Chapter 1

Riemann surfaces

1.1 Definition of a Riemann surface and basic examples

In its broadest sense a Riemann surface is a one dimensional complex manifold that locally looks like an open set of the complex plane, while its global topology can be quite different from the complex plane. The main reason why Riemann surfaces are interesting is that one can speak of complex functions on a Riemann surface as much as the complex function on the complex plane that one encounters in complex analysis.

Elementary example of Riemann surfaces are the complex plane $\mathbb{C}$, the disk

$$D = \{z \in \mathbb{C},\ |z| < 1\}$$

or the upper half space

$$H = \{z \in \mathbb{C},\ \Im(z) > 0\}.$$

B. Riemann introduced the concept of Riemann surface to make sense of multivalued functions like the square root or the logarithm. For the geometric representation of multi-valued functions of a complex variable $w = w(z)$ it is not convenient to regard z as a point of the complex plane. For example, take $w = \sqrt{z}$. On the positive real semiaxis $z \in \mathbb{R}, \quad z > 0$ the two branches $w_1 = +\sqrt{z}$ and $w_2 = -\sqrt{z}$ of this function are well defined by the condition $w_1 > 0$. This is no longer possible on the complex plane. Indeed, the two values $w_{1,2}$ of the square root of $z = r\,e^{i\psi}$

$$w_1 = \sqrt{r}\,e^{i\frac{\psi}{2}}, \quad w_2 = -\sqrt{r}\,e^{i\frac{\psi}{2}} = \sqrt{r}e^{i\frac{\psi+2\pi}{2}}, \tag{1.1}$$

interchange when passing along a path

$$z(t) = r\,e^{i\,(\psi+t)}, \quad t \in [0, 2\pi]$$

encircling the point $z = 0$. It is possible to select a branch of the square root as a function of z by restricting the domain of this function for example, by making a cut from zero to infinity. Namely the function $\sqrt{z}$ is single-valued in the cut plane $\mathbb{C}\backslash[0, +\infty)$. Riemann's idea was to combine the two branches of the function $\sqrt{z}$ in a geometric space in such a way that the function is well defined and single-valued. The rules are as follows: one has to take two copies of the complex plane cut along the positive real axis and join the two copies of the complex plane along the cuts. The different sheets have to be glue together in such a way that the branch of the function on one sheet joins continuously with the branch defined on the other sheet. The result of this operation is the surface in figure 1.1.

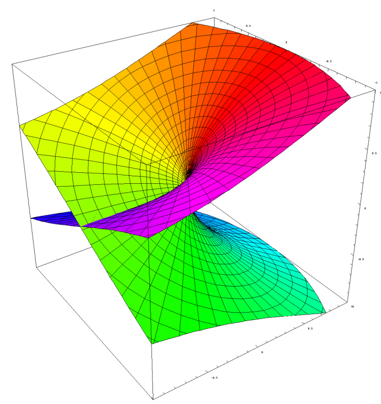

Figure 1.1: The two branches of the function $\sqrt{z}$

Note that such surface can be given for $(w, z) \in \mathbb{C}^2$ as the zero locus

$$F(z, w) = w^2 - z = 0.$$

A similar procedure of cutting and glueing can be repeated for any other analytic function. For example the logarithm $\log z$ is a single valued function on $\mathbb{C}\backslash[0, +\infty)$ with infinite branches. Each adjacent branch differs by an additive factor $2\pi i$. The infinite branches attached along the positive real line are shown in the figure 1.2.

Next we will give a more abstract definition of a Riemann surface and we will show how the surface defined by the graph of a multivalued function fits in this definition.

Definition 1.1.1. *A Riemann surface* Γ *is defined by the following data:*

- *a connected Hausdorff topological space* Γ*;*
- *an open cover* $\{U_\alpha\}_{\alpha \in A}$ *of* Γ*;*

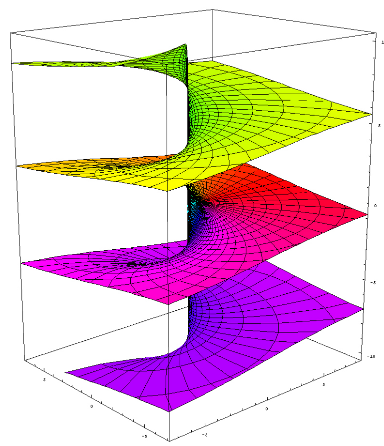

Figure 1.2: The infinite branches of the function log z

- *for each* $\alpha \in A$, *a homeomorphism* ϕ_α

$$\phi_\alpha : U_\alpha \to V_\alpha$$

to an open subset $V_\alpha \subset \mathbb{C}$ *in such a way that for each* $\alpha, \beta \in A$, *if* $U_\alpha \cap U_\beta \neq \emptyset$, *the transition functions*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta),$$

is bi-holomorphic, namely, holomorphic with inverse holomorphic.

Remark 1.1.2. Let us observe that the sets $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$, are subsets of the complex plane, and therefore the request of having holomorphic maps between these two subsets makes sense.

The pair $\{U_\alpha, \phi_\alpha\}$ is called complex chart. Complex charts are also called *local parameters* or *local coordinates*. Two charts (U_α, ϕ_α) and (U_β, ϕ_β) are compatible if either $U_\alpha \cap U_\beta = \emptyset$ or the transition function $\phi_\beta \circ \phi_\alpha^{-1}$ is bi-holomorphic. If all the complex charts $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ are compatible, they form a complex atlas $\mathcal{A}$ of Γ. Two complex atlas $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are compatible if their union $\mathcal{A} \cup \tilde{\mathcal{A}}$ is a complex atlas. The equivalence class of complex atlas is called a *complex structure* or also a *conformal structure.* With the definition of complex structure we can define a Riemann surface in the equivalent way.

Definition 1.1.3. *A Riemann surface is a connected one-complex dimensional analytic manifold, or a two real dimensional connected manifold with a complex structure on it.*

Let ϕ and $\tilde{\phi}$ be two local homeomorphism from two open sets U and $\tilde{U}$ of Γ with $U \cap \tilde{U} \neq \emptyset$. Let P and P_0 two points in $U \cap \tilde{U}$ and denote by $z = \phi(P)$ and $w = \tilde{\phi}(P)$ the two local coordinates with $z_0 = \phi(P_0)$ and $w_0 = \tilde{\phi}(P_0)$. Then the holomorphic transition function $T = \phi \circ \tilde{\phi}^{-1}$ must be of the form

$$z = T(w) = T(w_0) + \sum_{k>0} a_k (w - w_0)^k, \quad a_1 \neq 0 \tag{1.2}$$

with holomorphic inverse

$$w = T^{-1}(z) = T^{-1}(z_0) + \sum_{k>0} b_k (z - z_0)^k, \quad b_1 \neq 0,$$

namely the linear coefficient of the above Taylor expansions near the point w_0 or z_0 is necessarily nonzero.

Remark 1.1.4. We recall that that a manifold is called orientable if it has an atlas whose transition functions have positive Jacobian determinants. If Γ is a Riemann surface, then the manifold Γ is orientable. Indeed let $z = x + iy$ be a local coordinate in some open neighbourhood of z_0 in Γ. Another local coordinate $w = u + iv$ is connected with the first by a holomorphic change of variable $w = T(z)$ with $w_0 = T(z_0)$ which thus determines a smooth change of real coordinates. We want to show that the determinant

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = u_x v_y - u_y v_x$$

calculated in (x_0, y_0) is positive. We observe that $w = w(z)$ is a holomorphic function of z and $\frac{dw}{dz}|_{z=z_0} \neq 0$. We can use Cauchy Rieamnn equations $u_x = v_y$ and $u_y = -v_x$ to write $\frac{dw}{dz} = u_x - iu_v$ and $\frac{d\overline{w}}{d\bar{z}} = u_x + iu_v$ to conclude that

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\Bigg|_{\substack{x=x_0\\y=y_0}} = (u_x^2 + u_y^2)|_{\substack{x=x_0\\y=y_0}} = \left|\frac{dw}{dz}\right|^2_{z=z_0} > 0.$$

Example 1.1.5. Elementary examples of Riemann surfaces

(a) The complex plane $\mathbb{C}$. The complex atlas is define by one chart that is $\mathbb{C}$ itself with the identity map.

