Linear Algebra, lesson 5

L. Giacomazzi, 5/3/2020

Linear maps [Chapt. 7.3]

- *Ker(f)* and *Im(f)* are **vector subspaces** of V and W
- Kernel of a linear map ***f*:*V*** → ***W*** :

  Ker(f)={***v in V*** | ***f(v) = 0***$_W$}

  v,w are in the kernel, then f(av+bw)=af(v)+bf(w)=0
- *Image or range of a linear map*

  Im(f)={***w in W*** **| there is *v* in *V* with *w* =*f(v)*** }
- *w_1 ,w_2 in Im(f) means w_1=f(v_1) and w_2=f(v_2) then $aw_1+bw_2=af(v_1)+bf(v_2)=f(av_1+bv_2)$ is still in Im(f)*
- ***f* is surjective** *if and only if* ***f(v_1),….,f(v_n) generate W***

Linear maps [Chapt. 7.2]

- ***Composition of maps***

  $f: V \to W$ and $g: W \to Z$ then $g \circ f : V \to Z$ and takes a vector v to $g(f(v))$. If g and f are invertible then

  $[g \circ f]^{-1} = f^{-1} \circ g^{-1}$

- *Note that if A, B are the matrices [of $GL(n)$] associated to f and g, then BA is associated to $g \circ f$, and $A^{-1}B^{-1}$ is associated to $[g \circ f]^{-1}$, so that $(BA)^{-1} = A^{-1}B^{-1}$*

Linear maps [Lemma 7.3.5]

- Let f be a linear map $f:V \rightarrow W$, then
- ***f* is injective** if and only if $Ker(f)=\{0_V\}$

"=>" suppose f is injective and by absurd there is a vector v $\in$ ker(f) not zero. Then $f(v)=0_W$ *but also as f is a linear map, it is:* $f(0*v)=0*f(v)=0_W$ *. But* $f(0*v)=f(0_V)$ *Thus* $f(0_V)=0_W$

Now injective means that ***if f(x1)=f(x2) then x1=x2***

this implies that $v=0_V$

Linear maps [Lemma 7.3.5]

- Let f be a linear map $f: V \to W$, then
- ***f* is injective** if and only if $Ker(f)=\{0_V\}$

"<=" suppose $ker(f)=\{0_V\}$. Let's take v1 and v2 such that $f(v1)=f(v2)$, by linearity it is

$$f(v1)-f(v2)=f(v1-v2)=0_W$$

This means that v1-v2 belongs to the kernel i.e.

$$v1-v2=0_V$$

this implies that ***v1=v2, and thus f is injective***

Linear maps [Lemma 7.3.5]

- Let f be a linear map $f:V \to W$, then
- ***If f is injective and $\{v_1, \ldots, v_n\}$ is a basis for V, then the vectors $f(v_1), \ldots, f(v_n)$ are linearly independent***

"dim" *Let's consider a linear combination*

$a_1 f(v_1) + \ldots + a_n f(v_n) = 0_W$ with a_k real numbers. By linearity:

$f(a_1 v_1 + \ldots + a_n v_n) = 0_W$ in other words $a_1 v_1 + \ldots + a_n v_n \in \ker(f)$

and since f is injective it is $a_1 v_1 + \ldots + a_n v_n = 0_V$

But as $\{v_1, \ldots, v_n\}$ is a basis it means $a_k = 0$ for any k,

- *Thus $f(v_1), \ldots, f(v_n)$ are linearly independent*

Linear maps [dim theorem]

- Let f be a linear map ***f:V → W*** , then
- ***dim(V)=dim(Ker(f))+dim(Im(f))***
- i.e. ***dim(V)=dim(Ker(f))+rk(A)*** where A is the associated matrix to *f*

"dim" to be done ! Next lesson...

- ***Useful for exercises for finding dim(Ker(f)):***

dim(Ker(f))=dim(V)-rk(A) = n-rk(A)

- ***Note that rk(A) is usually straightforward to find (it's number of lin. indep. columns of A)***

Linear maps [dim. Theorem]

- Example with ***n=3=dim(V)***

$$A=\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$$

- ***rk(A)=2****=dim(R(A))=dim(row space)*
- ***dim(Ker(f))=3-2=1***
- *So you know that a basis of Ker(f) has only 1 vector*

  Ker(f)=L{(-1,2,-1)}

Linear maps and matrices

Example. Consider f: $\mathbf{R}^3 \to \mathbf{R}^2$ defined as

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + x_3 \\ x_1 - x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

dim(V)=3

rk(A)=2, so dim(ker(f))=3-2=1

ker(f)=L{(1,1,-3)}