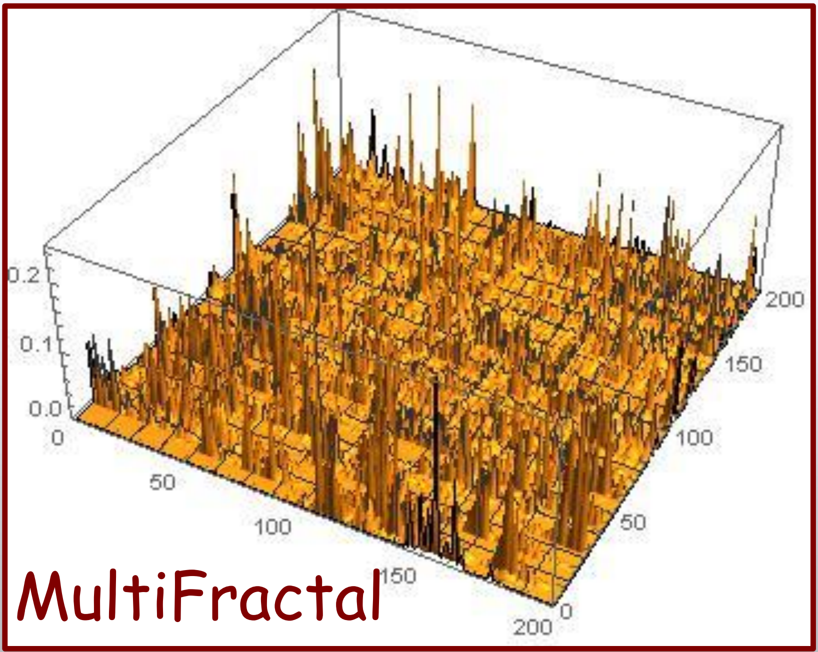
→ Multi-fractal spectrum from invariant matrix model!

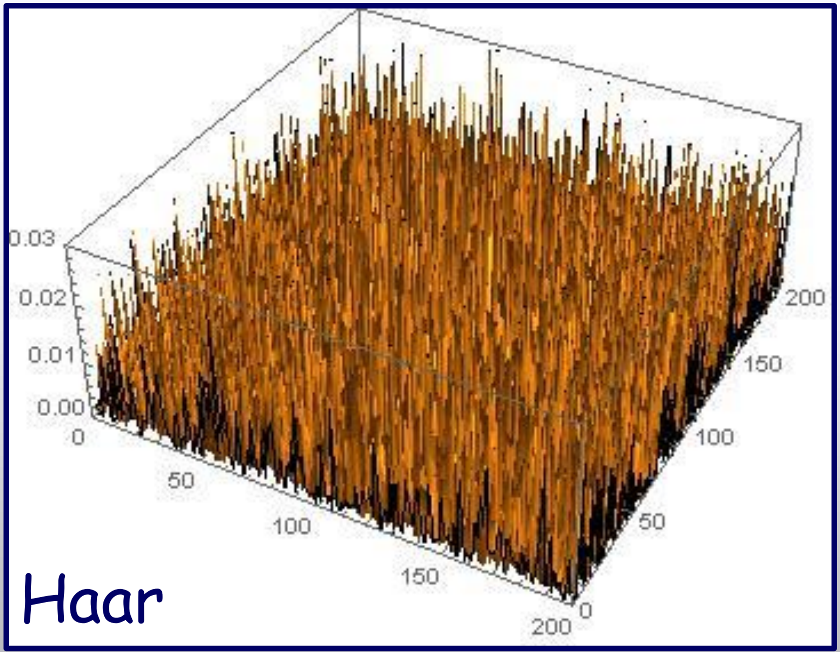




Connection to C-RBM

- From <u>conjectured</u> $\mathbf{U} = e^{i\mathbf{A}}$, reconstruct hermitian matrix $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$
- Distribution of entries in $\mathbf{M}$ similar to power-law critical random banded matrix!





