Spontaneous Ergodicity Breaking in Invariant Matrix Models

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MARIE CURIE ACTIONS

Hrzz Croatian Science Foundation

RBI T WINNING

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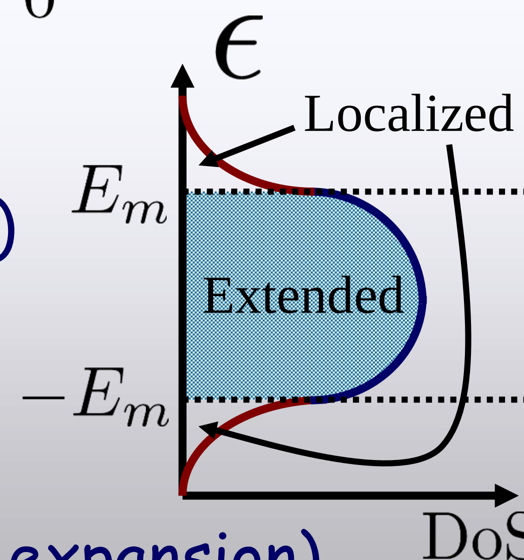
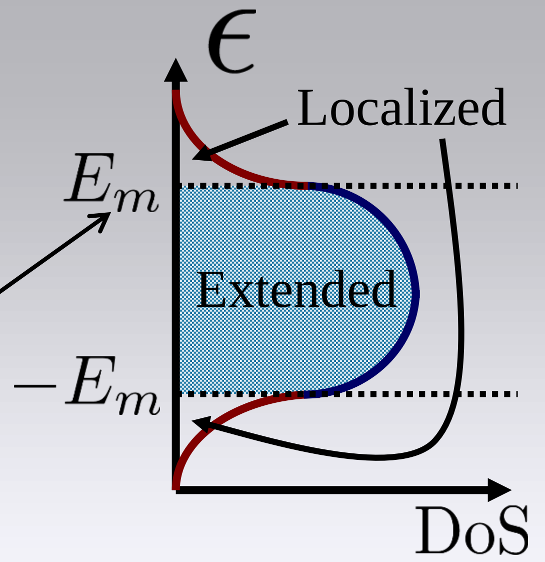
Disorder & Localization

- Anderson Model: $\mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[c_l^\dagger c_j + c_j^\dagger c_l \right]$ (Anderson. '58)
- Tight-binding model (nearest neighbor hopping)
- Random on-site energies: $\epsilon_j \in [-W, +W]$
- 1 (& 2) Dimensions: localized for any $W \neq 0$
- Higher D:
  - Small W : conducting (weak loc., Random Matrices)
  - $W > W_c$: insulating (localized at low energies)



- Hard problem (uncontrolled perturbation expansion)

Metal/Insulator Transition



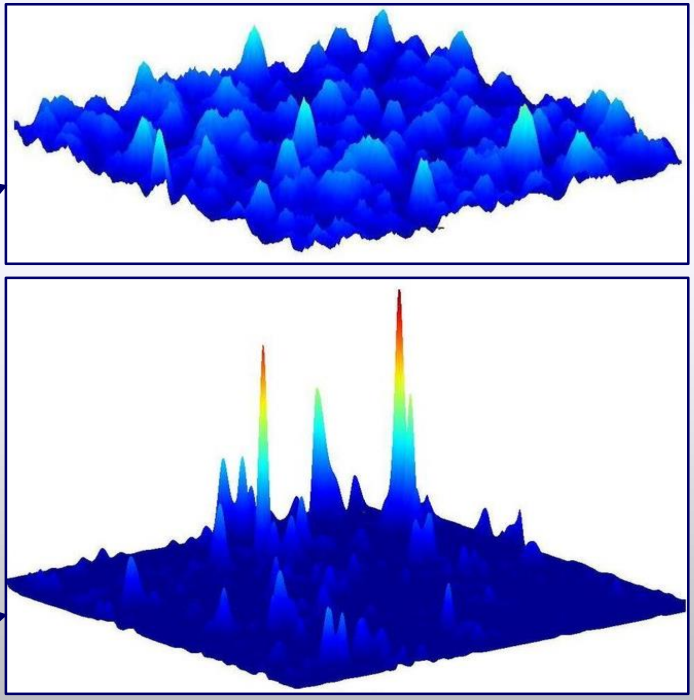
$$\mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[c_l^\dagger c_j + c_j^\dagger c_l \right]$$

$\epsilon_j \in [-W, +W]$

$W < W_c$

$D \geq 3$

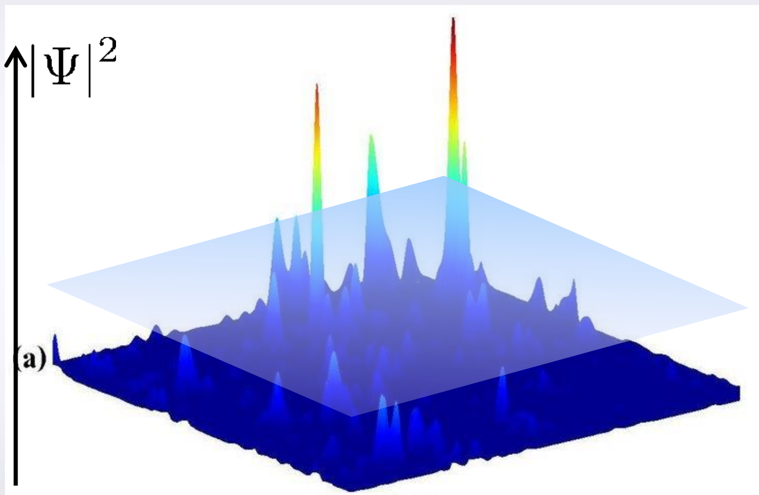
- Mobility Edge separates extended from localized states
- Transition as Intermediate state (multifractal)

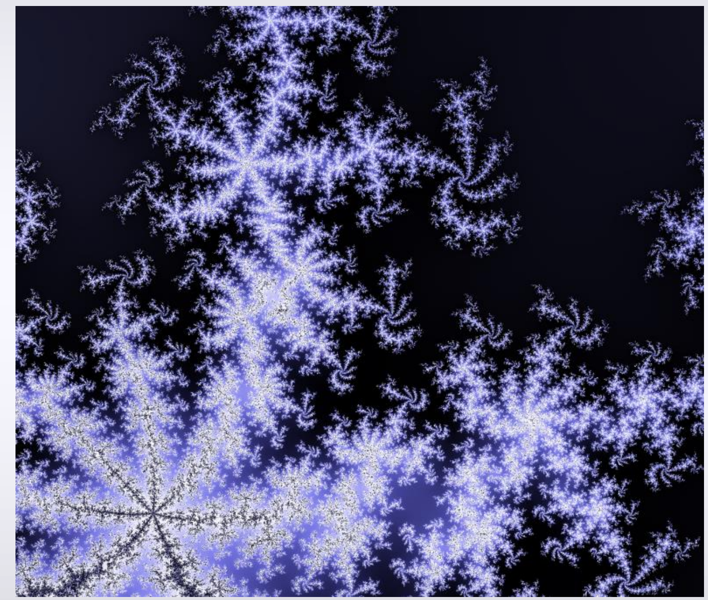


Van Tiggelen group (PRL 2009)

Multifractality

- At each height $|\Psi|^2 = \alpha$, the wavefunction's amplitude draws a "curve" with a different fractal dimension $f(\alpha)$





