Linear Algebra, lesson 4

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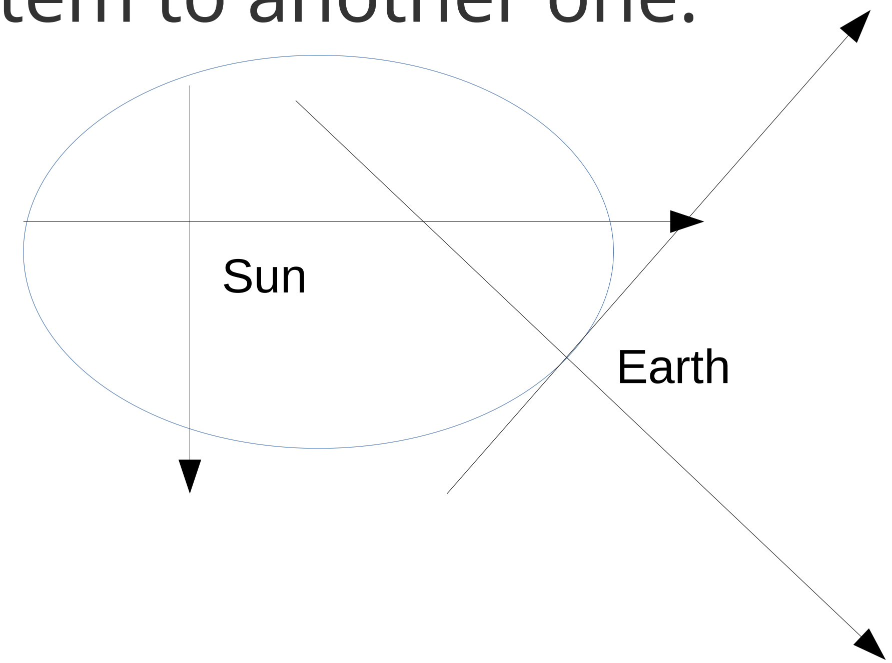
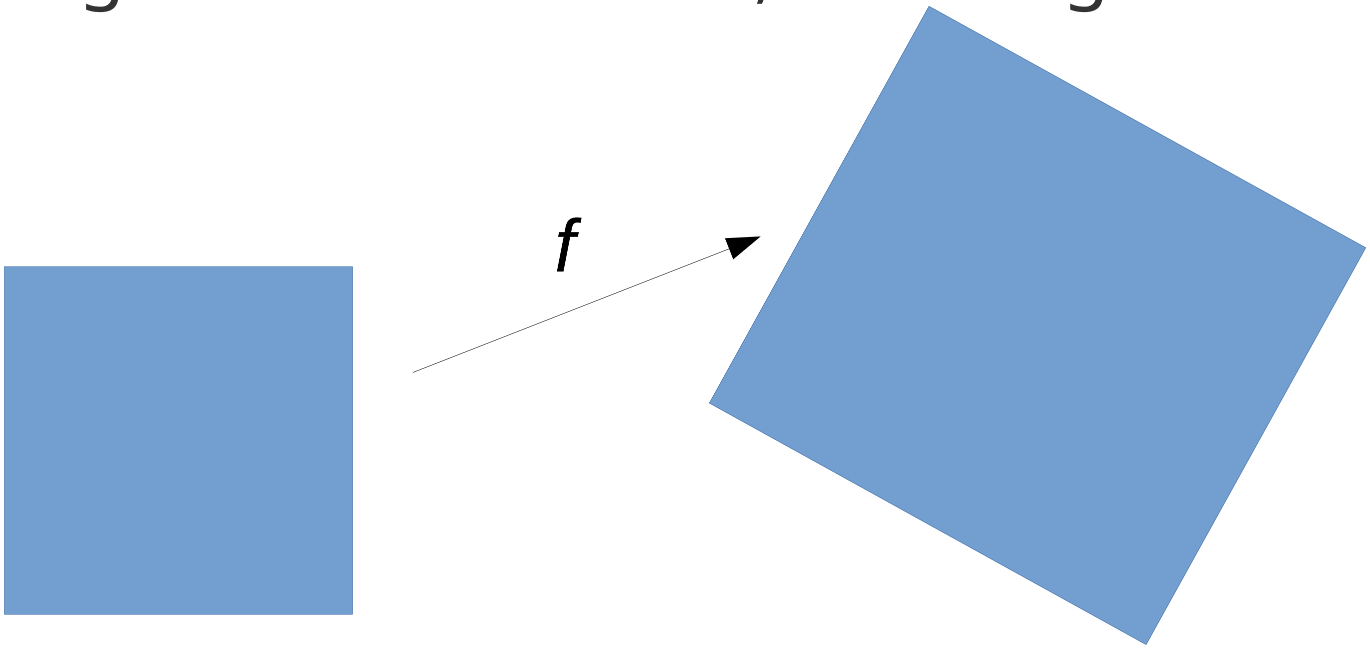
Linear transformations

- We need mathematical tools to describe the transformation of vectors from a reference system to another one:



Linear transformations

- We need tools to describe, for example, change of figures like rotations, rescaling:



Linear maps and matrices

- Consider f: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined as

  $f(x_1,x_2)=(ax_1+bx_2,\ cx_1+dx_2)$ with a,b,c,d real numbers
- Using the **row by column** product one could write it as:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1+bx_2 \\ cx_1+d\,x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- In short notation $f(X)=AX$, where X is a vector of $\mathbf{R}^2$ and A is an element of $\mathbf{R}^{2,2}$

Linear maps and matrices

- We denote with $\mathbf{R}^{m,n}$ the collection of all $m \times n$ matrices A whose entries are real numbers:

  [See Chapt 4.1 Landi's book]

$$A = \begin{pmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & ... & ... & a_{mn} \end{pmatrix}$$

- $\mathbf{R}^{m,n}$ is a **real vector space** with dim ***mn:***
- $A+B=(a_{ij}+b_{ij})$ *and* $\lambda A=(\lambda a_{ij})$

Linear maps and matrices

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Linear maps and matrices

Consider $X={}^t(x_1,x_2)$ *and* $Y={}^t(y_1,y_2)$ *then it is clear that* ***f(X+Y)=f(X)+f(Y):***

$$f\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}=\begin{pmatrix} a(x_1+y_1)+b(x_2+y_2) \\ c(x_1+y_1)+d(x_2+y_2) \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Also for any real number λ it is $f(\lambda X)=\lambda f(X)$

Such a *f* is a ***linear map or linear transformation or linear application***

Linear maps and matrices

Example. Consider f: $\mathbf{R}^3 \to \mathbf{R}^2$ defined as

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + x_3 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

f=AX is a linear map, with A belonging to $\mathbf{R}^{2,3}$

Linear maps and matrices

Consider a matrix *A=(a$_{ij}$)* of $\mathbf{R}^{mn}$ we can define a linear map f_A: **V** → **W** , with *B={v_1,…,v_n}* basis of V and *C={w_1, …,w_m}* basis of W , then taken a *vector v in V with* $v=x_1v_1+....+x_nv_n$ ***we have $f_A(v)=y$ ϵ W i.e.***

$f_A(v)=y_1w_1+...y_mw_m=[Av]_C$

Where:

$$f_A(v)=\begin{pmatrix} y_1 \\ \dots \\ \dots \\ y_m \end{pmatrix}_C = \left(\begin{pmatrix} a_{11} & a_{12} & \dots. & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & \dots & \dots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ \dots \\ \dots \\ x_n \end{pmatrix}_B\right)_C$$

Linear maps and matrices

Consider a linear map f: **V** → **W** we can define a matrix $M_f^{C,B}$ of $\mathbf{R}^{mn}$ with $B=\{v_1,\ldots,v_n\}$ basis of V and $C=\{w_1,\ldots,w_m\}$ basis of W. Taken a *vector v in V with* $v=x_1v_1+\ldots.+x_nv_n$ *we have* $\mathbf{y=f(v)=x_1f(v_1)+\ldots.+x_nf(v_n)} \in$ **W. Now** $\mathbf{f(v_j)=a_{1j}w_1+\ldots a_{mj}w_m}$ **can be thought as the column of a matrix** $\mathbf{M_f^{C,B}=[f(v_1),\ldots,f(v_n)]_C}$ **so that:**

$$\begin{pmatrix} y_1 \\ \ldots \\ \ldots \\ y_m \end{pmatrix}_C = \left(\begin{pmatrix} a_{11} & a_{12} & \ldots. & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \ldots \\ \ldots \\ x_n \end{pmatrix}_B \right)_C$$

Linear maps and matrices

Consider a linear map *f*: **V** → **W** we can define a matrix A=$M_f^{C,B}$ of $\mathbf{R}^{mn}$ with *B*={v_1,…,v_n} basis of V and *C*={w_1,…,w_m} basis of W. **It turns out** *[See Landi's book, chapt. 7, prop. 7.1.13]* **that f_A=*f* *and also for A in* $\mathbf{R}^{mn}$** by setting ***f*=f_A *it is* $M_f^{C,B}$=*A*.**

*Hence it is clear that a **linear map** corresponds to a **matrix** and viceversa.*

Linear maps and matrices

- Example: $id : V_B \rightarrow V_E$, $id: v \rightarrow id(v)=v$, with basis $B=\{v_1,...,v_n\}$ and $E=\{e_1,...,e_n\}$ *then it is*
- $M_{id}^{E,B}=[id(v_1),....,id(v_n)]_E=[v_1,....,v_n]_E$
- $M_{id}^{B,E}=[id(e_1),....,id(e_n)]_B=[e_1,....,e_n]_B$
- $e_1=a_{11}v_1+...a_{n1}v_n$
- $e_k=a_{1k}v_1+...a_{nk}v_n$

Linear maps [Chapt. 7.3]

- Kernel of a linear map $\boldsymbol{f:V \to W}$:

  Ker(f)={***v*** *in V* | ***f(v) = 0***$_W$}

  v,w are in the kernel, then f(av+bw)=af(v)+bf(w)=0

- *Image or range of a linear map*

  Im(f)={***w*** *in W* | there is *v* in *V* with ***w =f(v)*** }

- *Ker(f)* and *Im(f)* are **vector subspaces** of V and W
- ***f* is injective** if and only if ***Ker(f)={0***$_V$***}***
- ***f* is surjective** *if and only if* ***f(v***$_1$***),....,f(v***$_n$***) generate W***