Conclusions & Outlook

- Invariant Matrix Models usually applied only to extended/conducting states: eigenvectors discarded
- Eigenvalues deviate from Wigner-Dyson $\Rightarrow$ ergodicity loss: gaps in eigenvalues localize their eigenvectors / U(N) broken
- Invariant Models techniques for localization problems!
- WCMM has complex energy landscape $\to$ critical SSB

To Do List

- Critical exponents, machinery for new "eigenvector" observables
- Matrix SSB as Replica Symmetry Breaking?
- WCMM $\to$ full RSB as multi-fractal spectrum ?
- Implications of this $N \to \infty$ $U(N)$ symmetry breaking in string theories related to WCMM

Thank you!

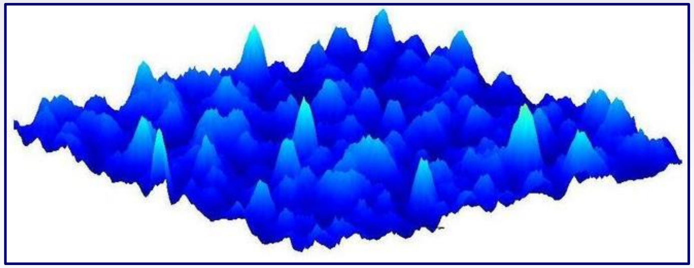
Multi-fractal Spectrum

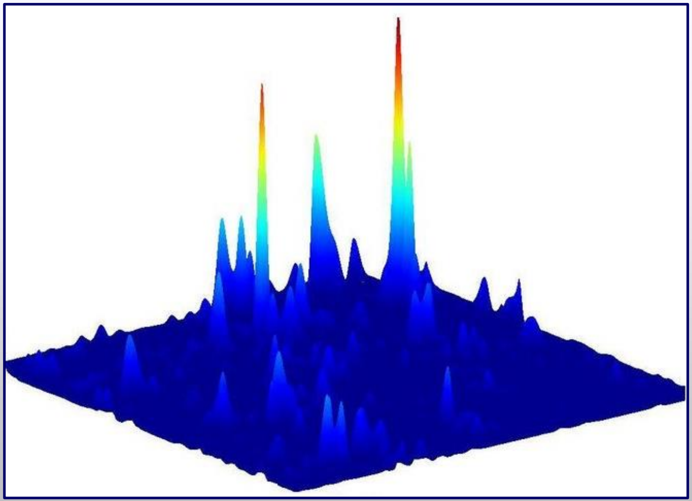
- To characterize localization: $\mathrm{IPR}_q = \sum_j^N |\Psi_j|^{2q}$, $N \propto L^d$
  - Extended: $\mathrm{IPR}_q \simeq N^{1-q} = L^{-d(q-1)}$
  - Localized: $\mathrm{IPR}_q \simeq \mathrm{const}$
  - Critical state:

$$\mathrm{IPR}_q \simeq L^{-d_q(q-1)}$$
$$= \int N^{-q\alpha + f(\alpha)} d\alpha$$

$0 < d_q < d$: fractal dimensions

$f(\alpha)$: multi-fractal spectrum



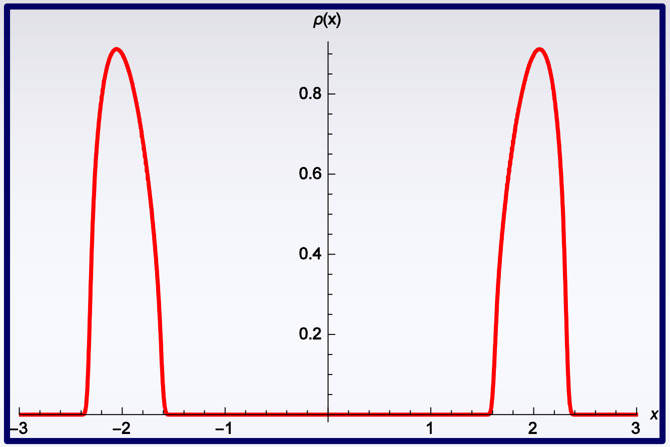


Van Tiggelen group (PRL 2009)

Brownian Motion Picture

- Level repulsion resolves degeneracy:

  $\Rightarrow$ each of the n cuts contains m_j eigenvalues



- Gap between cuts breaks rotational invariance: $U(N) \overset{N\to\infty}{\to} \prod_{j=1}^{n} U(m_j)$

- Dyson Brownian Motion for equilibrium distribution shows scale separation:

$$d\lambda_j = -\frac{dV(\lambda_j)}{d\lambda_j}\,dt + \frac{1}{N}\sum_{l\neq j}\frac{dt}{\lambda_j-\lambda_l} + \frac{1}{\sqrt{N}}\,dB_j(t)$$

$$d\vec{U}_j(t) = -\frac{1}{2N}\sum_{l\neq j}\frac{dt}{(\lambda_j-\lambda_l)^2}\,\vec{U}_j + \frac{1}{\sqrt{N}}\sum_{l\neq j}\frac{dW_{jl}(t)}{\lambda_j-\lambda_l}\,\vec{U}_l$$

dB_j, dW_{jl} delta-corr. stochastic sources

SSB Structure

- Each saddle point corresponds to a different SSB

- Unitary matrix from Hermitian matrix: $\mathbf{U} = e^{i\mathbf{A}}$

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2 \, |dA_{jl}|^2$$

$$d\mathbf{A} \equiv \mathbf{U}^\dagger d\mathbf{U}$$

- $U(1)^N$ saddle has all $dA_{ij} = 0$

- Conjecture:

  Each instanton $-q^n$ "turns on" one element: $dA_{i,i+n} \neq 0$

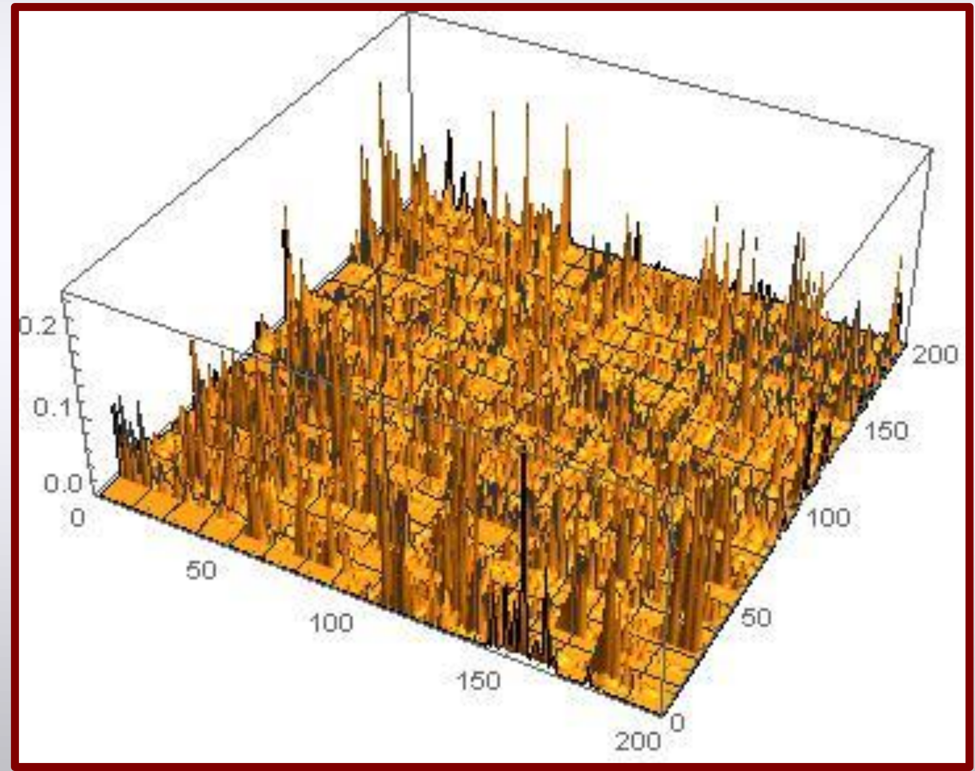
Multi-fractal Spectrum

- Numerical check of conjecture
- Unitary matrix from Hermitian matrix: $\mathbf{U} = e^{i\mathbf{A}}$
- Generate each element A_{jl}

with probability

q^{j-l} → sample A_{jl} uniformly

$1 - q^{j-l}$ → take $A_{jl} = 0$



$\Longrightarrow$ MULTIFRACTALITY!