- Behavior at mobility edge known in "perturbative" regimes

$\Rightarrow$ long-standing open problem

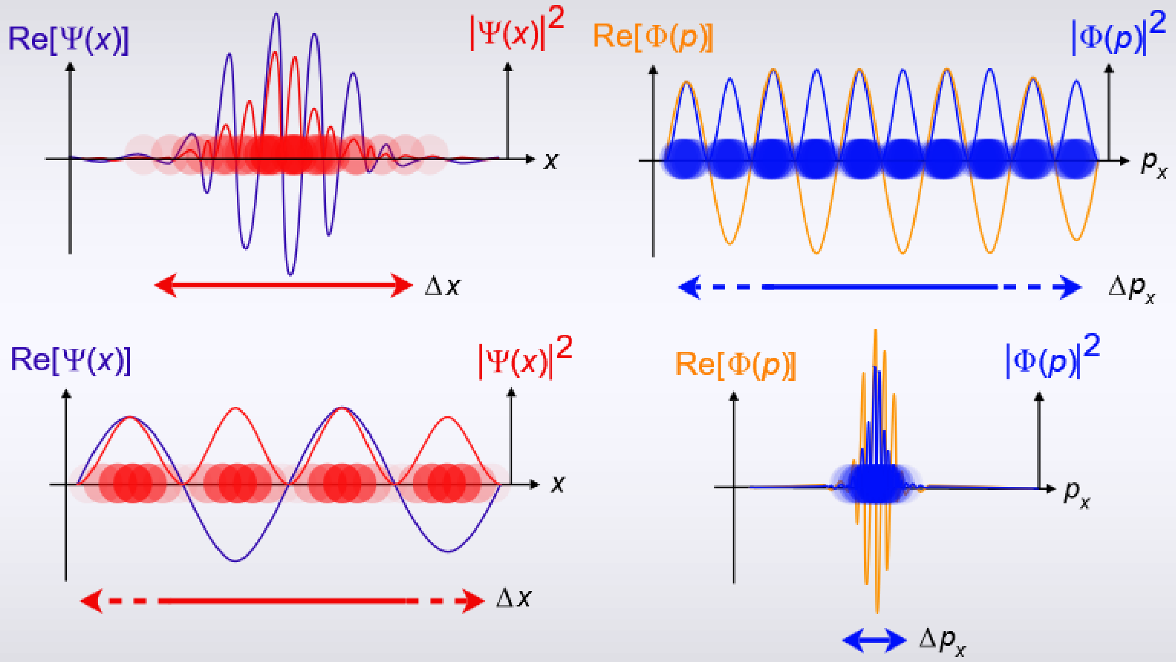
$$\boxed{\begin{array}{l} d = 2 + \epsilon \\ d \to \infty \end{array}}$$

Motivating Question for this work

- Localization/extendedness of wavefunctions is a basis-dependent property

Localized

Extended



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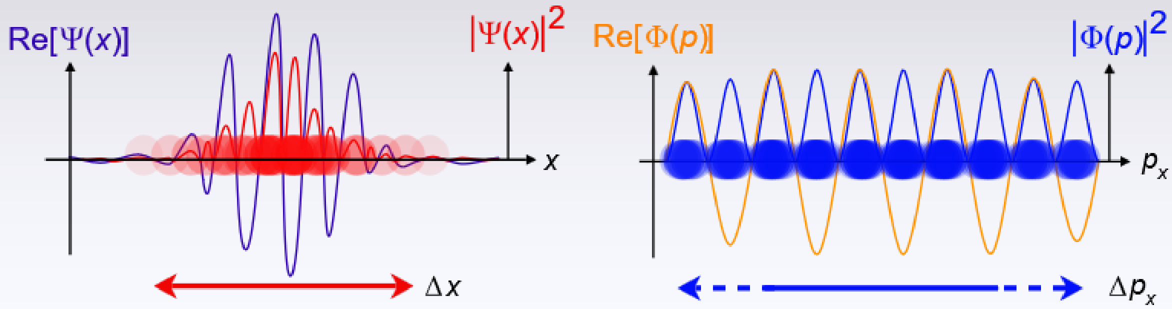
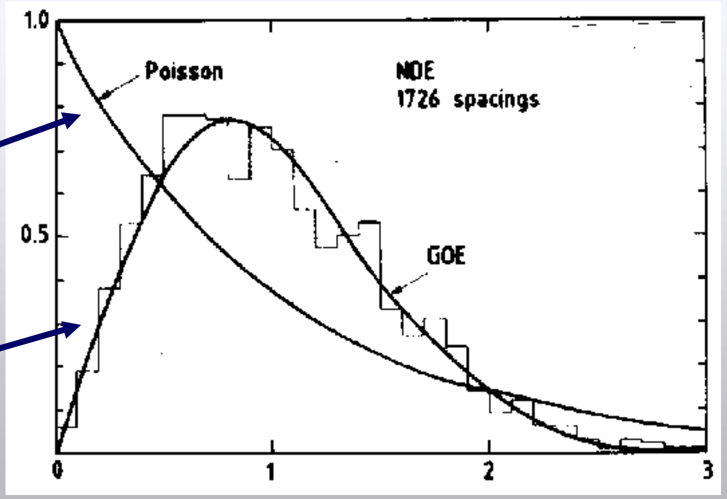


Extended

- However, level spacing statistics characterizes insulating/conducting systems

Poisson ↔ Localized

Wigner Dyson ↔ Extended



My Approach

- Spectral signature hints toward localization as basis independent property
- Random Matrix Theory ideal to test this hypothesis
- Similar ideas introduced before:
  - *Moshe, Neuberger, Shapiro:* **PRL '94**
  - *Canali, Kravtsov*: **PRE '95**; Bogomolny, Bohigas, Pato: **PRE '97**; Pato: **PRE '00**
  - *Bonnet, David, Eynard*: **JPA '00** …
- However: lack of analytical tools to study eigenstate behavior in RMT

(Allez & Bouchaud '11-'12; Allez & Guionnet '13)

My Claim

- Eigenvector and eigenvalue statistics are linked in RMT:

The U(N) symmetry matrix models are endowed with can be spontaneously broken

- Peculiar SSB: thermodynamic limit also takes symmetry's rank to infinity
- Conjecture 1: certain models break U(N) in a critical way (similar to Metal/Insulator Transition)
- Conj. 2: U(N) symmetry breaking as a replica breaking?

Outline

1. Introduction: Matrix Models
2. Spontaneous Symmetry Breaking:
   - Geometrical argument
   - Numerical finite size detection
   - Symmetry Breaking term
3. Conclusions & Outlook

Random Matrix Theory

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-W(\mathbf{M})} \qquad \mathcal{F} = \ln \mathcal{Z}$$

- Take W(M) real: statistical model
- Matrix M can represent different "objects"
- Consider M as a Hamiltonian:
  - M: Hermitean Matrix
  - Matrix entries randomly from a distribution
  - Interaction between every degree of freedom (no preconceived notion of locality)
- Common wisdom: RMT describes delocalized systems

Invariant Ensembles

- Action invariant under rotations: $\boxed{W(\mathbf{M}) = \mathrm{Tr} V(\mathbf{M})}$
- Switch to eigenvalues/eigenvectors: $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, \Delta^2(\{\lambda\}) \, e^{-\sum_j V(\lambda_j)}$$

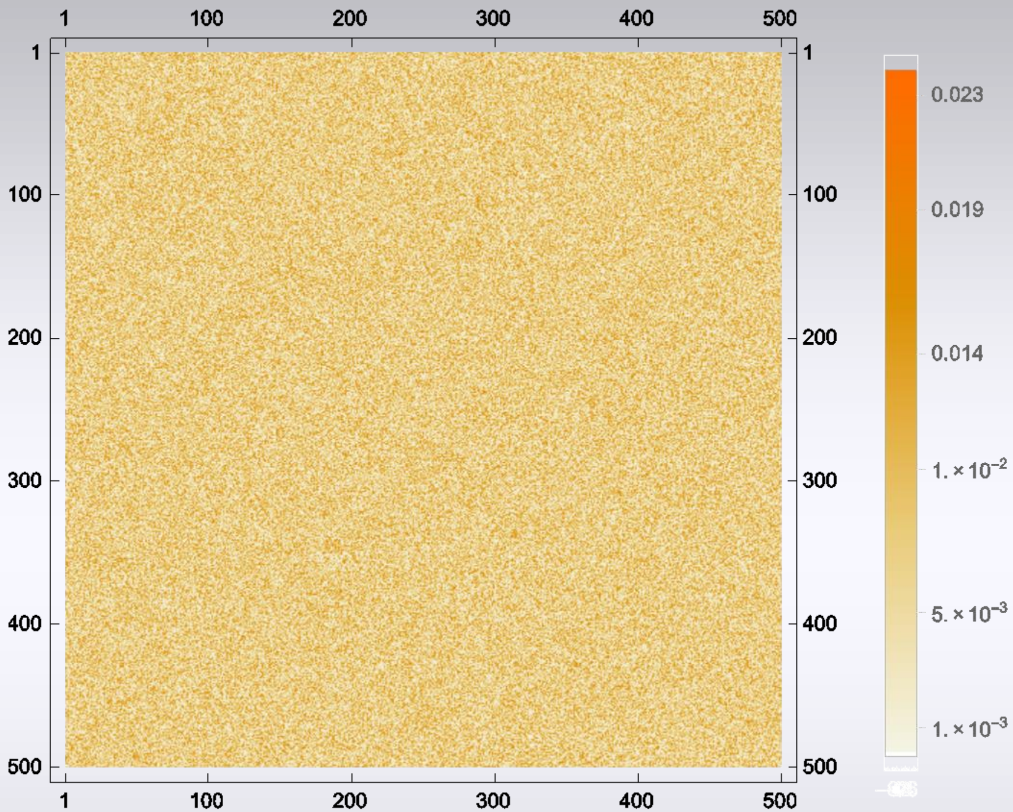
Eigenvectors uniformly distributed over the N-dimensional sphere (Hilbert space): independent from V(λ)

