Chapter 1

Riemann surfaces

1.1 Definition of a Riemann surface and basic examples

In its broadest sense a Riemann surface is a one dimensional complex manifold that locally looks like an open set of the complex plane, while its global topology can be quite different from the complex plane. The main reason why Riemann surfaces are interesting is that one can speak of complex functions on a Riemann surface as much as the complex function on the complex plane that one encounters in complex analysis.

Elementary example of Riemann surfaces are the complex plane $\mathbb{C}$, the disk

$$D = \{z \in \mathbb{C},\ |z| < 1\}$$

or the upper half space

$$H = \{z \in \mathbb{C},\ \Im(z) > 0\}.$$

B. Riemann introduced the concept of Riemann surface to make sense of multivalued functions like the square root or the logarithm. For the geometric representation of multivalued functions of a complex variable $w = w(z)$ it is not convenient to regard z as a point of the complex plane. For example, take $w = \sqrt{z}$. On the positive real semiaxis $z \in \mathbb{R}, \quad z > 0$ the two branches $w_1 = +\sqrt{z}$ and $w_2 = -\sqrt{z}$ of this function are well defined by the condition $w_1 > 0$. This is no longer possible on the complex plane. Indeed, the two values $w_{1,2}$ of the square root of $z = r\, e^{i\psi}$

$$w_1 = \sqrt{r}\, e^{i\frac{\psi}{2}}, \quad w_2 = -\sqrt{r}\, e^{i\frac{\psi}{2}} = \sqrt{r} e^{i\frac{\psi+2\pi}{2}}, \tag{1.1}$$

interchange when passing along a path

$$z(t) = r\, e^{i\,(\psi+t)}, \quad t \in [0, 2\pi]$$

encircling the point $z = 0$. It is possible to select a branch of the square root as a function of z by restricting the domain of this function for example, by making a cut from zero to infinity. Namely the function $\sqrt{z}$ is single-valued in the cut plane $\mathbb{C}\backslash[0, +\infty)$. Riemann's idea was to combine the two branches of the function $\sqrt{z}$ in a geometric space in such a way that the function is well defined and single-valued. The rules are as follows: one has to take two copies of the complex plane cut along the positive real axis and join the two copies of the complex plane along the cuts. The different sheets have to be glue together in such a way that the branch of the function on one sheet joins continuously with the branch defined on the other sheet. The result of this operation is the surface in figure 1.1.

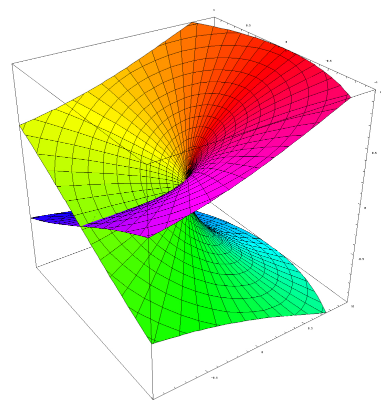


Figure 1.1: The two branches of the function $\sqrt{z}$

Note that such surface can be given for $(w, z) \in \mathbb{C}^2$ as the zero locus

$$F(z, w) = w^2 - z = 0.$$

A similar procedure of cutting and glueing can be repeated for any other analytic function. For example the logarithm $\log z$ is a single valued function on $\mathbb{C}\backslash[0, +\infty)$ with infinite branches. Each adjacent branch differs by an additive factor $2\pi i$. The infinite branches attached along the positive real line are shown in the figure 1.2.

Next we will give a more abstract definition of a Riemann surface and we will show how the surface defined by the graph of a multivalued function fits in this definition. Let us recall that a Hausdorff topological space is such that distinct points have distinct open neighbourhoods.

Definition 1.1. *A Riemann surface* Γ *is defined by the following data:*

- *a connected Hausdorff topological space* Γ*;*

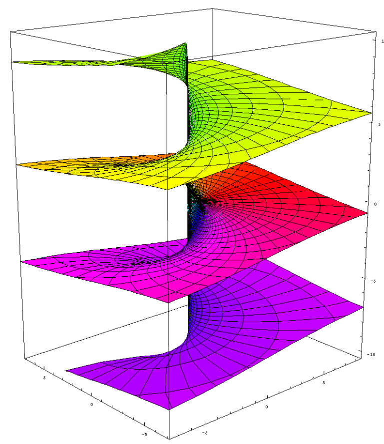


Figure 1.2: The infinite branches of the function log z

- *an open cover* $\{U_\alpha\}_{\alpha\in A}$ *of* Γ*;*
- *for each* $\alpha \in A$*, a homeomorphism* ϕ_α

  $$\phi_\alpha : U_\alpha \to V_\alpha$$

  to an open subset $V_\alpha \subset \mathbb{C}$ *in such a way that for each* $\alpha, \beta \in A$*, if* $U_\alpha \cap U_\beta \neq \emptyset$*, the transition functions*

  $$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta),$$

  is bi-holomorphic, namely, holomorphic with inverse holomorphic.

Remark 1.2. Let us observe that the sets $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$, are subsets of the complex plane, and therefore the request of having holomorphic maps between these two subsets makes sense.

The pair $\{U_\alpha, \phi_\alpha\}$ is called complex chart. Complex charts are also called *local parameters* or *local coordinates*. Two charts (U_α, ϕ_α) and (U_β, ϕ_β) are compatible if either $U_\alpha \cap U_\beta = \emptyset$ or the transition function $\phi_\beta \circ \phi_\alpha^{-1}$ is bi-holomorphic. If all the complex charts $\{U_\alpha, \phi_\alpha\}_{\alpha\in A}$ are compatible, they form a complex atlas $\mathcal{A}$ of Γ. Two complex atlas $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are compatible if their union $\mathcal{A} \cup \tilde{\mathcal{A}}$ is a complex atlas. The equivalence class of complex atlas is called a *complex structure* or also a *conformal structure*. With the definition of complex structure we can define a Riemann surface in the equivalent way.

Definition 1.3. *A Riemann surface is a connected one-complex dimensional analytic manifold, or a two real dimensional connected manifold with a complex structure on it.*

Let ϕ and $\tilde{\phi}$ be two local homeomorphism from two open sets U and $\tilde{U}$ of Γ with $U \cap \tilde{U} \neq \emptyset$. Let P and P_0 two points in $U \cap \tilde{U}$ and denote by $z = \phi(P)$ and $w = \tilde{\phi}(P)$ the two local coordinates with $z_0 = \phi(P_0)$ and $w_0 = \tilde{\phi}(P_0)$. Then the holomorphic transition function $T = \phi \circ \tilde{\phi}^{-1}$ must be of the form

$$z = T(w) = T(w_0) + \sum_{k>0} a_k (w - w_0)^k, \quad a_1 \neq 0 \tag{1.2}$$

with holomorphic inverse

$$w = T^{-1}(z) = T^{-1}(z_0) + \sum_{k>0} b_k (z - z_0)^k, \quad b_1 \neq 0,$$

namely the linear coefficient of the above Taylor expansions near the point w_0 or z_0 is necessarily nonzero.

Remark 1.4. We recall that that a manifold is called orientable if it has an atlas whose transition functions have positive Jacobian determinant. If Γ is a Riemann surface, then the manifold Γ is orientable. Indeed let $z = x + iy$ be a local coordinate in some open neighbourhood of z_0 in Γ. Another local coordinate $w = u + iv$ is connected with the first by a holomorphic change of variable $w = T(z)$ with $w_0 = T(z_0)$ which thus determines a smooth change of real coordinates. We want to show that the determinant

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = u_x v_y - u_y v_x$$

calculated in (x_0, y_0) is positive. We observe that $w = w(z)$ is a holomorphic function of z and $\frac{dw}{dz}|_{z=z_0} \neq 0$. We can use Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$ to write $\frac{dw}{dz} = u_x - iu_y$ and $\frac{d\overline{w}}{d\bar{z}} = u_x + iu_y$ to conclude that

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\Bigg|_{\substack{x=x_0 \\ y=y_0}} = (u_x^2 + u_y^2)|_{\substack{x=x_0 \\ y=y_0}} = \left|\frac{dw}{dz}\right|^2_{z=z_0} > 0.$$

Example 1.5. Elementary examples of Riemann surfaces

(a) The complex plane $\mathbb{C}$. The complex atlas is define by one chart that is $\mathbb{C}$ itself with the identity map.

(b) The extended complex plane $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$, namely the complex plane $\mathbb{C}$ with one extra point ∞. We make $\bar{\mathbb{C}}$ into a Riemann surface with an atlas with two charts:

$$U_1 = \mathbb{C}$$
$$U_2 = \bar{\mathbb{C}} \backslash \{0\},$$

with ϕ_1 the identity map and

$$\phi_2(z) = \begin{cases} 1/z, & \text{for } z \in \mathbb{C}\backslash\{0\} \\ 0, & \text{for } z = \infty. \end{cases}$$

1.1.1 Affine plane curves

Let us consider a polynomial $F(z,w) = \sum_{i=1}^{n} a_i(z)w^i$ of two complex variables z and w. The zero set $F(z,w)$ defines a n-valued function $w = w(z)$. The basic idea of Riemann surface theory is to replace the domain of the function $w(z)$ by its graph

$$\Gamma := \{(z,w) \in \mathbb{C}^2 \mid F(z,w) = \sum_{i=0}^{n} a_i(z)w^{n-i} = 0\} \tag{1.3}$$

and to study the function w as a single-valued function on Γ rather then a multivalued function of z. As in the example of $\sqrt{z}$, the multivalued function $w = w(z) = \sqrt{z}$ becomes a single-valued function $w = w(P)$ of a point P of the algebraic surface Γ: if $P = (z,w) \in \Gamma$, then $w(P) = w$ (the projection of the graph on the the w-axis). From the real point of view the algebraic curve (1.3) is a two-dimensional surface in $\mathbb{C}^2 = \mathbb{R}^4$ given by the two equations

$$\left. \begin{array}{r} \Re F(z,w) = 0 \\ \Im F(z,w) = 0 \end{array} \right\}.$$

In the theory of functions of a complex variable one encounters also more complicated (nonalgebraic) curves, where $F(z,w)$ is not a polynomial. For example, the equation $e^w - z = 0$ determines the surface of the logarithm or $\sin w - z = 0$ determines the surface of the arcsin. Such surfaces will not be considered here.

Definition 1.6. *An affine plane curve Γ is a subset in $\mathbb{C}^2$ defined by the equation (1.3) where $F(z,w)$ is polynomial in z and w. The curve Γ is nonsingular if for any point $P_0 = (z_0, w_0) \in \Gamma$ the complex gradient vector*

$$grad_{\mathbb{C}} F|_{P_0} = \left(\frac{\partial F(z,w)}{\partial z}, \frac{\partial F(z,w)}{\partial w} \right) \Bigg|_{(z=z_0, w=w_0)}$$

does not vanish. If the polynomial $F(z,w)$ is irreducible, the curve Γ is called irreducible affine plane curve.

Remark 1.7. A non trivial theorem states that an irreducible affine plane curve is connected (see Theorem 8.9 in O. Forster, Lectures on Riemann surfaces, Springer Verlag 1981).

In order to define a complex structure on Γ we need the following complex version of the implicit function theorem.

Lemma 1.8. *[Complex implicit function theorem] Let $F(z,w)$ be an analytic function of the variables z and w in a neighborhood of the point $P_0 = (z_0, w_0)$ such that $F(z_0,w_0) = 0$ and $\partial_w F(z_0,w_0) \neq 0$. Then there exists a unique function $\phi(z)$ such that $F(z,\phi(z)) = 0$ and $\phi(z_0) = w_0$. This function is analytic in z in some neighborhood of z_0.*

Proof. Let $z = x + iy$ and $w = u + iv$, $F = f + ig$. Then the equation $F(z,w) = 0$ can be written as the system

$$\begin{cases} f(x,y,u,v) &= 0 \\ g(x,y,u,v) &= 0 \end{cases} \tag{1.4}$$

The condition of the real implicit function theorem are satisfied for this system: the matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\[2ex] \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{pmatrix}_{(z_0,w_0)}$$

is nonsingular because

$$\det \begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\[2ex] \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{pmatrix} = \left| \frac{\partial F}{\partial w} \right|^2 > 0,$$

(we use only the analyticity in w of the function $F(z,w)$). Thus, in some neighbourhood of (z_0,w_0) there exist a smooth function $\phi(z,\bar{z}) = \phi_1(x,y) + i\phi_2(x,y)$ such that $F(z,\phi(z,\bar{z})) = 0$, with $\phi(z_0,\bar{z}_0) = w_0$. Differentiating with respect to $\bar{z}$

$$0 = \frac{d}{d\bar{z}} F(z,\phi(z,\bar{z})) = F_w \frac{d}{d\bar{z}} \phi(z,\bar{z}).$$

Since $F_w \neq 0$, the above relation implies that $\dfrac{d}{d\bar{z}}\phi(z,\bar{z}) = 0$ which shows that $\phi(z)$ is an analytic function of z. $\square$

Remark 1.9. A constructive way of obtaining the function $\phi(z)$ is to apply the Residue Theorem. Indeed let us consider the function $F(z, w)$ where z is treated as a parameter. Let D_0 be a small disk around w_0 where $F(z_0, w_0) = 0$ and $F_w(z_0, w)|_{w=w_0} \neq 0$. Then the number of solutions of the equation $F(z_0, w) = 0$ counted with multiplicity is given by the integral

$$\frac{1}{2\pi i}\int_{\partial D_0} \frac{F_w(z_0, w)}{F(z_0, w)} dw,$$

where ∂D_0 is the boundary of D_0. We assume D_0 sufficiently small so that the equation $F(z_0, w) = 0$ has only the solution w_0 in the closure of D_0. Then the above integral is equal to one. Furthermore by the residue theorem one has

$$\frac{1}{2\pi i}\int_{\partial D_0} w\frac{F_w(z_0, w)}{F(z_0, w)} dw = w_0.$$

By continuity, for z sufficiently close to z_0 there is a disk D centred at w such that the equation $F(z, w) = 0$ has only one solution $w = \phi(z)$ in the closure of D and

$$\frac{1}{2\pi i}\int_{\partial D} w\frac{F_w(z, w)}{F(z, w)} dw = \phi(z),$$

where $\phi(z_0) = z_0$ and $F(z, \phi(z)) = 0$. Clearly the function $\phi(z)$ is an analytic function of z.

Theorem 1.10. *Let Γ be an irreducible affine plane curve defined in (1.3). If Γ is non singular, then Γ is a Riemann surface.*

Proof. Γ is connected since $F(z, w)$ is irreducible. Let us define a complex structure on Γ. Let $P_0 = (z_0, w_0)$ be a nonsingular point of the surface Γ. Suppose, for example, that the derivative $\dfrac{\partial F}{\partial w}$ is nonzero at this point. Then by the lemma 1.8, in a neighborhood U_0 of the point P_0, the surface Γ admits a parametric representation of the form

$$(z, w(z)) \in U_0 \subset \Gamma, \quad w(z_0) = w_0, \tag{1.5}$$

where the function $w(z)$ is holomorphic. Therefore, in this case z is a complex local coordinate also called *local parameter* on Γ in a neighborhood U_0 of $P_0 = (z_0, w_0) \in \Gamma$. For this kind of local coordinate, the transition function is the identity.

Similarly, if the derivative $\dfrac{\partial F}{\partial z}$ is nonzero at the point $P_0 = (z_0, w_0)$, then we can take w as a local parameter (an obvious variant of the lemma), and the surface Γ can be represented in a neighborhood U_0 of the point P_0 in the parametric form

$$(z(w), w) \in \Gamma, \quad z(w_0) = z_0, \tag{1.6}$$

where the function $z(w)$ is, of course, holomorphic. For a local parameter of this second kind the transition function is the identity map. For a nonsingular surface it is possible to use both ways for representing the surface on the intersection of domains of the first and second types, i.e., at points of Γ where $\dfrac{\partial F}{\partial w} \neq 0$ and $\dfrac{\partial F}{\partial z} \neq 0$ simultaneously. The resulting *transition functions* $w = w(z)$ and, $z = z(w)$ are holomorphic and invertible. □

The preceding arguments show that such Riemann surfaces are complex manifolds (with complex dimension 1).

Let us consider a Riemann surface Γ defined in $\mathbb{C}^2$ by a monic polynomial

$$F(z,w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0. \tag{1.7}$$

Here the $a_1(z), \ldots, a_n(z)$ are polynomials in z. This Riemann surface is realized as an n-sheeted covering of the z-plane. The precise meaning of this is as follows: let $\pi : \Gamma \to \mathbb{C}$ be the projection of the Riemann surface onto the z-plane given by the formula

$$\pi(z,w) = z. \tag{1.8}$$

Then for almost all z the preimage $\pi^{-1}(z)$ consists of n distinct points

$$(z, w_1(z)), \quad (z, w_2(z)), \quad , \ldots (z, w_n(z)), \tag{1.9}$$

of the surface Γ where $w_1(z), \ldots, w_n(z)$ are the n roots of (1.7) for given value of z. For certain values of z, some of the points of the preimage can merge. This happens at the *branch points* (z_0, w_0) of the Riemann surface where the partial derivative $F_w(z,w)$ vanishes (recall that we consider only nonsingular curves so far).

If z_0 is a branch point then the polynomial $F(z_0, w)$ has multiple roots. The multiple roots can be determined from the system

$$\left.\begin{aligned} F(z_0,w) &= 0 \\ F_w(z_0,w) &= 0 \end{aligned}\right\}. \tag{1.10}$$

The ramification points on the z-plane can be determined, therefore, as the zeros of the *resultant* of $F(z,w)$ and $F_w(z,w)$ and denoted by $R(F,F_w)(z)$. Such quantity is also called the discriminant of $F(z,w)$ with respect to w.

Definition 1.11. *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ and $g(z) = b_0 + b_1 z + \cdots + b_m z^m$ be two polynomials of degree n and m respectively with $a_i,\ b_j \in \mathbb{C}$ with $a_n \neq 0$ and $b_m \neq 0$. The resultant*

$R(f,g)$ is given by the determinant of the $(n+m)\times(n+m)$ matrix

$$R(f,g)=\begin{pmatrix} a_0 & a_1 & \dots & a_n & 0 & 0 & & & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_n & 0 & 0 & & \dots & 0 \\ \dots & & & \dots & & & & & \dots & \\ 0 & 0 & \dots & \dots & a_0 & a_1 & a_2 & & \dots & a_n \\ b_0 & b_1 & \dots & \dots & b_{m-1} & b_m & 0 & & \dots & 0 \\ 0 & b_0 & b_1 & \dots & \dots & b_{m-1} & b_m & 0 & \dots & 0 \\ \dots & & & \dots & & & & & \dots & \\ 0 & \dots & b_0 & b_1 & \dots & & \dots & & b_{m-1} & b_m \end{pmatrix}. \tag{1.11}$$

Lemma 1.12. *$R(f,g)=0$ if and only if f and g have a common zero.*

Proof. The polynomials f and g have a non constant common root $r(z)$ if and only if there exists polynomials $\psi(z)$ and $\phi(z)$ such that $f(z)=r(z)\psi(z)$ and $g(z)=r(z)\phi(z)$. Here ψ and ϕ are polynomials of degree $n-1$ and $m-1$ respectively. This implies that

$$f(z)\phi(z)=g(z)\psi(z) \tag{1.12}$$

where

$$\phi(z)=\alpha_0+\alpha_1 z+\dots\alpha_{m-1}z^{m-1}$$

and

$$\psi(z)=\beta_0+\beta_1 z+\dots+\beta_{n-1}z^{n-1}.$$

Then (1.12) can be considered a system of equations for the coefficients $\alpha_0,\dots,\alpha_{m-1}$ and $\beta_0,\dots,\beta_{n-1}$. The solvability of such a system is equivalent to the vanishing of the determinant (1.11). □

Lemma 1.13.

$$R(f,g)=a_n^m b_m^n\prod(\mu_j-\nu_k)$$

where μ_j and ν_k are the roots of the polynomials f and g respectively.

For a proof of this lemma see [15].

The solutions of the system (1.10) are obtained by calculating the resultant of $F(z,w)$ and $F_w(z,w)$ with respect to z, which is also called the discriminant of F with respect to w. It can be computed as the determinant of a $(2n-1)\times(2n-1)$ matrix constructed from the coefficients of the polynomials

$$F=w^n+a_1w^{n-1}+\dots+a_{n-1}w+a_n$$

and

$$F_w = n\,w^{n-1} + (n-1)a_1 w^{n-2} + \cdots + a_{n-1}$$

$$R(F, F_w)(z) = \det \begin{pmatrix} 1 & a_1 & a_2 & \ldots & a_{n-1} & a_n & 0 & \ldots & 0 \\ 0 & 1 & a_1 & \ldots & \ldots & a_{n-1} & a_n & \ldots & 0 \\ \ldots & \ldots & & \ldots & & & \ldots & & \ldots \\ 0 & 0 & \ldots & & \ldots & & \ldots & a_{n-1} & a_n \\ n & (n-1)a_1 & (n-2)a_2 & \ldots & a_{n-1} & 0 & \ldots & \ldots & 0 \\ 0 & n & (n-1)a_1 & \ldots & 2a_{n-2} & a_{n-1} & 0 & \ldots & 0 \\ & \ldots & & \ldots & & & & & \ldots \\ 0 & 0 & \ldots & \ldots & & & \ldots & 2a_{n-2} & a_{n-1} \end{pmatrix}. \tag{1.13}$$

From lemma 1.13, the discriminant is also equal to

$$R(F, F_w)(z) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} \prod_{j=1}^{n-1} (w_i(z) - \tilde{w}_j(z)) \tag{1.14}$$

where $w_i(z)$, $i = 1, \ldots, n$, are the roots of the polynomials $F(z, w)$ and $\tilde{w}_j(z)$, $j = 1, \ldots, n-1$ are the roots of the polynomials $F_w(z, w)$ where z is considered as a parameter. Note that the total number of branch points is finite since $R(F, F_w)$ is a polynomial of finite degree.

The choice of the variables z or w as a local parameter is not always most convenient. We shall also encounter other ways of choosing a local parameter τ so that the point (z, w) of Γ can be represented locally in the form

$$z = z(\tau), \quad w = w(\tau) \tag{1.15}$$

where $z(\tau)$ and $w(\tau)$ are holomorphic functions of τ, and

$$\left(\frac{dz}{d\tau}, \frac{dw}{d\tau}\right) \neq 0. \tag{1.16}$$

We study the structure of the mapping π in (1.9) in a neighborhood of a branch point $P_0 = (z_0, w_0)$ of Γ defined in (1.3). Let τ be a local parameter on Γ in a neighborhood of P_0. It will be assumed that $z(\tau = 0) = z_0$, $w(\tau = 0) = w_0$. Then

$$\begin{aligned} z &= z_0 + a_k \tau^k + O(\tau^{k+1}), \quad a_k \neq 0 \\ w &= w_0 + b_q \tau^q + O(\tau^{q+1}), \quad b_q \neq 0, \end{aligned} \tag{1.17}$$

where a_k and b_q are nonzero coefficients. Since w can be taken as the local parameter in a neighborhood of P_0 it follows that $q = 1$. We get the form of the surface Γ in a neighborhood of a branch point:

$$\begin{aligned} z &= z_0 + a_k \tau^k + O(\tau^{k+1}), \\ w &= w_0 + b_1 \tau + O(\tau^2), \end{aligned} \tag{1.18}$$

where $k > 1$.