(b) The extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$, namely the complex plane $\mathbb{C}$ with one extra point ∞. We make $\bar{\mathbb{C}}$ into a Riemann surface with an atlas with two charts:

$$U_1 = \mathbb{C}$$
$$U_2 = \bar{\mathbb{C}} \backslash \{0\},$$

with ϕ_1 the identity map and

$$\phi_2(z) = \begin{cases} 1/z, & \text{for } z \in \mathbb{C}\backslash\{0\} \\ 0, & \text{for } z = \infty. \end{cases}$$

1.1.1 Affine plane curves

Let us consider a polynomial $F(z,w) = \sum_{i=1}^{n} a_i(z)w^i$ of two complex variables z and w. The zero set $F(z,w)$ defines a n-valued function $w = w(z)$. The basic idea of Riemann surface theory is to replace the domain of the function $w(z)$ by its graph

$$\Gamma := \{(z,w) \in \mathbb{C}^2 \mid F(z,w) = \sum_{i=0}^{n} a_i(z)w^{n-i} = 0\} \tag{1.3}$$

and to study the function w as a single-valued function on Γ rather then a multivalued function of z. As in the example of $\sqrt{z}$, the multivalued function $w = w(z) = \sqrt{z}$ becomes a single-valued function $w = w(P)$ of a point P of the algebraic surface Γ: if $P = (z,w) \in \Gamma$, then $w(P) = w$ (the projection of the graph on the the w-axis). From the real point of view the algebraic curve (1.3) is a two-dimensional surface in $\mathbb{C}^2 = \mathbb{R}^4$ given by the two equations

$$\left.\begin{array}{l} \Re F(z,w) = 0 \\ \Im F(z,w) = 0 \end{array}\right\}.$$

In the theory of functions of a complex variable one encounters also more complicated (nonalgebraic) curves, where $F(z,w)$ is not a polynomial. For example, the equation $e^w - z = 0$ determines the surface of the logarithm or $\sin w - z = 0$ determines the surface of the arcsin. Such surfaces will not be considered here.

Definition 1.1.6. *An affine plane curve Γ is a subset in $\mathbb{C}^2$ defined by the equation (1.3) where $F(z,w)$ is polynomial in z and w. The curve Γ is nonsingular if for any point $P_0 = (z_0, w_0) \in \Gamma$ the complex gradient vector*

$$grad_{\mathbb{C}} F|_{P_0} = \left.\left(\frac{\partial F(z,w)}{\partial z}, \frac{\partial F(z,w)}{\partial w}\right)\right|_{(z=z_0, w=w_0)}$$

does not vanish. If the polynomial $F(z,w)$ is irreducible, the curve Γ is called irreducible affine plane curve.

Remark 1.1.7. A non trivial theorem states that an irreducible affine plane curve is connected (see Theorem 8.9 in O. Forster, Lectures on Riemann surfaces, Springer Verlag 1981).

In order to define a complex structure on Γ we need the following complex version of the implicit function theorem.

Lemma 1.1.8. *[Complex implicit function theorem] Let $F(z,w)$ be an analytic function of the variables z, w in a neighborhood of the point $P_0 = (z_0, w_0)$ such that $F(z_0,w_0) = 0$ and $\partial_w F(z_0,w_0) \neq 0$. Then there exists a unique function $\phi(z)$ such that $F(z,\phi(z)) = 0$ and $\phi(z_0) = w_0$. This function is analytic in z in some neighborhood of z_0.*

Proof. Let $z = x+iy$ and $w = u+iv$, $F = f+ig$. Then the equation $F(z,w) = 0$ can be written as the system

$$\begin{cases} f(x,y,u,v) &= 0 \\ g(x,y,u,v) &= 0 \end{cases} \tag{1.4}$$

The condition of the real implicit function theorem are satisfied for this system: the matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\[2ex] \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{pmatrix}_{(z_0,w_0)}$$

is nonsingular because

$$\det \begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\[2ex] \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{pmatrix} = \left| \frac{\partial F}{\partial w} \right|^2 > 0,$$

(we use only the analyticity in w of the function $F(z,w)$). Thus, in some neighbourhood of (z_0,w_0) there exist a smooth function $\phi(z,\bar{z}) = \phi_1(x,y) + i\phi_2(x,y)$ such that $F(z,\phi(z,\bar{z})) = 0$, with $\phi(z_0,\bar{z}_0) = w_0$. Differentiating with respect to $\bar{z}$

$$0 = \frac{d}{d\bar{z}} F(z,\phi(z,\bar{z})) = F_w \frac{d}{d\bar{z}} \phi(z,\bar{z}).$$

Since $F_w \neq 0$, the above relation implies that $\frac{d}{d\bar{z}}\phi(z,\bar{z}) = 0$ which shows that $\phi(z)$ is an analytic function of z. A constructive way of obtaining the function $\phi(z)$ is to apply the Residue Theorem. Indeed let us consider the function $F(z,w)$ where z is considered a parameter. Let D_0 be a small disk around w_0 where $F(z_0, w_0) = 0$ and $F_w(z_0, w)|_{w=w_0} \neq 0$. Then the number of solutions of the equation $F(z_0, w) = 0$ counted with multiplicity is given by the integral

$$\frac{1}{2\pi i}\int_{\partial D_0} \frac{F_w(z_0, w)}{F(z_0, w)} dw,$$

where ∂D_0 is the boundary of D_0. We assume D_0 sufficiently small so that the equation $F(z_0, w) = 0$ has only the solution w_0 in the closure of D_0. Then the above integral is equal to one. Furthermore one has applying the residue theorem

$$\frac{1}{2\pi i}\int_{\partial D_0} w\frac{F_w(z_0, w)}{F(z_0, w)} dw = w_0.$$

By continuity, for z sufficiently close to z_0 there is a disk D centred at w such that the equation $F(z, w) = 0$ has only one solution $w = \phi(z)$ in the closure of D and

$$\frac{1}{2\pi i}\int_{\partial D} w\frac{F_w(z, w)}{F(z, w)} dw = \phi(z),$$

where $\phi(z_0) = z_0$ and $F(z, \phi(z)) = 0$. Clearly the function $\phi(z)$ is analytic function of z. □

Theorem 1.1.9. *Let Γ be an irreducible affine plane curve defined in (1.3). If Γ is non singular, then Γ is a Riemann surface.*

Proof. Γ is connected since $F(z, w)$ is irreducible. Let us define a complex structure on Γ. Let $P_0 = (z_0, w_0)$ be a nonsingular point of the surface Γ. Suppose, for example, that the derivative $\frac{\partial F}{\partial w}$ is nonzero at this point. Then by the lemma 1.1.8, in a neighborhood U_0 of the point P_0, the surface Γ admits a parametric representation of the form

$$(z, w(z)) \in U_0 \subset \Gamma, \quad w(z_0) = w_0, \tag{1.5}$$

where the function $w(z)$ is holomorphic. Therefore, in this case z is a complex local coordinate also called *local parameter* on Γ in a neighborhood U_0 of $P_0 = (z_0, w_0) \in \Gamma$. For this kind of local coordinate, the transition function is the identity.