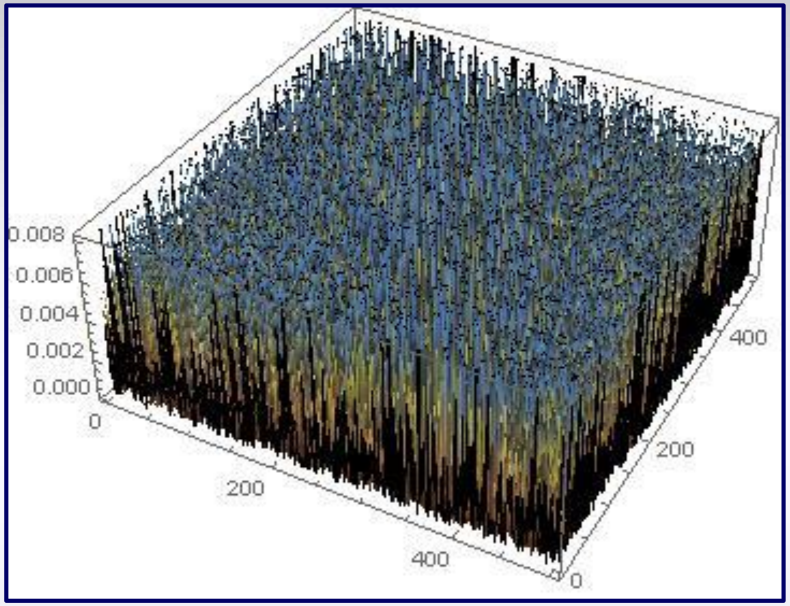
Van der Monde Determinant:

$$\Delta(\{\lambda\}) = \prod_{j>l}^{N} (\lambda_j - \lambda_l)$$

(from Jacobian)

The Haar Measure





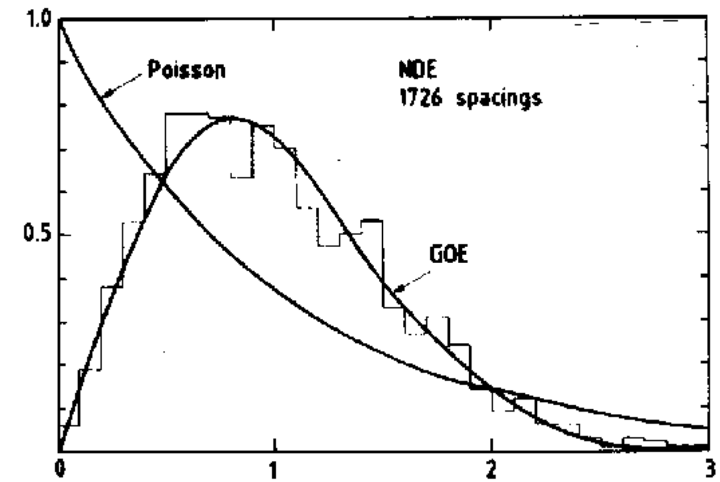
- Entries of Unitary matrix follow the Porther-Thomas

Distribution: $\mathcal{P}\left(\left|\tilde{U}_{ij}\right|^2\right) = N \exp\left[-N\left|\tilde{U}_{ij}\right|^2\right]$

Wigner-Dyson Universality

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, \Delta^2(\{\lambda\}) \, e^{-\sum_j V(\lambda_j)}$$

- Jacobian introduces interaction between eigenvalues
- Coulomb gas picture: $\mathcal{L} = -2 \sum_{j>l} \ln|\lambda_j - \lambda_l| + \sum_j V(\lambda_j)$
- Eigenvalues as 1-D particles with
  - logarithmic repulsion
  - external confining potential V(λ)
- Universal level spacing distribution
  (distance between n.n. eigenvalues) →



$$P(s) \propto s^{\beta} e^{-A(\beta)s^2}$$

- Valid for any polynomial V(λ)

Invariant Ensembles

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda\, \Delta^\beta(\{\lambda\})\, e^{-\sum_j V(\lambda_j)}$$

- Wigner Dyson distribution & level repulsion:
  Jacobian introduces interaction between eigenvalues
- Extended states/conducting phases:
  uniform distribution means eigenvectors typically have all non-vanishing entries
- Eigenvalues interact through their eigenvectors:

$$\boxed{\textbf{WD} \Leftrightarrow \textbf{extended states}}$$

Non-Invariant Ensembles

- To study localization problems, introduce non-invariant random matrix ensembles (Random Banded Matrices)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\sum_{j,l} A_{jl} \, |M_{jl}|^2}$$

$$\langle M_{nm}^2 \rangle = A_{nn}^{-1} \qquad \boxed{A_{nm} \propto (n-m)^{2\alpha}}$$

$\alpha > 1$ $\rightarrow$ Localized states (Poisson statistics)

(Mirlin et al. '96; …)

$\alpha = 1$ $\rightarrow$ Multi-Fractal states (Critical Statistics: Anderson Metal/Insulator transition)

(Evers & Mirlin, '00; …)

- Limited analytical tools (SUSY, cluster expansion…)

Eigenvalues/Eigenvectors Statistics

- Within Random Matrix Theory, localization problems studied through non-invariant ensembles
- Confirmation of eigenvalue/eigenvector statistics link
- However, limited analytical tools (perturbative)

New perspective:

- There are invariant models with non-WD statistics
- Invariant models are endowed with superior (non-perturbative) analytical techniques

Spontaneous Breaking of Rotational Invariance

- Consider <u>invariant</u> models with <u>non-WD</u> statistics
- If eigenvalue/eigenvector link holds

$\Rightarrow$ System undergoes a spontaneous breaking of rotational symmetry

$\Rightarrow$ Invariant machinery for localization problems!

- Recall a ferromagnet:
  - ➢ From partition function, rotational invariance
    $\rightarrow$ no spontaneous magnetization
  - ➢ Need, e.g., a symmetry breaking term

Weakly Confined Invariant Models

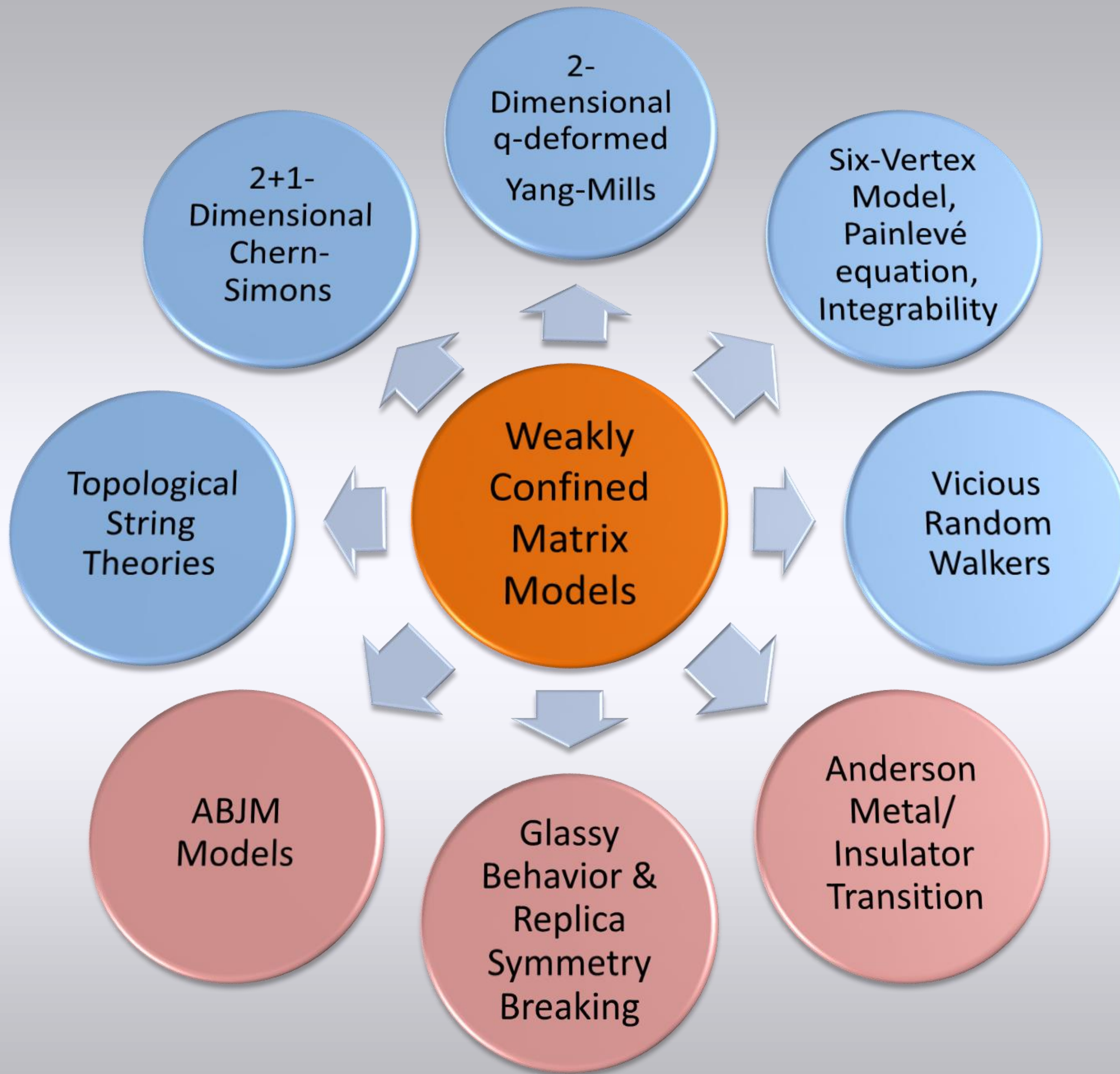
$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} \;, \; V(\lambda) \overset{|\lambda| \to \infty}{\simeq} \frac{1}{2\kappa} \ln^2 |\lambda|$$