Definition 1.14. *The number $b_z(P) = k - 1$ is called the multiplicity of the branch point, or the branching index of this point with respect to the projection $(z, w) \to z$.*

Exercise 1.15: Let $P_0 = (z_0, w_0)$ be a branch point for the curve (1.7) with respect to the projection $(z, w) \to z$. Suppose that the local parameter in the neighbourhood of P_0 is of the form (1.18) with $k > 1$. Show that

$$\left. \frac{d^j F(z, w)}{dw^j} \right|_{(z_0, w_0)} = 0, \quad j = 0, \dots, k - 1.$$

Lemma 1.16. *Let (z_0, w_0) be a branch point of a Riemann surface Γ defined in (1.3) with respect to the projection $(z, w) \to z$. Then there exists a positive integer $k > 1$ and k functions $w_1(z), \dots,$ $w_k(z)$ analytic on a sector $S_{\rho,\phi}$ of the punctured disc*

$$0 < |z - z_0| < \rho, \quad \arg(z - z_0) < \phi$$

for sufficiently small ρ and any positive $\phi < 2\pi$ such that

$$F(z, w_j(z)) \equiv 0 \quad \text{for} \quad z \in S_{\rho,\phi}, \quad j = 1, \dots, k.$$

The functions $w_1(z), \dots, w_k(z)$ are continuous in the closure $\bar{S}_{\rho,\phi}$ and

$$w_1(z_0) = \dots = w_k(z_0) = w_0.$$

Proof. By the nonsingularity assumption $F_z(z_0, w_0) \neq 0$. So the complex curve $F(z, w) = 0$ can be locally parametrized in the form $z = z(w)$ where the analytic function $z(w)$ is uniquely determined by the condition $z(w_0) = z_0$. Consider the first nontrivial term of the Taylor expansion of this function

$$z(w) = z_0 + \alpha_k (w - w_0)^k + \alpha_{k+1} (w - w_0)^{k+1} + \dots, \quad k > 1, \ \alpha_k \neq 0.$$

Introduce an auxiliary function

$$\begin{aligned} f(w) &= \beta(w - w_0) \left[1 + \frac{\alpha_{k+1}}{\alpha_k} (w - w_0) + O\left((w - w_0)^2\right) \right]^{\frac{1}{k}} \\ &= \beta(w - w_0) \left[1 + \frac{\alpha_{k+1}}{k\, \alpha_k} (w - w_0) + O\left((w - w_0)^2\right) \right] \end{aligned}$$

where the complex number β is chosen in such a way that $\beta^k = \alpha_k$. The function $f(w)$ is analytic for sufficiently small $|w - w_0|$. Observe that $f'(w_0) = \beta \neq 0$. Therefore the analytic inverse function f^{-1} locally exists. The needed k functions $w_1(z), \ldots, w_k(z)$ can be constructed as follows

$$w_j(z) = f^{-1}\left(e^{\frac{2\pi i\,(j-1)}{k}}(z - z_0)^{1/k}\right), \quad j = 1, \ldots, k \tag{1.19}$$

where we choose an arbitrary branch of the k-th root of $(z - z_0)$ for $z \in S_{\rho,\phi}$. □

Example 1.17. Elliptic and hyperelliptic Riemann surfaces have the form

$$\Gamma = \{(z, w) \in \mathbb{C}^2\ |w^2 = P_n(z)\}, \tag{1.20}$$

where $P_n(z)$ is a polynomial of degree n. These surfaces are two-sheeted coverings of the z-plane. Here $F(z, w) = w^2 - P_n(z)$. The gradient vector $\mathrm{grad}_{\mathbb{C}} F = (-P_n'(z), 2w)$. A point $(z_0, w_0) \in \Gamma$ is singular if

$$w_0 = 0, \quad P_n'(z_0) = 0. \tag{1.21}$$

Together with the condition (1.20) for a point (z_0, w_0) to belong to Γ we get that

$$P_n(z_0) = 0, \quad P_n'(z_0) = 0, \tag{1.22}$$

i.e. z_0 is a multiple root of the polynomial $P_n(z)$. Accordingly, the surface (1.20) is nonsingular if and only if the polynomial $P_n(z)$ does not have multiple roots:

$$P_n(z) = \prod_{i=1}^{n}(z - z_i), \quad z_i \neq z_j, \text{ for } i \neq j. \tag{1.23}$$

The curve Γ is called an elliptic curve for $n = 3, 4$ and it is called hyperelliptic for $n > 4$. We find the branch points of the surface (1.20). To determine them we have the system

$$w^2 = P_n(z), \ \ w = 0,$$

which gives us n branch points $P_i = (z = z_i, w = 0)$, $i = 1, \ldots, n$. All the branch points have multiplicity one. In a neighborhood of any point of Γ that is not a branch point it is natural to take z as a local parameter, and $w = \sqrt{P_n(z)}$ is a holomorphic function. In a neighborhood of a branch point P_i it is convenient to take

$$\tau = \sqrt{z - z_i}, \tag{1.24}$$

as a local parameter. Then for points of the Riemann surface (1.20) we get the local parametric representation

$$z = z_i + \tau^2, \quad w = \tau \sqrt{\prod_{j\neq i}(\tau^2 + z_i - z_j)} \tag{1.25}$$

where the radical is a single-valued holomorphic function for sufficiently small τ;(the expression under the root sign does not vanish), and $dw/d\tau \neq 0$ for $\tau = 0$.

Exercise 1.18: Prove that the total multiplicity of all the branch points on Γ over $z = z_0$ is equal to the multiplicity of $z = z_0$ as a root of the discriminant.

Exercise 1.19: Consider the collection of n-sheeted Riemann surfaces of the form

$$F(z, w) = \sum_{i+j\leqslant n} a_{ij} z^i w^j \tag{1.26}$$

for all possible values of the coefficients a_{ij} (so-called planar curves of degree n). Prove that for a general surface of the form (1.26) there are $n(n-1)$ branch points and they all have multiplicity 1. In other words, conditions for the appearance of branch points of multiplicity greater than one are written as a collection of algebraic relations on the coefficients a_{ij}.

1.1.2 Smooth projective plane curves

We recall the the projective space $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1}\backslash\{0\}$ by the equivalence relation that identifies vectors v and αv in $\mathbb{C}^{n+1}\backslash\{0\}$ with $\alpha \in \mathbb{C}^*$. Namely $\mathbb{P}^n = \mathbb{C}^{n+1}\backslash\{0\}/\mathbb{C}^*$. The space $\mathbb{P}^0$ is a singly point, $\mathbb{P}^1$ can be thought as the complex plane $\mathbb{C}$ plus a single point ∞ and it can be identified with the Riemann sphere. $\mathbb{P}^2$ can be thought as $\mathbb{C}^2$ together with a line at infinity, namely a copy of $\mathbb{P}^1$ and so on.

The projective line is the simplest example of a compact Riemann surface. The example of compact Riemann surfaces that we are going to considered are embedded in $\mathbb{P}^2$.

Definition 1.20. *The projective plane $\mathbb{P}^2$ is the set of one-dimensional subspaces in $\mathbb{C}^3$ or equivalently $\mathbb{P}^2 = \mathbb{C}^3\backslash\{0\}/\mathbb{C}^*$. Let (X, Y, Z) be a nonzero vector in $\mathbb{C}^3$. A point in $\mathbb{P}^2$ is denoted by $[X : Y : Z]$ and*

$$[X : Y : Z] = [\lambda X : \lambda Y : \lambda Z], \quad \lambda \neq 0, \ \lambda \in \mathbb{C}$$

As a quotient space, $\mathbb{P}^2$ is endowed with the quotient topology. Indeed let the projection map $\pi : \mathbb{C}^3\backslash\{0\} \to \mathbb{P}^2$ be defined as

$$\pi(X, Y, Z) = [X : Y : Z].$$

Then we can give to $\mathbb{P}^2$ the quotient topology induced from $\mathbb{C}^3\backslash\{0\}$, namely a subset U of $\mathbb{P}^2$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{C}^3\backslash\{0\}$. As a topological space, $\mathbb{P}^2$ is a Hausdorff space, namely two distinct points have disjoint open neighbourhoods.

Proposition 1.21. *The space $\mathbb{P}^2$ is compact.*

Proof. Let

$$S^5 = \{(X, Y, Z) \in \mathbb{C}^3 \mid |X|^2 + |Y|^2 + |Z|^2 = 1\}.$$

Then S^5 is a sphere of real dimension 5. It is a closed and bounded subset of $\mathbb{C}^3$ and by the Heine-Borel theorem is compact. The restriction of $\pi_{S^5} : S^5 \to \mathbb{P}^2$ is continuos. The image of a compact set under a continuous mapping is compact. Next let us show that π_{S^5} is also surjective. Let $[X : Y : Z] \in \mathbb{P}^2$, then

$$|X|^2 + |Y|^2 + |Z|^2 = \lambda, \quad \text{for some } \lambda > 0.$$

Then we also have

$$[X : Y : Z] = [\lambda^{-\frac{1}{2}}X : \lambda^{-\frac{1}{2}}Y : \lambda^{-\frac{1}{2}}Z].$$

Combining the above two relations one has that

$$|\lambda^{-\frac{1}{2}}X|^2 + |\lambda^{-\frac{1}{2}}Y|^2 + |\lambda^{-\frac{1}{2}}Z|^2 = 1$$

so that $[X : Y : Z] \in \pi(S^5)$. Namely the map $\pi : S^5 \to \mathbb{P}^2$ is surjective and continuos which implies that $\mathbb{P}^2$ is compact. □

Remark 1.22. The spaces $\mathbb{P}^n$, $n \geqslant 0$ are all compact. The proof of this statement is a simple generalisation of the proof of proposition 1.21.

The space $\mathbb{P}^2$ can be covered with three open sets homeomorphic to $\mathbb{C}^2$:

$$U_0 = \{[X : Y : Z] \in \mathbb{P}^2 \mid X \neq 0\}$$

$$U_1 = \{[X : Y : Z] \in \mathbb{P}^2 \mid Y \neq 0\}$$

$$U_2 = \{[X : Y : Z] \in \mathbb{P}^2 \mid Z \neq 0\}.$$

The homeomorphism on U_0 is given by the map $[X : Y : Z] \to (Y/X, Z/X) \in \mathbb{C}^2$ and similarly for the other open sets U_1 and U_2.

Definition 1.23. *Let $Q(X, Y, Z)$ be a homogeneous non constant polynomial of degree d, in the complex variables X, Y and Z with complex coefficients. The locus*

$$\Gamma = \{[X : Y : Z] \in \mathbb{P}^2 \mid Q(X, Y, Z) = 0\} \tag{1.27}$$

is the projective curve defined by the polynomial Q.

Remark 1.24. Observe that the curve Γ is well defined since the condition $Q(X,Y,Z) = 0$ is independent from the choice of homogeneous coordinates since $Q(\lambda X, \lambda Y, \lambda Z) = \lambda^d Q(X,Y,Z)$. Furthermore Γ is a closed subset of $\mathbb{P}^2$ and therefore it is compact.

The intersection of Γ with any of the U_i is an affine plane curve. For example

$$\Gamma_0 = \Gamma \cap U_0 = \{(u,v) \in \mathbb{C}^2 \mid Q(1,u,v) = 0\}.$$

Now we show that under non singularity assumptions, Γ is a Riemann surface.

Definition 1.25. *The curve (1.27) defined by the zeros of the homogeneous polynomial $Q(X,Y,Z)$ is nonsingular if there are no non zero solutions to the equations*

$$Q = \frac{\partial Q}{\partial X} = \frac{\partial Q}{\partial Y} = \frac{\partial Q}{\partial Z} = 0.$$

Exercise 1.26: Show that the projective curve Γ defined in (1.27) is non singular if and only if each of the affine components $\Gamma_i = \Gamma \cap U_i$, $i = 1,2,3$ is non singular. *Hint*: use Euler equation that is obtained differentiating the identity $Q(\lambda X, \lambda Y, \lambda Z) = \lambda^d Q(X,Y,Z)$ with respect to λ and setting $\lambda = 1$, namely

$$XQ_X + YQ_Y + ZQ_Z = Qd. \tag{1.28}$$

Suppose that Γ is a smooth projective curve. In order to give a complex structure on Γ let us recall that each Γ_i is a smooth irreducible affine plane curve and hence a Riemann surface. The coordinate charts are given by the projections. For example for the curve Γ_0 the coordinate charts are y/x or z/x and the transition functions are as the same as the one obtained for smooth affine plane curves. We have then to check that the complex structures given on each Γ_i are compatible. Let $P \in \Gamma_0 \cap \Gamma_1$ where $P = [X : Y : Z]$ and $X \neq 0$ and $Y \neq 0$. Since each affine plane curve is non singular (see exercise 1.26), we assume without loss of generality that F_X and F_Z are non zero. Let $\phi_0 : \Gamma_0 \to \mathbb{C}$ with $\phi_0(P) = Y/X$ and with inverse $\phi_0^{-1}(Y/X) = [1 : Y/X : h(Y/X)]$ where h is a holomorphic function. Let $\phi_1 : \Gamma_1 \to \mathbb{C}$ with $\phi_1(P) = Z/Y$ with inverse $\phi_1^{-1} = [g(\frac{Z}{Y}), 1, \frac{Z}{Y}]$ where $g(\frac{Z}{Y})$ is holomorphic for $Y \neq 0$ and non zero since we assume $X \neq 0$. Then $\phi_1 \circ \phi_0^{-1}(Y/X) = Xh(Y/X)/Y$ which is holomorphic because $Y \neq 0$, $X \neq 0$ and $h(Y/X)$ is holomorphic. In the same way $\phi_0 \circ \phi_1^{-1}(Z/Y) = \frac{1}{g(Z/Y)}$ which is holomorphic because $Y \neq 0$ and g is nonzero. Similar checks can be done with the other coordinate charts.

We summarise the above description in the following proposition.

Proposition 1.27. *Let $Q(X,Y,Z)$ be a homogeneous polynomial such that the projective plane curve Γ that is the zero locus of Q in $\mathbb{P}^2$ is a smooth compact Riemann surface. At every point of Γ one can take as a local coordinate a ratio of the homogeneous coordinates.*

Lemma 1.28. *Let $Q(X,Y,Z)$ and $F(X,Y,Z)$ be two homogeneous polynomials of degree d and m respectively. Suppose that $Q(0,0,Z) \neq 0$ and $F(0,0,Z) \neq 0$. Then the resultant*

$$R(Q_Z, F_Z)(X,Y)$$

is a homogeneous polynomial in X and Y of degree dm.

Proof. According to the assumptions, $Q(X,Y,Z) = q_0 Z^d + q_1(x,Y)Z^{d-1} + \cdots + q_d(X,Y)$ where $q_j(X,Y)$ are homogeneous polynomials of degree j in X and Y, $j = 0,\ldots,d$ and $F(X,Y,Z) = f_0 Z^m + f_1(X,Y)Z^{m-1} + \cdots + f_m(X,Y)$ where $f_j(X,Y)$ are homogeneous polynomials of degree j, $j = 0,\ldots,m$.

Then according to the definition of resultant in (1.11)

$$R(Q,F)(X,Y) = det \begin{pmatrix} q_0 & q_1 & \ldots & q_d & 0 & 0 & & \ldots & 0 \\ 0 & q_0 & q_1 & \ldots & q_d & 0 & 0 & \ldots & 0 \\ \ldots & & & \ldots & & & & \ldots & \\ 0 & 0 & \ldots & \ldots & q_0 & q_1 & q_2 & \ldots & q_d \\ f_0 & f_1 & \ldots & \ldots & f_{m-1} & f_m & 0 & \ldots & 0 \\ 0 & f_0 & f_1 & \ldots & \ldots & f_{m-1} & f_m \; 0 & \ldots & 0 \\ \ldots & & & \ldots & & & & \ldots & \\ 0 & \ldots & f_0 & f_1 & \ldots & & \ldots & f_{m-1} & f_m \end{pmatrix}. \quad (1.29)$$

We multiply the second row by $\lambda \neq 0$, the third row by λ^2 and so on till the $m-th$ row that is multiplied by λ^{m-1}. Then we multiply the $(m+2)-th$ row by λ, the $(m+3)-th$ by λ^2 and so on till the $(m+d)-th$ that is multiply by λ^{d-1} one has

$$\begin{aligned} R(Q,F)(\lambda X, \lambda Y) &= \frac{1}{\lambda^{\frac{1}{2}(d-1)d}\lambda^{\frac{1}{2}m(m-1)}} \\ &\times \det \begin{pmatrix} q_0 & \lambda q_1 & \ldots & \lambda^d q_d & 0 & 0 & & \ldots & 0 \\ 0 & \lambda q_0 & \lambda^2 q_1 & \ldots & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & & & \ldots & & & & \ldots & \\ 0 & 0 & \ldots & \ldots & \lambda^{m-1} q_0 & \lambda^m q_1 & \ldots & \ldots & \lambda^{d+m-1} q_d \\ f_0 & \lambda f_1 & \ldots & \ldots & \lambda^{m-1} f_{m-1} & \lambda^m f_m & 0 & \ldots & 0 \\ 0 & \lambda f_0 & \lambda^2 f_1 & \ldots & \ldots & \lambda^m f_{m-1} & \lambda^{m+1} f_m & \ldots & 0 \\ \ldots & & & \ldots & & & & \ldots & \\ 0 & \ldots & \lambda^{d-1} f_0 & \lambda^d f_1 & \ldots & & \ldots & \lambda^{m+d-2} f_{m-1} & \lambda^{m+d-1} f_m \end{pmatrix} \\ &= \lambda^{md} R(Q,F)(X,Y), \end{aligned}$$

where we use the fact that and $q_j(\lambda X, \lambda Y) = \lambda^j q_j(X,Y)$ and $f_j(\lambda X, \lambda Y) = \lambda^j f_j(X,Y)$. The above relation shows that the resultant $R(Q,F)(X,Y)$ is a homogeneous polynomial in X and Y of degree md. □

Theorem 1.29 (Bezout's theorem)**.** *Let Γ and M be two projective curves defined by the homogenous polynomials $Q(X,Y,Z)$ and $F(X,Y,Z)$ of degree d and m respectively. Then if Γ and M do not have a common component, then they intersect in dm points counting multiplicity.*

Proof. By Lemma 1.13, Γ and X have a common component if and only if their resultant is identically zero. Next we consider the case in which Γ and X do not have a common component. We assume that $[0:0:1]$ does not belong to both curves. With this assumption $Q(X,Y,Z) = q_0(X,Y)Z^d + q_1(x,Y)Z^{d-1} + \cdots + q_d(X,Y)$ where $q_j(X,Y)$ are homogeneous polynomials of degree j in X and Y, $j = 0,\ldots,d$ and $q_0 \neq 0$. In the same way $F(X,Y,Z) = f_0(X,Y)Z^m + f_1(X,Y)Z^{m-1} + \cdots + f_m(X,Y)$ where $f_j(X,Y)$ are homogeneous polynomials of degree j, $j = 0,\ldots,m$ and $f_0 \neq 0$. Therefore the resultant is a homogeneous polynomial of degree md by lemma 1.28 and it has md zeros counting their multiplicity. □

Lemma 1.30. *If the projective curve Γ defined in (1.27) is non singular, then the polynomial $Q(X,Y,Z)$ is irreducible. If Γ is irreducible, then it has at most a finite number of singular points.*

Proof. Let us suppose that the polynomial is reducible, namely $Q = Q_1Q_2$ where Q_1 and Q_2 are homogeneous polynomials in X, Y and Z of degree d_1 and $d-d_1$. The condition of Γ being singular takes the form

$$Q_2Q_1 = 0,\ \ Q_2\partial_X Q_1 + Q_1\partial_X Q_2 = 0,\ \ Q_2\partial_Y Q_1 + Q_1\partial_Y Q_2 = 0,\ \ Q_2\partial_Z Q_1 + Q_1\partial_Z Q_2 = 0.$$

Such system of equations has always a solution as long as there is a point P in the intersections of the curves defined by $Q_1 = 0$ and $Q_2 = 0$. But this is always the case. Indeed let us consider the resultant $R(Q_1,Q_2)(X,Y)$ of the polynomials $Q_1(X,Y,Z)$ and $Q_2(X,Y,Z)$ with respect to Z. Assuming that $Q_1(0,0,1) \neq 0$ and $Q_2(0,0,1) \neq 0$ the resultant $R(Q_1,Q_2)(X,Y)$ is a homogeneous polynomial of degree $d_1(d-d_1)$. Therefore the curves defined by the equations $Q_1(X,Y,Z) = 0$ and $Q_2(X,Y,Z) = 0$ intersects by Bezout's theorem in $d_1(d-d_1)$ points counted with multiplicity. We conclude that if Q is reducible, then Q is singular. Suppose that Γ is irreducible and defined by the polynomial Q of degree n. Then Q and Q_Z do not have a common component so that the resultant $R(Q,Q_Z)(X,Y)$ is a homogeneous polynomial of degree $n(n-1)$ not identically zero. Since the singular points of Γ are contained in the zeros of the resultant, the number is finite. □

The simplest example of projective curve is the projective line

$$\alpha X + \beta Y + \gamma Z = 0$$

where $(\alpha,\beta,\gamma) \neq (0,0,0)$. The tangent line to a projective curve Γ defined by a homogeneous polynomial $Q(X,Y,Z)$ at a non singular point (X_0,Y_0,Z_0) has the form

$$(X-X_0)Q_X(X_0,Y_0,Z_0) + (Y-Y_0)Q_Y(X_0,Y_0,Z_0) + (Z-Z_0)Q_Z(X_0,Y_0,Z_0) = 0.$$

Exercise 1.31: Let $Q(X, Y, Z)$ be an irreducible homogeneous polynomial of degree d defining a smooth projective curve Γ. Suppose that the equation $Q(X, Y, 1) = 0$ locally defines Y as a holomorphic function of X. Show that

$$\frac{d^2Y(X)}{dX^2} = \frac{1}{Q_Y^3} det \begin{pmatrix} Q_{XX} & Q_{XY} & Q_X \\ Q_{YX} & Q_{YY} & Q_Y \\ Q_X & Q_Y & 0 \end{pmatrix}.$$

Observe that a point $[X_0 : Y_0 : 1]$ is an inflection point for the curve Γ if and only if $\frac{d^2Y(X)}{dX^2}$ vanishes at X_0.

1.1.3 Compactification of affine plane curve

Complex affine plane curves $\Gamma := \{(z, w \in \mathbb{C}^2 \; |F(z, w) = 0\}$ where F is a nonsingular polynomial, are non compact Riemann surfaces. To compactify them one needs to add point(s) $\infty^1, \infty^2, \ldots \infty^N$ at infinity and introducing proper local parameters at these points in such a way that

$$\hat{\Gamma} = \Gamma \cup \infty^1 \cup \infty^2 \cup \cdots \cup \infty^N$$

is a compact Riemann surface.

The plane curve Γ, defined by the polynomial equation $F(z, w) = 0$, can be compactified by embedding it in $\mathbb{CP}^2$. The mappings

$$(X : Y : Z) \to \left(z = \frac{X}{Z}, \; w = \frac{Y}{Z} \right)$$

and the inverse mapping

$$(z, w) \to (z : w : 1)$$

establish an isomorphism between an affine part of $\mathbb{CP}^2$ and $\mathbb{C}^2$. The whole projective plane is obtained from the affine part $\mathbb{C}^2$ by adding the line at infinity of the form $(X : Y : 0) \simeq \mathbb{CP}^1 \simeq S^2$. An embedding of Γ in $\mathbb{CP}^2$ is defined as follows. Suppose that

$$F(z, w) = F_k(z, w) + F_{k-1}(z, w) + \cdots + F_0(z, w),$$

where each $F_j(z, w)$ is a homogeneous polynomial of degree j. Then we define the homogeneous polynomial

$$Q(X, Y, Z) = Z^k F\left(\frac{X}{Z}, \frac{Y}{Z} \right) \tag{1.30}$$

of degree k. A complex compact curve $\hat{\Gamma}$ is given in $\mathbb{CP}^2$ by the homogeneous equation

$$\hat{\Gamma} := \{[X : Y : Z] \in \mathbb{P}^2 \mid Q(X, Y, Z) = 0\}. \tag{1.31}$$

The affine part of the curve $\hat{\Gamma}$ (where $Z \neq 0$) coincides with Γ. The associated points at infinity have the form

$$Q(X, Y, 0) = 0.$$

The surface $\hat{\Gamma}$ is compact and is thus the desired compactification of the surface Γ.

Remark 1.32. Even if the curve Γ is non singular, the curve $\hat{\Gamma}$ might be singular. If this is the case, the compactification of the smooth affine plane curve as a singular projective curve is not a good compactification.

Example 1.33. $\Gamma = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z\}$. A local parameter at the branch point $(z = 0, w = 0)$ is given by $\tau = \sqrt{z}$, i.e. $z = \tau^2$, $w = \tau$. The compactification $\hat{\Gamma}$ has the form $\hat{\Gamma} = \{[X : Y : Z] \in \mathbb{P}^2 \mid Y^2 = XZ\}$. The point at infinity is given by solving the equation (1.31), that gives $P^\infty = [1 : 0 : 0]$. For $X \neq 0$ we introduce the coordinates u, v

$$u = \frac{Y}{X} = \frac{w}{z}, \quad v = \frac{Z}{X} = \frac{1}{z}, \tag{1.32}$$

which define the affine curve $u^2 = v$. The point at infinity is given by $(v = 0, u = 0)$ which is clearly a branch point for the curve defined by the equation $u^2 = v$ and $\sqrt{v}$ is a local parameter near this point. Therefore in a neighborhood of the point at infinity in $\hat{\Gamma}$ we have that

$$(z, w) \to \frac{1}{\sqrt{z}}$$

is a local homeomorphism.

Example 1.34. $\Gamma = \{w^2 = z^2 - a^2\}$. The branch points are $(z = \pm a, w = 0)$ and the corresponding local parameters are $\tau_\pm = \sqrt{z \pm a}$. The compactification has the form $\hat{\Gamma} = \{Y^2 = X^2 - a^2 Z^2\}$. The point at infinity is given by solving the equation (1.31), that gives $P^\infty_\pm = [1 : \pm 1 : 0]$. Making the substitution (1.32) we get the form of the curve $\hat{\Gamma}$ in a neighborhood of the ideal line: $u^2 = 1 - a^2 v^2$. For $v = 0$ we get that $u = \pm 1$. We can take $v = 1/z$ as a local parameter in a neighborhood of each of these points. The form of the surface $\hat{\Gamma}$ in a neighborhood of these points $P_\pm$ is as follows:

$$z = \frac{1}{v}, \quad w = \pm\frac{1}{v}\sqrt{1 - a^2 v^2}, \quad v \to 0 \tag{1.33}$$

where $\sqrt{1 - a^2 v^2}$ is, for small v, a single-valued holomorphic function, and the branch of the square root is chosen to have value 1 at $v = 0$.

Example 1.35. Let us consider the class of hyperelliptic Riemann surfaces

$$\Gamma = \{(z,w) \in \mathbb{C}^2 \mid F(z,w) = w^2 - P_N(z) = 0\}, \tag{1.34}$$

where $P_N(z) = \prod_{j=1}^{N}(z - a_j)$, and $a_i \neq a_j$ for $i \neq j$.