Landau Zener Picture

- Qualitative picture on eigenvalue/eigenvector connection
- 2-level system: $\begin{pmatrix} \epsilon_1 & V \\ V^* & \epsilon_2 \end{pmatrix} \longrightarrow \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$

$$\delta = E_1 - E_2 = \sqrt{(\epsilon_1 - \epsilon_2)^2 + |V|^2}$$

"Localized"

$$\boxed{V \ll \epsilon_1 - \epsilon_2}$$

$$\delta \simeq \epsilon_1 - \epsilon_2$$

$$\Psi_{1,2} \simeq \psi_{1,2} + \mathcal{O}\left(\frac{1}{\epsilon_1 - \epsilon_2}\right) \psi_{2,1}$$

"Extended"

$$\boxed{V \gg \epsilon_1 - \epsilon_2}$$

$$\delta \simeq |V|$$

$$\Psi_{1,2} \simeq \psi_{1,2} \pm \psi_{2,1}$$

Weakly Confined Matrix Model

- Unfolding to make density constant:

$$\boxed{\lambda_x = e^{\kappa|x|}\ \mathrm{sign}(x)}$$

$$\rho(\lambda) \equiv \mathrm{Tr}\left\{\delta\left(\lambda - \mathbf{H}\right)\right\} \longrightarrow \langle\tilde{\rho}(x)\rangle \equiv \langle\rho(\lambda_x)\rangle\,\frac{d\lambda_x}{dx} = 1$$

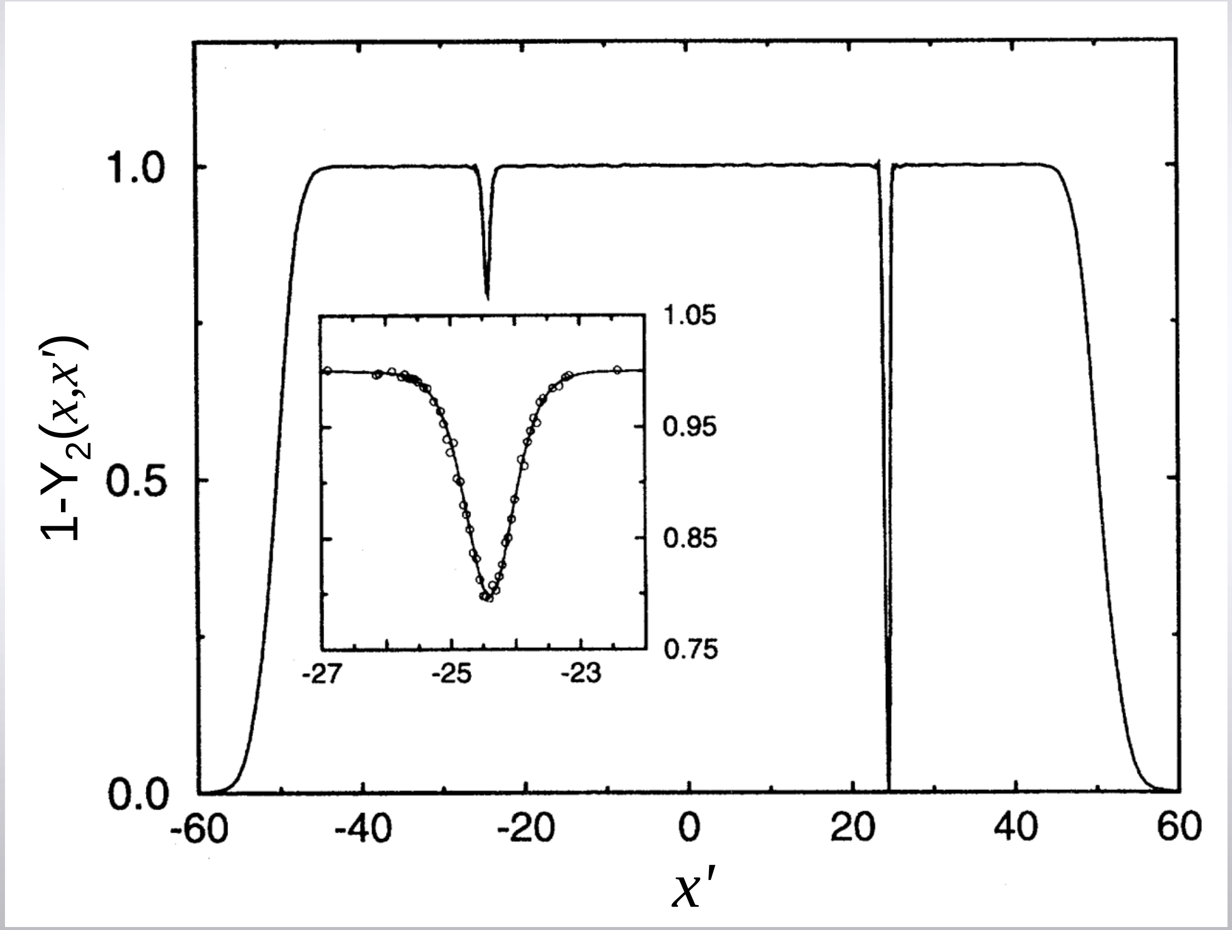
- For $e^{-2\pi^2/\kappa} << 1$ semiclassical analysis (Canali et al '95):

