UNG-FN, Študijsko leto 2019/20
1. letnik - 1. stopnja Fizika in astrofizika
Linear Algebra written exam
19/6/2020
Lecturers: L. Giacomazzi, Z. Benher.

Questions and exercise 1 are mandatory. You can then choose one other exercise among exercise 2 and 3.

Q. 1. [3 pt] Show that the set of symmetric square matrices 2x2 i.e.

$$\{A \in \mathbb{R}^{2,2} \quad | \quad {}^tA = A\}$$

equipped with the usual sum and product by a scalar, is a vector space of dimension 3.

Q. 2. [3 pt] In the affine space $\mathbb{A}^3$ two points are given $Q = (1, 2, 1)$ and $A = (3, -1, 2)$. Provide the cartesian equation of the plane π such that $Q \in \pi$ and the vector $A - Q$ is along the normal direction to π.

Q. 3. [2 pt] Say if the following bilinear form $p : \mathbb{R}[X]_2 \times \mathbb{R}[X]_2 \to \mathbb{R}$ defined as

$$p(a + bX + cX^2, a' + b'X + c'X^2) = cc'$$

(a, b, c, a', b', c' are real numbers) is a scalar product. Justify your answer.

Exercise 1. Let V be the vector space of symmetric square matrices in $\mathbb{R}^{2,2}$ and $\mathcal{B} = \{v_1, v_2, v_3\}$ a basis of V. An endomorphism $\phi : V \to V$ is such that $\phi(v_1) = 3v_1 + v_3$, $\phi(v_2) = 2v_2$, $\phi(v_3) = v_1 + 3v_3$.

i) [2 pt] Write the matrix $A = M_\phi^{\mathcal{B},\mathcal{B}}$

ii) [5 pt] Write the characteristic polynomial $p_\phi(\lambda)$

iii) [5 pt] Find the eigenvalues λ_j and eigenvectors ξ_j of ϕ.

iv) [2 pt] Write the eigenvectors ξ_j explicitly as matrices of $\mathbb{R}^{2,2}$

Exercise 2. Given the linear application $f : \mathbb{R}^3 \to \mathbb{R}^4$ which with respect to canonical basis is defined as:

$$f(x, y, z) = (x - 2z, 2y + 2z, x + 3y + z, y + z)$$

i) [2 pt] Write the matrix $A = M_f^{\mathcal{E}_4,\mathcal{E}_3}$

ii) [4 pt] Find the dimension of $Im(f)$ and a basis for $Im(f)^\perp$

iii) [3 pt] find the dimension and a basis of $ker(f)$.

iv) [2 pt] calculate $G = {}^tAA$

v) [2 pt] Write the matrix $A' = M_f^{\mathcal{E}_4,\mathcal{B}}$ where $\{\mathcal{B} = \{v_1, v_2, v_3\}$ and $v_1 = e_1 - e_2$, $v_2 = e_2 + e_3$, $v_3 = e_3 - 2e_1$.

Exercise 3. Consider the subspace of $\mathbb{R}^4$ given by $W = \mathcal{L}(v_1, v_2, v_3, v_4)$ where $v_1 = (2, 1, 0, 1)$, $v_2 = (-2, -1, 1, 1)$, $v_3 = (4, 2, 0, 2)$, $v_4 = (0, 0, 1, 2)$.

i) [2 pt] Do the vectors v_1, v_2, v_3, v_4 form a basis of $\mathbb{R}^4$?

ii) [5 pt] Using Gram-Schmidt, find an orthonormal basis $\mathcal{B}$ for the subspace W [hint: choose $\xi_1 = v_4/||v_4||$].

iii) [5 pt] Find the orthogonal projection of the vector $v = (1, 1, 0, 0)$ on the subspace U generated by the vectors of the set $\{\mathcal{B}\} \cup \{(0, 0, 0, 1)\}$. [hint: find $U^\perp$.]