Similarly, if the derivative $\frac{\partial F}{\partial z}$ is nonzero at the point $P_0 = (z_0, w_0)$, then we can take w as a local parameter (an obvious variant of the lemma), and the surface Γ can be represented in a neighborhood U_0 of the point P_0 in the parametric form

$$(z(w), w) \in \Gamma, \quad z(w_0) = z_0, \tag{1.6}$$

where the function $z(w)$ is, of course, holomorphic. For a local parameter of this second kind the transition function is the identity map. For a nonsingular surface it is possible to use both ways for representing the surface on the intersection of domains of the first and second types, i.e., at points of Γ where $\frac{\partial F}{\partial w} \neq 0$ and $\frac{\partial F}{\partial z} \neq 0$ simultaneously. The resulting *transition functions* $w = w(z)$ and, $z = z(w)$ are holomorphic and invertible. □

The preceding arguments show that such Riemann surfaces are complex manifolds (with complex dimension 1).

Let us consider a Riemann surface Γ defined in $\mathbb{C}^2$ by a monic polynomial

$$F(z,w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0. \tag{1.7}$$

Here the $a_1(z), \ldots, a_n(z)$ are polynomials in z. This Riemann surface is realized as an n-sheeted covering of the z-plane. The precise meaning of this is as follows: let $\pi : \Gamma \to \mathbb{C}$ be the projection of the Riemann surface onto the z-plane given by the formula

$$\pi(z,w) = z. \tag{1.8}$$

Then for almost all z the preimage $\pi^{-1}(z)$ consists of n distinct points

$$(z, w_1(z)), \quad (z, w_2(z)), \quad , \ldots (z, w_n(z)), \tag{1.9}$$

of the surface Γ where $w_1(z), \ldots, w_n(z)$ are the n roots of (1.7) for given value of z. For certain values of z, some of the points of the preimage can merge. This happens at the *branch points* (z_0, w_0) of the Riemann surface where the partial derivative $F_w(z,w)$ vanishes (recall that we consider only nonsingular curves so far).

If z_0 is a branch point then the polynomial $F(z_0, w)$ has multiple roots. The multiple roots can be determined from the system

$$\left.\begin{array}{r} F(z_0,w) = 0 \\ F_w(z_0,w) = 0 \end{array}\right\}. \tag{1.10}$$

The ramification points on the z-plane can be determined, therefore, as the zeros of the *resultant* $R(z)$ of $F(z,w)$ and $F_w(z,w)$. The resultant can be computed as the determinant of a $(2n-1) \times (2n-1)$ matrix called Sylvester matrix constructed from the coefficients of the polynomials

$$F = w^n + a_1 w^{n-1} + \cdots + a_{n-1} w + a_n$$

and

$$F_w = n\, w^{n-1} + (n-1) a_1 w^{n-2} + \cdots + a_{n-1}$$

in the following way

$$R(z) = (-1)^{\frac{n(n-1)}{2}} \det \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{n-1} & a_n & 0 & \dots & 0 \\ 0 & 1 & a_1 & \dots & \dots & a_{n-1} & a_n & \dots & 0 \\ \dots & \dots & & \dots & & & \dots & & \dots \\ 0 & 0 & \dots & & \dots & & \dots & a_{n-1} & a_n \\ n & (n-1)a_1 & (n-2)a_2 & \dots & a_{n-1} & 0 & \dots & \dots & 0 \\ 0 & n & (n-1)a_1 & \dots & 2a_{n-2} & a_{n-1} & 0 & \dots & 0 \\ & \dots & & \dots & & & & & \dots \\ 0 & 0 & \dots & \dots & & & \dots & 2a_{n-2} & a_{n-1} \end{pmatrix}. \tag{1.11}$$

For example, the discriminant of a cubic monic polynomial is given by the formula

$$R(z) = -\det \begin{pmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 3 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3 & 2a_1 & a_2 \end{pmatrix} = a_1^2 a_2^2 - 4a_2^3 - 4a_1^3 a_3 + 18 a_1 a_2 a_3 - 27 a_3^2. \tag{1.12}$$

The resultant is also equal to

$$R(z) = \prod_{i=1}^{n} \prod_{j=1}^{n-1} (w_i(z) - \tilde{w}_j(z)) \tag{1.13}$$

where $w_i(z)$, $i = 1, \dots, n$, are the roots of the polynomials $F(z, w)$ and $\tilde{w}_j(z)$, $j = 1, \dots, n-1$ are the roots of the polynomials $F_w(z, w)$ where z is considered as a parameter.

The choice of the variables z or w as a local parameter is not always most convenient. We shall also encounter other ways of choosing a local parameter τ so that the point (z, w) of Γ can be represented locally in the form

$$z = z(\tau), \quad w = w(\tau) \tag{1.14}$$

where $z(\tau)$ and $w(\tau)$ are holomorphic functions of τ, and

$$\left(\frac{dz}{d\tau}, \frac{dw}{d\tau} \right) \neq 0. \tag{1.15}$$

We study the structure of the mapping π in (1.9) in a neighborhood of a branch point $P_0 = (z_0, w_0)$ of Γ defined in (1.3). Let τ be a local parameter on Γ in a neighborhood of P_0. It will be assumed that $z(\tau = 0) = z_0$, $w(\tau = 0) = w_0$. Then

$$\begin{aligned} z &= z_0 + a_k \tau^k + O(\tau^{k+1}), \quad a_k \neq 0 \\ w &= w_0 + b_q \tau^q + O(\tau^{q+1}), \quad b_q \neq 0, \end{aligned} \tag{1.16}$$

where a_k and b_q are nonzero coefficients. Since w can be taken as the local parameter in a neighborhood of P_0 it follows that $q = 1$. We get the form of the surface Γ in a neighborhood of a branch point:

$$\begin{aligned} z &= z_0 + a_k\tau^k + O(\tau^{k+1}), \\ w &= w_0 + b_1\tau + O(\tau^2), \end{aligned} \tag{1.17}$$

where $k > 1$. Thus, the points of the form

$$P_1(z) = (z, w_0 + \epsilon_1 c\sqrt[k]{z - z_0} + \dots), \dots, P_k(z) = (z, w_0 + \epsilon_k c\sqrt[k]{z - z_0} + \dots), \tag{1.18}$$

where $\epsilon_1, \dots, \epsilon_k$ are the primitive kth roots of unity and $c = b_1 a_k^{-\frac{1}{k}}$ lie in the complete inverse image $\pi^{-1}(z)$ in any sufficiently small neighborhood of P_0 merging into a single point at this point itself (the dots stand for the terms of the form $o(\sqrt[k]{z - z_0})$.

Definition 1.1.10. *The number $b_z(P) = k - 1$ is called the multiplicity of the branch point, or the branching index of this point with respect to the projection $(z, w) \to z$.*

Exercise 1.1.11: Let $P_0 = (z_0, w_0)$ be a branch point for the curve (1.7) with respect to the projection $(z, w) \to z$. Suppose that the local parameter in the neighbourhood of P_0 is of the form (1.17) with $k > 1$. Show that

$$\left.\frac{d^j F(z, w)}{dw^j}\right|_{(z_0, w_0)} = 0, \quad j = 0, \dots, k - 1.$$

Lemma 1.1.12. *Let (z_0, w_0) be a branch point of a Riemann surface Γ defined in (1.3) with respect to the projection $(z, w) \to z$. Then there exists a positive integer $k > 1$ and k functions $w_1(z), \dots, w_k(z)$ analytic on a sector $S_{\rho,\phi}$ of the punctured disc*

$$0 < |z - z_0| < \rho, \quad \arg(z - z_0) < \phi$$

for sufficiently small ρ and any positive $\phi < 2\pi$ such that

$$F(z, w_j(z)) \equiv 0 \quad \text{for} \quad z \in S_{\rho,\phi}, \quad j = 1, \dots, k.$$