If we consider the projective curve defined by the zeros of homogeneous polynomial

$$Q(X,Y,Z) = Y^2 Z^{N-2} - Z^N P_N(X/Z) = 0$$

one can check that the curve is singular at the point $[0 : 1 : 0]$ if $N \geqslant 4$. Therefore, for $N \geqslant 4$, the embedding of Γ in $\mathbb{P}^2$ is not a good compactification. For $N = 3$ the projective curve

$$Y^2 Z = (X - a_1 Z)(X - a_2 Z)(X - a_3 Z)$$

is a compact smooth elliptic curve. By a projective transformation such curve can be reduced to the form

$$Y^2 Z = X(X - Z)(X - \lambda Z), \quad \lambda \in C\backslash\{0,1\}.$$

The point at infinity is given by $P^\infty = [0 : 1 : 0]$. For $Y \neq 0$ the substitution $u = X/Y$ and $v = Z/Y$ gives the curve

$$Q(u,1,v) = v - u(u - v)(u - \lambda v) = 0$$

The point $(0,0)$ is a branch point for the above curve. Indeed for $(u,v) \neq 0$ the projection $\pi : (u,v) \to v$ is a local coordinate. The preimage $\pi^{-1}(v)$ consists of three points. At the point $(0,0)$ one has $Q_u(0,1,0) = 0$ and $Q_{uu}(0,1,0) = 0$ so that the preimage of $\pi^{-1}(0)$ consists of a single point. A local coordinate near the point $(0,0)$ takes the form

$$u = \tau(1 + o(\tau)), \quad v = \tau^3(1 + o(\tau)).$$

We look for the holomorphic tail in the form

$$u = \tau g(\tau), \quad v = \tau^3 g(\tau)$$

with $g(\tau)$ analytic and invertible in a neighbourhood of $\tau = 0$. Plugging the above ansatz in the equation $Q(u,1,v) = v - u(u - v)(u - \lambda v) = 0$ one obtains that

$$g(\tau) = \frac{1}{\sqrt{(1-\tau^2)(1-\lambda\tau^2)}}.$$

Since

$$z = \frac{X}{Z} = \frac{u}{v}, \quad w = \frac{Y}{Z} = \frac{1}{v}$$

one has that a local coordinate near the point at infinity for the curve Γ is given by

$$z = \frac{1}{\tau^2}, \quad w = \frac{1}{\tau^3}\sqrt{(1-\tau^2)(1-\lambda\tau^2)}.$$

The above example shows that not all the affine plane curves can be compactified in a smooth way by embedding them in $\mathbb{P}^2$. Below we are going to illustrate another way of compactifying affine plane curves.

Definition 1.36. *Let Γ be a Riemann surface such $\overline{\Gamma} = \Gamma \cup \infty^1 \cup \dots \infty^N$ is a compact surface. Suppose that there exist open subsets*

$$U_{\infty^1} \cup U_{\infty^2} \cup \dots \cup U_{\infty^N} = U_\infty \subset \Gamma$$

such that U_{∞^n}, $n = 1, \dots, N$, are homeomorphic to puncture disks

$$\phi_n : U_{\infty^n} \to D \backslash \{0\} = \{z \in \mathbb{C} \mid 0 < |z| < c, \ c \in \mathbb{R}^+\},$$

and the homeomorphism ϕ_n are holomorphically compatible with the complex structure of Γ. Then Γ is called a compact Riemann surface with punctures.

The goal is to make the compact surface $\overline{\Gamma}$ a Riemann surface. Let us extend the homeomorphism ϕ_n to the whole neighbourhood $\overline{U}_{\infty^n} = U_{\infty^n} \cup \infty^n$ by defining

$$\phi_n(\infty^n) = 0, \ n = 1, \dots, N.$$

In order to make $\overline{\Gamma}$ a compact Riemann surface one needs to define a complex atlas on it as the union of the compatible coordinates charts on $\overline{U}_{\infty^n}$ and Γ. The result is a compact Riemann surface $\hat{\Gamma}$.

Example 1.37. We recall first how to compactify the complex z-plane $\mathbb{C}$. It is necessary to add to $\mathbb{C}$ a single "point at infinity" ∞. In this case $U_\infty = \mathbb{C}$ and the map $\phi : U_\infty \to D\backslash\{0\}$ is defined by $\phi(z) = \dfrac{1}{z}$ with $z \neq 0$ and $\phi(\infty) = 0$ A complex atlas on $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ is then defined as in example 1.5. We get a surface $\bar{\mathbb{C}}$ with the topology of a sphere (the "Riemann sphere"). Topological equivalence to the standard sphere is given by stereographic projection, with one of the poles of the sphere passing into the point ∞. Another description of $\bar{\mathbb{C}}$ is the complex projective line $\mathbb{P}^1 := \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 \neq 0, \ \ (z_1 : z_2) \simeq (\lambda z_1 : \lambda z_2), \ \lambda \in \mathbb{C}, \ \lambda \neq 0\}$. The equivalence $\mathbb{P}^1 \to \bar{\mathbb{C}}$ is established as follows: $(z_1 : z_2) \to z = \dfrac{z_1}{z_2}$. The affine part $\{z_2 \neq 0\}$ of $\mathbb{P}^1$ passes into $\mathbb{C}$ and the point $(1:0)$ into ∞.

Example 1.38. Let us consider the class of hyperelliptic Riemann surfaces

$$\Gamma = \{(z, w) \in \mathbb{C}^2 \mid F(z, w) = w^2 - P_N(z) = 0\}, \tag{1.35}$$

where $P_N(z) = \prod_{j=1}^N (z - a_j)$, $N \geqslant 4$ and $a_i \neq a_j$ for $i \neq j$. We need to consider separately the case of N odd or even. Let us rewrite the curve in the form

$$\left(\frac{w}{z^{n+1}}\right)^2 - \frac{1}{z}\prod_{j=1}^N (1 - \frac{a_j}{z}) = 0, \quad N = 2n+1,$$

$$\left(\frac{w}{z^{n+1}}\right)^2 - \prod_{j=1}^{N}(1 - \frac{a_j}{z}) = 0, \quad N = 2n + 2.$$

For N odd the map

$$\psi : (z, w) \to \left(\frac{1}{z}, \frac{w}{z^{n+1}}\right) \tag{1.36}$$

describes a biholomorphic map from a punctured neighbourhood of infinity

$$U_\infty = \{(z, w) \in \Gamma \mid |z| > c > |a_j|, \ j = 1, \ldots, 2n + 1\}$$

where $c > 0$, to the punctured neighbourhood

$$V = \{(x, y) \in \tilde{\Gamma} \mid |0 < |x| < 1/c\}$$

of the point $(x, y) = (0, 0)$ of the curve $\tilde{\Gamma}$ defined by the equation

$$\tilde{\Gamma} = \{(x, y) \in \mathbb{C}^2 \mid y^2 - x\prod_{j=1}^{N}(1 - xa_j) = 0\}, \ \ N = 2n + 1. \tag{1.37}$$

For $N = 2n + 2$ even, the map (1.36) describes a biholomorphic map from punctured neighbourhoods of infinity $\infty^\pm$

$$U_\infty^\pm = \{(z, w) \in \Gamma \mid |z| > c > |a_j|, \ j = 1, \ldots, 2n + 2, \lim \frac{w}{z^{n+1}} = \pm 1\}$$

to the punctured neighbourhoods

$$V^\pm = \{(x, y) \in \tilde{\Gamma} \mid 0 < |x| < 1/c\}$$

of the points $(0, \pm 1)$ of the curve

$$\tilde{\Gamma} = \{(x, y) \in \mathbb{C}^2 \mid y^2 - \prod_{j=1}^{N}(1 - xa_j) = 0\}, \ \ N = 2n + 2. \tag{1.38}$$

The local coordinate near $(0, 0)$ of the curve $\tilde{\Gamma}$ in (1.37) is defined by the homeomorphism $(x, y) \to \sqrt{x}$, while the local coordinate near the point $(0, \pm 1)$ of the curve (1.38) is given by $(x, y) \to x$. Therefore for $N = 2n + 1$ the curve (1.35) has one puncture at infinity and the local parameter in its neighbourhood is given by

$$\phi(z, w) = \frac{1}{\sqrt{z}}, \quad \phi(\infty) = 0$$

while for $N = 2n + 2$, the curve (1.35) has two punctures $\infty^{\pm} = (\infty, \pm\infty)$ distinguished by the conditions

$$(z, w) \to \infty^{\pm} \leftrightarrow \frac{w}{z^{n+1}} \to \pm 1$$

and the local parameter near these points is given by the homeomorphism

$$\phi_{\pm}(z, w) \to \frac{1}{z}, \quad \phi_{\pm}(\infty^{\pm}) = 0.$$

Proposition 1.39. *The local parameters*

$$(z, w) \to z \ \text{ near an ordinary point}$$
$$(z, w) \to \sqrt{z - z_j} \ \text{ near a branch point } (z_j, 0)$$
$$(z, w) \to \begin{cases} 1/\sqrt{z} \ \text{ near the point at infinity, } N \text{ odd} \\ 1/z \ \text{ near the point at infinity, } N \text{ even} \end{cases}$$

describe a compact Riemann surface $\hat{\Gamma} = \Gamma \cup \infty$ of the hyperelliptic curve (1.35) for N odd and $\hat{\Gamma} = \Gamma \cup \infty^{\pm}$ for N even.

Quotients under Group action

Complex Tori. Let ω_1 and ω_2 be two complex numbers which are linearly independent over the real numbers. Define the lattice

$$L_{\omega_1,\omega_2} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}. \tag{1.39}$$

Two complex numbers z and $\tilde{z}$ are equivalent mod L_{ω_1,ω_2} if $z - \tilde{z} \in L_{\omega_1,\omega_2}$. The set of all equivalence classes is denoted by $\mathbb{C}/L_{\omega_1,\omega_2}$ and an element in $\mathbb{C}/L_{\omega_1,\omega_2}$ is denoted by $[z]$.

Proposition 1.40. *The quotient $\Gamma = \mathbb{C}/L_{\omega_1,\omega_2}$ is a compact Riemann surface that is topologically a torus.*

Proof. To prove the statement one needs to construct a complex structure on Γ. Let $\pi : \mathbb{C} \to \Gamma$ be the projection map. Let us endowed Γ with the quotient topology namely a set $U \subset \Gamma$ is open if $\pi^{-1}(U)$ is open in $\mathbb{C}$. This definition makes π continuous and since $\mathbb{C}$ is connected so is Γ. Furthermore, it is easy to check that π is an open mapping. Indeed let U be an open set in $\mathbb{C}$, then $\pi(U)$ is open if $\pi^{-1}(\pi(U))$. But this is certainly the case since $\pi^{-1}(\pi(U)) = \bigcup_{\omega \in L}(\omega + U)$ is open. In order to define a complex structure on Γ, let $D_\alpha = D_{z_\alpha,\epsilon}$ be a disk centered at $z_\alpha \in \mathbb{C}$ and of radius ϵ where ϵ is chosen in such a way that $|\omega| > \epsilon$ for every non zero $\omega \in L$. Then the map $\pi|_{D_\alpha} : D_\alpha \to \pi(D_\alpha)$ is a homeomorphism. Let $\phi_\alpha : \pi(D_\alpha) \to D_\alpha$ be the inverse of the map $\pi|_{D_\alpha}$. The pairs $(\pi(D_{\alpha)}), \phi_\alpha)_{\alpha \in A}$ defines a complex chart. We now must check that the charts are compatible. Chose two distinct points z_1 and z_2 and consider two charts $\phi_1 : \pi(D_1) \to D_1$ and $\phi_2 : \pi(D_2) \to D_2$ with $U := \pi(D_1) \cap \pi(D_2) \neq \emptyset$. We need to check that the transition function $T(z) = \phi_2(\phi_1^{-1}(z))$ is holomorphic for $z \in \phi_1(U)$. Observe that $\pi(T(z)) = \pi(z)$ for all $z \in \phi_1(U)$. Therefore $T(z) - z = \omega(z) \in L$. Since $T(z)$ is continuos and L is discrete, $\omega(z)$ is constant. Therefore $T(z) = z + \omega$ for some $\omega \in L$, namely the transition function $T(z)$ is holomorphic. The collection of charts $\{(D_\alpha, \phi_\alpha) \mid z_\alpha \in \mathbb{C}\}$ is a complex atlas on Γ. We conclude that Γ is a compact Riemann surface. ◻

Remark 1.41. Let $A \in SL(2, \mathbb{Z})$ namely A is 2×2 matrix with integer entries and $\det A = 1$. Suppose that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Then the $L_{\omega_1,\omega_2} = L_{\omega_1',\omega_2'}$. Indeed for $m, n \in \mathbb{Z}$ one has

$$L_{\omega_1,\omega_2} \ni m\omega_1 + n\omega_2 = (n, m)A^{-1} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = m'\omega_1' + n'\omega_2' \in L_{\omega_1',\omega_2'},$$

because $m', n' \in \mathbb{Z}$ since the matrix A has integer entries and determinant equal to one.

The above relation shows that $L_{\omega_1,\omega_2} \subseteq L_{\omega_1',\omega_2'}$. Repeating the same reasoning for a point in $L_{\omega_1',\omega_2'}$ one obtains that $L_{\omega_1',\omega_2'} \subseteq L_{\omega_1,\omega_2}$ which shows that $L_{\omega_1,\omega_2} = L_{\omega_1',\omega_2'}$.

Remark 1.42. Let us consider an automorphism of the complex plane, namely a map $F : \mathbb{C} \to \mathbb{C}$ of the form $F(z) := \alpha z + \beta$ with $\alpha \neq 0$. We choose $\beta = 0$ so that $F(0) = 0$. A lattice L_{ω_1,ω_2} is transformed under F to the lattice $L_{\alpha\omega_1,\alpha\omega_2}$. The corresponding tori are isomorphic, with the isomorphism given by $[z] \to [\alpha z]$. The map F projects to an automorphism of the torus if $|\alpha| = 1$. In general

- $\alpha = \pm 1$, for a generic torus;
- $\alpha = i$, for the square torus;
- $\alpha = e^{i\frac{\pi}{3}}$, for the rhombi torus.

Let us define $\tau = \dfrac{\omega_1}{\omega_2}$ with $\Im(\tau) > 0$. Then the lattice L_{ω_1,ω_2} defined in (1.39) and

$$L_{\tau,1} = \{n + m\tau \mid m, n \in \mathbb{Z}\}, \quad \tau = \frac{\omega_1}{\omega_2}$$

defined isomorphic tori $\mathbb{C}/L_{\omega_1,\omega_2}$ and $\mathbb{C}/L_{\tau,1}$ respectively. Combining the above remarks one arrives to the following theorem.

Theorem 1.43. *Let T_τ and $T_{\tau'}$ be two tori defined by the lattices $L_{\tau,1}$ and $L_{\tau',1}$ with $\Im(\tau) > 0$ and $\Im(\tau') > 0$. The tori are isomorphic if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{1.40}$$

The proof is left as an exercise.

Exercise 1.44: Consider the group $2\pi\mathbb{Z}$ under addition and consider the quotient $\mathbb{C}/2\pi\mathbb{Z}$. This surface is clearly homeomorphic to the cylinder $S^1 \times \mathbb{R}$. Show that $\mathbb{C}/2\pi\mathbb{Z}$ is a Riemann surface.

Exercise 1.45: Let G be the multiplicative group $G := \{a^n \mid n \in \mathbb{Z}\}$ and $a \in \mathbb{R}^+$. The quotient

$$\Gamma := \mathbb{C}^*/G$$

is defined as the set of equivalence class with respect to the equivalence relation

$$z \simeq \tilde{z} \longleftrightarrow z\tilde{z}^{-1} \in G.$$

(i) Prove that there exist a unique structure of a Riemann surface on Γ such that the canonical projection $\pi : \mathbb{C}^* \to \Gamma$ is locally biholomorphic.

(ii) Show that the Rieamann surface constructed in (i) is isomorphic to a torus

$$\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), \quad \tau \in \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

Calculate τ.

The above construction of Riemann surface as quotients can be generalized

Definition 1.46. *Let Δ be a domain of $\mathbb{C}$. A group $G : \Delta \to \Delta$ of holomorphic transformations acts discontinuously and fixed point free on Δ if for any $P \in \Delta$ there exists a neighbourhood $V \ni P$ such that*

$$gV \cap V = \emptyset, \quad \forall g \in G, \quad g \neq I$$

The action of G is called proper if the inverse image of compact subset is compact.

Introducing an equivalent relation between points of Δ, namely $P \simeq P'$ if $\exists g \in G$ so that $P' = gP$, one can define the quotient space Δ/G of equivalent classes.

Theorem 1.47. *If a group G acts on a domain Δ of the complex plane properly discontinuously and the action is fixed point free, then the quotient space Δ/G has the structure of a Riemann surface.*

The proof of the above theorem is very similar to the proof given above for obtaining a complex structure on the complex one-dimensional tori. In the frame of the uniformization theory, it is proven that all compact Riemann surfaces can be described as quotients Δ/G.

Chapter 2

Topological properties of Riemann surfaces

2.1 The genus of a a compact Riemann surface

An arbitrary Riemann surface is a two-dimensional manifold. What can be said about the topology of this surface? We have showed in the previous chapter that a Riemann surface is orientable manifold.

The following result can be found in [18].

Theorem 2.1. *Any compact connected and orientable two dimensional surface is topologically equivalent to a sphere with $g \geqslant$ handles.*

The number g is called the genus of the surface.

Each surface of genus g can be obtained from a genus $g-1$ surface by removing a disc and attaching a torus.

Let us compute the genus of the surfaces in the examples 1.33-1.35. We begin with Example 1.34 namely the curve $w^2 = z^2 - a^2$. Delete the segment $[-a, a]$ with endpoints at the branch points from the z-plane $\bar{\mathbb{C}}$. Off this segment it is possible to distinguish the two branches $w_\pm = \pm\sqrt{z^2 - a^2}$ of the two-values function $w(z) = \sqrt{z^2 - a^2}$, that do not get mixed up with each other. In other words, the complete image $\pi^{-1}(\bar{\mathbb{C}} \backslash [-a, a])$ on Γ splits into two pieces, with

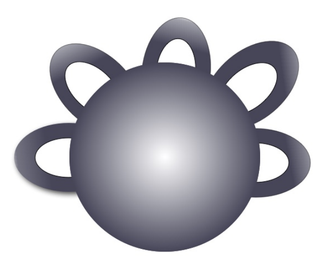


Figure 2.1: A sphere with five handles

the mapping π an isomorphism on each of them.
The branches $w_+(z)$ and $w_-(z)$ are interchanged in passing from one edge of the cut $[-a, a]$ to the other. Therefore, the surface is glued together from two identical copies of spheres with cuts according to the rule indicated in the figure 2.2

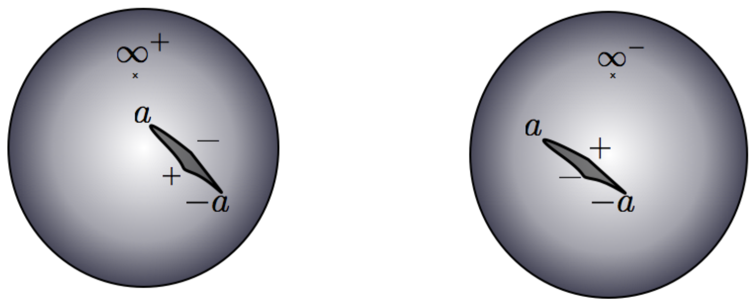


Figure 2.2: The cuts of the algebraic function $\sqrt{z^2 - a^2}$

After the gluing we again obtain a sphere, i.e., the genus g is equal to zero. Example 1.33 is analogous to Example 1.34, but the cut must be made between the points 0 and ∞, i.e. the point at infinity must be regarded as a branch point. Again the genus is equal to zero.

In Example 1.35 it is necessary to split up the branch points arbitrarily into pairs and make cuts (arcs) in $\bar{\mathbb{C}}$ joining the paired branch points n cuts for n even. The surface Γ is glued together from two identical copies of a sphere with such cuts, with the edges of the corresponding cuts glued together in "cross-wise" fashion (see the figure for $n = 4$).

It is not hard to see that a sphere with $n/2$ handles is obtained after the gluing, i.e., the genus g is $n/2$, see figure 2.4 for $n = 4$. For n odd the situation is analogous, except that in making the cuts it is necessary to take ∞ as one of the branch points. The genus g is equal to $(n + 1)/2$.

Exercise 2.2: Suppose that all the zeros $z_1 < \cdots < z_{2n+1}$ of the polynomial $P_{2n+1}(z)$ are real. We choose the segments $[z_1, z_2], [z_3, z_4], \ldots, [z_{2n+1}, \infty]$ of the real axis as the cuts for the surface $\Gamma = \{w^2 = P_{2n+1}(z)\}$. The function $w(z) = \sqrt{P_{2n+1}(z)}$ which is single-valued on each sheet of Γ formed after removal of the cycles $\pi^{-1}([z_1, z_2]), \ldots, \pi^{-1}([z_{2n+1}, \infty])$ is real on the edges of these cuts on each of the sheets. Show that on each sheet the sign of the square root $\sqrt{P_{2n+1}(z)}$ on the upper edge of the cut alternates.

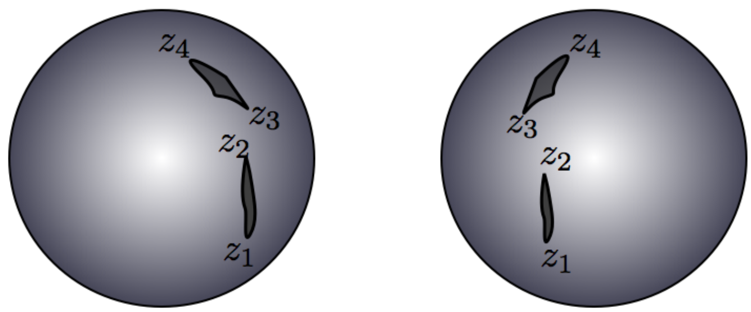


Figure 2.3: Opening of the cuts of the two branches of the function $\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

2.1.1 Triangulation of a Riemann surface and Riemann-Hurwitz formula

We derive a formula for the computation of the genus of a compact connected Riemann surface.

A triangulation of a two-dimensional surface Γ is a decomposition of Γ into closed subsets homeomorphic to triangles such that each couple of them is

- disjoint
- meet at a vertex
- meet at an edge.

Theorem 2.3. *[18] Every compact connected orientable* 2*-dimensional manifold can be triangulated.*

Given a 2-dimensional compact manifold M (possibly with boundary) and a triangulation of the manifold with

- $e =$ # of edges;
- $v =$ # of vertices;
- $t =$ # of triangles;

the number

$$E(M) = v - e + t \tag{2.1}$$

is called the Euler number of the manifold M with respect to the given triangulation.

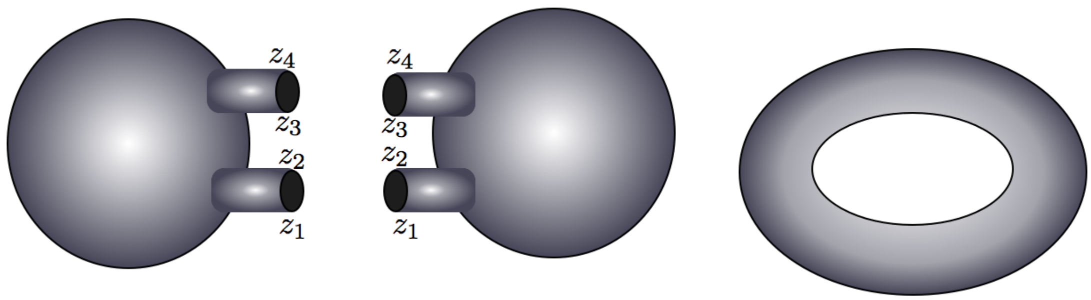


Figure 2.4: The Riemann surface of $w^2 = (z - a_1)(z - a_2)(z - a_3)(z - a_4)$ is glued from two copies of the extended complex plane cut along the intervals $[z_1, z_2]$ and $[z_3, z_4]$. The resulting surface is topological a torus.

Proposition 2.4. *The Euler number is independent from the choice of the triangulation. For a compact Riemann surface* Γ *of topological genus* g *the Euler number is*

$$E(\Gamma) = 2 - 2g. \tag{2.2}$$

Proof. We give a sketch of the proof. We consider compact surfaces with no boundaries. Given a triangulation, one can *refine* the triangulation by adding a vertex inside a triangle and three edges. This operation replaces one triangle with three triangles an it is easy to check that the Euler number remains unchanged. Another way to refine the triangulation is to add a point on an edge, so that two triangles are replaced by four triangles. Also in this case the Euler number remains unchanged. These operations define elementary refinements. A general refinement is obtained by making a sequence of elementary refinements. Therefore a given triangulation and any of its refinement have the same Euler number. Now the main point is to show that two triangulations have a common refinement. It is sufficient to superimpose two triangulations and add the necessary number for points to make the union of these two triangulations a triangulation. Then the triangulation obtained in this way is a refinement of both the triangulations. This is enough to show that the Euler number does not depend on the triangulation. Now let us make the computation of the Euler characteristic for a compact Riemann surface of genus g. We use an inductive argument. For the sphere the Euler number is equal to 2. For the disc $\bar{D} = \{z \in \mathbb{C} \,|\, |z| \leqslant 1\}$, the Euler number is equal to $E(\bar{D}) = 1$ and for the cylinder $C_{ylinder}$ of finite length the Euler number $E(C_{ylinder}) = 0$, see figure 2.5

The torus can be obtained from the sphere by removing two discs and connecting them with a cylinder. It is simple to check that the Euler number of the torus Γ_1 can be

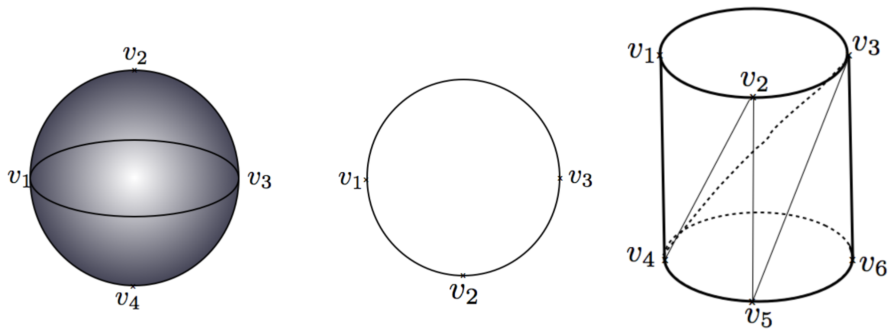


Figure 2.5: Triangulation of the sphere with 4 vertices, 6 edges and 4 triangles. Triangulation of the disc with 3 vertices, 3 edges and one triangle.Triangulation of the cylinder with 6 vertices, 12 edges and 6 triangles.

obtained as

$$E(\Gamma_1) = E(\Gamma_0) - 2E(\bar{D}) + E(C_{ylinder}) = 2 - 2 + 0 = 0. \tag{2.3}$$

Indeed removing two disks from a genus zero surface, the Euler number decreases by two, because it is just sufficient to subtract from the Euler formula the two discs that are homeomorphic to two triangles. Next we add a cylinder to connect the two discs. In order to compute the Euler number of the resulting surface, it is sufficient to add the contribution of the cylinder. The resulting Euler characteristics then can be written as in (2.3).

