UNG-FN, Študijsko leto 2018/19
1. letnik - 1. stopnja Fizika in astrofizika

Linear Algebra: Exam (02/09/2019)

Each exercise has a value of 1 point. Each answer has to be properly justified.

Exercise 1. Let $\phi \in \text{End}(\mathbb{R}^3)$ be defined in the canonical basis $\mathcal{E}$ as:

$$\phi(x, y, z) = (5x + 2y + 3z, 3x + 4y + 3z, -4x - 2y - 2z)$$

i (0.1 pt) Write the matrix $A = M_{\phi}^{\mathcal{E},\mathcal{E}}$.

ii (0.25 pt) Write the characteristic polynomial $p_{\phi}(T)$ where $T \in \mathbb{R}$, and provide the value of $det(A)$.

iii (0.5 pt) Find the eigenvalues and a basis $\mathcal{B}$ of eigenvectors of ϕ. Show that $\mathcal{B}$ is not an orthonormal basis.

iv (0.15 pt) Find the eigenvalues of A^2.

Exercise 2. A system of linear equations

$$\Sigma : \begin{cases} 2x_1 + x_2 + 2x_3 + x_4 &= 1 \\ 2x_1 + 3x_2 - x_3 &= 3 \\ x_1 + x_3 &= 0 \end{cases}$$

is given where the unknowns $X = (x_1, x_2, x_3, x_4)$ are real numbers (in short notation $\Sigma : \ AX = b$).

i (0.5 pt) Find the solutions S_0 of the associated homogeneous system $\Sigma_0 : \ AX = 0$

ii (0.3 pt) Write all the solutions of the system Σ.

iii (0.2 pt) Find a basis for the orthogonal complement $S_0^{\perp}$.

Exercise 3. The atoms of an ammonia molecule (NH_3) are found at certain moment to have, in a cartesian orthogonal reference system, the following coordinates:

$$\begin{aligned} \mathrm{N} &= (3, 0, 0) \\ \mathrm{H}^i &= (0, 0, 0) \\ \mathrm{H}^{ii} &= (4, 3, 0) \\ \mathrm{H}^{iii} &= (4, -2, -2) \end{aligned}$$

resulting in a pyramid shape with N being at the peak and the three H on the corners.

i (0.3 pt) Find the cartesian equation of the plane π which contains the nitrogen N, H^i and H^{ii} hydrogen atoms.

ii (0.3 pt) Find the angle between the edges NH^i and NH^{ii}

iii (0.3 pt) Find the cartesian equation of the line r passing through N and H^{ii}.

iv (0.1 pt) Find the parametric equation of the line r' passing through N and H^{iii}.

Exercise 4. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map:

$$f(x, y, z) = (x + y - z, 2x - y + z, 3x)$$

defined with respect to the canonical basis $\mathcal{E}$.

i (0.1 pt) write the matrix $A = M_f^{\mathcal{E},\mathcal{E}}$.

ii (0.1 pt) calculate the $det(A)$ and trace of A.

iii (0.4 pt) say if the vector $u = (3, 0, 3)$ belongs to $Im(f)$ i.e. $u \in Im(f)$?

iv (0.4 pt) find a basis for $ker(f)$.

Exercise 5. In $\mathbb{R}^4$ let's consider the following vectors

$$v_1 = (2, 1, 1, 1), \qquad v_2 = (2, 2, 1, 4), \qquad v_3 = (0, 0, 0, 1)$$

i (0.6 pt) By applying the Grahm-Schmidt method find an orthonormal basis $\{\xi_1, \xi_2, \xi_3\}$ for the subspace $W = \mathcal{L}\{v_1, v_2, v_3\}$ (Hint: use $\xi_1 = v_3$).

ii (0.3) Find the orthogonal complement $W^\perp$.

iii (0.1) Find the orthogonal projection on W and on $W^\perp$ of the vector $v = (6, 4, 7, 5)$.