The functions $w_1(z), \dots, w_k(z)$ are continuous in the closure $\bar{S}_{\rho,\phi}$ and

$$w_1(z_0) = \cdots = w_k(z_0) = w_0.$$

Proof. By the nonsingularity assumption $F_z(z_0, w_0) \neq 0$. So the complex curve $F(z, w) = 0$ can be locally parametrized in the form $z = z(w)$ where the analytic function $z(w)$ is uniquely determined by the condition $z(w_0) = z_0$. Consider the first nontrivial term of the Taylor expansion of this function

$$z(w) = z_0 + \alpha_k (w - w_0)^k + \alpha_{k+1}(w - w_0)^{k+1} + \dots, \quad k > 1, \ \alpha_k \neq 0.$$

Introduce an auxiliary function

$$\begin{aligned} f(w) &= \beta(w - w_0)\left[1 + \frac{\alpha_{k+1}}{\alpha_k}(w - w_0) + O\left((w - w_0)^2\right)\right]^{\frac{1}{k}} \\ &= \beta(w - w_0)\left[1 + \frac{\alpha_{k+1}}{k\,\alpha_k}(w - w_0) + O\left((w - w_0)^2\right)\right] \end{aligned}$$

where the complex number β is chosen in such a way that $\beta^k = \alpha_k$. The function $f(w)$ is analytic for sufficiently small $|w - w_0|$. Observe that $f'(w_0) = \beta \neq 0$. Therefore the analytic inverse function f^{-1} locally exists. The needed k functions $w_1(z), \dots, w_k(z)$ can be constructed as follows

$$w_j(z) = f^{-1}\left(e^{\frac{2\pi i\,(j-1)}{k}}(z - z_0)^{1/k}\right), \quad j = 1, \dots, k \tag{1.19}$$

where we choose an arbitrary branch of the k-th root of $(z - z_0)$ for $z \in S_{\rho,\phi}$. □

Example 1.1.13. Elliptic and hyperelliptic Riemann surfaces have the form

$$\Gamma = \{(z, w) \in \mathbb{C}^2 \; |w^2 = P_n(z)\}, \tag{1.20}$$

where $P_n(z)$ is a polynomial of degree n. These surfaces are two-sheeted coverings of the z-plane. Here $F(z, w) = w^2 - P_n(z)$. The gradient vector $\mathrm{grad}_{\mathbb{C}} F = (-P_n'(z), 2w)$. A point $(z_0, w_0) \in \Gamma$ is singular if

$$w_0 = 0, \quad P_n'(z_0) = 0. \tag{1.21}$$

Together with the condition (1.20) for a point (z_0, w_0) to belong to Γ we get that

$$P_n(z_0) = 0, \quad P_n'(z_0) = 0, \tag{1.22}$$

i.e. z_0 is a multiple root of the polynomial $P_n(z)$. Accordingly, the surface (1.20) is nonsingular if and only if the polynomial $P_n(z)$ does not have multiple roots:

$$P_n(z) = \prod_{i=1}^{n}(z - z_i), \quad z_i \neq z_j, \ \text{for } i \neq j. \tag{1.23}$$

The curve Γ is called an elliptic curve for $n = 3, 4$ and it is called hyperelliptic for $n > 4$. We find the branch points of the surface (1.20). To determine them we have the system

$$w^2 = P_n(z), \quad w = 0,$$

which gives us n branch points $P_i = (z = z_i, w = 0)$, $i = 1, \ldots, n$. All the branch points have multiplicity one. In a neighborhood of any point of Γ that is not a branch point it is natural to take z as a local parameter, and $w = \sqrt{P_n(z)}$ is a holomorphic function. In a neighborhood of a branch point P_i it is convenient to take

$$\tau = \sqrt{z - z_i}, \tag{1.24}$$

as a local parameter. Then for points of the Riemann surface (1.20) we get the local parametric representation

$$z = z_i + \tau^2, \quad w = \tau \sqrt{\prod_{j \neq i} (\tau^2 + z_i - z_j)} \tag{1.25}$$

where the radical is a single-valued holomorphic function for sufficiently small τ;(the expression under the root sign does not vanish), and $dw/d\tau \neq 0$ for $\tau = 0$.

Exercise 1.1.14: Prove that the total multiplicity of all the branch points on Γ over $z = z_0$ is equal to the multiplicity of $z = z_0$ as a root of the discriminant $R(z)$.

Exercise 1.1.15: Consider the collection of n-sheeted Riemann surfaces of the form

$$F(z, w) = \sum_{i+j \leqslant n} a_{ij} z^i w^j \tag{1.26}$$

for all possible values of the coefficients a_{ij} (so-called planar curves of degree n). Prove that for a general surface of the form (1.26) there are $n(n-1)$ branch points and they all have multiplicity 1. In other words, conditions for the appearance of branch points of multiplicity greater than one are written as a collection of algebraic relations on the coefficients a_{ij}.

1.1.2 Smooth projective plane curves

We recall the the projective space $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1}\backslash\{0\}$ by the equivalence relation that identifies vectors v and αv in $\mathbb{C}^{n+1}\backslash\{0\}$ with $\alpha \in \mathbb{C}^*$. Namely $\mathbb{P}^n = \mathbb{C}^{n+1}\backslash\{0\}/\mathbb{C}^*$. The space $\mathbb{P}^0$ is a singly point, $\mathbb{P}^1$ can be thought as the complex plane $\mathbb{C}$ plus a single point ∞ and it can be identified with the Riemann sphere. $\mathbb{P}^2$ can be thought as $\mathbb{C}^2$ together with a line at infinity, namely a copy of $\mathbb{P}^1$ and so on.

The projective line is the simplest example of a compact Riemann surface. The example of compact Riemann surfaces that we are going to considered are embedded in $\mathbb{P}^2$.

Definition 1.1.16. *The projective plane $\mathbb{P}^2$ is the set of one-dimensional subspaces in $\mathbb{C}^3$ or equivalently $\mathbb{P}^2 = \mathbb{C}^3\backslash\{0\}/\mathbb{C}^*$. Let (X, Y, Z) be a nonzero vector in $\mathbb{C}^3$. A point in $\mathbb{P}^2$ is denoted by $[X : Y : Z]$ and*

$$[X : Y : Z] = [\lambda X : \lambda Y : \lambda Z], \quad \lambda \neq 0,\ \lambda \in \mathbb{C}$$

As a quotient space, $\mathbb{P}^2$ is endowed with the quotient topology. Indeed let the projection map $\pi : \mathbb{C}^3\backslash\{0\} \to \mathbb{P}^2$ be defined as

$$\pi(X, Y, Z) = [X : Y : Z].$$

Then we can give to $\mathbb{P}^2$ the quotient topology induced from $\mathbb{C}^3\backslash\{0\}$, namely a subset U of $\mathbb{P}^2$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}^3\backslash\{0\}$. As a topological space, $\mathbb{P}^n$ is a Hausdorff space, namely two distinct points have disjoint open neighbourhoods.

Proposition 1.1.17. *The space $\mathbb{P}^2$ is compact.*

Proof. Let

$$S^5 = \{(X, Y, Z) \in \mathbb{C}^3 \mid |X|^2 + |Y|^2 + |Z|^2 = 1\}.$$

Then S^5 is a sphere of real dimension 5. It is a closed and bounded subset of $\mathbb{C}^3$ and by the Heine-Borel theorem is compact. The restriction of $\pi_{S^5} : S^5 \to \mathbb{P}^2$ is continuos. The image of a compact set under a continuous mapping is compact. Next let us show that π_{S^5} is also surjective. If $[X : Y : Z] \in \mathbb{P}^2$ then

$$|X|^2 + |Y|^2 + |Z|^2 = \lambda > 0$$

and

$$[X : Y : Z] = [\lambda^{-\frac{1}{2}} X : \lambda^{-\frac{1}{2}} Y : \lambda^{-\frac{1}{2}} Z].$$

Combining the above two relations one has that

$$|\lambda^{-\frac{1}{2}} X|^2 + |\lambda^{-\frac{1}{2}} Y|^2 + |\lambda^{-\frac{1}{2}} Z|^2 = 1$$

so that $[X : Y : Z] \in \pi(S^5)$. Namely the map $\pi : S^5 \to \mathbb{P}^2$ is surjective and continuos which implies that $\mathbb{P}^2$ is compact. □

Remark 1.1.18. The spaces $\mathbb{P}^n$, $n \geqslant 0$ are all compact. The proof of this statement is a simple generalisation of the proof of proposition 1.1.17.