This procedure can be iterate. Indeed the surface Γ_g of genus g can be obtained from the surface of genus Γ_{g-1} by removing two discs and connecting them with a cylinder. Therefore one has

$$E(\Gamma_g) = E(\Gamma_{g-1}) - 2E(\bar{D}) + E(C_{ylinder})$$

which implies

$$E(\Gamma_g) = 2 - 2g.$$

□

We apply this result to calculate the genus of an affine plane curve.

Proposition 2.5. *Let* $\Gamma = \{(z,w) \in \mathbb{C}^2 \mid F(z,w) = w^n + a_1(z)w^{n-1} + \dots a_n(z) = 0\}$ *an irreducible non singular affine plane curve and let* $\bar{\Gamma}$ *be the compactification of* Γ*. Let* $z_1, \dots, z_M$

be the branch point for $\bar{\Gamma}$ with respect to the projection $\pi(z,w) \to z$ with multiplicity $b_1, \dots, b_m$ respectively. Then the genus of $\bar{\Gamma}$ is equal to

$$g = \frac{1}{2}\sum_{j=1}^{m} b_j - n + 1. \tag{2.4}$$

Proof. Consider a triangulation of $\bar{\mathbb{C}}$ so that the set of vertices of the triangulation contains the points $z_1, \dots, z_M$. Suppose that for each triangle T in $\bar{\mathbb{C}}$, the projection π restricted to the interior of each preimage $\pi^{-1}(T)$ is homeomorphic to the interior of T. In this way the triangulation on $\bar{\mathbb{C}}$ can be lifted to a triangulation on $\bar{\Gamma}$. Suppose the triangulation of $\mathbb{C}$ has v vertices, t triangles and e edges. Then the triangulation of $\bar{\Gamma}$ has

- $\tilde{t} = nt$ triangles
- $\tilde{e} = ne$ edges
- $\tilde{v} = nv - b$ vertices,

where $b = \sum_{j=1}^{m} b_j$. The Euler characteristic of the surface $\bar{\Gamma}$ gives

$$2 - 2g = nv - b - ne + nt = n(v - e + t) - b$$

so that one obtains the statement. □

The relation (2.4) is a particular case of a more general formula known as Riemann-Hurwitz formula. As an application of the proposition 2.5 we calculate the genus of a smooth projective curve

$$\Gamma = \{[X : Y : Z] \in \mathbb{P}^2 \,|\, Q(X,Y,Z) = 0\}$$

where Q is a homogeneous polynomial of degree n. Suppose that $[0 : 0 : 1] \notin \Gamma$ so that $Q(0,0,Z) = c\, Z^n \neq 0$ with $c \neq 0$. Then the map

$$\phi : \Gamma \to \mathbb{P}^1, \quad \phi(X,Y,Z) = [X : Y]$$

realised Γ as a n-sheeted covering of $\mathbb{P}^1$. Let us calculate the total branching number of this map. The branch points are obtained by solving the equations

$$Q(X,Y,Z) = 0, \quad Q_Z(X,Y,Z) = 0$$

The solution of the above two equations are given by the zeros of the resultant $R(Q, Q_Z)$ with respect to Z. Since $R(Q, Q_Z)$ is a homogeneous polynomial of degree $n(n-1)$ in X and Y, the total number of branch points counting their multiplicity is $n(n-1)$.

Recall that the branching number of a branch point $P_0 = [X_0 : Y_0 : Z_0]$ indicated as $b_\phi(P_0)$ is the order of the zero of $Q(X_0, Y_0, Z)$ at $Z = Z_0$ minus one. We can write

$$Q(X_0, Y_0, Z) = \prod_{0 \leqslant j \leqslant s} (Z - Z_j)^{m_j}$$

where $\sum_j m_j = n$ and $Z_0, \ldots, Z_s$ are distinct complex numbers, $Z_j = Z_j(X_0, Z_0)$. Here we assume that $Q(0, 0, Z) = Z^n$. With the above notation the branching number of each branch point $P_j = [X_0 : Y_0 : Z_j]$ is $b_\phi(P_j) = m_j - 1$. So a regular point is simple zero of $Q(X_0, Y_0, Z)$ a branch point with branching number one is a double zero, and a branch point with branching number $m - 1$ is a zero of order m of $Q(X_0, Y_0, Z)$. So if the number of distinct roots of the discriminant is $n(n - 1)$ it means that the curve has $n(n - 1)$ branch points with multiplicity one, so that the total branching number is $n(n - 1)$. If the discriminant has for example $n(n - 1) - k$ distinct roots, $k > 0$, it means that some of the branch points have branching number bigger then one. However the total branching number remains equal to $n(n - 1)$. Then we can apply formula 2.4 to obtain

$$g = \frac{1}{2}(n - 1)n - n + 1$$

which gives $g = \dfrac{1}{2}(n - 2)(n - 1)$.

Lemma 2.6. *The genus of a smooth projective curve of degree n is given by the relation*

$$g = \frac{1}{2}(n - 2)(n - 1). \tag{2.5}$$

Exercise 2.7: Calculate the genus of the following surfaces

$$w^3 = (z - 1)(z - 2)(z - 3)(z - 4),$$

$$w^n = z^n + a^n, \quad a \neq 0.$$

Theorem 2.8. *[14] Any compact connected orientable two-dimensional surface of genus g that admits a triangulation, can be made into a Riemann surface.*

Any surface of genus zero is topologically equivalent to the sphere $\mathbb{P}^1$. The surfaces of genus one are one-dimensional tori. The complex structure is unique only for $g = 0$ (see [14] 16.13). The set of complex structures has one complex parameter for $g = 1$ and $3g - 3$ complex parameters for $g \geqslant 2$. The space of these parameters is called Teichmuller space. We observe that the number $3g - 3$ coincides with the dimension of the moduli space of Riemann surfaces of genus $g \geqslant 2$.

2.2 Monodromy of a surface

In order to define the monodromy of a surface we recall the definition of fundamental group. Let M be a topological space and P, Q two points on M. A curve $u : I \to M$ starting in P and ending in Q is a continuous map from $I[0,1]$ to M such $u(0) = P$ and $u(1) = Q$. If two curves can be deformed continuously one into the other, the curves are called homotopic.

Definition 2.9. *Two curves u and w are homotopic if there is a continuos map $A : I \times I \to M$ such*

- $A(t,0) = u(t)$,
- $A(t,1) = w(t)$,
- $A(0,s) = P$ *and* $A(1,s) = Q$, *for all* $s \in [0,1]$.

The notion of homotopic is an equivalence relation. It is easy to construct homotopic curves. For example given a smooth map $f : I \to I$, the curves u and $u \circ f$ are homotopic. In the space of curves we can define a group structure.

Definition 2.10. *Given two curves $u : I \to M$ and $w : I \to M$, $I = [0,1]$, such that $u(0) = P$ and $u(1) = Q$ and $w(0) = Q$ and $w(1) = R$ the product curve is*

$$(u \circ w)(t) := \begin{cases} u(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ w(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases},$$

the inverse of a curve is

$$u^{-1}(t) := u(1-t), \quad t \in I,$$

the constant curve is

$$Id : I \to M, \quad Id(t) = P.$$

Clearly $u \circ u^{-1}$ is homotopic to Id. Now let us consider curves starting and ending in P, namely close loops.

Definition 2.11. *Let M be a topological space. The set of homotopic classes of loops starting and ending in $P \in M$ is denoted by $\pi_1(M,P)$.*

The set $\pi_1(M,P)$ forms a group under the operation induced by the product of curves. We denote its elements by $[\gamma]$. It is easy to check that for arc-wise connected spaces M, the group $\pi_1(M,P)$ is independent from the base point P. Indeed let $\pi_1(M,Q)$ be the fundamental group with base point Q, and let w be a path from P to Q. Then for any element $[\gamma] \in \pi_1(M,P)$ the loop $[w^{-1} \circ \gamma \circ w] \in \pi_1(M,Q)$ and this map is an isomorphism.

Definition 2.12. *An arc-wise topological space M is called simply connected if* $\pi_1(M) = 0$.

Remark 2.13. We remind that the only Riemann surfaces with trivial fundamental group are the sphere the complex plane and the disk. The only Riemann surface M with $\pi_1(M) = \mathbb{Z}$ is the punctured disk or the punctured complex plane. The only compact Riemann surface M with $\pi_1(M) = \mathbb{Z} \times \mathbb{Z}$ is the torus.

Although the trivial element of the group is the identity, it has become standard notation to write $\pi_1(M) = 0$ for the fundamental group that contains only the identity. In a simply connected space all loops are homotopic to the identity loop. The sphere $\mathbb{P}^1$ is a simply connected space.

Now we are ready to define the monodromy group of a surface. Consider a compact Riemann surface $\bar{\Gamma}$ realised as the compactification of a smooth affine plane curve

$$\Gamma = \{(z, w) \in \mathbb{C}^2 | F(z, w) = z^2 + a_1(z)w^{n-1} + \cdots + a_n(z) = 0\}$$

and consider the projection $\pi : \bar{\Gamma} \to \bar{\mathbb{C}}$, $\pi(z, w) = z$ and denote by $z_1, \ldots, z_M$ the branch point of such map. Let delete from $\bar{\mathbb{C}}$ the branch points $z_1, \ldots, z_M$ and delete from Γ the complete inverse images $\pi^{-1}(z_1), \ldots, \pi^{-1}(z_M)$ of these points. We get a surface Γ_0 that is a n-sheeted covering of the punctured sphere $\bar{\mathbb{C}} \backslash (z_1 \cup \cdots \cup z_M)$. The monodromy group of the Riemann surface is the monodromy group of this covering. We recall the general definition of the monodromy group of a covering in connection with this case (see [9] for more details). Fix a point $z_0 \in \bar{\mathbb{C}} \backslash (z_1 \cup \cdots \cup z_M)$ and number the points $P_1, \ldots, P_n$ in the fiber $\pi^{-1}(z_0)$ arbitrarily (these points are all distinct). Any closed contour in $\bar{\mathbb{C}} \backslash (z_1 \cup \cdots \cup z_M)$ beginning and ending at z_0 gives rise to a permutation of the points $P_1, \ldots, P_n$ of the fiber after being lifted to Γ_0. We get a representation of the fundamental group $\pi_1(\bar{\mathbb{C}} \backslash (z_1 \cup \cdots \cup z_M), z_0)$ in the group S_n of permutations of n elements; this is called the monodromy representation. Let γ_k, $k = 1, \ldots, M$ be a loop starting and ending in z_0 and encircling the point z_k, $k = 1, \ldots, M$. We denote by $[\gamma_k]$ the homotopy class of this loop. The loops $[\gamma_1], \ldots, [\gamma_M]$ are generators of $\pi_1(\bar{\mathbb{C}} \backslash (z_1 \cup \cdots \cup z_M), z_0)$ with the constraint

$$[\gamma_1] \circ [\gamma_2] \circ \cdots \circ [\gamma_n] = Id \tag{2.6}$$

namely the trivial loop. The mondromy representation

$$\rho : \pi_1(\bar{\mathbb{C}} \backslash (z_1 \cup \cdots \cup z_M), z_0) \to S_n, \quad \rho([\gamma_k]) = \sigma_k$$

is a group homomorphism namely

$$\rho([\gamma_k] \circ [\gamma_j]) = \sigma_k \sigma_j, \tag{2.7}$$

for any set of generators. The homotopy relation (2.6) implies

$$\sigma_1 \sigma_2 \ldots \sigma_M = Id$$

the identity in S_n.

Definition 2.14. *The image of the map ρ defined in (2.7) in S_n is called the monodromy group of the surface Γ.*

Remark 2.15. For connected surfaces, the image of the monodromy group is a transitive subgroup in S_n. Indeed a transitive subgroup $G \in S_n$ has the property that for every number $i, j \in \{1, \ldots, n\}$ there exists a permutation $\tau \in G$ such that $j = \tau(i)$. If the Riemann surface is connected, for any points P_i and P_j in the fiber $\pi^{-1}(z)$, $z \in \mathbb{C}$ it is possible to find a path connecting these points.

For hyperelliptic Riemann surfaces the monodromy group coincides $S_2 = Z_2$.

Exercise 2.16: For curves of the form

$$w^n = \prod_{j=1}^{N}(z - z_i)$$

show that the monodromy group coincides with Z_n.

In the general case the action of the generators of the monodromy group that correspond to circuits about branch points is determined by the branching indices.

Exercise 2.17: Let z be a branch point, and let the complete inverse image $\pi^{-1}(z)$ on Γ consist of the ramification points $P_1, \ldots, P_k$ of multiplicity $b_1, ..., b_k$, respectively (if some point P_i is not a branch point, then we set $b_i = 0$). Prove that to a cycle in $\bar{\mathbb{C}}$ encircling z_0 once there corresponds an element in the monodromy group splitting into cycles of length $b_1 + 1, ..., b_k + 1$. This assertion gives a purely topological definition of the multiplicities (indices) of branch points.

Remark 2.18. Suppose that one of the branch points, let say $z_M = \infty$. Then the monodromy corresponding to circuits about the point $z = \infty$ is uniquely determined by the monodromy corresponding to circuits about the images of the finite branch points. Indeed, a contour encircling only the point $z = \infty$ splits into a product of contours encircling all the finite branch points, and we get the monodromy at infinity by multiplying the corresponding elements of the monodromy groups at the finite points. For example, for the surface $w^2 = P_{2n+2}(z)$ the monodromy at infinity is trivial (the corresponding contour in the z-plane encircles an even number of branch points), i.e., this surface has no branch points at infinity. But for the surface $w^2 = P_{2n+1}(z)$ the monodromy at infinity is nontrivial, because here a contour encircling $z = \infty$ encircles an odd number of branch points. We thus see once more that the point at infinity of the surface $w^2 = P_{2n+1}(z)$ is a branch point.

Exercise 2.19: Prove that for a general surface of the form (1.26)) the monodromy group coincides with the complete symmetric group S_n . *Hint. Show that the branch points of such* *a surface can be labeled by pairs of distinct numbers $i \neq j, (i, j = 1, ..., n)$ in such a way that a circuit about the images of the points P_{ij} and P_{ji} gives rise to a transposition of the ith and jth points of the fiber (when these points are suitably numbered).*

2.3 Singular curves

Let us consider an affine plane curve

$$\Gamma = \{(z, w) \in \mathbb{C}^2 | F(z, w) = z^2 + a_1(z)w^{n-1} + \cdots + a_n(z) = 0\}.$$

A point $P_0 = (z_0, w_0) \in \Gamma$ is called singular if

$$F(z_0, w_0) = F_z(z_0, w_0) = F_w(z_0, w_0) = 0.$$

If the polynomial F is irreducible then the set of singular points is finite.

Nodes of an affine plane curve. The singular point $P_0 = (z_0, w_0) \in \Gamma$ is called a node if the Hessian

$$\det \begin{pmatrix} F_{zz}(z_0, w_0) & F_{zw}(z_0, w_0) \\ F_{zw}(z_0, w_0) & F_{ww}(z_0, w_0) \end{pmatrix} \neq 0.$$

We can expand in Taylor series the equation of the curve near the node $P_0 = (z_0, w_0)$ to obtain

$$F(z, w) = \alpha_1(z - z_0)^2 + \alpha_2(z - z_0)(w - w_0) + \alpha_3(w - w_0)^2 + \text{higher order tems}$$

where $\alpha_1 = F_{zz}(z_0, w_0)/2$, $\alpha_2 = F_{zw}(z_0, w_0)$ and $\alpha_3 = F_{ww}(z_0, w_0)$. The quadratic term is a homogenous polynomial that can be factor in the product of two first order homogeneous polynomials namely

$$F(z, w) = (c_1(z - z_0) + c_2(w - w_0))(b_1(z - z_0) + b_2(w - w_0)) + \text{higher order tems},$$

Defining $x = c_1(z - z_0) + c_2(w - w_0)$ and $y = b_1(z - z_0) + b_2(w - w_0)$ one has

$$F(z, w) = xy + \sum_{j \geqslant 2} f_j(x, y)$$

where f_j are homogenous polynomials of degree j in x and y. Applying Hensel's Lemma, which say that if the lower order term of a power series factor, then the entire power series can factor compatibly, we can write the above expansion in the form

$$F(z, w) = r(x, y)s(x, y)$$

with

$$r(x,y) = x + \sum_{j\geqslant 2} r_j(x,y), \quad s(x,y) = y + \sum_{j\geqslant 2} s_j(x,y)$$

where each $r_j(x,y)$ and $s_j(x,y)$ is a homogeneous polynomial of degree j that can be obtained uniquely from the polynomials f_k. Since $F(z,w)$ is a polynomial the power series for the function $r(x,y)$ and $s(x,y)$ are clearly convergent.

Near the node (z_0, w_0) the curve is the locus of zeros of $r(x,y)s(x,y)$ which is the union of the locus of zeros of the functions $r(x,y)$ and the locus of zeros of $s(x,y)$. Each locus corresponds to a curve Γ_r and Γ_s respectively. These curves are nonsingular in P_0. Now we call $\hat{\Gamma}$ the curve obtained from the singular curve Γ by removing the point P_0. The curve $\hat{\Gamma}$ looks locally as the union $\Gamma_r\backslash\{P_0\}$ and $\Gamma_s\backslash\{P_0\}$. Let us consider open sets U_r and U_s on $\hat{\Gamma}$ which are equal to open set on $\Gamma_r\backslash\{P_0\}$ and $\Gamma_s\backslash\{P_0\}$. Such open sets are homeomorphic to puncture disks. According to definition 1.36, the surface $\hat{\Gamma}$ is a Riemann surface with two punctures. Compactifying the curve $\hat{\Gamma}$ according to section 1.1.3, one obtains a smooth compact Riemann surface S. The whole process is called *resolving the nodes of* Γ. The smooth compact Riemann surface obtained in this way is called also the normalisation of Γ.

Genus of a projective curve with nodes. Let us consider a projective curve with k nodes defined by the zeros $F(X,Y,Z) = 0$ of the homogeneous polynomial F of degree n. In order to compute the genus of the smooth curve $\tilde{\Gamma}$,obtained from Γ by resolving the nodes, it is necessary to observe that for each node the total branching number of the curve decreases by two, indeed perturbing slightly the polynomial equation near the node, two branch points with multiplicity one are obtained. Then using Riemann-Hurwitz formula (2.4) one obtains the genus of a projective curve with nodes, namely Plücker's formula.

Proposition 2.20. *Let Γ be a projective curve of degree n with k nodes and no other singularities. Then the genus of $\tilde{\Gamma}$ the curve obtained by resolving the nodes of Γ is*

$$g = \frac{1}{2}(n-1)(n-2) - k.$$

Monomial singularities

A curve Γ defined by the zero of the polynomial $F(z,w) = 0$ has a singular point in $(0,0)$ called a monomial singularity if locally the polynomial $F(z,w)$ is of the form

$$F(z,w) = w^n - z^m,$$

with m and n integers. If $m = n = 2$ the singular point is a node, and for $n = 2$ and $m = 2k$ it is a higher order node. In the case $n = 2$ and $m = 3$ the singularity is a cusp and for

$m = 2k + 1$ it is called a higher order cusp. In general when n and m are co-prime the singularity is called a monomial singularity of type (m, n). If $m/n = kp/kq$ with k, p and q integers and p and q relatively prime, then the monomial singularity can be factored as

$$F(z,w) = \prod_{j=1}^{k}(w^q - \xi^j z^p), \quad \xi = e^{\frac{2\pi i}{k}}.$$

Let us consider a monomial singularity with (m, n) co-prime. Then near such singularity the curve has a parametrisation $t \to (t^n, t^m)$. Let us consider the puncture neighbourhood of $(0, 0)$ in Γ, namely the set

$$U = \{(z,w) \in \mathbb{C}^2 : \ 0 < |z| < \rho, \text{ and } F(z,w) = 0\}$$

and the disc

$$D = \{t \in \mathbb{C} : \ |t| < \rho^{\frac{1}{n}}\}.$$

The map

$$\Psi : D\backslash\{0\} \to U, \quad \Phi(t) = (t^n, t^m)$$

is a biholomorphic map from $D\backslash\{0\}$ to U. The inverse map is given by

$$\Phi(z,w) \to z^a w^b = t, \quad an + mb = 1$$

with a and b integers. The map Φ is compatible with the complex structure of Γ. So the curve $\Gamma\backslash\{(0,0\}$ is a Riemann surface with punctures according to definition 1.36. We can extend the map $\Phi : U \cup \{(0,0)\} \to D$ by defining $\Phi(0,0) = 0$. The Riemann surface that we obtain is a smooth compact Riemann surface S.

Resolution of singularities of general curves and Puiseux expansion

Resolution of singularities of curves was essentially first proved by Newton (1676), who showed the existence of Puiseux series for a curve from which the resolution of singularities follows easily. Puiseux series are a generalisation of powers series and they were first introduced by Newton and then they were rediscovered by Puiseux in 1850. A Puiseux series in the variable z is a power series of the form $\sum_{j=k}^{\infty} a_j z^{j/n}$ where k is an integer and n is a positive integer.

Let us consider the polynomial equation $F(z, w) = 0$. When

$$\text{grad}F = (F_z(z_0, w_0), F_w(z_0, w_0)) \neq (0, 0)$$

the implicit function theorem gives a local parametrisation of the curve

$$z \to (z, \psi(z))$$

in the case $F_w(z_0, w_0) \neq 0$, and $\psi(z)$ is an analytic function of z in the neighbourhood of $z = z_0$. Therefore the curve looks locally like the graph of a function which is locally like its tangent line. For singular curves such parametrisation does not exist, like for example for the curve $F(z, w) = w^2 - z^3$. However there is a parametrisation of the form

$$t \to (t^2, t^3), \quad \text{or} \quad z \to (z, z^{\frac{3}{2}}).$$

Locally any singular branch of a curve has a parametrisation of the form

$$t \to (t^n, \psi(t)), \quad \text{or} \quad (z, \psi(z^{\frac{1}{n}})), \quad k > 1,$$

for some power series ψ. Such series is called Puiseux series. The next theorem, called Puiseux's theorem, asserts that, given a polynomial equation $F(z, w) = 0$, its solutions in w, viewed as functions of z, may be expanded as convergent Puiseux series. Suppose the the point (z_0, w_0) is a singular point of the curve defined by $F(z, w) = 0$. Furthermore we assume for simplicity that the pre-image of the point z_0 with respect to the projection $\pi(z, w) \to z$ consists only of one point, namely $\pi^{-1}(z_0) = w_0$.

Theorem 2.21. *Let $F(z, w)$ be a polynomial such that $F(0, w) \neq 0$ and $\deg F(0, w) = n$. For each point near z_0, there are homolorphic functions $\psi_1(t), \ldots, \psi_l(t)$ defined near $t = 0$, such that $\psi_j(0) = w_0$ and positive integers $m_1, \ldots, m_l$ with $m_1 + \cdots + m_l = n$ such that*

$$F(z_0 + t^{m_j}, \psi_j(t)) = 0, \quad j = 1, \ldots, l.$$

In other words for every z sufficiently close to z_0 the polynomial $F(z, w)$ can be factored in the form

$$F(z, w) = c \prod_{j=1}^{l} \prod_{s=1}^{m_j} \left(w - \psi_j(e^{2\pi i s/m_j} (z - z_0)^{\frac{1}{m_j}}) \right).$$

Two Puiseux expansions with indices $j \neq \tilde{j}$ are essentially different. Newton gave an algorithm to construct such parametrisations that it is know as Newton polygon technique. We are not going to enter the details of this technique. We give only some examples.

Example 2.22. Suppose that $F(z, w)$ is a polynomials with $\deg F(0, w) = k$, such that there are integers numbers p and q such that

$$F(z, w) = \sum_{qi + pj = kp} a_{ij} z^i w^j.$$

Then we can look for a parametrisation of the form $F(t^q, \lambda t^p) = 0$, namely

$$F(z, w) = F(t^q, \lambda t^p) = t^{kp} \sum_{iq+pj=kp} a_{ij}\lambda^j := t^{kp} h(\lambda)$$

We can always find $\lambda_0 \in \mathbb{C}$ such that $h(\lambda_0) = 0$.

In general for a polynomial

$$F(z, w) = \sum_{ij} a_{ij} z^i w^j$$

the *carrier* C of F is defined as

$$C(F) = \{(i, j) \in \mathbb{Z}^2 \,|\, a_{ij} \neq 0\}.$$

The Newton polygon is the convex hull of the points in the carrier. We can assume without loss of generality that the Newton polygon touches both axis. Suppose that there are rational number μ and ν such that

$$F(z, w) = \sum_{i+\mu j \geqslant \nu} a_{ij} z^i w^j.$$

Then the the line

$$z + \mu w = \nu$$

lies below the carrier $C(F)$. Now substituting $w = tz^\mu$ into F we get

$$F(z, tz^\mu) = z^\nu \sum_{i+\mu j=\nu} a_{ij} t^j + \sum_{i+\mu j>\nu} a_{ij} z^{i+\mu j} = z^\nu \sum_{i+\mu j=\nu} a_{ij} t^j + o(z^\nu).$$

Let t_0 be a solution of the equation $\sum_{i+\mu j=\nu} a_{ij} t^j = 0$. Note that this equation has a solution if there are at least two points of the carrier $C(F)$ on the line $z + \mu w = \nu$. Then we can consider $(z, t_0 z^\mu)$ as an approximate solution of the equation $F(z, w) = 0$ near the singular point $(0, 0)$. The next step is to improve the above approximation. Assuming $\mu = \dfrac{p}{q}$ with p and q integers that do not have a common factor, one can look for an expansion of the form

$$z_1 = z^q, \quad w = z_1^p (t_0 + w_1).$$

Then plugging the above ansatz into $F(z, w)$ one obtains

$$F(z_1^q, z_1^t (t_0 + w_1)) = z_1^{q\nu} F_1(z_1, w_1).$$

The next step is to study the singularity structure of the polynomial $F_1(z_1, w_1)$. By iterating this procedure, one obtains the Puiseux expansion near the point $(0, 0)$.