- Soft confinement sets them apart from usual polynomial potentials
  $\to$ WD universality does not apply
  $\to$ Indeterminate moment problem

- Arise in localization limit of Chern-Simons/ABJM: $\kappa \propto \frac{i}{g_s}$
  (Marino '02; Kapustin et a. '10; …)

- Solvable through orthogonal polynomials:
  q-deformed Hermite/Laguerre Polynomials
  (Muttalib et al. '93; Tierz'04)

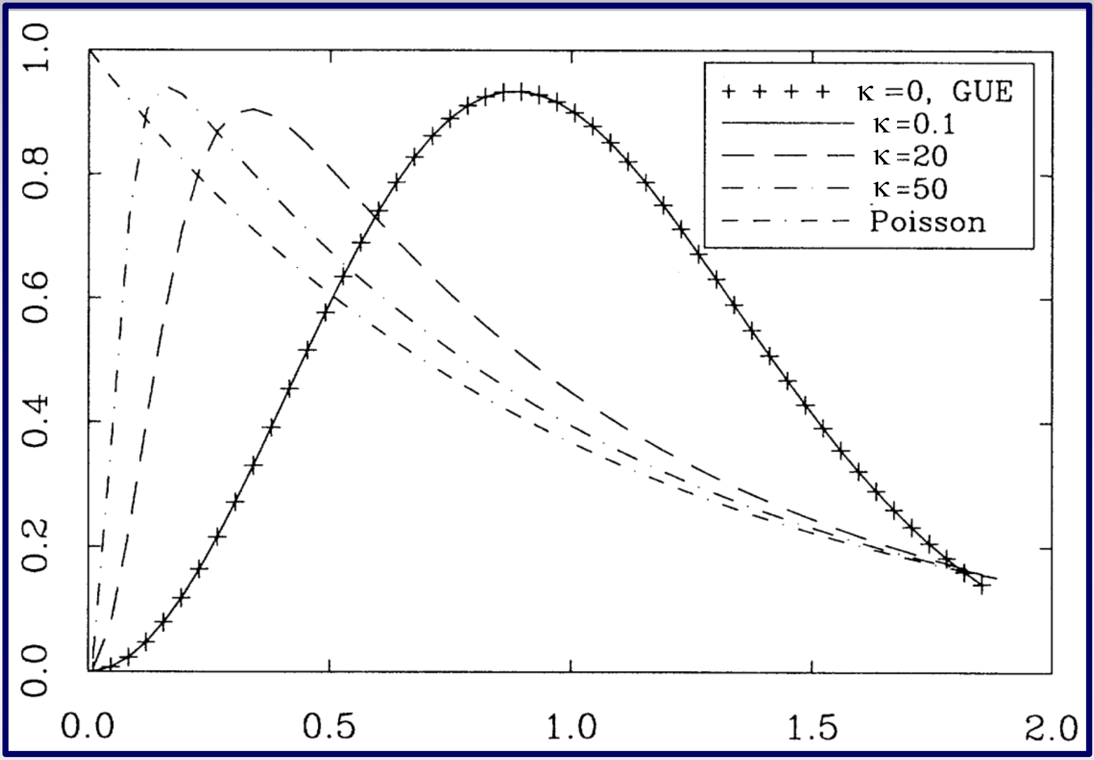
Weakly Confined Matrix Models & their applica-tions



Weakly Confined Matrix Models

$$V(\lambda) \overset{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa} \ln^2 |\lambda|$$

- Intermediate level spacing statistics
- Same eigenvalue correlations as Critical Random Banded Matrices



(Muttalib et al. '93)

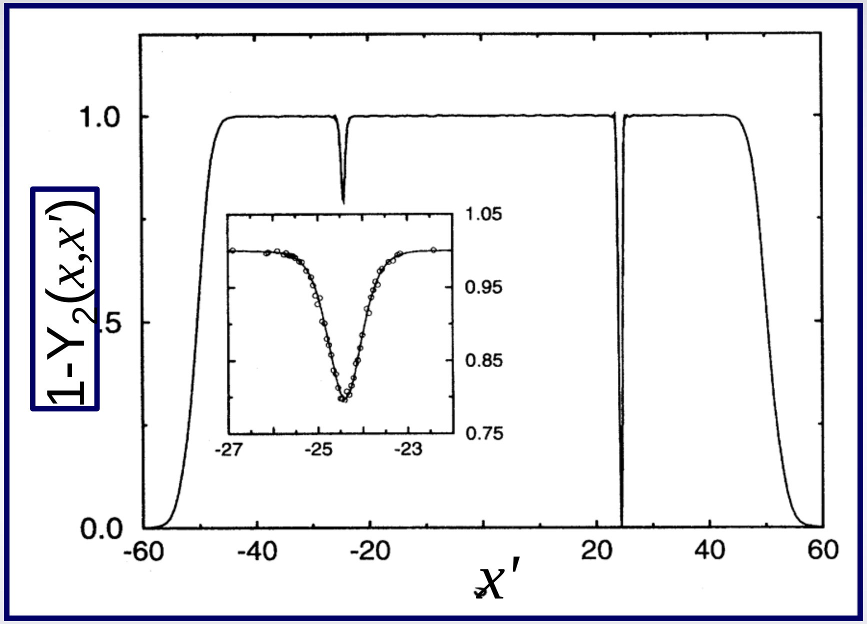
- Critical level statistics signals fractal eigenstates?
- Critical Spontaneous Breaking of U(N) Invariance?