The space $\mathbb{P}^2$ can be covered with three open set homeomorphic to $\mathbb{C}^2$:

$$U_0 = \{[X : Y : Z] \in \mathbb{P}^2 \mid X \neq 0\}$$

$$U_1 = \{[X : Y : Z] \in \mathbb{P}^2 \mid Y \neq 0\}$$

$$U_3 = \{[X : Y : Z] \in \mathbb{P}^2 \mid Z \neq 0\}.$$

The homeomorphism on U_0 is given by the map $[X : Y : Z] \to (Y/X, Z/X) \in \mathbb{C}^2$ and similarly for the other open sets U_1 and U_2.

Definition 1.1.19. *Let $F(X, Y, Z)$ be a homogeneous non constant polynomial of degree d, in the complex variables X, Y and Z with complex coefficients. The locus*

$$\Gamma = \{[X : Y : Z] \in \mathbb{P}^2 \mid F(X, Y, Z) = 0\} \tag{1.27}$$

is the projective curve defined by the polynomial F.

Remark 1.1.20. Observe that the curve Γ is well defined since the condition $F(X, Y, Z) = 0$ is independent from the choice of homogeneous coordinates since $F(\lambda X, \lambda Y, \lambda Z) = \lambda^d F(X, Y, Z)$. Furthermore Γ is a closed subset of $\mathbb{P}^2$ and therefore it is compact.

The intersection of Γ with any of the U_i is an affine plane curve. For example

$$\Gamma_0 = \Gamma \cap U_0 = \{(u, v) \in \mathbb{C}^2 \mid F(1, u, v) = 0\}.$$

Now we show that under non singularity assumptions, Γ is a Riemann surface.

Definition 1.1.21. *The curve (1.27) defined by the zeros of the homogeneous polynomial $F(X, Y, Z)$ is nonsingular if there are no non zero solutions to the equations*

$$F = \frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0.$$

Exercise 1.1.22: Show that the projective curve Γ defined in (1.27) is non singular if and only if each of the affine components $\Gamma_i = \Gamma \cap U_i$, $i = 1, 2, 3$ is non singular. *Hint*: use Euler equation that is obtained differentiating the identity $F(\lambda X, \lambda Y, \lambda Z) = \lambda^d F(X, Y, Z)$ with respect to λ and setting $\lambda = 1$, namely

$$XF_X + YF_Y + ZF_Z = Fd. \tag{1.28}$$

Lemma 1.1.23. *If the projective curve Γ defined in (1.27) is non singular, then the polynomial $F(X, Y, Z)$ is irreducible.*

Proof. Let us suppose that the polynomial is reducible, namely $F = F_1F_2$ where F_1 and F_2 are homogeneous polynomials in X, Y and Z of degree d_1 and $d - d_1$. The condition of Γ being singular takes the form

$$F_2F_1 = 0,\quad F_2\partial_X F_1 + F_1\partial_X F_2 = 0,\quad F_2\partial_Y F_1 + F_1\partial_Y F_2 = 0,\quad F_2\partial_Z F_1 + F_1\partial_Z F_2 = 0.$$

Such system of equations has always a solution as long as there is a point P in the intersections of the curves defined by $F_1 = 0$ and $F_2 = 0$. The resultant $R(X, Y)$ of the polynomials $F_1(X, Y, Z)$ and $F_2(X, Y, Z)$ with respect to Z is a homogeneous polynomial of degree $d_1(d - d_1)$. Indeed it is easy to show that $R(\lambda X, \lambda Y) = \lambda^{d_1(d-d_1)}R(X, Y)$. Therefore the curves defined by the equations $F_1(X, Y, Z) = 0$ and $F_2(X, Y, Z) = 0$ intersects in $d_1(d - d_1)$ points counted with multiplicity, (such result is called Bezout's theorem). We conclude that if F is reducible, then F is singular. □

In order to give a complex structure on Γ let us recall that each Γ_i is a smooth irreducible affine plane curve and hence a Riemann surface. The coordinate charts are given by the projections. For example for the curve Γ_0 the coordinate charts are y/x or z/x and the transition functions are as the same as the one obtained for smooth affine plane curves. We have then to check that the complex structures given on each Γ_i are compatible. Let $P \in \Gamma_0 \cap \Gamma_1$ where $P = [X : Y : Z]$ and $X \neq 0$ and $Y \neq 0$. Since each affine plane curve is non singular (see exercise 1.1.22), we assume without loss of generality that F_X and F_Z are non zero. Let $\phi_0 : \Gamma_0 \to \mathbb{C}$ with $\phi_0(P) = Y/X$ and with inverse $\phi_0^{-1}(Y/X) = [1 : Y/X : h(Y/X)]$ where h is a holomorphic function. Let $\phi_1 : \Gamma_1 \to \mathbb{C}$ with $\phi_1(P) = Z/Y$ with inverse $\phi_1^{-1} = [g(\frac{Z}{Y}), 1, \frac{Z}{Y}]$ where $g(\frac{Z}{Y})$ is holomorphic for $Y \neq 0$ and non zero since we assume $X \neq 0$. Then $\phi_1 \circ \phi_0^{-1}(Y/X) = Xh(Y/X)/Y$ which is holomorphic because $Y \neq 0$, $X \neq 0$ and $h(Y/X)$ is holomorphic. In the same way $\phi_0 \circ \phi_1^{-1}(Z/Y) = \frac{1}{g(Z/Y)}$ which is holomorphic because $Y \neq 0$ and g is nonzero. Similar checks can be done with the other coordinate charts.

Since $\mathbb{P}^2$ is compact and Γ is a closed subset of $\mathbb{P}^2$, it follows that Γ is compact.

Proposition 1.1.24. *Let $F(X, Y, Z)$ be an irreducible homogeneous polynomial. Then the projective plane curve Γ that is the zero locus of F in $\mathbb{P}^2$ is a smooth compact Riemann surface. At every point of Γ one can take as a local coordinate a ratio of the homogeneous coordinate.*

The simplest example of projective curve is the projective line

$$\alpha X + \beta Y + \gamma Z = 0$$

where $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. The tangent line to a projective curve Γ defined by a homogeneous polynomial $F(X, Y, Z)$ at a non singular point (X_0, Y_0, Z_0) has the form

$$(X - X_0)F_X(X_0, Y_0Z_0) + (Y - Y_0)F_Y(X_0, Y_0Z_0) + (Z - Z_0)F_Z(X_0, Y_0Z_0) = 0$$

Observe that using the Euler identity (1.28) one can write the tangent line in the form

$$X_0F_X(X_0, Y_0Z_0) + Y_0F_Y(X_0, Y_0Z_0) + Z_0F_Z(X_0, Y_0Z_0) = 0.$$

Exercise 1.1.25: Let $F(X, Y, Z)$ be an irreducible homogeneous polynomial of degree d defining a smooth projective curve Γ. The Hessian of this polynomial is defined as

$$H(X, Y, Z) = \det \begin{pmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{pmatrix}.$$

A non singular point $[X_0 : Y_0 : Z_0] \in \Gamma$ is called and inflection point if $H(X_0, Y_0, Z_0) = 0$. Show that

$$Z^2H(X, Y, Z) = (d-1)^2 \det \begin{pmatrix} F_{XX} & F_{XY} & F_X \\ F_{YX} & P_{YY} & P_Y \\ F_X & F_Y & F\frac{d}{d-1} \end{pmatrix}.$$

Suppose that $Z \neq 0$ and F_Y is not zero. Then the equation $F(X, Y, 1) = 0$ locally defines Y as a holomorphic function of X. Show that

$$\frac{d^2Y(X)}{dX^2} = \frac{H(X, Y, Z)}{(d-1)^2P_Y^3},$$

namely a point $[X_0 : Y_0 : 1]$ is an inflection point for the curve Γ if and only if $\frac{d^2Y(X)}{dX^2}$ vanishes at X_0.