Example 2.23. Let $F(z,w) = w^5 - zw^2 + 4z^5 + z^4 - 3w^3z^3$. By considering the Newton polygon (see figure 2.6), one can see that there are two lines that lie below it

$$z + \frac{1}{3}w = \frac{5}{3}, \quad z + \frac{3}{2}w = 4.$$

We first analyse the the first line, namely $w = tz^{\frac{1}{3}}$, which gives

$$F(z, tz^{\frac{1}{3}}) = z^{\frac{5}{3}}(t^5 - t^2) + o(z^{\frac{5}{3}}),$$

so that $t^3 = 1$, namely t is one of the three roots of unity. For simplicity let us consider $t = 1$. Next we consider the parametrisation

$$z = z_1^3, \quad w = z_1(1 + w_1)$$

so that

$$F(z_1^3, z_1(1+w_1)) = z_1^5 F_1(z_1, w_1),$$
$$F_1(z_1, w_1) = w_1^5 + 5w_1^4 + (z_1^7 + 10)w_1^3 + (3z_1^7 + 9)w_1^2 + (3z_1^7 + 3)w_1 - z_1^{10}. \quad (2.8)$$

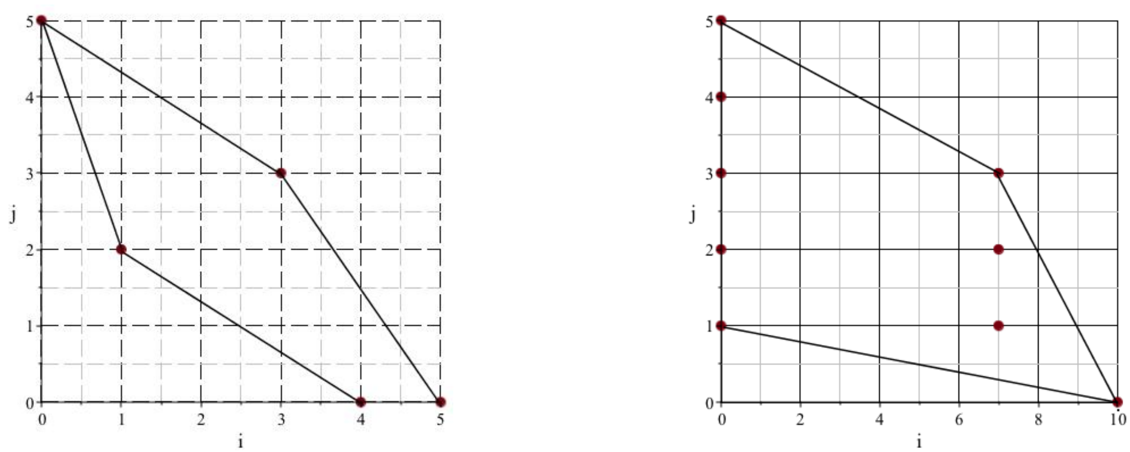


Figure 2.6: The Newton polygon for $F(z,w)$ on the left and $F_1(z_1, w_1)$ on the right.

The Newton polygon of the polynomial $F_1(z_1, w_1)$ is show in figure 2.6 and one can see that the line $z_1 + 10w_1 = 10$ is at the boundary of the Newton polygon. So we look for $w_1 = t_1 z_1^{10}$

$$F_1(z_1, t_1 z_1^{10}) = z_1^{10}(3t_1 - 1) + o(z_1^{10}),$$

which gives $t_1 = \frac{1}{3}$. We conclude that the first two terms of the Puiseux expansion are

$$w = z^{\frac{1}{3}} + \frac{1}{3} z^{\frac{11}{3}} + \dots .$$

Repeating the same procedure for the coefficient $\mu = \dfrac{3}{2}$ one obtains

$$w = (-z)^{\frac{3}{2}} - \frac{1}{2}(-z)^{\frac{5}{2}} + \dots .$$

Summarizing we have obtained two essentially different Puiseux expansions near the point $(0,0)$.

Once the Puiseux series for a curve near a singular point has been found the resolution of the singularity follows easily.

Theorem 2.24. *For every irreducible algebraic curve $\Gamma \in \mathbb{P}^2$ there exists a compact Riemann surface S and a holomorphic map*

$$\phi : S \to \Gamma$$

with the properties

- *let $\hat{\Gamma} := \Gamma \backslash Sing\, \Gamma$ be the smooth part of Γ and let $\hat{S} := \phi^{-1}(\hat{\Gamma})$. Then*

  $$\hat{\phi} := \phi|_{\hat{S}} : \ \hat{S} \to \hat{\Gamma}$$

  is bi-holomorphic

- *$\phi : S \to \Gamma$ is surjective.*

For a singular point $P \in \mathrm{Sing}\, \Gamma$, the number of points in the preimage of $\phi^{-1}(P)$ is given the by the number of essentially different Puiseux expansions of Γ near P. In the example 2.23 the number of pre-images of the singular point $(0,0)$ consists of two points.

Exercise 2.25: Calculate the genus of the singular curves

$$w^3 = (z - a_1)^2 (z - a_2)(z - a_3)^2 (z - a_4)$$

and

$$w^3 = z^3 (z - a_3)^2 (z - a_4).$$

For each singular point calculate the number of points in the preimage of the map ϕ defined in theorem 2.24.

Exercise 2.26: For which value of λ the following curves are non singular?

$$X^3 + Y^3 + Z^3 + 3\lambda XYZ = 0$$

and

$$X^3 + Y^3 + Z^3 + \lambda(X + Y + Z)^3 = 0.$$

Describe the singularities when they exist and calculate the genus of the corresponding Riemann surface.

2.4 Homology

In this section we define the homology of a Riemann surface.

Given a triangulation of a Rieamnn surface we define the vertex as 0-*simplex*, the edges as 1-*simplex* and the triangles as 2-*simplex*. The orientation on the manifold induces an orientation on the triangles that can be used to orient the edges bounding each triangle.

Definition 2.27. *Given a triangulation of a Riemann surface, a simplicial* 0, 1 *and* 2 *chain is a formal sum of vertices, P_j, edges e_j or triangles t_j*

$$c_0 = \sum_j n_j P_j, \quad c_1 = \sum_j m_j e_j, \quad c_2 = \sum_j k_j t_j.$$

The set of n-chain C_n has a natural structure of additive abelian group.

The element $-c_1$ is the edge with opposite orientation and $-t$ is the triangle with opposite orientation. The vertices P_1, P_2, $P_3, \dots$ can be used to identify edges and triangles. For example $\langle P_1 P_2 \rangle$ is the oriented edge from P_1 to P_2 and $\langle P_1, P_2, P_3 \rangle$ is the oriented triangle with sides the oriented edges $\langle P_1 P_2 \rangle$, $\langle P_2 P_3 \rangle$ and $\langle P_3 P_1 \rangle$. With this notation we define the boundary operator δ

Definition 2.28. *The boundary operator $\delta : C_n \to C_{n-1}$ with $n = 0, 1, 2$ is defined as follows:*

$$\delta c_0 = 0, \quad c_0 \in C_0$$

$$\delta \langle P_1 P_2 \rangle = P_2 - P_1$$

$$\delta \langle P_1, P_2, P_3 \rangle = \langle P_1 P_2 \rangle + \langle P_2 P_3 \rangle + \langle P_3 P_1 \rangle.$$

The above relation defines δ on 1 *and* 2*-simplex and it can be extend to* 1 *and* 2*-chain by linearity.*

With the above definition it is immediate to observe that $\delta^2 = 0$. Furthermore, δ is a group homomorphism from C_n to C_{n-1}. We define

$$Z_n = \{c_n \in C_n \,|\, \delta c_n = 0\}, \quad B_n = \{c_n \in C_n \,|\, \exists c_{n+1} \in C_{n+1}, \;\; c_n = \delta c_{n+1}\}.$$

From the above definition it is clear that

$$B_n \subseteq Z_n \subseteq C_n$$

Since all the groups are abelian, B_n is a normal subgroup. We are interested in the following quotient groups.

Definition 2.29. *The n-homology group of* Γ *is defined as*

$$H_n(\Gamma, \mathbb{Z}) = \frac{Z_n}{B_n}, \quad n = 0, 1, 2.$$

We remark that the homology groups are independent from the triangulation. A basis for $H_n(\Gamma, \mathbb{Z})$ is the set of elements of $H_n(\Gamma, \mathbb{Z})$ such that any other element can be written as a linear integer combination of elements of the basis. It is clear from the definition that $\dim H_0(\Gamma, \mathbb{Z}) = 1$. Regarding $H_2(\Gamma, \mathbb{Z})$, with a little thought it is easy to see that for a compact Riemann surface $\dim H_2(\Gamma, \mathbb{Z}) = 1$. The only nontrivial group is $H_1(\Gamma, \mathbb{Z})$. We have the following result.

Theorem 2.30. *The first homology group* $H_1(\Gamma, \mathbb{Z})$ *is isomorphic to the abelianization of the first homotopy group* $\pi(\Gamma)$*. The group* $H_1(\Gamma, \mathbb{Z})$ *is a free abelian group with* $2g$ *generators. Any cycle can be written as a sum of of generators.*

The theorem above gives the following. Let Γ be a compact Riemann surface of genus g and let $[\gamma_1], \dots, [\gamma_{2g}]$ be the set of generators of $\pi_1(\Gamma)$. Then any element $[\gamma] \in \pi_1(\Gamma)$ can be uniquely written as

$$[\gamma]_{\pi_1} = [\gamma_{k_1}]_{\pi_1}^{j_1} \circ [\gamma_{k_2}]_{\pi_1}^{j_2} \circ \dots [\gamma_{k_n}]_{\pi_1}^{j_n}, \quad k_1, \dots k_n \in \{1, 2, \dots, 2g\}$$

with $j_1, \dots, j_n \in \mathbb{Z}$ and where we put an under script of π_1 to denote an element of the fundamental group. Then the corresponding element $[\gamma]_{H_1}$ in the homology is obtained as

$$[\gamma]_{H_1} = j_1[\gamma_{k_1}]_{H_1} + j_2[\gamma_{k_2}]_{H_1} + \dots + j_n[\gamma_{k_n}]_{H_1}, \quad k_1, \dots k_n \in \{1, 2, \dots, 2g\}.$$

In the rest of this section we simply denote as γ an element in the homology basis. Let $a_1, \dots, a_g, b_1, \dots, b_g$ be a basis in $H_1(\Gamma, \mathbb{Z})$. Then any cycle γ is homologous to a linear combination of the basis with integer coefficients:

$$\gamma \simeq \sum_{i=1}^{g} m_i a_i + \sum_{i=1}^{g} n_i b_i, \quad m_i, \, n_i \in \mathbb{Z}.$$

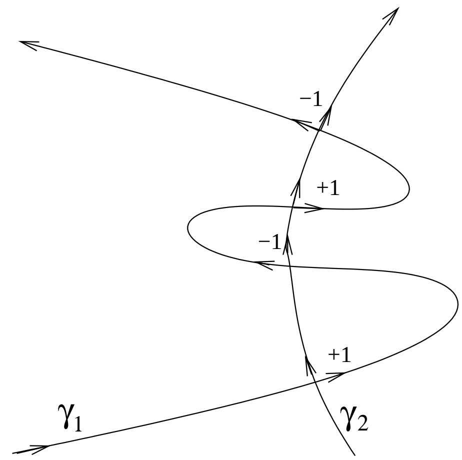


Figure 2.7: Intersection of γ_1 and γ_2.

The intersection number $\gamma_1 \circ \gamma_2$ is defined for any two cycles γ_1 and γ_2 on Γ. By a small deformations of the cycle it is possible to reduce the intersection of two cycles in such a way that

- the intersection is finite;
- the intersection occurs transversally, namely the tangent lines of the two cycle in the point of intersection are not parallel.

At each intersection point there is an ordered reference frame consisting of the tangent vectors to the respective cycles γ_1 and γ_2 with the direction of the tangent vectors chosen to correspond to the orientation of the cycles. The intersection points are assigned the number $\nu(P)$ which is equal to $+1$ if the orientation of this frame coincides with that of the surface, and -1 otherwise (see the figure). The sum of these numbers ± 1, taken over all points of intersection of γ_1 and γ_2 is the intersection number $\gamma_1 \circ \gamma_2$:

$$\gamma_1 \circ \gamma_2 = \sum_{P \in \gamma_1 \cap \gamma_2} \nu(P).$$

Properties of the intersection number are:

1) $\gamma_1 \circ \gamma_2$ depends only on the homology classes of γ_1 and γ_2;

2) the map

$$\circ : H_1(\Gamma, \mathbb{Z}) \times H_1(\Gamma, Z) \to \mathbb{Z}$$

is bilinear, skew-symmetric, and nondegenerate.

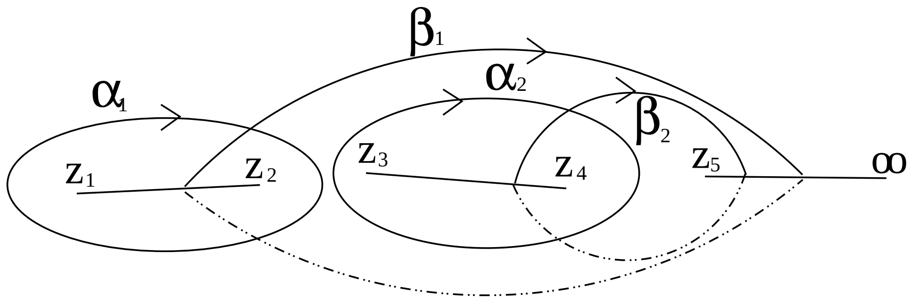


Figure 2.8: Homology basis.

Nondegenerate means that if $\gamma_1 \circ \gamma_2 = 0$ for every cycle γ_2, then the cycle γ_1 is homologous to zero. A basis of cycles $\alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g$ on a surface Γ of genus g can be chosen so that the pairwise intersection numbers have the form

$$\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0, \quad \alpha_i \circ \beta_j = \delta_{ij}, \ \ i, j = 1 \dots, g. \tag{2.9}$$

Such a basis will be called canonical. Note that if for a cycle γ and a canonical basis $\alpha_1, \dots \alpha_g, \beta_1, \dots, \beta_g$ the intersection numbers are $\gamma \circ \alpha_i = n_i$, $\gamma \circ \beta_j = m_j$, $i, j = 1 \dots, g$, then the decomposition of γ in the basis has the form

$$\gamma = \sum_{i=1}^{g} m_i \alpha_i - \sum_{i=1}^{g} n_i \beta_i.$$

This simple consideration is useful in practical computations with cycles on Riemann surfaces.

Example 2.31. Let us construct a canonical basis of cycles on the hyperelliptic surface $w^2 = \prod_{i=1}^{2g+1}(z - z_i)$, $g \geqslant 1$. We represent this surface in the form of two copies of $\mathbb{C}$ (sheets) with cuts along the segments $[z_1, z_2]$, $[z_3, z_4]$, $\dots$, $[z_{2g+1}, \infty]$. A canonical basis of cycles can be chosen as indicated on the figure for $g = 2$ (the dashed lines represent the parts of a_1 and a_2 lying on the lower sheet).

Let $\alpha = (\alpha_1, \dots, \alpha_g)^t$ and $\beta = (\beta_1, \dots, \beta_g)^t$, then the condition (2.9) can be written in the form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \circ \begin{pmatrix} \alpha^t & \beta^t \end{pmatrix} = J, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2g, \mathbb{Z}). \tag{2.10}$$

Let us consider a change of the homology basis of the form

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad S := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2g, \mathbb{Z}).$$

The matrix S is in $SL(2g,\mathbb{Z})$ so that the inverse transformation has integer entries. The new basis $\tilde{\alpha}$ and $\tilde{\beta}$ is canonical if

$$\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} \circ \begin{pmatrix}\tilde{\alpha}^t & \tilde{\beta}^t\end{pmatrix} = J$$

with J defined in (2.10). This implies that the matrix $S := \begin{pmatrix} a & b\\ c & d\end{pmatrix}$ satisfies the relation

$$SJS^t = J$$

or in other words the matrix S is in the symplectic group $Sp(2g,\mathbb{Z})$. So we have shown that canonical homology basis are related by symplectic transformations.

Poincaré polygon

A canonical basis of cycles on a Riemann surface Γ of genus g has another remarkable property. Let us construct the cycles a_i and b_j so that they all begin and end at a particular point $*$ of Γ and otherwise do not have common points, and let us make cuts along these cycles. As a result the surface Γ becomes a $(4g)$-gon $\tilde{\Gamma}$ – a so-called Poincarè polygon of Γ. Indeed, the domain $\tilde{\Gamma}$ obtained as a result of the cutting is bounded by a closed contour $\partial\tilde{\Gamma}$ made up of $4g$ segments, and any cycle in $\tilde{\Gamma}$ is homologous to zero by property 2 of intersection number. Therefore, $\tilde{\Gamma}$ is a simply connected planar domain. Conversely, it is possible to glue the surface Γ together from the $(4g)$-gon $\tilde{\Gamma}$ by identifying its sides of the same name in the way indicated in the figure. In the figure, we write a_i^{-1} and b_i^{-1} the

2

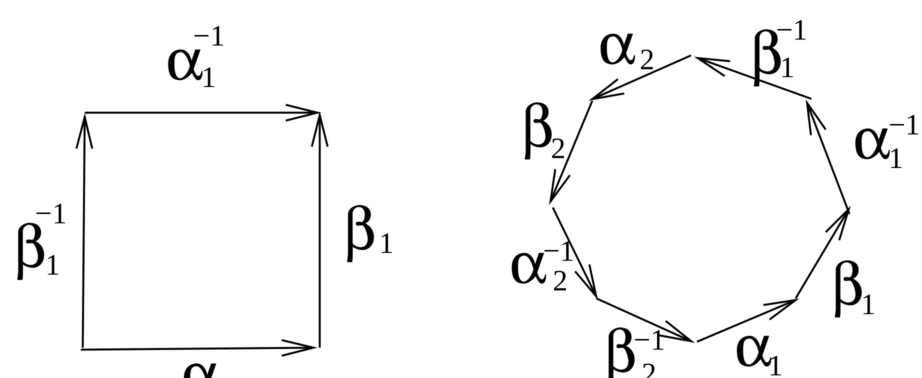


Figure 2.9: Poincaré polygon for surfaces of genus one and two.

edges of the cut along the cycles a_i and b_i, respectively, if these edges occur in the oriented boundary $\partial\tilde{\Gamma}$ with a minus sign. The segment a_i is glued together with the segment a_i^{-1} and b_i with the segment b_i^{-1} in the direction indicated by the arrows.

Chapter 3

Meromorphic functions on a Riemann surface.

3.1 Holomorphic mappings of Riemann surfaces

Definition 3.1. *Let* Γ *be a Riemann surface. A function* $f : \Gamma \to \mathbb{C}$ *is said to be holomorphic, if for each local chart the function*

$$\begin{aligned} f \circ \phi_\alpha^{-1} &: \phi_\alpha(U_\alpha) \to V_\alpha \subset \mathbb{C} \\ z_\alpha &\to f_\alpha(z_\alpha) := f(\phi_\alpha^{-1}(z_\alpha)), \end{aligned}$$

is holomorphic on the open subset $\phi_\alpha(U_\alpha)$.

The following theorem is inherited from complex analysis.

Theorem 3.2. *If* Γ *is a connected compact Riemann surface, then the only holomorphic functions are constants.*

Proof. Since f is holomorphic, $|f|$ is continuos on Γ compact. Therefore $|f|$ achieves its maximum value at some point of Γ. By the maximum modulus Theorem, f must be constant on Γ since Γ is connected. $\square$

In the same way one can define meromorphic functions.

Definition 3.3. *A function* f *is a meromorphic function on a Riemann surface* Γ *if it is holomorphic in a neighborhood of any point of* Γ *except for finitely many points* $Q_1, \ldots, Q_m$. *At the points* $Q_1, \ldots, Q_m$ *the function* f *has poles of respective multiplicities* $q_1, \ldots, q_m$ *i.e., in a neighborhood of the point* Q_j, $j = 1, \ldots, m$, *it can be represented in the form*

$$f = \tau_j^{-q_i} \tilde{f}_j(\tau_j), \tag{3.1}$$

where τ_j is a local parameter centred at the point Q_j, and $\tilde{f}_j(\tau_j)$ is a holomorphic function for small τ_j and $\tilde{f}_j(\tau_j)|_{\tau_j=0} \neq 0$. The order of f in Q_j denoted as $ord_{Q_j}(f)$ is the first nonzero exponent in the Laurent series of f in Q_j, namely

$$ord_{Q_j}(f) = -q_j.$$

It is easy to verify that Definition 3.4 is unambiguous. i.e., is independent from the choice of the local parameter, and also that the definition of the multiplicity of a pole is unambiguous.

Definition 3.4. *Let Γ be a compact Riemann surface defined as $\Gamma = \{(z, w) \in \mathbb{C}^2 \mid F(z, w) = 0\}$, $F(z, w)$ polynomial. A function $f = f(z, w)$ is meromorphic on Γ if it is a rational function of z and w, i.e., it has the form*

$$f(z, w) = \frac{P(z, w)}{Q(z, w)}, \tag{3.2}$$

where $P(z, w)$ and $Q(z, w)$ are polynomials, and $Q(z, w)$ is not identically zero on Γ.

The meromorphic functions on the surface Γ form a field whose algebraic structure actually bears in itself all the information about the geometry of the Riemann surface.

A similar definition of meromorphic functions can be given for a projective curve $\Gamma := \{[X : Y : Z] \in \mathbb{P}^2 | Q(X, Y, Z) = 0\}$ where now $Q(X, Y, Z)$ is a homogeneous polynomial. Meromorphic functions on the projective curve Γ take the form

$$R(X, Y, Z) = \frac{G(X, Y, Z)}{H(X, Y, Z)}$$

where G and H are homogeneous polynomials of the same degree and Q does not divide H.

It is not hard to verify that the conditions of Definition 3.3 follow from the conditions of Definition 3.4. The following result turns out to be true.

Theorem 3.5. *Definitions 3.4 and 3.3 are equivalent.*

We do not give a proof of this theorem; see, for example, [**?**] or [6].

Holomorphic mappings of Riemann surfaces are defined by analogy with meromorphic functions on Riemann surfaces.

Definition 3.6. *Let Γ and $\tilde{\Gamma}$ be Riemann surfaces. A map $f : \Gamma \to \tilde{\Gamma}$ is called holomorphic at a point $P \in \Gamma$ if and only if there is exists charts from a neighbourhood U of P and a neighbourhood $\widetilde{U}$ of $f(P)$, namely $\phi : U \to V \subset \mathbb{C}$ and $\widetilde{\phi} : \widetilde{U} \to \widetilde{V} \subset \mathbb{C}$ such that the composition*

$$\widetilde{\phi} \circ f \circ \phi^{-1}$$

is holomorphic. The map f is holomorphic, if it is holomorphic everywhere on Γ.

In other words, if τ is a local parameter on Γ and $\widetilde{\tau}$ a local parameter in a neighborhood of the point $f(P)$, then f must be written locally in the form $\widetilde{\tau} = \psi(\tau)$, where ψ is a holomorphic function of τ.

If $\Gamma = \{(z,w) \in \mathbb{C}^2 \mid F(z,w) = 0\}$, $\widetilde{\Gamma} = \{(\widetilde{z},\widetilde{w}) \in \mathbb{C}^2 \mid \widetilde{F}(\widetilde{z},\widetilde{w}) = 0\}$, then a holomorphic mapping $f : \Gamma \to \widetilde{\Gamma}$ is defined by a pair of meromorphic functions $\widetilde{z} = f_1(z,w), \widetilde{w} = f_2(z,w)$. It follows from Theorem 3.5 that this definition is equivalent to (3.6).

Remark 3.7. Let $f : \Gamma \to \mathbb{C}$ be a meromorphic function on Γ. Then f can be extended to an holomorphic function from Γ to $\bar{\mathbb{C}}$ in the following way:

$$F(P) = \begin{cases} f(P), & \text{if } P \text{ is not a pole for } f \\ \infty & \text{if } P \text{ is a pole for } f. \end{cases}$$

Let us verify that the map F is holomorphic. This is obvious in a neighborhood of regular points. Let z be a local coordinate in the finite part of $\mathbb{C}$, and $\zeta = 1/z$ the local coordinate at $\infty \in \bar{\mathbb{C}}$. Assume that the function has a pole of order k at the point $P_0 \in \Gamma$, i.e., it can be written in terms of a local coordinate τ centred in P_0 in the form

$$z = f(P) = \frac{c}{\tau^k} + O(\tau^{-k+1}), \quad c \neq 0,$$

Then $\zeta = \dfrac{1}{f(P)} = c^{-1}\tau^k + O(\tau^{k+1})$, i.e., the mapping has a zero of multiplicity k at P_0.