(Canali, Kravtsov, '95)

Non-Local Correlations

$$Y_2(x,x') \equiv \delta(x-x') - \frac{\langle \rho(E_x)\rho(E_{x'})\rangle - \langle \rho(E_x)\rangle\langle \rho(E_{x'})\rangle}{\langle \rho(E_x)\rangle\langle \rho(E_{x'})\rangle}$$

$$= \frac{\kappa^2}{4\pi^2}\,\frac{\sin^2[\pi(x-x')]}{\sinh^2[\kappa(x-x')/2]}\,\theta(x\,x') + \frac{\kappa^2}{4\pi^2}\,\frac{\sin^2[\pi(x-x')]}{\cosh^2[\kappa(x+x')/2]}\,\theta(-\,x\,x')$$



(Canali & Kravtsov ’95)

- Level repulsion also at antipodal points!
- Effective description as Luttinger Liquid in Rindler space

(Franchini & Kravtsov ’09)

WCMM and Anderson Transition

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} \ , \ V(\lambda) \overset{|\lambda| \to \infty}{\simeq} \frac{1}{2\kappa} \ \ln^2 |\lambda|$$

- Same spectral signatures as C-RBM :

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\sum_{j,l} A_{jl} \ |M_{jl}|^2}, \ A_{nm} = 1 + \frac{(n-m)^2}{B^2}$$

- C-RBM toy model for the Anderson Transition: reproduce multifractal spectrum (analytical for $d = 2 + \epsilon$)

$$B \sim \frac{1}{\kappa} \sim d, \mathrm{Connectivity}$$

- Conjecture: SSB of WCMM to calculate analytically multifractal spectrum of Anderson MIT

(Canali, Kravtsov, '95)

Part 2

Spontaneous Breaking of Rotational Symmetry in Invariant Multi-Cuts Matrix Model

Multi-Cut Solutions

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, e^{-\sum_j V(\lambda_j) + 2\sum_{j>l} \ln|\lambda_j - \lambda_l|}$$

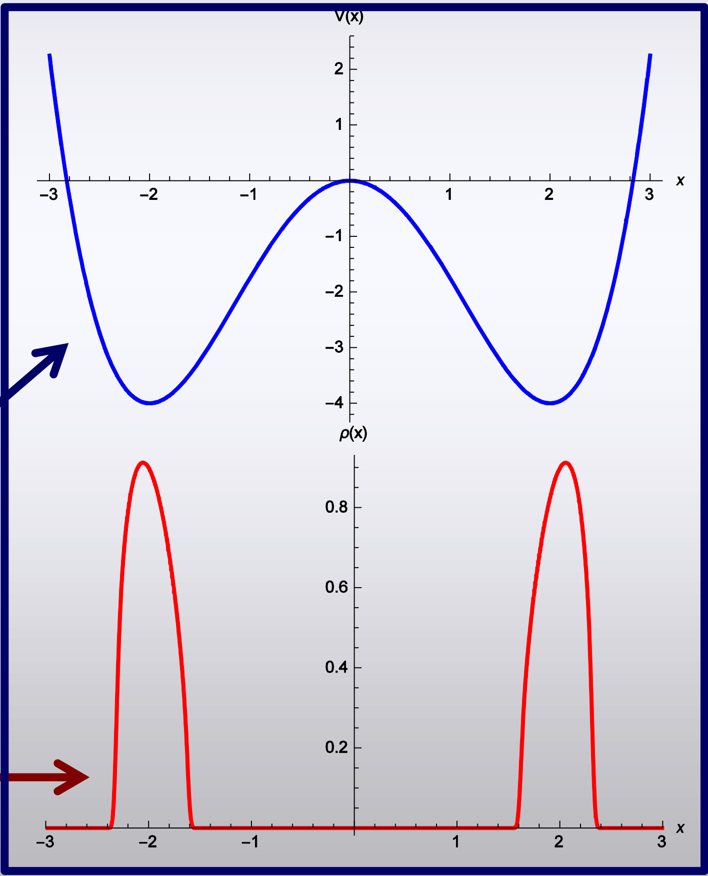
- V(x) with several, well separated, minima

  $\Rightarrow$ disconnected support for eigenvalues (multi-cuts)

- For example: double well potential

$$V_{2W}(x) = \frac{1}{4}x^4 - \frac{t}{2}x^2$$

(2-cuts for $t > 2$)



Level Density: $\rho(x) = \sum_j^N \delta(x - \lambda_j)$

Understanding the matrix SSB

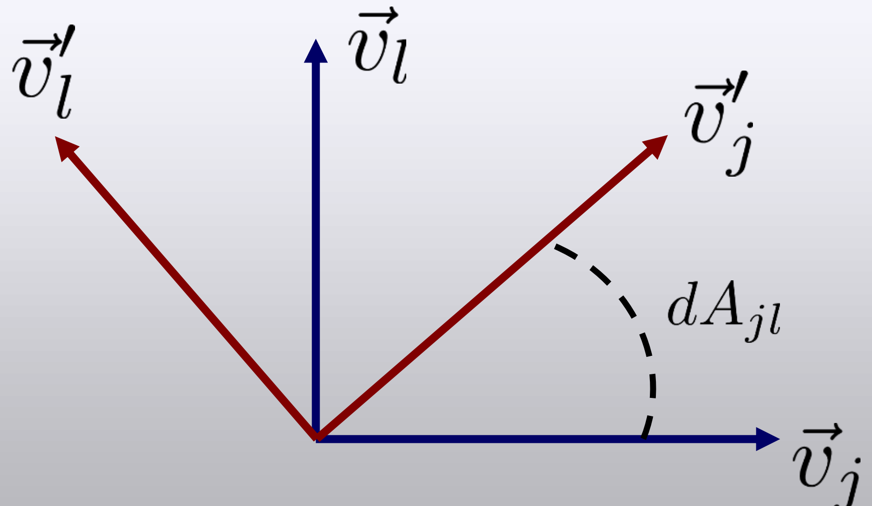
- Geometrical argument: line element

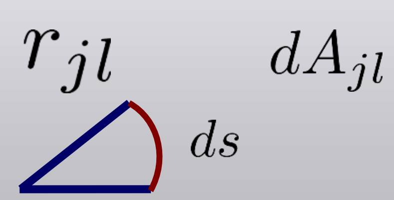
$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2\, |dA_{jl}|^2$$

$$d\mathbf{A} \equiv \mathbf{U}^\dagger d\mathbf{U}$$

$$\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$

- Angular degrees of freedom live on spheres of radii $r_{jl} = |\lambda_j - \lambda_l|$





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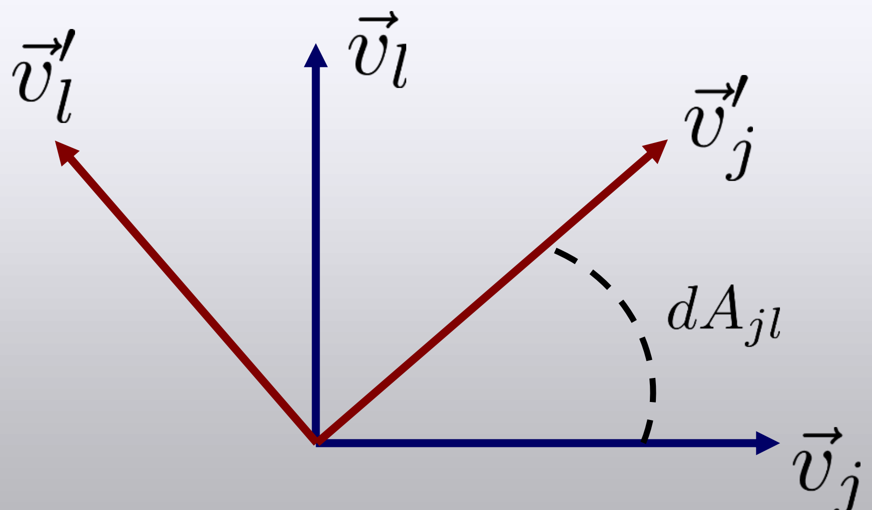

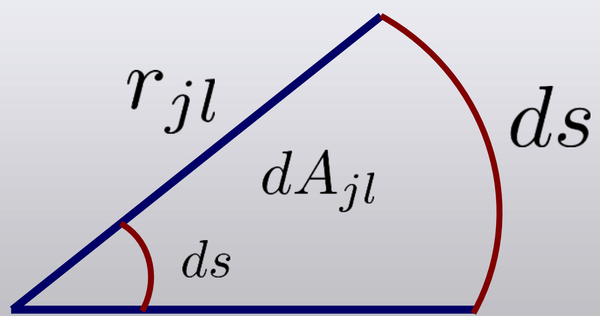

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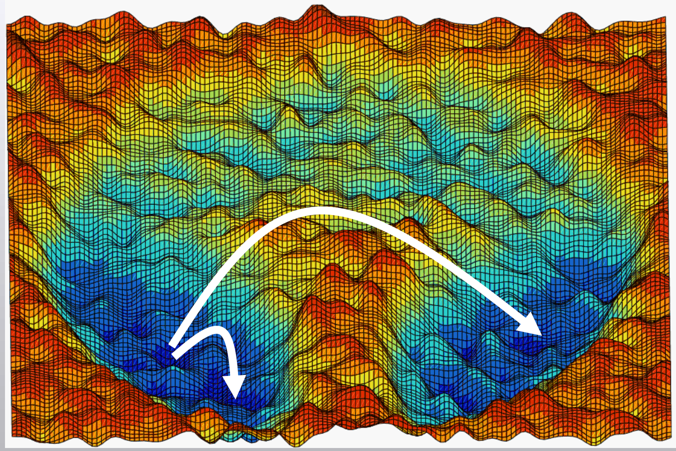
- Angular degrees of freedom live on spheres of radii $r_{jl} = |\lambda_j - \lambda_l|$
- For large r_{jl}, rotations generate large ds

