UNG-FN, Študijsko leto 2018/19

1. letnik - 1. stopnja Fizika in astrofizika

Linear Algebra: Exam (03/07/2019)

Each exercise has a value of 1 point. Each answer has to be properly justified.

Exercise 1. Let $\phi \in \text{End}(\mathbb{R}^3)$ be defined in the canonical basis $\mathcal{E}$ as:

$$\phi(x, y, z) = (4x + y + 2z, 2x + 3y + 2z, -3x - y - z)$$

- i (0.1 pt) Write the matrix $A = M_\phi^{\mathcal{E},\mathcal{E}}$.
- ii (0.25 pt) Write the characteristic polynomial $p_\phi(T)$ where $T \in \mathbb{R}$, and provide the value of $det(A)$.
- iii (0.5 pt) Find the eigenvalues and a basis $\mathcal{B}$ of eigenvectors of ϕ. Show that $\mathcal{B}$ is not an orthonormal basis.
- iv (0.15 pt) Find the eigenvalues of ϕ^{-1}.

Exercise 2. A system of linear equations

$$\Sigma : \begin{cases} x_1 - 2x_2 + x_4 = -1 \\ x_2 + 2x_3 - x_4 = 1 \\ x_1 - x_2 + 2x_3 = \alpha \\ x_1 + 4x_3 - x_4 = 1 \end{cases}$$

is given where the unknowns $X = (x_1, x_2, x_3, x_4)$ and the parameter α are real numbers (in short notation $\Sigma : \ AX = b$).

- i (0.2 pt) Find the value of α so that a solution of Σ is (1,1,0,0)
- ii (0.6 pt) Find the solutions of the associated homogeneous system $\Sigma_0 : \ AX = 0$
- iii (0.2 pt) Write all the solutions of the system Σ for the α value found in the first request (i).

Exercise 3. The atoms of an ammonia molecule (NH_3) are found at certain moment to have, in a cartesian orthogonal reference system, the following coordinates:

$$\begin{aligned} \mathrm{N} &= (3, 0, 0) \\ \mathrm{H}^{i} &= (0, 0, 0) \\ \mathrm{H}^{ii} &= (4, 3, 0) \\ \mathrm{H}^{iii} &= (4, -2, -2) \end{aligned}$$

resulting in a pyramid shape with N being at the peak and the three H on the corners.

i (0.3 pt) Find the cartesian equation of the plane π which contains the three hydrogen atoms.

ii (0.2 pt) Find the distance between the plane π and the N atom.

iii (0.2 pt) Find the parametric equation of the line r passing through H^{ii} and H^{iii}.

iv (0.3) Find the volume of the parellelepiped with sides (H^iH^{ii},H^iH^{iii},H^iN).

Exercise 4. Let $f : \mathbb{R}[X]_{\leq 2} \to \mathbb{R}[X]_{\leq 2}$ be the linear map defined in the basis $\mathcal{B} = \{1, X, X^2\}$ as:

$$f(a + bX + cX^2) = a - b + (2a - 2b)X + (a + b - 2c)X^2$$

Find:

i (0.25 pt) the matrix $M_f^{\mathcal{B},\mathcal{B}}$ and $det(M_f^{\mathcal{B},\mathcal{B}})$;

ii (0.25 pt) a basis for $ker(f)$. Is f injective ?

iii (0.25 pt) a basis for $Im(f)$; Is f surjective ?

iv (0.25 pt) the matrix of $g \circ f$ with respect to $\mathcal{B}$, for a g defined as:

$$g(a + bX + cX^2) = a + b + (a + b)X + (a - b + 2c)X^2$$

Exercise 5. In $\mathbb{R}^3$ let's consider the following basis $\mathcal{B} = \{b_1, b_2, b_3\}$:

$$\begin{cases} b_1 &= \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2 \\ b_2 &= -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2 \\ b_3 &= ae_3 \end{cases}$$

where $a \in \mathbb{R}$ and where $\mathcal{E} = \{e_1, e_2, e_3\}$ is the canonical basis.

i (0.2 pt) Write the matrix $R_a = M^{\mathcal{E},\mathcal{B}}$ and find $det(M^{\mathcal{E},\mathcal{B}})$;

ii (0.4) For which a does R_a belongs to the group SO(3) ?

iii (0.15) For the value of a as in the previous point, write R_a^{-1}

iv (0.25) Calculate, for a as in (ii), the length of the vector $v = 2e_1 - e_2 + 4e_3$ in the basis $\mathcal{E}$. Write then v and $||v||$ in the basis $\mathcal{B}$.