UNG-FN, Študijsko leto 2018/19
1. letnik - 1. stopnja Fizika in astrofizika

Linear Algebra: Exam (18/06/2019)

Each exercise has a value of 1 point. Each answer has to be properly justified.

Exercise 1. Let $\phi \in \text{End}(\mathbb{R}^3)$ be defined in the canonical basis $\mathcal{E}$ as:

$$\phi(x, y, z) = (3x - 2y + 4z, -2x + 6y + 2z, 4x + 2y + 3z)$$

- i (0.1 pt) Write the matrix $A = M_{\phi}^{\mathcal{E},\mathcal{E}}$.
- ii (0.2 pt) Write the characteristic polynomial $p_{\phi}(T)$ where $T \in \mathbb{R}$.
- iii (0.4 pt) Find the eigenvalues and a basis of eigenvectors of ϕ.
- iv (0.3 pt) Find an orthonormal basis of eigenvectors $\mathcal{B}$. Show by explicitly computing the matrix product that $D = {}^tPAP$ is the diagonal matrix with the eigenvalues of ϕ on the diagonal, provided that $P = M^{\mathcal{E},B}$ where $\mathcal{B}$ is the orthonormal basis of eigenvectors.

Exercise 2. A system of linear equation

$$\Sigma : \begin{cases} x_1 + ax_2 + (1 + 4a)x_3 & = 1 + 4a \\ 2x_1 + (a + 1)x_2 + (2 + 7a)x_3 & = 1 + 7a \\ 3x_1 + (a + 2)x_2 + (3 + 9a)x_3 & = 1 + 9a \end{cases}$$

is given where the unknowns $X = (x_1, x_2, x_3)$ and the parameter a are real numbers (in short notation $\Sigma : \ AX = b$).

- i (0.5 pt) Find the solution of Σ when $a = -1$ and provide the value of $det(A)$.
- ii (0.5 pt) Discuss the number of solutions of the system Σ depending on the parameter $a \in \mathbb{R}$.

Exercise 3. In the Euclidean affine space $\mathbb{E}^3$, a point P$= (1, -1, 2)$ and a line r:

$$r : \quad \{(x, y, z) = (t, 2t, 3t), \quad t \in \mathbb{R}\}.$$

are given.

- i (0.25 pt) Find the carthesian equation of the plane π which is orthogonal to the line r and contains the point P.
- ii (0.3 pt) Find the parametric equation of the line r' knowing that P$\in r'$ and $r' \perp r$ and $r'_{\cap} r \neq \emptyset$.

iii (0.25 pt) Find the carthesian equation of the plane π' which contains both lines r and r'.

iv (0.2 pt) Find the distance between the plane π and the point $T = (4, 1, 3)$.

Exercise 4. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ the linear map defined by:

$$f(x, y, z) = (x - y, -x + y, x + y + 2z).$$

Find:

i (0.1 pt) the matrix $M_f^{\mathcal{E},\mathcal{E}}$ where $\mathcal{E}$ is the canonical basis of $\mathbb{R}^3$;

ii (0.3 pt) a basis for $ker(f)$;

iii (0.2 pt) a basis for $Im(f)$;

iv (0.4 pt) the matrix $M_f^{\mathcal{B},\mathcal{E}}$ where $\mathcal{B}$ is the basis $\mathcal{B} = \{(1, 1, -1), (-1, 1, 0), (0, 0, 1)\}$

Exercise 5. Let's consider the bilinear symmetric form $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ of which the matrix w.r.t. the canonical basis is:

$$G = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

i (0.1 pt) Is g degenerate ?

ii (0.2 pt) Is g a scalar product ? (hint: check if g is positive definite).

iii (0.4 pt) Find a g-orthonormal basis $\mathcal{E}'$ for the bilinear symmetric form g.

iv (0.3 pt) If $H = M^{\mathcal{E}',\mathcal{E}}$ is the change of basis matrix, write how G transforms under the change of basis, and find the matrix G' of g with respect to the basis $\mathcal{E}'$.