Example 3.8. A meromorphic function f from $\mathbb{P}^1$ to $\mathbb{C}$ is of the form

$$f(X,Y) = \frac{P(X,Y)}{Q(X,Y)}$$

where P and Q are homogeneous polynomials of the same degree. One can extend f to a holomorphic function $F : \mathbb{P}^1 \to \mathbb{P}^1$ in the form

$$F(X,Y) := [P(X,Y) : Q(X,Y)].$$

Theorem 3.9. *Let Γ and $\tilde{\Gamma}$ be connected Riemann surfaces and Γ be compact. Let $f : \Gamma \to \tilde{\Gamma}$ be a non constant holomorphic map. Then $\tilde{\Gamma}$ is compact and f is onto.*

Proof. Since f is holomorphic, it is also an open mapping. Therefore, $f(\Gamma)$ is open in $\tilde{\Gamma}$. Since Γ is compact, $f(\Gamma)$ is compact in $\tilde{\Gamma}$. Since $\tilde{\Gamma}$ is Hausdorff and connected, $f(\Gamma)$ is open and close in $\tilde{\Gamma}$, therefore $f(\Gamma) = \tilde{\Gamma}$ and $\tilde{\Gamma}$ is compact. □

The following lemma characterizes the local behaviour of a holomorphic mapping.

Lemma 3.10. *let $f : \Gamma \to \widetilde{\Gamma}$ be a non constant holomorphic function between compact Riemann surfaces. Then there exists local parameters τ and $\widetilde{\tau}$ centered in $P \in \Gamma$ and $Q = f(P) \in \widetilde{\Gamma}$ respectively, such that the map f takes the form*

$$\widetilde{\tau} = \tau^k, \quad k \in \mathbb{N}. \tag{3.3}$$

Proof. Let s and $\tilde{s}$ be local coordinates centered at $P \in \Gamma$ and $f(P) \in \widetilde{\Gamma}$. Then in local coordinates the holomorphic non constant function $f : \Gamma \to \widetilde{\Gamma}$ takes the form

$$\widetilde{s} = \psi(s)$$

with ψ holomorphic and $\psi(0) = 0$. The function ψ can be written in the form

$$\psi(s) = s^k h(s) \tag{3.4}$$

with h holomorphic, $h(0) \neq 0$ and k non negative integer. The number k does not depend on the choice of the local parameters s and $\widetilde{s}$. Let us define the new local coordinate τ as

$$\tau = sg(s), \quad g^s(s) = h(s).$$

Such map is biholomorphic. In terms of the local coordinate τ, the map f takes the form (3.3). □

Definition 3.11. *The number k defined (3.4) is called the multiplicity of f in P, and denoted by $mult_P(f)$. A point $P \in \Gamma$ is called ramification point for f if $mult_P(f) \geqslant 2$. The point $f(P) = Q \in \tilde{\Gamma}$ is called branch point. The number*

$$b_f(P) = mult_P(f) - 1$$

is called the branch number of f in P. The map $f : \Gamma \to \tilde{\Gamma}$ is called a holomorphic unramified (ramified) covering if f does not (does) have branch points.

Lemma 3.12. *Non constant holomorphic mappings $f : \Gamma \to \widetilde{\Gamma}$ are discrete. Namely the pre-image of a point $Q \in \tilde{\Gamma}$ is a discrete set $f^{-1}(Q)$ in Γ. In particular, if Γ and $\widetilde{\Gamma}$ are compact, $f^{-1}(Q)$ is finite.*

Proof. Let $Q \in \widetilde{\Gamma}$ and $P \in f^{-1}(Q)$. Let τ and $\widetilde{\tau}$ local coordinates centered at P and Q respectively. In these coordinates the function f takes the form $\widetilde{\tau} = h(\tau)$ with $h(0) = 0$ and h holomorphic. Since the set of zeros of a non constant holomorphic function is discrete, it follows that P is the only pre-image of Q. Therefore $f^{-1}(Q)$ forms a discrete subset. The second statement follows from the fact that discrete subsets of compact space are finite. □

Lemma 3.13. *Let $f : \Gamma \to \widetilde{\Gamma}$ be a non constant holomorphic map. Then the set of branch points*

$$B = \{P \in \Gamma \mid b_f(P) > 0\}$$

is discrete and it is finite if Γ is compact.

The proof of the Lemma is similar to the proof of Lemma 3.12

Example 3.14. A hyperelliptic nonsingular Riemann surface $w^2 = P_{2n+1}(z)$, $P_{2n+1}(z) = \prod_{i=1}^{2n+1}(z - z_i)$. Here the coordinates z and w are single-valued functions on Γ and holomorphic in the finite part of Γ. These functions have poles at the point of Γ at infinity: z has a double pole, and w has a pole of multiplicity $2n + 1$. This follows immediately from the proposition (1.39). The function $1/(z - z_i)$ has for each i a unique second order pole on Γ at the branch points. This follows from (1.25). We mention also that the function z has on Γ two simple zeros at the points $z = 0$, $w = \pm\sqrt{P_{2n+1}(0)}$ which merges into a single double zero if $P_{2n+1}(0) = 0$. The function w has $2n + 1$ simple zeros on Γ at the branch points. (The multiplicity of a zero of a meromorphic function is defined by analogy with the multiplicity of a pole.)

Example 3.15. A hyperelliptic Riemann surface $w^2 = P_{2n+2}(z)$. Here again the functions z and w are holomorphic in the finite part of Γ. But these functions have two poles at infinity (in the infinite part of the surface Γ): z has two simple poles, and w has two poles of multiplicity $n + 1$. This follows from proposition (1.39).

Exercise 3.16: Prove Theorem 3.5 for $\mathbb{P}^1$.

Exercise 3.17: Prove Theorem 3.5 for hyperelliptic Riemann surfaces. *Hint.* Let $f = f(z, w)$ be a meromorphic (in the sense of Definition 3.3) function on the hyperelliptic Riemann surface Γ defined by the equation $w^2 = P(z)$. Show that the functions $f_+ = f(z, w) + f(z, -w)$ and $f_- = \dfrac{f(z, w) - f(z, -w)}{w}$ are rational functions of z, so that any meromorphic function on on Γ is of the form $f(z, w) = r_1(z) + r_2(z)w$ where r_1 and r_2 are rational functions.

To prove the simplest properties of meromorphic functions on Riemann surfaces it is useful to employ arguments connected with the concept of the degree of a mapping.

Proposition 3.18. *Let $f : \Gamma \to \tilde{\Gamma}$ be a nonconstant holomorphic mapping between compact Riemann surfaces. For each $Q \in \tilde{\Gamma}$ let us define $deg_Q(f)$ to be the sum of the multiplicities of f at the point of Γ mapping to Q:*

$$deg_Q(f) = \sum_{P \in f^{-1}(Q)} mult_P(f).$$

Then $deg_Q(f)$ is constant independent from Q.

Proof. We show that the function $Q \to \deg_Q(f)$ is locally constant. Let $P_1, \ldots P_j$ be the number of pre-images of Q under f. Let τ_i be local coordinates centered at P_i and $\tilde{\tau}$ local coordinate centered in Q so that locally near P_i the function f takes the form

$$\tilde{\tau} = \tau_i^{m_i}, \quad i = 1, \ldots, j.$$

The above map has constant degree in a small neighbourhood of $\tau_i = 0$ for $i = 1, \ldots, j$. What is left to prove is that near Q there are no other pre-images of Q left unaccounted which are not in a neighbourhood of $P_1, \ldots P_j$. Suppose by contradiction that arbitrary close to Q there are pre-images which are not contained in any of the neighbourhood of the P_i. Since Γ is compact we may extract a convergent sub-sequence of points in Γ, say P'_n which are not contained in any of the neighbourhood of the P_i. This subsequence has the property that $f(P'_n) \to Q$ because f is holomorphic, therefore, the limit point of P'_n must be one of the P_i, $i = 1, \ldots j$. We obtained a contradiction since we assumed that none of the P_n's lie in a neighbourhood of the P_i, $i = 1, \ldots, j$. □

Exercise 3.19: Prove that for any meromorphic function on a Riemann surface Γ the number of zeros is equal to the number of poles (zeros and poles are taken with multiplicity counted).

Remark 3.20. A single non constant meromorphic function on a Riemann surface Γ completely determines the complex structure of Γ. Indeed let $P \in \Gamma$ and $n = b_f(P) + 1$. Then a local coordinate vanishing at P is given by

$$\begin{aligned} &(f - f(P))^{1/n} \quad \text{if } f(P) \neq \infty \\ &f(P)^{-1/n} \quad \text{if } f(P) = \infty. \end{aligned} \tag{3.5}$$

Exercise 3.21 (Riemann-Hurwitz formula)**:** Let $f : \Gamma \to \tilde{\Gamma}$ be a nonconstant holomorphic map between compact Riemann surfaces. Prove the following generalization of the Riemann-Hurwitz formula (see Lecture 2)

$$2 - 2g(\Gamma) = \deg f(2 - 2g(\tilde{\Gamma}) - \sum_{P \in \Gamma} (\mathrm{mult}_P f - 1) \tag{3.6}$$

where $g(\Gamma)$ and $g(\tilde{\Gamma})$ is the genus of the Riemann surface Γ and $\widetilde{\Gamma}$ respectively and deg is the degree of the function f.

Exercise 3.22: Let Γ be a nonsingular projective curve defined as $\Gamma := \{[X : Y : Z] \in \mathbb{P}^2 \,|\, Q(X, Y, Z) = 0\}$ where Q is an irreducible homogenueos polynomial of degree n. Show that the map

$$[X : Y : Z] \to [Q_X : Q_Y : Q_Z]$$

from $\mathbb{P}^2$ to $\mathbb{P}^2$ is well defined. The image of such map is called the dual curve $\hat{\Gamma}$ to Γ. Show that the map is holomorphic but it does not have a holomorphic inverse if $n \geqslant 3$.

Definition 3.23. *A map $f : \Gamma \to \widetilde{\Gamma}$ is called a biholomorphic isomorphism if it is a bijective holomorphic map with holomorphic inverse. If $\widetilde{\Gamma} = \Gamma$, then the map is called an automorphism.*

It is not hard to derive from Theorem 3.5 that the class of biholomorphic isomorphisms of Riemann surfaces coincides with the class of birational isomorphisms (the mapping itself and its inverse are given by rational functions. Namely let $\Gamma := \{(z, w) \in \mathbb{C}^2 \,|\, F(z, w) = 0\}$ and $\widetilde{\Gamma} := \{(\widetilde{z}, \widetilde{w}) \in \mathbb{C}^2 \,|\, \widetilde{F}(\widetilde{z}, \widetilde{w}) = 0\}$, then a birational isomorphism is of the form $\widetilde{z} = r_1(z, w)$, $\widetilde{w} = s_1(z, w)$ and $z = r_2(\widetilde{z}, \widetilde{w})$, $w = s_2(\widetilde{z}, \widetilde{w})$, with $r_1(z, w)$, $r_2(\widetilde{z}, \widetilde{w})$, $s_1(z, w)$ and $s_2(\widetilde{z}, \widetilde{w})$ rational functions. In what follows we use the terms bi-holomorphic isomorphism and birational isomorphism interchangeably.

The following is obvious but important.

Lemma 3.24. *If the surfaces Γ and $\tilde{\Gamma}$ are biholomorphically (birationally) isomorphic, then they have the same genus.*

Proof. A biholomorphic isomorphism is clearly a homeomorphism. But the genus is invariant under homeomorphisms [10]. The assertion is proved. □

Definition 3.25. *A Riemann surface Γ is said to be rational if it is biholomorphically isomorphic to $\mathbb{P}^1$.*

The genus of a rational surface is equal to zero. It turns out that this condition is also sufficient for rationality.

Exercise 3.26: Let Γ be a Riemann surface of genus $g > 1$. Prove that there is no meromorphic function on Γ with a single simple pole.

Example 3.27. The surface $w^2 = z$. This surface is rational. A birational isomorphism onto $\mathbb{P}^1$ is given by the projection $(z, w) \to w$.

Exercise 3.28: Consider the Riemann surface $\Gamma := \{(z, w) \in \mathbb{C}^2 \,|w^n = P_m(z)\}$ where $P_m(z)$ is a polynomial of degree m in z with distinct roots. Consider the automorphism

$$J : (z, w) \to (z, e^{2\pi j/n} w), \quad j = 1, \ldots, n$$

and define the equivalence relation $(z_1, w_1) \simeq (z_2, w_2)$ if $z_1 = z_2$ and $w_1 = e^{2\pi j/n} w_2$ for some j. Show that the quotient surface Γ/J is well defined and it is rational. Determine the branch points of the projection map

$$\pi : \Gamma \to \Gamma/J.$$

Example 3.29. A surface with $w^2 = P_{2g+2}(z)$ with $g > 1$ is nonrational. We show that any such surface is birationally isomorphic to some surface of the form $\widetilde{w}^2 = \widetilde{P}_{2g+1}(\widetilde{z})$. Let z_0 be one of the zeros of the polynomial $P_{2g+2}(z)$, and let

$$\widetilde{z} = \frac{1}{z - z_0}, \quad \widetilde{w} = \frac{w}{(z - z_0)^{g+1}}.$$

The inverse mapping has the form

$$z = z_0 + \frac{1}{\tilde{z}}, \quad w = \frac{\widetilde{w}}{\widetilde{z}^{g+1}}.$$

If $P_{2g+2}(z) = (z - z_0)\prod_{i=1}^{2g+1}(z - z_i)$, then $\widetilde{P}_{2g+1}(\widetilde{z}) = \prod_{i=1}^{2g+1}(1 + (z_0 - z_i)\widetilde{z})$. Thus, both ”types” of hyperelliptic Riemann surfaces considered in Lecture 1 give the same class of surfaces.

Chapter 4

Differentials on a Riemann surface.

4.1 Holomorphic differentials

We consider a complex-one dimensional manifold M with with an atlas of charts $\{U_\alpha, \phi_\alpha\}$ with

$$\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}$$

and $\phi_\alpha(P) = z_\alpha \in V_\alpha$ and $P \in U_\alpha$. Here we are identifying $\mathbb{C}$ with $\mathbb{R}^2$ by writing $z_\alpha = x_\alpha + iy_\alpha$ with x_α and y_α standard coordinates on $\mathbb{R}^2$.

A smooth 0-form on M is a smooth function on M.

Definition 4.1. *A smooth one 1-form (also called differential) ω on M is an assignment of a collection of two smooth functions $h_\alpha(z_\alpha, \bar{z}_\alpha)$ and $g_\alpha(z_\alpha, \bar{z}_\alpha)$ to each local coordinate z_α in U_α such that*

$$\omega = h_\alpha(z_\alpha, \bar{z}_\alpha)\, dz_\alpha + g_\alpha(z_\alpha, \bar{z}_\alpha)\, d\bar{z}_\alpha, \tag{4.1}$$

is invariant under coordinate change. Namely if $z_\beta = z_\beta(z_\alpha, \bar{z}_\alpha)$ and $\bar{z}_\beta = \bar{z}_\beta(z_\alpha, \bar{z}_\alpha)$ are another local coordinates such that $U_\alpha \cap U_\beta \neq \emptyset$ then

$$\omega = \left(h_\alpha(z_\alpha, \bar{z}_\alpha) \frac{\partial z_\alpha}{\partial z_\beta} + g_\alpha(z_\alpha, \bar{z}_\alpha) \frac{\partial \bar{z}_\alpha}{\partial z_\beta} \right) dz_\beta + \left(h_\alpha(z_\alpha, \bar{z}_\alpha) \frac{\partial z_\alpha}{\partial \bar{z}_\beta} + g_\alpha(z_\alpha, \bar{z}_\alpha) \frac{\partial \bar{z}_\alpha}{\partial \bar{z}_\beta} \right) d\bar{z}_\beta.$$

The two parts $h(z_\alpha, \bar{z}_\alpha)\, dz_\alpha$ and $g(z_\alpha, \bar{z}_\alpha)\, d\bar{z}_\alpha$ of the expression (4.1) will be called $(1,0)$- and $(0,1)$-forms respectively. The above expression shows that the decomposition of ω in $(1,0)$ and $(0,1)$ form is invariant under local change of coordinates, if and only if the change of coordinates is holomorphic, namely

$$\frac{\partial \bar{z}_\alpha}{\partial z_\beta} = 0, \quad \frac{\partial z_\alpha}{\partial \bar{z}_\beta}.$$

The above conditions in real coordinates are equivalent to the Cauchy-Riemann equation. For a one-complex dimensional manifold M that has a complex structure (namely a Riemann surface), the decomposition of a one form in $(1,0)$ and $(0,1)$ form is invariant under local change of coordinates. From now on we will consider only holomorphic change of coordinates.

Definition 4.2. *A one form ω is called holomorphic is the functions $h_\alpha(z_\alpha, \bar{z}_\alpha)$ in (4.1) are all holomorphic functions and $g_\alpha \equiv 0$, namely*

$$\omega = h(z_\alpha)dz_\alpha.$$

A one form ω is called antiholomorphic if

$$\omega = g(\bar{z}_\alpha)d\bar{z}\alpha.$$

In a similar way to one form we can define two-forms.

Definition 4.3. *A smooth two form η on M is an assignment of a smooth function $f_\alpha(z_\alpha, \bar{z}_\alpha)$ such that*

$$\eta = f_\alpha(z_\alpha, \bar{z}_\alpha)dz_\alpha \wedge d\bar{z}_\alpha$$

is invariant under coordinate change.

The exterior multiplication satisfies the conditions

$$dz_\alpha \wedge dz_\alpha = 0, \quad d\bar{z}_\alpha \wedge d\bar{z}_\alpha = 0, \quad dz_\alpha \wedge d\bar{z}_\alpha = -d\bar{z}_\alpha \wedge dz_\alpha.$$

Under holomorphic change of coordinates $z_\beta = z_\beta(z_\alpha)$, $\bar{z}_\beta = \bar{z}_\beta(\bar{z}_\alpha)$ one has

$$\eta = f_\beta(z_\beta, \bar{z}_\beta)dz_\beta \wedge d\bar{z}_\beta = f_\alpha(z_\alpha, \bar{z}_\alpha)dz_\alpha \wedge d\bar{z}_\alpha$$

where

$$f_\beta(z_\beta, \bar{z}_\beta) = f_\alpha(z_\alpha, \bar{z}_\alpha)\left|\frac{dz_\alpha}{dz_\beta}\right|^2.$$

We define Ω^k for $k = 0, 1, 2$ as the set of smooth functions, smooth one forms and smooth two-forms on M respectively. We define the exterior derivative

$$d : \Omega^k \to \Omega^{k+1}, \quad k = 0, 1, 2$$

as follows. For $f \in \Omega^0$,

$$df(z, \bar{z}) = f_z dz + f_{\bar{z}} d\bar{z},$$

For one forms $\omega \in \Omega^1$, with $\omega = h(z,\bar{z})dz + g(z,\bar{z})d\bar{z}$ in a given coordinate chart, the exterior derivative takes the form

$$d\omega = dh \wedge dz + dg \wedge d\bar{z}$$

and for two forms, $\eta \in \Omega^2(M)$

$$d\eta = 0.$$

Clearly the fundamental property of the exterior differentiation is

$$d^2 = 0.$$

We can decompose the exterior derivative operator d according to the decomposition of 1-form in $(0,1)$ and $(1,0)$ forms

$$d = \partial + \bar{\partial}$$

so that for $h \in \Omega^{0,0} := \Omega^0$ in a local chart

$$\partial : \Omega^0 \to \Omega^{1,0}, \ \ \partial h(z,\bar{z}) = h_z dz,$$

and

$$\bar{\partial} : \Omega^0 \to \Omega^{0,1}, \ \ \bar{\partial} h(z,\bar{z}) = h_{\bar{z}} d\bar{z}.$$

In general we get the diagram

$$\begin{tikzcd}
\Omega^{0,1} \arrow[r, "\partial"] & \Omega^2 \\
\Omega^0 \arrow[u, "\bar{\partial}"'] \arrow[r, "\partial"'] & \Omega^{1,0} \arrow[u, "\bar{\partial}"]
\end{tikzcd}$$

where $\Omega^2 = \Omega^{1,1}$. Also in this case $\partial^2 = 0$ and $\bar{\partial}^2 = 0$.

Definition 4.4. *A one form ω is called exact if there is a function $f \in \Omega^0$ such that $df = \omega$. A one form $\omega \in \Omega^1$ is called closed if $d\omega = 0$.*

Lemma 4.5. *A $(1,0)$-form $\omega = h(z,\bar{z})\, dz$ is closed if and only if the function $h(z,\bar{z})$ is holomorphic.*

It follows that all the *holomorphic differentials*, locally can be written in the form $\omega = h(z)dz$ where $h(z)$ is a holomorphic function. Holomorphic differentials are closed differentials.

Definition 4.6. *The first de Rham cohomology group is defined as*

$$H^1_{deRham}(\Gamma) = \frac{\textit{Closed 1-forms}}{\textit{Exact 1-forms}} = \frac{\ker(d : \Omega^1 \to \Omega^2)}{Im(d : \Omega^0 \to \Omega^1)}.$$

A similar definition can be obtained for the Dolbeault cohomology groups $H^{1,0}(\Gamma)$ and $H^{0,1}(\Gamma)$ with respect to the operator $\bar{\partial}$:

$$H^{1,0}(\Gamma) := \frac{\ker(\bar{\partial} : \Omega^{1,0} \to \Omega^2)}{(\bar{\partial} : \Omega^0 \to \Omega^{1,0})} = \ker(\bar{\partial} : \Omega^{1,0} \to \Omega^2),$$

$$H^{0,1}(\Gamma) := \frac{\ker(\bar{\partial} : \Omega^{0,1} \to \Omega^2)}{(\bar{\partial} : \Omega^0 \to \Omega^{0,1})} = \frac{\Omega^{0,1}}{\mathrm{Image}(\bar{\partial} : \Omega^0 \to \Omega^{0,1})}.$$

A non trivial result shows that there are isomorphisms among the above three groups [17]. By denoting $\overline{H^{0,1}(\Gamma)}$ the complex conjugate of the group $H^{0,1}(\Gamma)$, one has the following theorem.

Theorem 4.7. *The Dolbeault cohomology groups $H^{1,0}(\Gamma)$ and $\overline{H^{0,1}(\Gamma)}$ are isomorphic*

$$H^{1,0}(\Gamma) \simeq \overline{H^{0,1}(\Gamma)} \tag{4.2}$$

and the first de-Rham cohomology group is isomorphic to

$$H^1_{deRham}(\Gamma) \simeq H^{1,0}(\Gamma) \oplus H^{0,1}(\Gamma). \tag{4.3}$$

The relation (4.2) shows that the complex vector spaces $H^{1,0}(\Gamma)$ and $H^{0,1}(\Gamma)$ have the same dimension. The relation (4.3) shows that the dimension of the complex vector space $H^{1,0}(\Gamma)$ and $H^{0,1}(\Gamma)$ is half the dimension of the complex vector space $H^1_{deRham}(\Gamma)$.

4.1.1 Integration

We can integrate one forms on curves of the Rieamnn surface Γ, two-forms on domains of Γ and 0-forms on zero dimensional domains of Γ, namely points. Let c_0 be a 0-chain,

$$c_0 = \sum_i n_i P_i, \quad P_i \in \Gamma$$

then for $f \in \Omega^0(\Gamma)$ the integral of f over a 0-chain c_0 is

$$\int_{c_0} f = \sum_i n_i f(P_i)$$

A one form ω can be integrated over a one-chain c. If the piece-wise differentiable path $c : [0,1] \to \Gamma$ is contained in a single coordinate disc with coordinates $z = x + iy$, then the integral of ω over the one-chain c takes the form

$$\int_c \omega = \int_0^1 h(z(t), \bar{z}(t)) \frac{dz}{dt} dt + \int_0^1 g(z(t), \bar{z}(t)) \frac{d\bar{z}(t)}{dt} dt$$

By the transition formula for ω the above integral is independent from the choice of the coordinate chart z. In a similar way a two-form η can be integrated over two chains D. Again restricting to a single coordinate chart one has

$$\int\int_D \eta = \int\int_D f(z, \bar{z}) dz d\bar{z}.$$

The integral is well defined and extends in a obvious way to an arbitrary two-chain.

Theorem 4.8 (Stokes theorem)**.** *Let D be a domain of Γ with a piece-wise smooth boundary ∂D and let ω be a smooth one-form. Then*

$$\int_D d\omega = \int_{\partial D} \omega. \tag{4.4}$$

As a consequence of Stokes theorem, the integral of closed forms ω on any closed oriented contour (cycle) γ on Γ does not depend on the homology class of γ. Recall that two cycles γ_1 and γ_2 are said to be homologous if their difference $\gamma_1 - \gamma_2 = \gamma_1 \cup (-\gamma_2)$ (where $(-\gamma_2)$ is the cycle with the opposite orientation) is the oriented boundary of some domain D on Γ with $\partial D = \gamma_1 - \gamma_2$. Then for a close differential ω and from Stokes theorem we obtain

$$0 = \int_D d\omega = \int_{\partial D} \omega = \int_{\gamma_1 - \gamma_2} \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega.$$

In addition, the integral of a close differential ω on a close cycle γ is independent from the cohomology class. Let $\omega' = \omega + df$ for some smooth function f, then

$$\int_\gamma \omega = \int_\gamma (\omega' - df) = \int_\gamma \omega'.$$

We summarise the above discussion with the following proposition.

Proposition 4.9. *The integration is a paring between the first homology group $H_1(\Gamma, \mathbb{Z})$ and the first cohomology group $H^1_{deRham}(\Gamma, \mathbb{C})$*

$$\int : H_1(\Gamma, \mathbb{Z}) \times H^1_{deRham}(\Gamma, \mathbb{C}) \to \mathbb{C}$$

The pairing is non-degenerate.

Proof. We need to prove that the pairing is non-degenerate. Consider a smooth one-form ω such that

$$\int_\gamma \omega = 0$$

for all $\gamma \in H_1(\Gamma, \mathbb{Z})$. It follows that the function

$$f(P) = \int_{P_0}^{P} \omega$$

is well defined and it does not depend on the path of integration between P_0 and P. Therefore $df = \omega$, namely the equivalent class of ω in the de-Rham cohomology is zero, $[\omega] = 0$ in $H^1_{deRham}(\Gamma, \mathbb{C})$. □

As a consequence of the above proposition we have the following lemma.