1.1.3 Compactification of affine plane curve

Complex affine plane curves $\Gamma := \{(z, w \in \mathbb{C}^2 \; |F(z, w) = 0\}$ where F is a nonsingular polynomial, are non compact Riemann surfaces. To compactify them one needs to add point(s) $\infty^1, \infty^2, \dots \infty^N$ at infinity $z \to \infty, w \to \infty$ and introducing proper local parameters at these points in such a way that

$$\hat{\Gamma} = \Gamma \cup \infty^1 \cup \infty^2 \cup \cdots \cup \infty^N$$

is a compact Riemann surface.

The plane curve Γ, defined by the polynomial equation $F(z, w) = 0$, can be compactified by embedding it in $\mathbb{CP}^2$. The mappings

$$(X : Y : Z) \to \left(z = \frac{X}{Z}, \; w = \frac{Y}{Z}\right)$$

and the inverse mapping

$$(z, w) \to (z : w : 1)$$

establish an isomorphism between an affine part of $\mathbb{CP}^2$ and $\mathbb{C}^2$. The whole projective plane is obtained from the affine part $\mathbb{C}^2$ by adding the part at infinity of the form $(X : Y : 0) \simeq \mathbb{CP}^1 \simeq S^2$. An embedding of Γ in $\mathbb{CP}^2$ is defined as follows. Suppose that

$$F(z, w) = F_k(z, w) + F_{k-1}(z, w) + \cdots + F_0(z, w),$$

where each $F_j(z, w)$ is a homogeneous polynomial of degree j. Then we define the homogeneous polynomial

$$Q(X, Y, Z) = Z^k F\left(\frac{X}{Z}, \frac{Y}{Z}\right) \tag{1.29}$$

of degree k. A complex compact curve $\hat{\Gamma}$ is given in $\mathbb{CP}^2$ by the homogeneous equation

$$\hat{\Gamma} := \{[X : Y : Z] \in \mathbb{P}^2 \mid Q(X, Y, Z) = 0\}. \tag{1.30}$$

The affine part of the curve $\hat{\Gamma}$ (where $Z \neq 0$) coincides with Γ. The associated points at infinity have the form

$$Q(X, Y, 0) = 0. \tag{1.31}$$

Since $X \neq 0$ or $Y \neq 0$, the equation (1.31) has a finite set of solutions. The surface $\hat{\Gamma}$ is compact and is thus the desired compactification of the surface Γ.

Remark 1.1.26. Even if the curve Γ is non singular, the curve $\hat{\Gamma}$ might be singular. If this is the case, the compactification of the smooth affine plane curve as a singular projective curve is not a good compactification.

Example 1.1.27. $\Gamma = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z\}$. A local parameter at the branch point $(z = 0, w = 0)$ is given by $\tau = \sqrt{z}$, i.e. $z = \tau^2$, $w = \tau$. The compactification $\hat{\Gamma}$ has the form $\hat{\Gamma} = \{[X : Y : Z] \in \mathbb{P}^2 \mid Y^2 = XZ\}$. The point at infinity is given by solving the equation (1.31), that gives $P^\infty = [1 : 0 : 0]$. For $X \neq 0$ we introduce the coordinates u, v

$$u = \frac{Y}{X} = \frac{w}{z}, \quad v = \frac{Z}{X} = \frac{1}{z}, \tag{1.32}$$

which define the affine curve $u^2 = v$. The point at infinity is given by $(v = 0, u = 0)$ which is clearly a branch point for the curve defined by the equation $u^2 = v$ and $\sqrt{v}$ is a local parameter near this point. Therefore in a neighborhood of the point at infinity in $\hat{\Gamma}$ we have that

$$(z, w) \to \frac{1}{\sqrt{z}}$$

is a local homeomorphism.

Example 1.1.28. $\Gamma = \{w^2 = z^2 - a^2\}$. The branch points are $(z = \pm a, w = 0)$ and the corresponding local parameters are $z = \pm a + \tau^2$ and $w = \tau\sqrt{\tau^2 \pm 2a^2}$. The compactification has the form $\hat{\Gamma} = \{Y^2 = X^2 - a^2 Z^2\}$. The point at infinity is given by solving the equation (1.31), that gives $P^\infty_\pm = [1 : \pm 1 : 0]$. Making the substitution (1.32) we get the form of the curve $\hat{\Gamma}$ in a neighborhood of the ideal line: $u^2 = 1 - a^2 v^2$. For $v = 0$ we get that $u = \pm 1$. We can take $v = 1/z$ as a local parameter in a neighborhood of each of these points. The form of the surface $\hat{\Gamma}$ in a neighborhood of these points $P_\pm$ is as follows:

$$z = \frac{1}{v}, \quad w = \pm\frac{1}{v}\sqrt{1 - a^2 v^2}, \quad v \to 0 \tag{1.33}$$

where $\sqrt{1 - a^2 v^2}$ is, for small v, a single-valued holomorphic function, and the branch of the square root is chosen to have value 1 at $v = 0$.

Example 1.1.29. Let us consider the class of hyperelliptic Riemann surfaces

$$\Gamma = \{(z, w) \in \mathbb{C}^2 \mid F(z, w) = w^2 - P_N(z) = 0\}, \tag{1.34}$$

where $P_N(z) = \prod_{j=1}^N (z - a_j)$, and $a_i \neq a_j$ for $i \neq j$.

If we consider the projective curve defined by the zeros of homogeneous polynomial

$$Q(X, Y, Z) = Y^2 Z^{N-2} - Z^N P_N(X/Z) = 0$$

one can check that the curve is singular at the point $[0 : 1 : 0]$ if $N \geqslant 4$. Therefore, for $N \geqslant 4$, the embedding of Γ in $\mathbb{P}^2$ is not a good compactification. For $N = 3$ the projective curve

$$Y^2 Z = (X - a_1 Z)(X - a_2 Z)(X - a_3 Z)$$

is a compact smooth elliptic curve. By a projective transformation such curve can be reduced to the form

$$Y^2 Z = X(X - Z)(X - \lambda Z), \quad \lambda \in C\backslash\{0, 1\}.$$

The point at infinity is given by $P^\infty = [0 : 1 : 0]$. For $Y \neq 0$ the substitution $u = X/Y$ and $v = Z/Y$ gives the curve

$$Q(u, 1, v) = v - u(u - v)(u - \lambda v) = 0$$

The point $(0, 0)$ is a branch point for the above curve. Indeed for $(u, v) \neq 0$ the projection $\pi : (u, v) \to v$ is a local coordinate. The preimage $\pi^{-1}(v)$ consists of three points. At the point $(0, 0)$ one has $Q_u(0, 1, 0) = 0$ and $Q_{uu}(0, 1, 0) = 0$ so that the preimage of $\pi^{-1}(0)$ consists of a single point. A local coordinate near the point $(0, 0)$ takes the form

$$u = \tau(1 + o(\tau)), \quad v = \tau^3(1 + o(\tau)).$$

We look for the holomorphic tail in the form

$$u = \tau g(\tau), \quad v = \tau^3 g(\tau)$$

with $g(\tau)$ analytic and invertible in a neighbourhood of $\tau = 0$. Plugging the above ansatz in the equation $Q(u, 1, v) = v - u(u - v)(u - \lambda v) = 0$ one obtains that

$$g(\tau) = \frac{1}{\sqrt{(1 - \tau^2)(1 - \lambda\tau^2)}}.$$

Since

$$z = \frac{X}{Z} = \frac{u}{v}, \quad w = \frac{Y}{Z} = \frac{1}{v}$$

one has that a local coordinate near the point at infinity for the curve Γ is given by

$$z = \frac{1}{\tau^2}, \quad w = \frac{1}{\tau^3}\sqrt{(1 - \tau^2)(1 - \lambda\tau^2)}.$$

The above example shows that not all the affine plane curves can be compactified by embedding them in $\mathbb{P}^2$. Below we are going to illustrate another way of compactifying affine plane curves.