$\Rightarrow$ move to far point in conf. space

- Entropic (fine tuning) origin of SSB (same as level repulsion)



Understanding the matrix SSB

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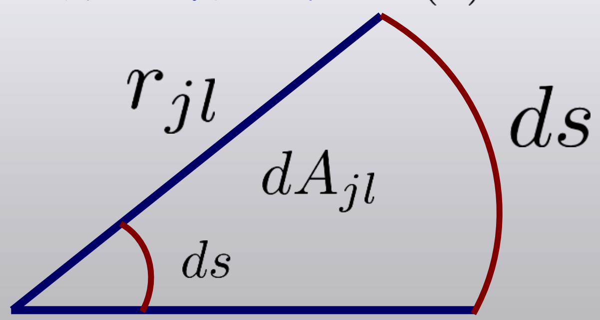
$$\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$

- Angular degrees of freedom live on spheres of radii $r_{jl} = |\lambda_j - \lambda_l|$

  - Two lengths scales:
    - Eigenvalues spacing: $\mathcal{O}\left(\frac{1}{N}\right)$
    - Support of distribution: $\mathcal{O}(1)$

- Eigenvectors of eigenvalues in different cuts cannot mix



Generating a Random Matrix

- Gaussian Models: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}\mathbf{M}^2} = \int \prod dM_{jl} e^{-\sum_{jl} M_{jl}^2}$

  $\rightarrow$ each matrix entries sampled independently

- One-Cut Models: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}V(\mathbf{M})} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}\sum_k g_k \mathbf{M}^k}$

  $\rightarrow$ entries correlated: generated as perturbation of Gaussian case in a Metropolis scheme

- Multi-Cut Solutions: Gaussian case unstable

  $\rightarrow$ start from initial seed and evolve it to equilibrium

  $\rightarrow$ SSB: final configuration has memory of eigenvectors of initial seed

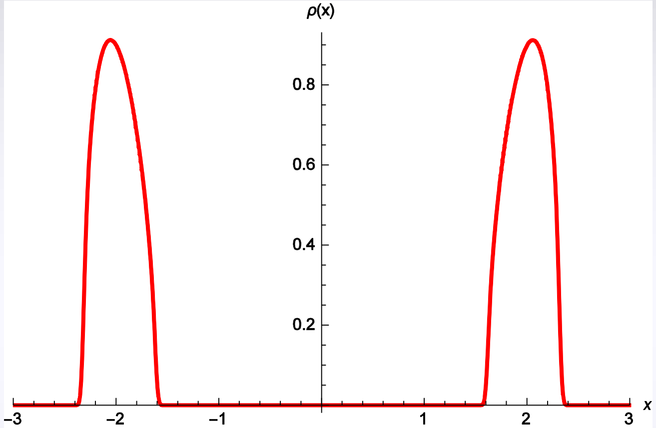
Multi-Cuts SSB

- Level repulsion resolves degeneracy:

  $\Rightarrow$ each of the n cuts contains m_j eigenvalues

- Gap between cuts breaks rotational invariance: $$\boxed{U(N) \overset{N\to\infty}{\to} \prod_{j=1}^{n} U(m_j)}$$

- Three Arguments:
  - ★ Brownian motion;
  - ★ Numerical finite size analysis;
  - ★ Symmetry Breaking Term



Double well

$$U(N) \overset{N\to\infty}{\to} U(N/2) \times U(N/2)$$

(assume N even)

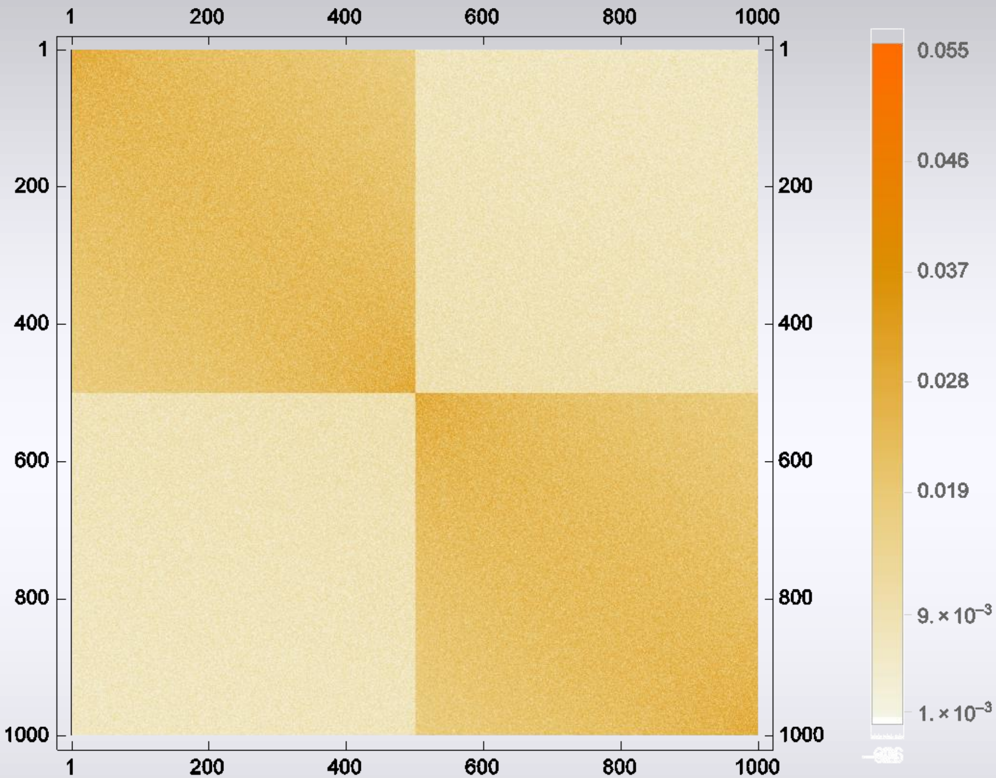
F.F. arXiv:1412.6523

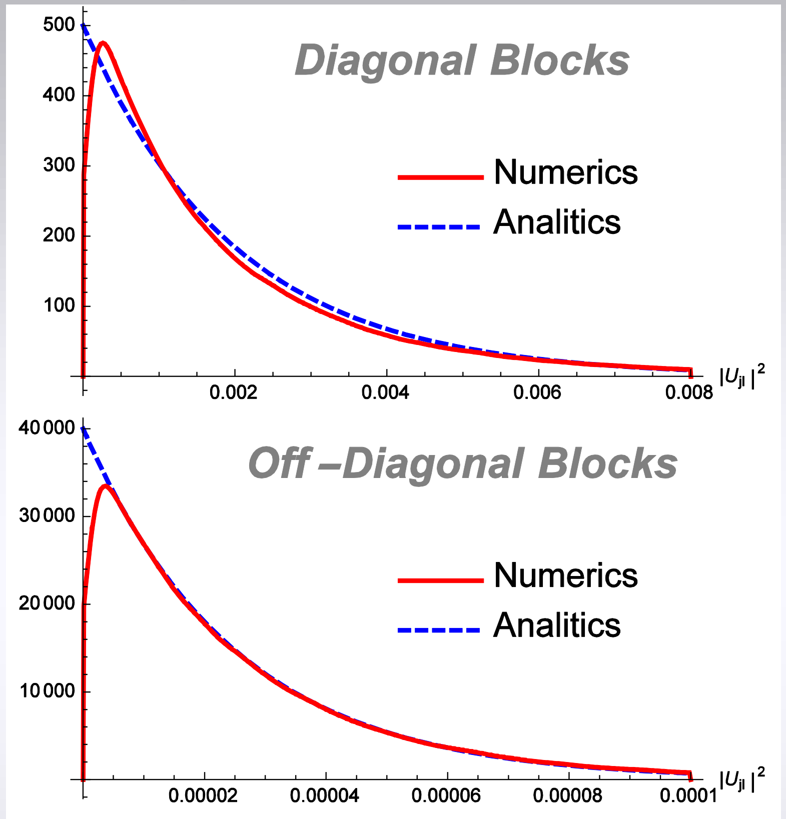
Finite Size Analysis

- Without preferred, reference basis; localization means rigidity of eigenvectors under perturbations
- Take double well matrix model: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N \mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$
- Generate a representative matrix: $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$
- Apply perturbation $\mathbf{\Delta}\mathrm{M}$ (sparse Gaussian Matrix)
  (Order of N non-zero elements)
- Find eigenvectors of perturbed matrix: $\mathbf{M} + \mathbf{\Delta M} = \mathbf{U}'^\dagger \mathbf{\Lambda}' \mathbf{U}'$
- Eigenvector rotation induced by perturbation: $\tilde{\mathbf{U}} = \mathbf{U}' \mathbf{U}^\dagger$