Lemma 4.10. *The dimension of the space $H^1_{deRham}(\Gamma, \mathbb{C})$ is less then or equal to $2g$ where g is the genus of the compact Riemann surface Γ.*

Proof. Suppose by contradiction, that there are $\omega_1, \ldots, \omega_s$, $s > 2g$ independent closed differentials in $H^1_{deRham}(\Gamma, \mathbb{C})$. Then let us consider a basis of the homology γ_j, $j = 1 \ldots, 2g$ and construct the matrix with entries

$$c_{jk} = \int_{\gamma_j} \omega_k, \quad j = 1, \ldots 2g, \;\; k = 1, \ldots s.$$

Such matrix has rank at most equal to $2g$, and therefore one can find nonzero constants $a_1, \ldots, a_s$ such that the differential $\omega = \sum_{k=1}^{s} a_k \omega_s$ has all its periods equal to zero, namely

$$\int_{\gamma_j} \omega, \quad j = 1, \ldots 2g.$$

By proposition 4.9 it follows that $[\omega] = 0$ and we arrive to a contradiction. □

As a consequence of the above lemma we have the following corollary for the dimension of the space of holomorphic differentials.

Corollary 4.11. *The space of holomorphic differentials on a Riemann surface of genus g is no more than g-dimensional.*

Actually the number of independent holomorphic differentials is indeed equal to g.

Theorem 4.12. *The space of holomorphic differentials on a Riemann surface Γ of genus g has dimension g.*

We do not give a proof of the above theorem that is constructive (see [18] or [17]). However for a Riemann surface given as the zeros of a polynomial equation one can determine explicitly the holomorphic differentials.

Example 4.13. Let us consider holomorphic differentials on a hyperelliptic Riemann surface

$$\Gamma = \{w^2 = P_{2g+1}(z)\}, \quad P_{2g+1}(z) = \prod_{k=1}^{2g+1}(z - z_k)$$

of genus $g \geqslant 1$. Let us check that the differentials

$$\eta_k = \frac{z^{k-1}dz}{w} = \frac{z^{k-1}dz}{\sqrt{P_{2g+1}(z)}}, \quad k = 1, \dots, g \tag{4.5}$$

are holomorphic. Indeed, holomorphicity at any finite point but branch point is obvious as the denominator does not vanish. We verify holomorphicity in a neighborhood of the i-th branch point $P_i = \{z = z_i, \quad w = 0\}$. Choosing the local parameter τ in a neighborhood of P_i in the form $\tau = \sqrt{z - z_i}$, we get from (1.25) that $\eta_k = \psi_k(\tau)d\tau$, where the function

$$\psi_k(\tau) = \frac{2(z_i + \tau^2)^{k-1}}{\sqrt{\prod_{j\neq i}(\tau^2 + z_i - z_j)}}$$

is holomorphic for small τ.

At the point at infinity the differentials η_k can be written in terms of the local parameter $\tau = z^{-\frac{1}{2}}$ in the form $\eta_k = \phi_k(\tau)d\tau$, where the functions

$$\phi_k(\tau) = -2\tau^{2(g-k)}\left[\prod_{i=1}^{2g+1}(1 - z_i\tau)\right]^{-\frac{1}{2}}, \quad k = 1, \dots, g$$

are holomorphic for small τ.

In the same way it can be verified that the differentials $\eta_k = z^{k-1}dz/w$, $k = 1, \dots, g$ are holomorphic on the Riemann surface $w^2 = P_{2g+2}(z)$ with $P_{2g+2}(z)$ an even polynomial with $2g + 2$ distinct roots.

In general for a nonsingular Riemann surface $\Gamma := \{(z, w) \in \mathbb{C}^2, | F(z, w) = 0\}$, where $F(z, w)$ is a polynomial in z and w, the differential

$$\omega = \frac{z^i w^j dz}{F_w(z, w)}, \quad i, j \geqslant 0, \tag{4.6}$$

is holomorphic for all finite values of z and w. Indeed the only possible points where such differential might have poles are the zeros of F_w, namely the branch points with respect to the projection $\pi : \Gamma \to \mathbb{C}$ such that $\pi(z,w) = z$. At the branch points with respect to the projection π one needs to take w as local coordinate. Since $F_z dz + F_w dw = 0$ one has

$$\frac{dz}{F_w} = -\frac{dw}{F_z}.$$

Therefore at the branch points where $F_w = 0$ one can write the differential ω in the form $\omega = -\dfrac{z^j w^k dw}{F_z}$. Since we assume that the surface Γ is nonsingular, $F_z \neq 0$ at the branch points.

In order to determine for which coefficients (i,j) the differential ω in (4.6) remains holomorphic also at infinity, we explain the following rule, that is true for nonsingular Riemann surfaces. Consider the carrier of the polynomial $F(z,w) = \sum_{i,j} a_{ij} z^i w^j$, namely the set of all integral points in $\mathbb{Z}^2$ such that

$$C(F) = \{(i,j) \in \mathbb{Z}^2 | a_{ij} \neq 0\}.$$

The Newton polygon $N(F)$ of $F(z,w)$ is defined as the convex hull of the carrier $C(F)$. Then the holomorphic differentials associated to the curve given by the equation $F(z,w) = 0$ are

$$\frac{z^{i-1} w^{j-1} dz}{F_w(z,w)}, \quad (i,j) \in N(F)$$

where (i,j) are the points strictly inside the Newton polygon $N(F)$.

This fact can be easily verified for hyperelliptic Riemann surfaces. Now let us check it for a smooth projective curves.

Consider the smooth compact Riemann surface

$$\Gamma := \{[X:Y:Z] \in \mathbb{P}^2, | \, Q(X,Y,Z) = \sum_{0 \leqslant i+j \leqslant n} a_{ij} X^i Y^j Z^{n-i-j} = 0\}.$$

Let us consider the affine part of Γ given by the equation $F(z,w) = \sum_{i+j \leqslant n} a_{ij} z^i w^j$. The point(s) at infinity of the affine curve are determined by the equation $Q(X,Y,0) = \sum_{\leqslant i+j=n} a_{ij} X^i Y^j = 0$. For simplicity we assume that there are no branch points at infinity so that the homogeneous equation $Q(X,Y,0) = 0$ has n distinct roots. From this it follows that $\deg Q(X,0,0) = \deg Q(0,Y,0) = n$.

Then the holomorphic differentials are

$$\eta_{ij} = \frac{z^{i-1} w^{j-1} dz}{\partial F(z,w)/\partial w}, \quad i+j \leqslant n-1. \tag{4.7}$$

Indeed the above expression is holomorphic for finite values of z and w. The only points we need to consider are the points at infinity $\infty^1, \ldots, \infty^n$. By the above assumptions we have that a local coordinate at infinity is

$$z = \frac{1}{\xi}, \quad w = \frac{c_j}{\xi} \quad j = 1, \ldots, n$$

where c_j are the solutions of the homogeneous equation $Q(c_j, 1, 0) = 0$. In these coordinates ω takes the form

$$\omega = -c\frac{d\xi}{\xi^{i+j}} \frac{1}{F_w\left(\frac{1}{\xi}, \frac{c_j}{\xi}\right)} = -c\frac{\xi^{n-1}(1 + O(\xi))d\xi}{\xi^{i+j}}$$

where c is a nonzero constant. The above differential is holomorphic if $i + j \leqslant n - 1$. If $F(0,0) \neq 0$ then the Newton polygon associated the F is the triangle with vertices $(0,0)$, $(0,n)$ and $(n,0)$. Then all the integral points strictly inside the triangle satisfy the rule $0 < i + j \leqslant n - 1$. Therefore the integral points inside the triangle are in one to one correspondence with the holomorphic differentials (4.7).

Exercise 4.14: Show that the differentials obtained using the Newton polygon formula for the polynomiil $F(z,w)$ are holomorphic without assuming that $F(0,0) = 0$ and that at infinity there are no branch points. (Study the conditions on the shape of the Newton polygon so that the curve Γ is non singular in $(0,0)$ or at infinity.)

4.1.2 Riemann bilinear relations

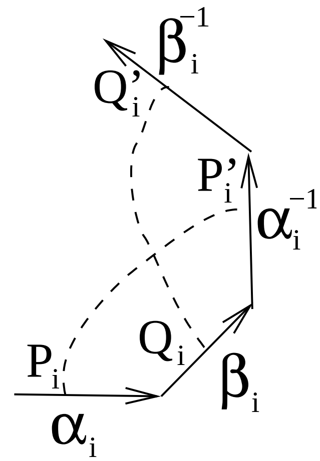
In this section we prove several technical assertions regarding the periods of close differential and holomorphic differentials. Such relations are known as Riemann bilinear relations

Lemma 4.15. *Let ω_1 and ω_2 be two closed differentials on a surface Γ of genus $g \geqslant 1$. Denote their periods with respect to a canonical basis of cycles $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, by A_i, B_i and A'_i, B'_i:*

$$A_i = \int_{\alpha_i} \omega, \;\; B_i = \int_{\beta_i} \omega, \quad A'_i = \int_{\alpha_i} \omega', \;\; B'_i = \int_{\beta_i} \omega'. \tag{4.8}$$

Denote by $f = \int \omega$ the primitive of ω, which is single-valued on the surface $\tilde{\Gamma}$ cut along a_i, b_j, then

$$\int\int_\Gamma \omega \wedge \omega' = \oint_{\partial\tilde{\Gamma}} f\omega' = \sum_{i=1}^{g} (A_i B'_i - A'_i B_i). \tag{4.9}$$



Proof. The first of the equalities in (4.9) follows from Stokes' formula, since $d(f\omega') = \omega \wedge \omega'$. Let us prove the second. We have that

$$\oint_{\partial\tilde{\Gamma}} f\omega' = \sum_{i=1}^{g} \left(\int_{\alpha_i} + \int_{\alpha_i^{-1}} \right) f\omega' + \sum_{i=1}^{g} \left(\int_{\beta_i} + \int_{\beta_i^{-1}} \right) f\omega'.$$

To compute the i-th term in the first sum we use the fact that $f(P) = \int_{P_0}^{P} \omega$ where P_0 is a point in the interior of $\tilde{\Gamma}$:

$$f(P_i) - f(P_i') = \int_{P_0}^{P_i} \omega - \int_{P_0}^{P_i'} \omega = \int_{P_i'}^{P_i} \omega = -B_i \tag{4.10}$$

since the cycle $P_i'P_i$, which is closed on Γ, is homologous to the cycle β_i (see the figure; a fragment of the boundary $\partial\tilde{\Gamma}$ is pictured). Similarly, the jump of the function f in crossing the cut β_i has the form

$$f(Q_i) - f(Q_i') = \int_{Q_i'}^{Q_i} \omega = A_i \tag{4.11}$$

since the cycle $Q_i'Q_i$ on Γ is homologous to the cycle a_i. Moreover, $\omega'(P_i') = \omega'(P_i)$ and

$\omega'(Q_i') = \omega'(Q_i)$ because the differential ω' is single-valued on Γ. We have that

$$\int_{\alpha_i} f(P_i)\omega'(P_i) + \int_{\alpha_i^{-1}} f(P_i')\omega'(P_i') = \int_{\alpha_i} f(P_i)\omega'(P_i) - \int_{\alpha_i} (f(P_i) + B_i)\omega'(P_i)$$
$$= -B_i \int_{\alpha_i} \omega'(P_i) = -B_i A_i'$$

where the minus sign appears because the edge a_i^{-1} occurs in $\partial\tilde{\Gamma}$ with a minus sign. Similarly,

$$\left(\int_{\beta_i} + \int_{\beta_i^{-1}} \right) f\omega' = A_i B_i'.$$

Summing these equalities, we get (4.9). The lemma is proved. □

We derive some important consequences for periods of holomorphic differentials from the lemma 4.15. Everywhere we denote by $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ the canonical basis of cycles on Γ.

Corollary 4.16. . *Let ω be a nonzero holomorphic differential on Γ, and $A_1, \ldots, A_g, B_1, \ldots, B_g$ its corresponding periods with respect to the canonical homology basis $\alpha_1 \ldots, \alpha_g$ and $\beta_1 \ldots, \beta_g$, then*

$$\Im \left(\sum_{i=1}^{g} A_k \bar{B}_k \right) < 0. \tag{4.12}$$

Proof. Take $\omega' = \bar{\omega}$ in the lemma 4.15. Then $A_i' = \bar{A}_i$ and $B_i' = \bar{B}_i$ for $i = 1, \ldots, g$. We have that

$$\frac{i}{2} \int\int_{\Gamma} \omega \wedge \omega' = \frac{i}{2} \int\int |f|^2 dz \wedge d\bar{z} = \int\int_{\Gamma} |f|^2 dx \wedge dy > 0.$$

Here $z = x + iy$ is a local parameter, and $\omega = f(z)dz$. In view of (4.9) this integral is equal to

$$\frac{i}{2} \sum_{k=1}^{g} A_k \bar{B}_k - \bar{A}_k B_k = -\Im \left(\sum_{k=1}^{g} A_k \bar{B}_k \right).$$

The corollary is proved. □

Corollary 4.17. *If all the α-periods of a holomorphic differential are zero, then $\omega = 0$.*

This follows immediately from Corollary 4.16.

Corollary 4.18. *On a surface Γ of genus g there exists a basis $\omega_1, \dots, \omega_g$ of holomorphic differentials such that*

$$\oint_{\alpha_j} \omega_k = \delta_{jk}, \quad j, k = 1, \dots, g. \tag{4.13}$$

Proof. Let $\eta_1, \dots, \eta_g$ be an arbitrary basis of holomorphic differentials on Γ. The matrix

$$A_{jk} = \oint_{\alpha_j} \eta_k \tag{4.14}$$

is nonsingular. Indeed, otherwise there are constants $c_l, \dots, c_g$ such that $\sum_k A_{jk} c_k = 0$. But then $\sum_k c_k \eta_k = 0$, since this differential has zero a-periods. This contradicts the independence of the differentials $\eta_i, \dots, \eta_k$.

$$\omega_j = \sum_{k=1}^{g} \tilde{A}_{kj} \eta_k, \quad j = 1, \dots, g, \tag{4.15}$$

where the matrix $(\tilde{A}_{kj})$ is the inverse of the matrix (A_{jk}), $\sum_k \tilde{A}_{ik} A_{kj} = \delta_{ij}$, we get the desired basis. The corollary is proved. □

A basis $\omega_1, \dots, \omega_g$ satisfying the conditions (4.13) will be called a normal basis of holomorphic differentials (with respect to a canonical basis of cycles $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$) .

Corollary 4.19. *Let $\omega_1, \dots \omega_g$ be a normalized basis of holomorphic differentials, and let*

$$B_{jk} = \oint_{\beta_j} \omega_k, \quad j, k = 1, \dots, g. \tag{4.16}$$

Then the matrix (B_{jk}) is symmetric and has positive-definite imaginary part.

Proof. Let us apply the lemma 4.15 to the pair $\omega = \omega_j$ and $\omega' = \omega_k$. Then $\omega \wedge \omega'$, $A_i = \delta_{ij}$, $B_i = B_{ij}$, $A'_i = \delta_{ik}$, $B'_i = B_{ik}$. By (4.9) we have that

$$0 = \sum_i (\delta_{ij} B_{ik} - \delta_{ik} B_{ij}) = (B_{jk} - B_{kj}).$$

The symmetry is proved. Next, we apply Corollary 4.16 to the differential $\sum_{j=1}^{g} x_j \omega_j$ where all the coefficients $x_1, \dots, x_g$ are real. We have that $A_k = x_k$, $B_k = \sum_j x_j B_{kj}$ which implies

$$\Im(\sum_k x_k \sum_j x_j \bar{B}_{kj}) = \sum_{k,j} \Im(\bar{B}_{kj}) x_k x_j < 0.$$

The lemma is proved. □

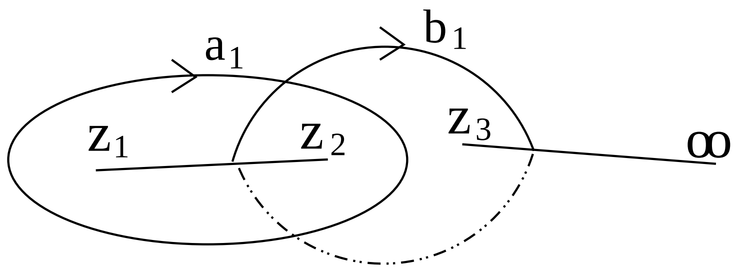


Figure 4.1: Homology basis.

Definition 4.20. *The matrix (B_{jk}) is called a period matrix of the Riemann surface Γ.*

Example 4.21. We consider a surface Γ of the form $w^2 = P_3(z)$ of genus $g = 1$ (an elliptic Riemann surface). Let $P_3(z) = (z - z_1)(z - z_2)(z - z_3)$ and choose a basis of cycles as shown in the figure 2.7. We have that

$$\omega_1 = \omega = \frac{adz}{\sqrt{P_3(z)}}, \quad a = \left(\oint_{\alpha_1} \frac{dz}{\sqrt{P_3(z)}} \right)^{-1}.$$

Note that

$$\oint_{\alpha_1} \frac{dz}{\sqrt{P_3(z)}} = 2 \int_{z_1}^{z_2} \frac{dz}{\sqrt{P_3(z)}}.$$

The period matrix is the single number

$$B = \oint_{\beta_1} \frac{adz}{\sqrt{P_3(z)}} = \frac{\int_{z_2}^{z_3} \frac{dz}{\sqrt{P_3(z)}}}{\int_{z_1}^{z_2} \frac{dz}{\sqrt{P_3(z)}}}, \quad \Im(B) > 0. \tag{4.17}$$

Example 4.22. . Consider a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z) = \prod_{i=1}^{2g+1}(z - z_i)$ for genus $g \geqslant 2$, and choose a basis of cycles as indicated in the figure 4.2 (there $g = 2$). A normal basis of holomorphic differentials has the form

$$\omega_j = \frac{\prod_{k=1}^{g} c_{jk} z^{k-l} dz}{\sqrt{P_{2g+1}(z)}}, \quad j = 1, \dots, g. \tag{4.18}$$

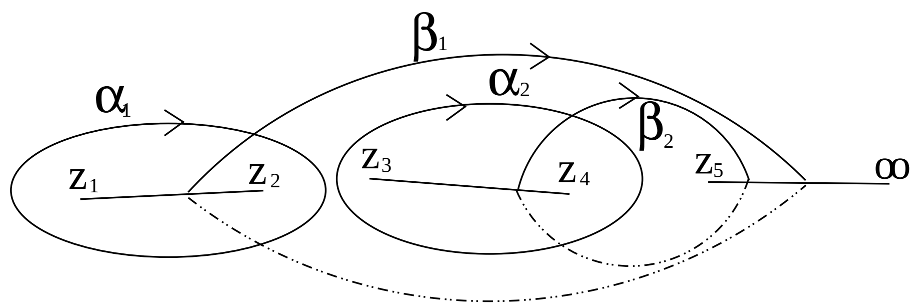


Figure 4.2: Homology basis.

Here (c_{jk}) is the matrix inverse to the matrix (A_{jk}) where

$$A_{jk} = 2\int_{z_{2j-1}}^{z_{2j}} \frac{z^{k-1}dz}{\sqrt{P_{2g+1}(z)}}, \quad j,k = 1,\ldots,g. \tag{4.19}$$

4.1.3 Meromorphic differentials, their residues and periods

Meromorphic (Abelian) differentials on a Riemann surface differ from holomorphic differentials by the possible presence of singularities of pole type. If a surface is given in the form $F(z,w) = 0$, then the Abelian differentials have the form $\omega = R(z,w)dz$ or, equivalently, $\omega = R_1(z,w)dw$, where $R(z,w)$ and $R_1(z,w)$ are rational functions. For example, on a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z)$ the differential $w^{-1}z^{k-1}dz$ has for $k > g$ a unique pole at infinity of multiplicity $2(k-g)$ (see Example 4.13). Suppose that the differential ω has a pole of multiplicity k at the point P_0 i.e., can be written in terms of a local parameter z, $z(P_0) = 0$, in the form

$$\omega = \left(\frac{c_{-k}}{z^k} + \cdots + \frac{c_{-1}}{z} + O(1)\right) dz \tag{4.20}$$

(the multiplicity of the pole does not depend on the choice of the local parameter z).

Definition 4.23. *The residue* $\mathrm{Res}_{P=P_0}\,\omega(P)$ *of the differential* ω *at a point* P_0 *is defined to be the coefficient* c_{-1}.

Lemma 4.24. *The residue* $\mathrm{Res}_{P=P_0}\,\omega(P)$ *does not depend on the choice of the local parameter* z.

Proof. This residue is equal to

$$c_{-1} = \frac{1}{2\pi i}\oint_C \omega$$

where C is an arbitrary small contour encircling P_0. The independence of this integral on the choice of the local parameter is obvious. The lemma is proved. □

Theorem 4.25 (The Residue Theorem)**.** . *The sum of the residues of a meromorphic differential ω on a Riemann surface, taken over all poles of this differential, is equal to zero.*

Proof. Let $P_1, \dots, P_N$ be the poles of ω. We encircle them by small contours $C_1, ..., C_N$ such that

$$\mathrm{Res}\omega = \frac{1}{2\pi i} \oint_{C_i} \omega, \quad i = 1, \dots, N,$$

(the contours C_i run in the positive direction), and cut out the domains bounded by $C_1, \dots, C_N$ from the surface Γ. This gives a domain Γ' with oriented boundary of the form $\partial\Gamma' = -C_1 - \cdots - C_N$ (the sign means reversal of orientation). The differential ω is holomorphic on Γ'. By Stokes' formula,

$$\sum_{j=1}^{N} \operatorname*{Res}_{P_j} \omega = \frac{1}{2\pi i} \sum_{j=1}^{N} \oint_{C_j} \omega = -\frac{1}{2\pi i} \oint_{\partial\Gamma'} \omega = -\frac{1}{2\pi i} \int\int_{\Gamma'} d\omega = 0,$$

since $d\omega = 0$. The theorem is proved. □

We present the simplest example of the use of the residue theorem: we prove that the number of zeros of a meromorphic function is equal to its number of poles (counting multiplicity). Let $P_1, \dots, P_k$, be the zeros of the meromorphic function f, with multiplicities $m_1, \dots, m_k$ a nd let $Q_1, ..., Q_l$ be the poles of this function, with multiplicities $n_1, \dots, n_k$. Consider the logarithmic differential $d(lnf)$. This is a meromorphic differential on Γ with simple poles at $P_1, \dots, P_k$ with residues $m_1, \dots, m_k$ and at the points $Q_1, \dots, Q_l$ with residues $-n_1, \dots, -n_l$. By the residue theorem: $m_1 + \cdots + m_k - n_1 - \cdots - n_k = 0$, which means that the assertion to be proved is valid. One more example. For any elliptic function $f(z)$ on the torus $T^2 = \mathbb{C}/\{2m\omega + 2n\omega'\}$ the residues at the poles are defined with respect to the complex coordinate z (in $\mathbb{C}$). These are the residues of the meromorphic differential $f(z)dz$, since dz is holomorphic everywhere. Conclusion: the sum of the residues of any elliptic function (over all poles in a lattice parallelogram) is equal to zero. We formulate an existence theorem for meromorphic differentials on a Riemann surface Γ (see [**?**] for a proof).

Theorem 4.26. *Suppose that $P_1, \dots, P_N$ are points of a Riemann surface Γ and $z_1, \dots, z_N$ are local parameters centered at these points, $z_i(P_i) = 0$, and the collection of principal parts is*

$$\left(\frac{c^{(i)}_{-k_i}}{z_i^{k_i}} + \cdots + \frac{c^{(i)}_{-1}}{z_i} \right) dz_i, \quad i = 1, \dots, N. \tag{4.21}$$

Assume the condition

$$\sum_{i=1}^{N} c_{-1}^{i} = 0. \tag{4.22}$$

Then there exists on Γ *a meromorphic differential with poles at the points* $P_1, \dots, P_N$*, and principal parts (4.21).*

Any meromorphic differential can be represented as the sum of a holomorphic differential and the following elementary meromorphic differentials.

1. Abelian differential of the second kind Ω_P^n has a unique pole of multiplicity $n+1$ at P and a principal part of the form

   $$\Omega_P^n = \left(\frac{1}{z^{n+1}} + O(1)\right) dz \tag{4.23}$$

   with respect to some local parameter z, $z(P) = 0$, $n = 1, 2, \dots$.

2. An Abelian differential of the third kind Ω_{PQ} has a pair of simple poles at the points P and Q with residues $+1$ and -1 respectively.

Example 4.27. We construct elementary Abelian differentials on a hyperelliptic Riemann surface $w^2 = P_{2g+1}(z)$. Suppose that a point P which is not a branch point takes the form $P = (a, w_a = \sqrt{P_{2g+1}(a)})$. An Abelian differential of the second kind $\Omega_P^{(1)}$ has the form

$$\Omega_P^{(1)} = \left(\frac{w + w_a}{(z-a)^2} - \frac{P'_{2g+1}(a)}{2w_a(z-a)}\right)\frac{dz}{2w} \tag{4.24}$$

(with respect to the local parameter z-a). The differentials $\Omega_P^{(n)}$ can be obtained as follows:

$$\Omega_P^n = \frac{1}{n!}\frac{d^{n-1}}{da^{n-1}}\Omega_P^1. \tag{4.25}$$

If $P = (z_i, 0)$ is one of the branch points, then

$$\Omega_P^n = \frac{dz}{2(z-z_i)^{k+1}} \quad \text{for} \quad n = 2k, \quad \Omega_P^n = \frac{dz}{2(z-z_i)^{k+1}w} \quad \text{for} \quad n = 2k+1. \tag{4.26}$$

Finally, if $P = \infty$, then

$$\Omega_P^{(n)} = -\frac{1}{2}z^{k-1}dz \quad \text{for} \quad n = 2k, \quad \Omega_P^n = -\frac{1}{2}z^{g+k-1}\frac{dz}{w} \quad \text{for} \quad n = 2k+1. \tag{4.27}$$

We now construct differentials of the third kind. Suppose that the point P and Q have the form $P = (a, w_a = \sqrt{P_{2g+1}(a)})$ and $Q = (b, w_b = \sqrt{P_{2g+1}(b)})$. Then

$$\Omega_{PQ} = \left(\frac{w + w_a}{z - a} - \frac{w + w_b}{z - b} \right) \frac{dz}{2w} \tag{4.28}$$

If $Q = +\infty$ then

$$\Omega_{PQ} = \frac{w + w_a}{z - a} \frac{dz}{2w}. \tag{4.29}$$

Accordingly, we see that for a hyperelliptic Riemann surface it is possible to represent all the Abelian differentials without appealing to Theorem 4.26.