Definition 1.1.30. *Let Γ be a non compact Riemann surface such that there exists open subsets*

$$U_{\infty^1} \cup U_{\infty^2} \cup \cdots \cup U_{\infty^N} = U_\infty \subset \Gamma$$

such that $\Gamma \backslash U_\infty$ is compact and U_{∞^j}, $j = 1, \ldots, N$, are homeomorphic to puncture disks

$$\phi_j : U_{\infty^j} \to D\backslash\{0\} = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$$

and the homeomorphism ϕ_j, $j = 1, \ldots, N$ are holomorphically compatible with the complex structure of Γ. Then Γ is called a Riemann surface with punctures.

We extend the homeomorphism ϕ_j to the whole disk D

$$\phi_j : \bar{U}_{\infty^j} = U_{\infty^j} \cup \infty^j \to D \tag{1.35}$$

by defining $\phi_j(\infty^j) = 0$, $j = 1, \ldots, N$. A complex atlas on $\hat{\Gamma} = \Gamma \cup \infty^1 \cup \cdots \cup \infty^n$ is defined as the union of the complex atlas on Γ and the coordinate charts defined in (1.35) which are compatible with the complex structure of Γ.

Example 1.1.31. We recall first how to compactify the complex z-plane $\mathbb{C}$. It is necessary to add to $\mathbb{C}$ a single "point at infinity" ∞. In this case $U_\infty = \mathbb{C}\backslash\{0\}$ and $\bar{U}_\infty = U_\infty \cup \infty$ and the map $\phi : \bar{U}_\infty \to D$ is defined by $\phi(z) = \dfrac{1}{z}$ and $\phi(\infty) = 0$. A complex atlas on $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ is then defined as in example 1.1.5. We get a surface $\bar{\mathbb{C}}$ with the topology of a sphere (the "Riemann sphere"). Topological equivalence to the standard sphere is given by stereographic projection, with one of the poles of the sphere passing into the point ∞. Another description of $\bar{\mathbb{C}}$ is the complex projective line $\mathbb{P}^1 := \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 \neq 0, \quad (z_1 : z_2) \simeq (\lambda z_1 : \lambda z_2),\ \lambda \in \mathbb{C},\ \lambda \neq 0\}$. The equivalence $\mathbb{P}^1 \to \bar{\mathbb{C}}$ is established as follows: $(z_1 : z_2) \to z = \dfrac{z_1}{z_2}$. The affine part $\{z_2 \neq 0\}$ of $\mathbb{P}^1$ passes into $\mathbb{C}$ and the point $(1 : 0)$ into ∞.

Example 1.1.32. Let us consider the class of hyperelliptic Riemann surfaces

$$\Gamma = \{(z, w) \in \mathbb{C}^2 \mid F(z, w) = w^2 - P_N(z) = 0\}, \tag{1.36}$$

where $P_N(z) = \prod_{j=1}^{N}(z - a_j)$, $N \geqslant 4$ and $a_i \neq a_j$ for $i \neq j$. We need to consider separately the case of N odd or even. Let us rewrite the curve in the form

$$\left(\frac{w}{z^{n+1}}\right)^2 - \frac{1}{z}\prod_{j=1}^{N}(1 - a_j z) = 0, \quad N = 2n + 1,$$

$$\left(\frac{w^2}{z^{n+1}}\right)^2 - \prod_{j=1}^{N}(1 - a_j z) = 0, \quad N = 2n + 2,$$

The map

$$\psi : (z, w) \to \left(\frac{1}{z}, \frac{w}{z^{n+1}}\right)$$

describes a biholomorphic map from a neighbourhood of infinity

$$U_\infty = \{(z, w) \in \Gamma \mid |z| > c > |a_j|,\ j = 1, \ldots, 2n + 1\}$$

where $c > 0$, to the punctured neighbourhood

$$V = \{(x, y) \in \tilde{\Gamma} \mid |0 < |x| < 1/c\}$$

of the point $(x, y) = (0, 0)$ of the curve $\tilde{\Gamma}$ defined by the equation

$$\tilde{\Gamma} = \{(x, y) \in \mathbb{C}^2 \mid y^2 - x\prod_{j=1}^{N}(1 - x a_j) = 0\}, \quad N = 2n + 1, \tag{1.37}$$

or the points $(0, \pm 1)$ of the curve

$$\tilde{\Gamma} = \{(x, y) \in \mathbb{C}^2 \mid y^2 - \prod_{j=1}^{N}(1 - xa_j) = 0\}, \quad N = 2n + 2. \tag{1.38}$$

The local coordinate near $(0, 0)$ of the curve $\tilde{\Gamma}$ in (1.37) is defined by the homeomorphism $(x, y) \to \sqrt{x}$, while the local coordinate near the point $(0, \pm 1)$ of the curve (1.38) is given by $(x, y) \to x$. Therefore for $N = 2n + 1$ the curve (1.36) has one puncture at infinity and the local parameter in its neighbourhood is given by the homeomorphism

$$\phi(z, w) = \frac{1}{\sqrt{z}}, \quad \phi(\infty) = 0$$

while for $N = 2n + 2$, the curve (1.36) has two punctures $\infty^{\pm} = (\infty, \pm\infty)$ distinguished by the conditions

$$(z, w) \to \infty^{\pm} \leftrightarrow \frac{w}{z^{n+1}} \to \pm 1 \text{ for } z \to \infty \text{ and } w \to \infty\}$$

and the local parameter near these points is given by the homeomorphism

$$\phi_{\pm}(z, w) \to \frac{1}{z}, \quad \phi_{\pm}(\infty^{\pm}) = 0.$$

Proposition 1.1.33. *The local parameters*

$$(z, w) \to z \text{ near an ordinary point}$$
$$(z, w) \to \sqrt{z - z_j} \text{ near a branch point } (z_j, 0)$$
$$(z, w) \to \begin{cases} 1/\sqrt{z} & \text{near the point at infinity, } N \text{ odd} \\ 1/z & \text{near the points at infinity, } N \text{ even} \end{cases}$$

describe a compact Riemann surface $\hat{\Gamma} = \Gamma \cup \infty$ *of the hyperelliptic curve (1.36) for* N *odd and* $\hat{\Gamma} = \Gamma \cup \infty^{\pm}$ *for* N *even.*

Exercise 1.1.34: Compactify the curves defined by the equations

$$w^4 = z^4 + a^4, \quad w^3 = z^4 + a^4, \quad a \in \mathbb{R}^+.$$

Find the local coordinates in the neighbourhood of the points at infinity.