Finite Size Analysis

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N\mathrm{Tr}\left[\frac{1}{4}\,\mathbf{M}^4 - \frac{t}{2}\,\mathbf{M}^2\right]}$$



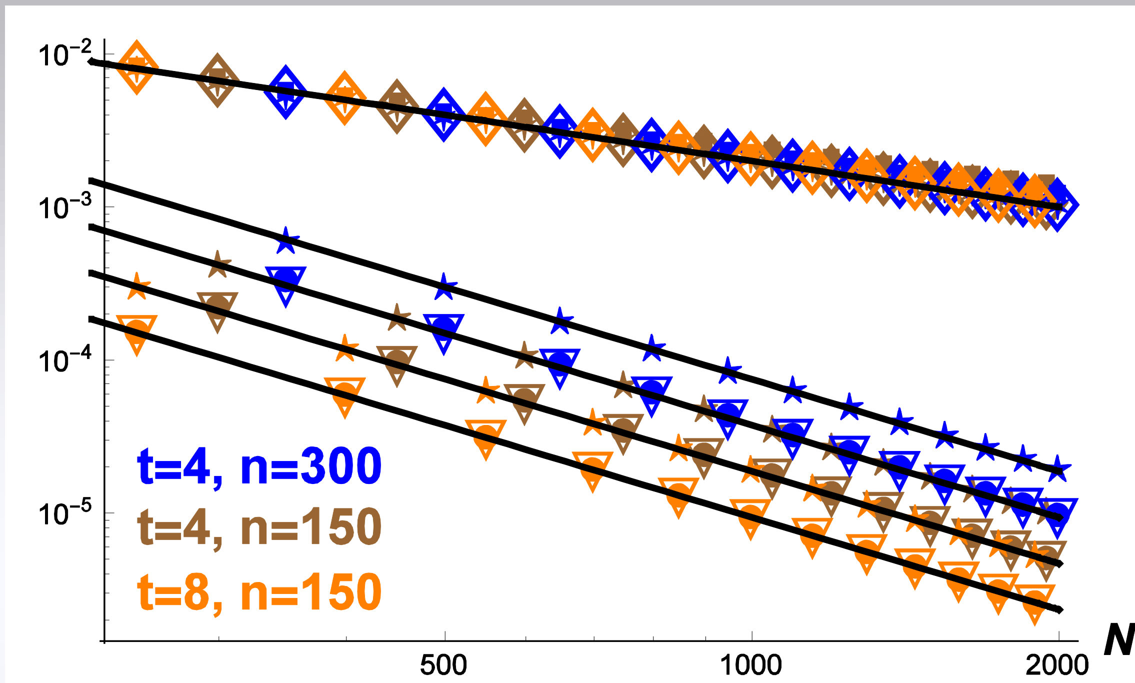


t=4, N=1000, sparse matrix with n=200 non zero elements, drawn from Gaussian with zero mean and variance N)

$$\mathcal{P}\left(\left|\tilde{U}_{ij}\right|^2\right) = \chi \exp\left[-\chi \left|\tilde{U}_{ij}\right|^2\right]$$

$$\chi_{\mathrm{D}} = \frac{N}{2} \qquad \chi_{\mathrm{OD}} = \frac{2tN^2}{n}$$

Finite Size Analysis



$$O_{jl} = \sum_m^N \left|\tilde{U}_{mj}\right|^2 \left|\tilde{U}_{ml}\right|^2$$

Overlaps between eigenstates

$$\langle |O_{jl}| \rangle_{\mathrm{D}} = \langle \left|\tilde{U}_{jl}\right| \rangle_{\mathrm{D}} = \langle \left|\Delta\tilde{U}_{jl}\right| \rangle_{\mathrm{D}} = \frac{1}{\chi_{\mathrm{D}}} = \frac{2}{N}$$

$$\langle |O_{jl}| \rangle_{\mathrm{OD}} = 2\langle \left|\tilde{U}_{jl}\right| \rangle_{\mathrm{OD}} = 2\langle \left|\Delta\tilde{U}_{jl}\right| \rangle_{\mathrm{OD}} = \frac{2}{\chi_{\mathrm{OD}}} = \frac{n}{tN^2}$$

Off-diagonal blocks suppressed as 1/N compared to diagonal ones

Onset of localizations!

Symmetry Breaking Term

- To detect SSB introduce symmetry breaking term
- Most natural one is $\mathrm{Tr}\left([\mathbf{M},\mathbf{S}]\right)^2$, but too hard to handle

$\mathbf{S}$: given Hermitian Matrix
Favors alignment of eigenvectors

- We introduce: $W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}$

J : source strength

Absolute value can be removed by sorting eigenvalues in increasing order

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

Symmetry Breaking: Double Well

$$W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}$$

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

- Double well: $U(N) \stackrel{N\to\infty}{\to} U(N/2) \times U(N/2)$

  (assume N even)

- Take $\mathbf{S}$ with 2 sets of N/2-degenerate eigenvalues: $t = \pm 1$ to induce correct symmetry breaking

- Use (<u>regularized</u>) Itzykson-Zuber formula: (Itzykson & Zuber, '80)

$$\int d\mathbf{U} e^{JN \mathrm{Tr} \mathbf{M}\,\mathbf{S}} \propto \frac{1}{\Delta(\{\lambda\})} \sum_{\{\alpha\}\cup\{\alpha'\}=\{\lambda\}}{}^{'} e^{-JN \sum_j (\alpha_j - \alpha'_j)} \Delta(\{\alpha\}) \Delta(\{\alpha'\})$$

Sum over ways to partition eigenvalues of M according to degeneracies of S

Symmetry Breaking Term

$$\boxed{W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}}$$

$$\boxed{\begin{aligned} \mathrm{M} &= \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U} \\ \mathrm{S} &= \mathbf{V}^\dagger \mathbf{T} \mathbf{V} \end{aligned}}$$

- Calculate (dis-)order parameter:
  - $\frac{d}{dJ} \lim_{N\to\infty} W(J)\Big|_{J=0} = 0 \longrightarrow$ Symmetry is Broken!
  - Finite N: $\frac{dW(J)}{dJ}\Big|_{J=0} = \langle \mathrm{Tr}\,(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S}) \rangle \neq 0$

Eigenvectors misaligned

Symmetry Breaking Term

$$\boxed{W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\, \mathbf{T} - \mathbf{M}\, \mathbf{S})|}}$$

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

- Calculate (dis-)order parameter:
  - $\frac{d}{dJ} \lim_{N \to \infty} W(J) \Big|_{J=0} = 0$
  - Finite N: $\frac{dW(J)}{dJ} \Big|_{J=0} \neq 0$

Instantons:

- Pairs of eigenvalues tunneling between cuts
- Restore broken symmetries

$$\int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\, \mathbf{T} - \mathbf{M}\, \mathbf{S})|} \propto \mathcal{Z}_0 + \mathcal{Z}_1 \left(e^{-2JN|\lambda_j - \lambda_l'|} \right) + \ldots$$

Perspective: WCMM energy landscape

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} \; , \; V(\lambda) \overset{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa} \ln^2 |\lambda|$$

- WCMM partition function can be expanded on a large number of saddle point configurations (1 for WD models)
  - ➢ Each corresponds to a different multi-cut solution
  - ➢ Each corresponds to a different pattern of U(N) breaking
- Critical behavior of the model from interference between different saddles (instantons)
- Glassy behavior?

Conclusions

- Gap in the eigenvalue distribution (deviation from WD)
  $\Rightarrow$ Spontaneous breaking of rotational symmetry
- Not "localization", but loss of ergodicity: $\boxed{U(N) \overset{N\to\infty}{\to} \prod_{j=1}^{n} U(m_j)}$
- Criticality at gap opening as SSB phenomenon

Outlook

- Critical exponents, machinery for new "eigenvector" observables in invariant models
- Invariant models as toy model of Anderson MIT?
- Matrix SSB as Replica Symmetry Breaking?
- Generality of mechanism / string theory / additional applications?

Thank you!