$$Y_2(x,x') = \frac{\kappa^2}{4\pi^2}\,\frac{\sin^2[\pi(x-x')]}{\sinh^2[\kappa(x-x')/2]}\,\theta(x\,x') + \frac{\kappa^2}{4\pi^2}\,\frac{\sin^2[\pi(x-x')]}{\cosh^2[\kappa(x+x')/2]}\,\theta(-\,x\,x')$$

$$\boxed{Y_2(x,x') \equiv \delta(x-x') - \frac{\langle\rho(E_x)\rho(E_{x'})\rangle - \langle\rho(E_x)\rangle\langle\rho(E_{x'})\rangle}{\langle\rho(E_x)\rangle\langle\rho(E_{x'})\rangle}}$$

Weakly Confined Invariant Ensemble

- Numerical check (Canali et al ’95):



Luttinger theory for RME

$$\rho(x,\tau) = \rho_0 - \frac{1}{\pi}\,\partial_x\Phi + \frac{A_K}{\pi}\cos\left[2\pi\rho_0 x - 2\Phi\right] + \ldots$$

Unfolding:

$\rho_0 = 1$

- Two-Point function (Kravtsov et al. '00):

$$\begin{aligned} Y_2 \quad &= \quad -\frac{1}{\pi^2}\langle\partial_x\Phi(x)\partial_{x'}\Phi(x')\rangle \\ &\quad -\frac{A_K^2}{2\pi^2}\cos(2\pi(x-x'))\langle e^{i2\Phi(x)}e^{-i2\Phi(x')}\rangle + \ldots \end{aligned}$$

- In flat space: $\langle\Phi(x,t)\Phi(x',t')\rangle \propto \ln\left(\Delta x^2 + \Delta t^2\right)$