Quotients under Group action

Complex Tori. Let ω_1 and ω_2 be two complex numbers which are linearly independent over the real numbers. Define the lattice

$$L_{\omega_1,\omega_2} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}. \tag{1.39}$$

Two complex numbers z and $\tilde{z}$ are equivalent mod L_{ω_1,ω_2} if $z - \tilde{z} \in L_{\omega_1,\omega_2}$. The set of all equivalence classes is denoted by $\mathbb{C}/L_{\omega_1,\omega_2}$ and an element in $\mathbb{C}/L_{\omega_1,\omega_2}$ is denoted by $[z]$.

Proposition 1.1.35. *The quotient $\Gamma = \mathbb{C}/L_{\omega_1,\omega_2}$ is a compact Riemann surface that is topologically a torus.*

Proof. To prove the statement one needs to construct a complex structure on Γ. Let $\pi : \mathbb{C} \to \Gamma$ be the projection map. Let us endowed Γ with the quotient topology namely a set $U \subset \Gamma$ is open if $\pi^{-1}(U)$ is open in $\mathbb{C}$. This definition makes π continuous and since $\mathbb{C}$ is connected so is Γ. Furthermore, it is easy to check that π is an open mapping. Indeed let U be an open set in $\mathbb{C}$, then $\pi(U)$ is open if $\pi^{-1}(\pi(U))$. But this is certainly the case since $\pi^{-1}(\pi(U)) = \bigcup_{\omega\in L}(\omega + U)$ is open. In order to define a complex structure on Γ, let $D_\alpha = D_{z_\alpha,\epsilon}$ be a disk centered at $z_\alpha \in \mathbb{C}$ and of radius ϵ where ϵ is chosen in such a way that $|\omega| > \epsilon$ for every non zero $\omega \in L$. Then the map $\pi|_{D_\alpha} : D_\alpha \to \pi(D_\alpha)$ is a homeomorphism. Let $\phi_\alpha : \pi(D_\alpha) \to D_\alpha$ be the inverse of the map $\pi|_{D_\alpha}$. The pairs $(\pi(D_{\alpha)}), \phi_\alpha)_{\alpha\in A}$ defines a complex chart. We now must check that the charts are compatible. Chose two distinct points z_1 and z_2 and consider two charts $\phi_1 : \pi(D_1) \to D_1$ and $\phi_2 : \pi(D_2) \to D_2$ with $U := \pi(D_1) \cap \pi(D_2) \neq \emptyset$. We need to check that the transition function $T(z) = \phi_2(\phi_1^{-1}(z))$ is holomorphic for $z \in \phi_1(U)$. Observe that $\pi(T(z)) = \pi(z)$ for all $z \in \phi_1(U)$. Therefore $T(z) - z = \omega(z) \in L$. Since $T(z)$ is continuos and L is discrete, $\omega(z)$ is constant. Therefore $T(z) = z + \omega$ for some $\omega \in L$, namely the transition function $T(z)$ is holomorphic. The collection of charts $\{(D_\alpha, \phi_\alpha) \mid z_\alpha \in \mathbb{C}\}$ is a complex atlas on Γ. We conclude that Γ is a Riemann surface. The surface Γ is compact because it is covered by the image under π of the compact set

$$\{\alpha\omega_1 + \beta\omega_2, \quad \alpha, \beta \in [0, 1]\}$$

□

Remark 1.1.36. Let $A \in SL(2, \mathbb{Z})$ namely A is 2×2 matrix with integer entries and $\det A = 1$. Suppose that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Then the $L_{\omega_1,\omega_2} = L_{\omega'_1,\omega'_2}$. Indeed for $m, n \in \mathbb{Z}$ one has

$$L_{\omega_1,\omega_2} \ni m\omega_1 + n\omega_2 = (n, m)A^{-1}\begin{pmatrix}\omega'_1\\ \omega'_2\end{pmatrix} = m'\omega'_1 + n'\omega'_2 \in L_{\omega'_1,\omega'_2},$$

because $m', n' \in \mathbb{Z}$ since the matrix A has integer entries and determinant equal to one.

The above relation shows that $L_{\omega_1,\omega_2} \subseteq L_{\omega'_1,\omega'_2}$. Repeating the same reasoning for a point in $L_{\omega'_1,\omega'_2}$ one obtains that $L_{\omega'_1,\omega'_2} \subseteq L_{\omega_1,\omega_2}$ which shows that $L_{\omega_1,\omega_2} = L_{\omega'_1,\omega'_2}$.

Remark 1.1.37. Any automorphism of the complex plane, $F : \mathbb{C} \to \mathbb{C}$ is of the form $F(z) := \alpha z + \beta$ with $\alpha \neq 0$. We choose $\beta = 0$ so that $F(0) = 0$. A lattice L_{ω_1,ω_2} is transformed under F to the lattice $L_{\alpha\omega_1,\alpha\omega_2}$. The corresponding tori are isomorphic, with the automprhism given by $[z] \to [\alpha z]$.

Let us define $\tau = \dfrac{\omega_1}{\omega_2}$ with $\Im(\tau) > 0$. Then the lattice L_{ω_1,ω_2} in (1.39) and

$$L_{\tau,1} = \{n + m\tau \mid m, n \in \mathbb{Z}\}, \quad \tau = \frac{\omega_1}{\omega_2}$$

defined tori $\mathbb{C}/L_{\omega_1,\omega_2}$ and $\mathbb{C}/L_{\tau,1}$ that are isomorphic. Combining the above remarks one arrive to the following theorem.

Theorem 1.1.38. *Let T_τ and $T_{\tau'}$ two tori defined by the lattices $L_{\tau,1}$ and $L_{\tau',1}$ with $\Im(\tau) > 0$ and $\Im(\tau') > 0$. These tori are isomorphic if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL(2, \mathbb{Z}). \tag{1.40}$$

The proof is left as an exercise.

Exercise 1.1.39: Consider the group $2\pi\mathbb{Z}$ under addition and consider the quotient $\mathbb{C}/2\pi\mathbb{Z}$. This surface is clearly homeomorphic to the cylinder $S^1 \times \mathbb{R}$. Show that $\mathbb{C}/2\pi\mathbb{Z}$ is a Riemann surface.

Exercise 1.1.40: Let G be the multiplicative group $G := \{a^n \mid n \in \mathbb{Z}\}$ and $a \in \mathbb{R}^+$. The quotient

$$\Gamma := \mathbb{C}^*/G$$

is defined as the set of equivalence class with respect to the equivalence relation

$$z \simeq \tilde{z} \longleftrightarrow z\tilde{z}^{-1} \in G.$$

(i) Prove that there exist a unique structure of a Riemann surface on Γ such that the canonical projection $\pi : \mathbb{C}^* \to \Gamma$ is locally biholomorphic.

(ii) Show that the Rieamann surface constructed in (i) is isomorphic to a torus

$$\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \quad \tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

Calculate τ.

The above construction of Riemann surface as quotients can be generalized

Definition 1.1.41. *Let Δ be a domain of $\mathbb{C}$. A group $G : \Delta \to \Delta$ of holomorphic transformations acts discontinuously and fixed point free on Δ if for any $P \in \Delta$ there exists a neighbourhood $V \ni P$ such that*

$$gV \cap V = \varnothing, \ \forall g \in G, \ g \neq I$$

The action of G is called proper if the inverse image of compact subset is compact.

Introducing an equivalent relation between points of Δ, namely $P \simeq P'$ if $\exists g \in G$ so that $P' = gP$, one can define the quotient space Δ/G of equivalent classes.

Theorem 1.1.42. *If a group G acts on a domain Δ of the complex plane properly discontinuously and the action is fixed point free, then the quotient space Δ/G has the structure of a Riemann surface.*

The proof of the above theorem is very similar to the proof given above for obtaining a complex structure on the complex one-dimensional tori. In the frame of the uniformization theory, it is proven that all compact Riemann surfaces can be described as quotients Δ/G.