Luttinger theory for RME

$$\rho(x,\tau) = \rho_0 - \frac{1}{\pi}\,\partial_x\Phi + \frac{A_K}{\pi}\cos\left[2\pi\rho_0 x - 2\Phi\right] + \ldots$$

Unfolding:

$\rho_0 = 1$

- Two-Point function (Kravtsov et al. '00):

$$\begin{aligned} Y_2 \quad = \quad & -\frac{1}{\pi^2}\langle\partial_x\Phi(x)\partial_{x'}\Phi(x')\rangle \\ & -\frac{A_K^2}{2\pi^2}\cos(2\pi(x-x'))\langle \mathrm{e}^{\mathrm{i}2\Phi(x)}\mathrm{e}^{-\mathrm{i}2\Phi(x')}\rangle + \ldots \end{aligned}$$

- In flat space: $\langle\Phi(x,t)\Phi(x',t')\rangle \propto \ln\left(\Delta x^2 + \Delta t^2\right)$

$$Y_2 \propto \frac{\sin^2\left[\pi(x-x')\right]}{(x-x')^2}$$

2-Point Function for Gaussian RME (K=1: Unitary)

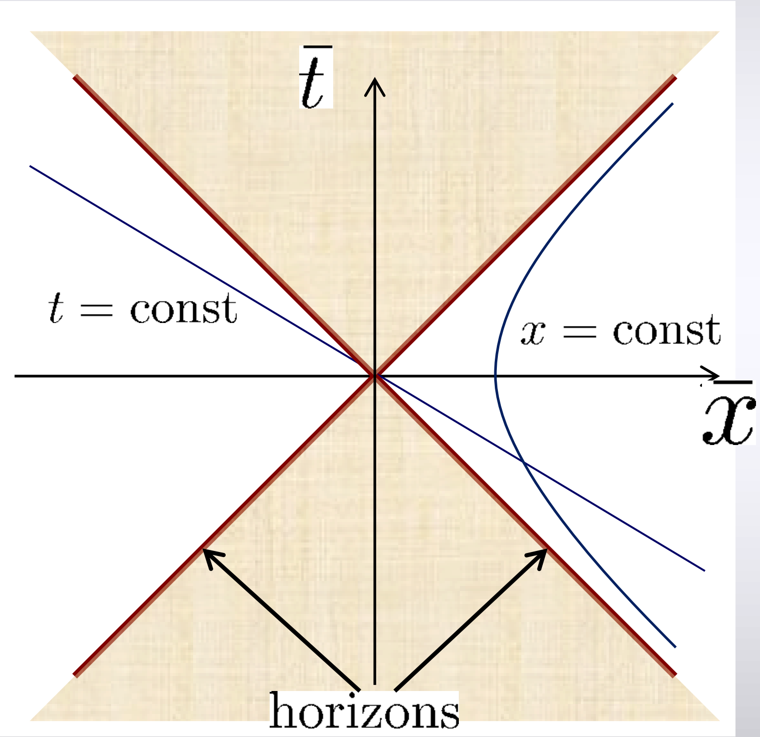
Luttinger theory in Rindler space

$$\begin{cases} \bar{t} & \equiv & \frac{1}{\kappa} \sinh \kappa x \, \sinh \kappa t \\ \bar{x} & \equiv & \frac{1}{\kappa} \sinh \kappa x \, \cosh \kappa t \end{cases}$$

Periodic in imaginary time

$\rightarrow$ finite **temperature**

$$\begin{aligned} ds^2 &= -\sinh^2(\kappa x) \, du^+ \, du^- \\ &= -\, d\bar{u}^+ \, d\bar{u}^- \\ \bar{u}^\pm &\equiv \bar{t} \pm \bar{x} \end{aligned}$$



- Far from the origin:

$$\bar{u}^\pm \simeq \begin{cases} \pm \frac{e^{\pm \kappa u^\pm}}{2\kappa}, & x \gg 1 \\ \mp \frac{e^{\pm \kappa u^\mp}}{2\kappa}, & x \ll -1 \end{cases}$$

Luttinger Liquid in Rindler Space

- Remind two-Point function:

$$Y_2 = -\frac{1}{\pi^2}\langle \partial_x \Phi(x) \partial_{x'} \Phi(x') \rangle - \frac{A_K^2}{2\pi^2} \cos(2\pi(x - x')) \langle \mathrm{e}^{\mathrm{i}2\Phi(x)} \mathrm{e}^{-\mathrm{i}2\Phi(x')} \rangle + \dots$$

- With the new coordinates: $\left(\bar{x} = \frac{\mathrm{e}^{\kappa |x|}}{2\kappa} \operatorname{sgn}(x) \right)$

$$\langle \Phi(x) \Phi(x') \rangle \overset{|x|,|x'| \gg 1}{\propto} \begin{cases} \ln \left[\frac{2}{\kappa} \sinh \frac{\kappa(x-x')}{2} \right], & x\, x' > 0 \\ \ln \left[\frac{2}{\kappa} \cosh \frac{\kappa(x+x')}{2} \right], & x\, x' < 0 \end{cases}$$

Luttinger Liquid in Rindler Space

- We recover exactly the RME correlation (K=1):

$$Y_2^a(x,x') = \frac{\kappa^2}{4\pi^2} \, \frac{\sin^2\left[\pi(x-x')\right]}{\cosh^2\left[\kappa(x+x')/2\right]} \,, \qquad \text{for} \;\; x\,x' < 0$$

(Anomalous: non-translational invariant)

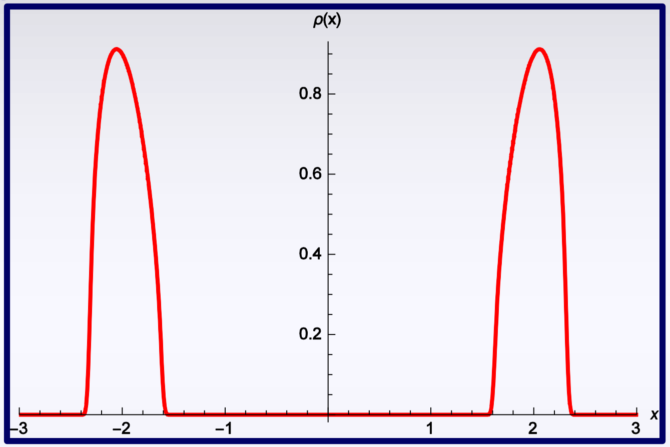
$$Y_2^n(x,x') = \frac{\kappa^2}{4\pi^2} \, \frac{\sin^2\left[\pi(x-x')\right]}{\sinh^2\left[\kappa(x-x')/2\right]} \,, \qquad \text{for} \;\; x\,x' > 0$$

(Normal: translational invariant)

Brownian Motion Picture

- Level repulsion resolves degeneracy:

  $\Rightarrow$ each of the n cuts contains m_j eigenvalues



- Gap between cuts breaks rotational

  invariance: $U(N) \overset{N\to\infty}{\to} \prod_{j=1}^{n} U(m_j)$

- Dyson Brownian Motion for equilibrium distribution shows scale separation:

$$d\lambda_j = -\frac{dV(\lambda_j)}{d\lambda_j}\, dt + \frac{1}{N}\sum_{l\neq j}\frac{dt}{\lambda_j-\lambda_l} + \frac{1}{\sqrt{N}}\, dB_j(t)$$

$$d\vec{U}_j(t) = -\frac{1}{2N}\sum_{l\neq j}\frac{dt}{(\lambda_j-\lambda_l)^2}\,\vec{U}_j + \frac{1}{\sqrt{N}}\sum_{l\neq j}\frac{dW_{jl}(t)}{\lambda_j-\lambda_l}\,\vec{U}_l$$

dB_j, dW_{jl} delta-corr. stochastic sources

Landau Zener Picture

- Qualitative picture on eigenvalue/eigenvector connection
- 2-level system: $\begin{pmatrix} \epsilon_1 & V \\ V^* & \epsilon_2 \end{pmatrix} \longrightarrow \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$

$$\delta = E_1 - E_2 = \sqrt{(\epsilon_1 - \epsilon_2)^2 + |V|^2}$$

"Localized"	"Extended"
$\boxed{V \ll \epsilon_1 - \epsilon_2}$	$\boxed{V \gg \epsilon_1 - \epsilon_2}$
$\delta \simeq \epsilon_1 - \epsilon_2$	$\delta \simeq \lvert V \rvert$
$\Psi_{1,2} \simeq \psi_{1,2} + \mathcal{O}\left(\frac{1}{\epsilon_1 - \epsilon_2}\right) \psi_{2,1}$	$\Psi_{1,2} \simeq \psi_{1,2} \pm \psi_{2,1}$