$$Y_2 \propto \frac{\sin^2\left[\pi(x-x')\right]}{(x-x')^2}$$

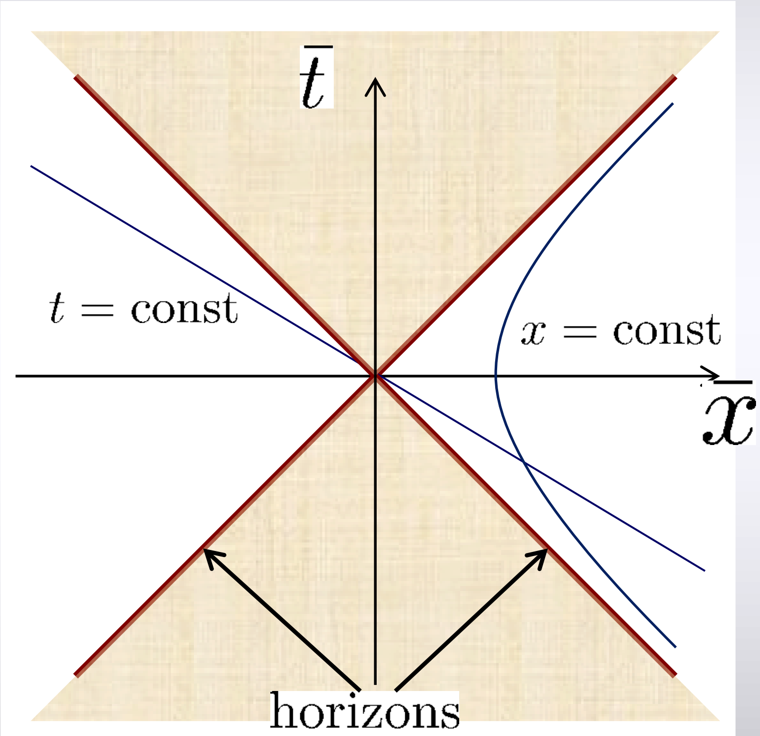
2-Point Function
for Gaussian RME
(K=1: Unitary)

Luttinger theory in Rindler space

$$\begin{cases} \bar{t} & \equiv & \frac{1}{\kappa} \sinh \kappa x \ \sinh \kappa t \\ \bar{x} & \equiv & \frac{1}{\kappa} \sinh \kappa x \ \cosh \kappa t \end{cases}$$

Periodic in imaginary time $\to$ finite **temperature**

$$\begin{aligned} ds^2 &= -\sinh^2(\kappa x)\, du^+\, du^- \\ &= -\, d\bar{u}^+\, d\bar{u}^- \\ \bar{u}^\pm &\equiv \bar{t} \pm \bar{x} \end{aligned}$$



- Far from the origin:

$$\bar{u}^\pm \simeq \begin{cases} \pm \frac{e^{\pm \kappa u^\pm}}{2\kappa}, & x \gg 1 \\ \mp \frac{e^{\pm \kappa u^\mp}}{2\kappa}, & x \ll -1 \end{cases}$$

Luttinger Liquid in Rindler Space

- Remind two-Point function:

$$Y_2 \quad = \quad -\frac{1}{\pi^2}\langle \partial_x \Phi(x) \partial_{x'} \Phi(x') \rangle$$
$$- \frac{A_K^2}{2\pi^2} \cos(2\pi(x - x')) \langle \mathrm{e}^{\mathrm{i}2\Phi(x)} \mathrm{e}^{-\mathrm{i}2\Phi(x')} \rangle + \ldots$$

- With the new coordinates: $\left(\bar{x} = \frac{\mathrm{e}^{\kappa|x|}}{2\kappa} \operatorname{sgn}(x) \right)$

$$\langle \Phi(x)\Phi(x') \rangle \overset{|x|,|x'|\gg 1}{\propto} \begin{cases} \ln\left[\frac{2}{\kappa}\sinh\frac{\kappa(x-x')}{2}\right], & x\,x' > 0 \\ \ln\left[\frac{2}{\kappa}\cosh\frac{\kappa(x+x')}{2}\right], & x\,x' < 0 \end{cases}$$

Luttinger Liquid in Rindler Space

- We recover exactly the RME correlation (K=1):

$$Y_2^a(x,x') = \frac{\kappa^2}{4\pi^2}\,\frac{\sin^2\left[\pi(x-x')\right]}{\cosh^2\left[\kappa(x+x')/2\right]}\,, \qquad \text{for} \quad x\,x' < 0$$

(Anomalous: non-translational invariant)

$$Y_2^n(x,x') = \frac{\kappa^2}{4\pi^2}\,\frac{\sin^2\left[\pi(x-x')\right]}{\sinh^2\left[\kappa(x-x')/2\right]}\,, \qquad \text{for} \quad x\,x' > 0$$

(Normal: translational invariant)