Exercise 4.28: Deduce from Theorem 4.26 that a Riemann surface Γ of genus 0 is rational. *Hint.* Show that for any points $P, Q \in \Gamma$ the function $f = \exp \int \Omega_{PQ}$ is single valued and meromorphic on Γ and gives a biholomorphic isomorphism $f : \Gamma \to \mathbb{CP}^1$.

The period of a meromorphic differential ω along the cycle γ is defined if the cycle does not pass through poles of this differential. The period $\int_\gamma \omega$ depends only on the homology class of γ on the surface Γ, with the poles of ω with nonzero residue deleted. For example, the periods of the differential Ω_{PQ} of the third kind along a cycle not passing through the points P and Q are determined to within integer multiples of $2\pi i$. In speaking of the periods of meromorphic differentials we shall assume that the cycles do not pass through the poles of the differential, and we also recall that the dependence of the period on the homology class of Γ is not single-valued (for differentials of the third kind).

Lemma 4.29. *Suppose that the differentials Ω_1 and Ω_2 on a Riemann surface Γ have the same poles and principal parts, and the same periods with respect to the cycles $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$. Then these differentials coincide.*

Proof. The difference $\omega_1 - \omega_2$ is a holomorphic differential that has zero α-periods. Therefore, it is identically zero (see Lecture 4.1.2). The lemma is proved. □

Definition 4.30. *A meromorphic differential ω is said to be normalized with respect to a basis of cycles $\alpha_1, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ if it has zero α-periods.*

Any meromorphic differential ω can be turned into a normalized differential by adding a holomorphic differential $\sum_{k=1}^{g} c_k \omega_k$. Indeed the condition that $\Omega = \omega + \sum c_k \omega_k$ is normalised, namely

$$\int_{\alpha_j} \omega + \sum_{k=1}^{g} c_k \int_{\alpha_j} \omega_k = 0, \quad j = 1, \ldots, g,$$

defines the constants $c_1, \dots, c_g$ uniquely.

By Lemma 4.29, a normalized meromorphic differential is uniquely determined by its poles and by the principal parts at the poles. In what follows we assume that meromorphic differentials are normalized. We obtain formulas that will be useful for the β-periods of such differentials by arguments like those in the proof of Lemma 4.15.

Lemma 4.31. *The following formulas hold for the β-periods of normalized differentials $\Omega_P^{(n)}$ and Ω_{PQ}*

$$\oint_{\beta_k} \Omega_P^{(n)} = 2\pi i \frac{1}{n!} \frac{d^{n-1}}{dz^{n-1}} \psi_i(z)|_{z=0}, \;\; k = 1, \dots, g, \; n = 1, 2, \dots, \tag{4.30}$$

where z is a particular local parameter in a neighborhood of P, $z(P) = 0$, and the functions $\psi_k(z)$ are determined by the equality $\omega_k = \psi_k(z)dz$ and $\omega_1, \dots, \omega_g$ is a normalized basis of holomorphic differentials with respect to the canonical homology basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$,

$$\oint_{\beta_k} \Omega_{PQ} = 2\pi i \int_Q^P \omega_k, \;\; i = 1, \dots, g, \tag{4.31}$$

where the integration from Q to P in the last integral does not intersect the cycles $\alpha_1, \dots, \alpha_g$, $\beta_1, \dots, \beta_g$.

Proof. We encircle the point P with a small circle C oriented anti-clockwise; deleting the interior of this circle from the surface Γ, we get a domain Γ' with $\partial\Gamma' = -C$. Let us apply the arguments of Lemma 4.15 to the pair of differentials $\omega = \omega_k$, $\omega' = \Omega_P^{(n)}$. Denote by u_i the primitive

$$u_k(Q) = \int_{P_0}^{Q} \omega_k \tag{4.32}$$

which is single-valued on the Poincare′ polygon $\tilde{\Gamma}$ of the surface Γ. We have that

$$0 = \int\int_{\Gamma'} \omega \wedge \omega' = \int_{\partial\tilde{\Gamma}'} u_k \Omega_P^{(n)} = \sum_{j=1}^{g} (A_j B'_j - A'_j B_j) - \oint_C u_k \Omega_P^{(n)} \tag{4.33}$$

(the boundary $\partial\tilde{\Gamma}'$ differs from the boundary $\partial\tilde{\Gamma}$ by $(-C)$). Here the α and β-periods of ω_k and Ω_P^N have the form

$$A_j = \delta_{kj}, \;\; B_j = B_{kj}, \quad A'_j = 0, \;\; B'_j = \oint_{\beta_j} \Omega_P^{(n)}.$$

From this,

$$\oint_{\beta_k} \Omega_P^{(n)} = \oint_C u_k \Omega_P^{(n)} = 2\pi i \operatorname*{Res}_P (u_k \Omega_P^{(n)}) = 2\pi i \operatorname*{Res}_{z=0} \left[\left(\int_{P_0}^{P} + \int_0^z \psi_k(\tau) d\tau \right) \frac{dz}{z^{n+1}} \right] \tag{4.34}$$

Computation of the residue on the right-hand side of this equality leads to (4.30).
We now prove (4.31). Let C and C' small circles around P and Q respectively. Deleting the interior of this circles from the surface Γ, we get a domain Γ' with $\partial\Gamma' = -C - C'$. Let us apply the arguments of Lemma 4.15 to the pair of differentials $\omega = \omega_k$, $\omega' = \Omega_{PQ}$. Denote by u_i the primitive of ω_i. By analogy with (4.33) and (4.34) we have that

$$\oint_{\beta_k} \Omega_{PQ} = 2\pi i \oint_C u_k \Omega_{PQ} + 2\pi i \oint_{C'} u_k \Omega_{PQ}$$

Since the differential Ω_{PQ} has a simple pole in P and Q with residue ± 1 respectively, the above integrals are equal to

$$\oint_{\beta_k} \Omega_{PQ} = u_k(P) - u_k(Q) = \int_{P_0}^{P} \omega_k - \int_{P_0}^{Q} \omega_k = \int_{Q}^{P} \omega_k$$

where we assume that the point P_0 lies in the interior of Γ'. The lemma is proved. □

Exercise 4.32: Prove the following equality, which is valid for any quadruple of distinct points $P_1, \ldots, P_4$ on a Riemann surface:

$$\int_{P_2}^{P_1} \Omega_{P_3 P_4} = \int_{P_4}^{P_3} \Omega_{P_1 P_2}. \tag{4.35}$$

Exercise 4.33: Consider the series expansion of the differentials $\Omega_P^{(n)}$ in a neighborhood of the point P

$$\Omega_P^{(n)} = \left(\frac{1}{z^{n+1}} + \sum_{j=0}^{\infty} c_j^{(n)} z^j \right) dz. \tag{4.36}$$

Prove the following symmetry relations for the coefficients $c_j^{(k)}$:

$$k c_{j-1}^{(k)} = j c_{k-1}^{(j)}, \quad k, j = 1, 2 \ldots. \tag{4.37}$$

Exercise 4.34: Prove that a meromorphic differential of the second kind ω is uniquely determined by its poles, principal parts, and the real normalization condition

$$\Im \oint_{\gamma} \omega = 0 \tag{4.38}$$

for any cycle γ. Formulate and prove an analogous assertion for differentials of the third kind (with purely imaginary residues).

Elliptic curve and elliptic functions

Let's come back to the example 4.21 and consider the function ("elliptic integral")

$$u(P) = \int_{P_0}^{P} \omega_1, \tag{4.39}$$

which is single-valued and holomorphic on the surface $\tilde{\Gamma}$ which is obtained by cutting Γ along the cycles α_1 and β_1. This function is not single-valued on Γ. When the path of integration in the integral (4.39) is changed, the integral changes according to the law $u(P) \to u(P) + \int_\gamma \omega_i$ where γ is a closed contour (cycle). Decomposing it with respect to the basis of cycles, $\gamma = m\alpha_1 + n\beta_1$, m and n integers we rewrite the last formula in the form

$$u(P) \to u(P) + m + Bn, \quad \Im(B) > 0. \tag{4.40}$$

We define the two-dimensional torus T^2 as the quotient of the complex plane $\mathbb{C} = \mathbb{R}^2$ by the integer lattice generated by the vectors 1 and B,

$$T^2 = \mathbb{C}/\{2\pi i m + Bn \mid m, n \in \mathbb{Z}\} \tag{4.41}$$

(the vectors 1 and B are independent over $\mathbb{R}$ because $\Im(B) > 0$). The torus T^2 is a one-dimensional compact complex manifold. By (4.40) the function $u(P)$ unambiguously defines a mapping $\Gamma \to T^2$. It is holomorphic everywhere on Γ: $du = \omega$ and du vanishes nowhere (verify!). It is easy to see that this is an isomorphism. The meromorphic functions on the Riemann surface Γ are thereby identified with the so-called elliptic functions – the meromorphic functions on the torus T^2. The latter functions can be regarded as doubly periodic meromorphic functions of a complex variable. The absence of nonconstant holomorphic functions on Γ (see Lecture 3) leads to the well-known assertion that there are no nonconstant doubly periodic holomorphic functions. For comparison with the standard notation of the theory of elliptic functions we note that usually B is denoted with the letter τ and $\Im\tau > 0$. We give the construction of the mapping $T^2 \to \Gamma$ inverse to

(4.39). Let ω' and ω'' be two complex numbers linearly independent over the real numbers and consider the torus T^2 defined as

$$T^2 = \mathbb{C}/L, \quad L = \{2m\omega' + 2n\omega'' \mid m, n \in \mathbb{Z}\}. \tag{4.42}$$

The Weierstrass elliptic function, $\wp(u)$, $u \in \mathbb{C}$ is defined by

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega \in L \backslash \{0\}} \left[\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2} \right] \tag{4.43}$$

It is not hard to verify that the function $\wp(u)$ converges absolutely and uniformly on compact sets not containing nodes of the period lattice. Therefore, it defines a meromorphic function of u having double poles at the lattice nodes. Its derivative $\wp'(u)$ can be obtained by differentiating the series term by term (check!)

$$\wp'(u) = -2 \sum_{\omega \in L} \frac{1}{(u-\omega)^3}.$$

The function $\wp(u)$ is obviously doubly periodic: $\wp(u + 2m\omega' + 2n\omega'') = \wp(u)$, $m, n \in \mathbb{Z}$. The Laurent expansions of the functions $\wp(u)$ and $\wp'(u)$ have the following forms as $u \to 0$

$$\wp(u) = \frac{1}{u^2} + \frac{g_2 u^2}{20} + \frac{g_3 u^4}{28} + \dots, \tag{4.44}$$

$$\wp'(u) = -\frac{2}{u^3} + \frac{g_2 u}{10} + \frac{g_3 u^3}{7} + \dots, \tag{4.45}$$

where

$$\begin{aligned} g_2 &= 60 \sum_{\omega \in L \backslash \{0\}} \omega^{-4} \\ g_2 &= 140 \sum_{\omega \in L \backslash \{0\}} \omega^{-6}, \end{aligned} \tag{4.46}$$

(verify!). This gives us that the Laurent expansion of the function $(\wp')^2 - 4\wp^3 + g_2\wp + g_3$ has the form $O(u)$ as $z \to 0$. Hence, this doubly periodic function is constant, and thus equal to zero. Conclusion: the Weierstrass function $\wp(u)$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3. \tag{4.47}$$

Proposition 4.35. *The function $\wp : \mathbb{C} \backslash L \to \mathbb{C}$ is surjective. If*

$$\wp(u) = \wp(u_0), \quad \textit{then } u \in L \pm u_0. \tag{4.48}$$

Proof. For any $c \in \mathbb{C}$ consider the function $f(u) = \wp(u) - c$. This function is meromorphic with a double pole on the lattice points. Consider the parallelogram

$$\Pi := \{\xi + 2s\omega' + 2t\omega'', \quad s, t \in [0,1]\}.$$

Since the function f has only a double pole in Π, it has two zeros counting multiplicity. Let u_0 be one of the two zeros, $f(u_0) = \wp(u_0) - c = 0$. Since $\wp(-u) = \wp(u)$, it follows that $0 = f(-u_0) = \wp(-u_0) - c$ and this shows that the function $\wp(u)$ is surjective. From the above argument and the periodicity of $\wp$, it follows that for any $u \in L \pm u_0$, one has $\wp(u) = \wp(u_0)$. □

Let us now consider the curve

$$\Gamma_L := \{[X : Y : Z] \in \mathbb{P}^2 \,|\, ZY^2 = 4X^3 - g_2XZ^2 - g_3Z^3\} \tag{4.49}$$

Lemma 4.36. *The curve Γ_L is non singular.*

Proof. Consider the affine curve (4.47). By the periodicity properties of $\wp(u)$ one has

$$\wp'(u + 2\omega') = \wp'(u)$$

which is true in particular for $u = -\omega'$ so that $\wp'(\omega') = \wp'(-\omega')$. Since $\wp'(u)$ is odd it follows that

$$\wp'(\omega') = 0.$$

Repeating the same reasoning for ω'' one has

$$\wp'(\omega'') = 0, \quad \wp'(\omega'' + \omega') = 0.$$

Using (4.47) the zeros of the polynomial $4\wp^3(u) - g_2\wp(u) - g_3$ are given by $u = \omega'$, $u = \omega''$ and $u = \omega' + \omega''$ so that one has

$$4\wp^3(u) - g_2\wp(u) - g_3 = 4(\wp(u) - \wp(\omega'))(\wp(u) - \wp(\omega''))(\wp(u) - \wp(\omega' + \omega'')).$$

By proposition 4.35 the values $\wp(\omega')$, $\wp(\omega'')$ and $\wp(\omega' + \omega'')$ are distinct so that the curve (4.47) is non singular. □

The following theorem can be proved as an exercise

Theorem 4.37. *The map*

$$\phi : T^2 \to \Gamma_L$$

defined by

$$\phi(u + L) = \begin{cases} [\wp(u) : \wp'(u) : 1] & u \in \mathbb{C}\backslash L \\ [0 : 1 : 0] & u \in L, \end{cases} \tag{4.50}$$

is biholomorphic.

In particular the map (4.50) is the inverse of the map (4.39). We observe that from lemma 4.36 the discriminant $\Delta(\omega', \omega'')$ of the curve (4.47) is different from zero, namely

$$\Delta(\omega', \omega'') = g_2^3(\omega', \omega'') - 27g_3^2(\omega', \omega'') \neq 0$$

furthermore under the dilatation $\omega' \to \lambda\omega'$ and $\omega'' \to \lambda\omega''$ the discriminant scales as

$$\Delta(\lambda\omega', \lambda\omega'') = \frac{1}{\lambda^{12}}\Delta(\omega', \omega'').$$

In particular, choosing $\lambda = \frac{1}{2\omega'}$ and defining $\tau = \dfrac{2\omega''}{2\omega'}$, with $\Im(\omega''/\omega') > 0$, we obtain that $g_2 = g_2(\tau)$, and $g_3 = g_3(\tau)$, $\Delta = \Delta(\tau)$ with $\tau \in \mathbb{H}$, $\mathbb{H} := \{\tau \in \mathbb{C},\ \Im\tau > 0\}$. Regarding the Weierstrasse $\wp$ function it is easy to check that

$$\wp(\lambda u; \lambda\omega', \lambda\omega'') = \frac{1}{\lambda^2}\wp(u, ; \omega', \omega'')$$

so that choosing $\lambda = \frac{1}{2\omega'}$ one can consider the Weierstrasse function normalised as

$$\wp(\tilde{u}; \tau) = \frac{1}{\tilde{u}^2} + \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} \left[\frac{1}{(\tilde{u} - m - n\tau)^2} - \frac{1}{(m + n\tau)^2}\right], \quad \tilde{u} = \frac{u}{2\omega'}.$$

Exercise 4.38: Show that

$$\wp(\frac{\tilde{u}}{c\tau + d}; \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^2\wp(\tilde{u}; \tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{4.51}$$

Definition 4.39. *The Klein J function $J : \mathbb{H} \to \mathbb{C}$ is defined as*

$$J(\tau) = 1728\frac{g_2(\tau)^3}{\Delta(\tau)}. \tag{4.52}$$

The Klein J function is an an analytic function from $\mathbb{H}$ to $\mathbb{C}$. The choice of the number 1728 is due to the fact that defining $q = e^{2\pi i\tau}$ the expansion of J as $q \to 0$ takes the form

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots$$

namely all the coefficients of the expansion are integers. We consider the action of the modular group

$$PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{I, -I\}$$

namely the set of 2×2 matrices with integer entries and determinant equal to one where the matrices A and $-A$ are identified. Such group has two generators

$$\tau \to \tau + 1, \quad \tau \to -\frac{1}{\tau}.$$

In order to determine isomorphism classes of elliptic curves given by 4.49), the following lemma and theorem will be usefull.

Lemma 4.40. *Let τ and $\tau' \in \mathbb{H}$. Then*

$$J(\tau') = J(\tau).$$

if and only if

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{Z}). \tag{4.53}$$

Proof. From the definition one has

$$g_2(\tau') = 60 \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \left(\frac{1}{m + n\frac{a\tau + b}{c\tau + d}} \right)^4 = 60(c\tau + d)^4 \sum_{m',n' \in \mathbb{Z}, (m',n') \neq (0,0)} \frac{1}{(m' + n'\tau)^4}$$
$$= (c\tau + d)^4 g_2(\tau).$$

In the same way we obtain

$$g_3(\tau') = (c\tau + d)^6 g_3(\tau)$$

so that

$$J(\tau') = 1728 \frac{g_2^3(\tau')}{g_2^3(\tau') - 27g_3^2(\tau')} = 1728 \frac{(c\tau + d)^{12} g_2^3(\tau)}{(c\tau + d)^{12}(g_2^3(\tau) - 27g_3^2(\tau)} = J(\tau).$$

Viceversa, let us assume that $J(\tau) = J(\tau') = \mu$. Suppose $\mu \neq 0$ and $\mu \neq 1728$. Then

$$\mu - 1728 = 1728 \frac{27g_3^2(\tau)}{\Delta(\tau)} = 1728 \frac{27g_3^2(\tau')}{\Delta(\tau')}$$

so that

$$\frac{\mu}{\mu - 1728} = \frac{27g_3^2(\tau')}{g_2^3(\tau')} = \frac{27g_3^2(\tau)}{g_2^3(\tau)}$$

which shows that

$$\left(\frac{g_3(\tau)}{g_3(\tau')}\right)^2 = \left(\frac{g_2(\tau)}{g_2(\tau')}\right)^3.$$

Defining $\sigma^2 := \dfrac{g_2(\tau)}{g_2(\tau')}\dfrac{g_3(\tau')}{g_3(\tau)}$, it is straightforward to obtain the identity

$$\sigma^4 = \left(\frac{g_2(\tau)}{g_2(\tau')}\frac{g_3(\tau')}{g_3(\tau)}\right)^2 = \frac{g_2(\tau')}{g_2(\tau)}$$

and

$$\sigma^6 = \frac{g_3(\tau')}{g_3(\tau)}.$$

Therefore the curves defined by $w^2 = 4z^3 - g_2(\tau)z - g_3(\tau)$ and $y^2 = 4x^3 - g_2(\tau')x - g_3(\tau')$ are isomorphic. Indeed the dilatation

$$x = z\sigma^2, \quad y = w\sigma^3$$

maps one curve into the other one. Therefore the two tori defined by the above two curves are isomorphic. By theorem 1.43 it follows that their corresponding periods τ and τ' are related by a modular transformation (4.53). In the case $\mu = 1728$ one has $g_3(\tau) = g_3(\tau') = 0$. In this case defining σ in such a way that $\sigma^4 = \dfrac{g_2(\tau')}{g_2(\tau)}$ one can prove the statement in a similar way. For the case $\mu = 0$ one has $g_2(\tau) = g_2(\tau') = 0$. In this case defining σ in such a way that $\sigma^6 = \dfrac{g_3(\tau')}{g_3(\tau)}$ one can prove the statement in a similar way. $\square$

The above lemma shows that the Klein J function is a modular function of weight zero. We recall that an analytic function $f : \mathbb{H} \to \mathbb{C}$ is a modular function of weight k with respect to the modular group $PSL(2,\mathbb{Z})$ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}).$$

Remark 4.41. The upper half space $\mathbb{H}$ can be naturally identified with the Teichmüller space $T(1,0)$ of compact surfaces of genus one. The quotient $\mathbb{H}/PSL(2,\mathbb{Z})$ is the moduli space of Riemann surfaces of genus one. Below we will see that this moduli space can be identified with $\bar{\mathbb{C}}$ with three marked points modulo the permutation group S_3.

Combining theorem 1.43 and lemma 4.40 we conclude that

Theorem 4.42. *Given two lattices* $L = \{n + m\tau, \quad m, n \in \mathbb{Z}\}$ *and* $L' = \{n + m\tau', \quad m, n \in \mathbb{Z}\}$ *with* $\tau, \tau' \in \mathbb{H}$, *the tori*

$$\mathbb{C}/L, \quad \mathbb{C}/L'$$

are isomorphic if and only if

$$J(\tau) = J(\tau').$$

Doing some algebra we can express the Klein J invariant using the branch points $\wp(\tau/2)$, $\wp(1/2)$ and $\wp(\frac{1+\tau}{2})$ of the elliptic curve (4.47). For simplicity we define

$$e_1 = \wp(\tau/2), \quad e_2 = \wp(1/2), \quad e_3 = \wp(\frac{1+\tau}{2}). \tag{4.54}$$

It is easy to check that

$$\Delta = 16(e_2 - e_1)^2(e_3 - e_1)^2(e_3 - e_2)^2, \quad g_2 = \frac{4}{3}\left((e_2 - e_1)^2 - (e_3 - e_1)(e_2 - e_1) + (e_3 - e_1)^2\right)$$

so that $J(\tau)$ can be written in the form

$$J(\tau) = 256 \frac{\left(1 - \frac{e_3 - e_1}{e_2 - e_1} + \frac{(e_3 - e_1)^2}{(e_2 - e_1)^2}\right)^3}{\frac{(e_3 - e_1)^2}{(e_2 - e_1)^2}\frac{(e_3 - e_2)^2}{(e_2 - e_1)^2}}. \tag{4.55}$$

Introducing the function $\lambda : \mathbb{H} \to \mathbb{C}\backslash\{0, 1\}$

$$\lambda = \frac{e_3 - e_1}{e_2 - e_1} = \frac{\wp(\frac{1+\tau}{2}) - \wp(\tau/2)}{\wp(1/2) - \wp(\tau/2)} \tag{4.56}$$

and the function $j : \mathbb{C}\backslash\{0, 1\} \to \mathbb{C}$ defined as

$$j(\lambda) = 256 \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2} \tag{4.57}$$

it follows that the Klein J invariant is the composition of the maps

$$J = j \circ \lambda.$$

Remark 4.43. Since the function J as defined in (4.52) is invariant under the action of the permutation group S_3 on e_1, e_2 and e_3, such invariance must be preserved for the function $j(\lambda)$. Indeed one has the following relations between the action of S_3 on e_1, e_2 and e_3 and transformations of λ:

$$123 \to 213 \text{ then } \lambda \to 1-\lambda, \quad 123 \to 321 \text{ then } \lambda \to \frac{\lambda}{1-\lambda}, \quad 123 \to 132 \text{ then } \lambda \to \frac{1}{\lambda}$$
$$123 \to 231 \text{ then } \lambda \to \frac{1}{1-\lambda}, \quad 123 \to 312 \text{ then } \lambda \to 1-\frac{1}{\lambda}$$

and the function $j(\lambda)$ is invariant under the above five transformations of λ (six including the identity).

The curve $w^2 = 4(z-e_1)(z-e_2)(z-e_3)$ is mapped under the linear transformation

$$x = \frac{z-e_1}{e_2-e_1}, \quad y = \frac{w}{2(e_2-e_1)^{\frac{3}{2}}}$$

to the curve

$$y^2 = x(x-1)(x-\lambda).$$

So using the j-invariant (4.57), we have the following corollary.

Corollary 4.44. *Two curves $y^2 = x(x-1)(x-\lambda)$ and $y^2 = x(x-1)(x-\lambda')$ are isomorphic if and only $j(\lambda) = j(\lambda')$.*

We will see later that any Riemann surface of genus one can be realised as a double covering of the sphere branched over four points e_1, e_2, e_3 and ∞. We can use a linear transformation to map the points e_1, e_2 and e_3 to $0, 1$ and λ respectively. Any other linear transformation obtained from the permutation of the points e_1, e_2 and e_3 will give an isomorphic Riemann surface. So we can identify the moduli space of genus one Riemann surface as the quotient $(\mathbb{C}\backslash\{0,1\})/S_3$. In remark (4.41) we identify the moduli space of Riemann surfaces of genus one with $H/PSL(2,\mathbb{Z})$. Below we are going to sketch an argument which shows that the spaces

$$(\mathbb{C}\backslash\{0,1\})/S_3 \quad \text{and} \quad H/PSL(2,\mathbb{Z})$$

are isomorphic.

Lemma 4.45. *The map $\lambda : \mathbb{H} \to \mathbb{C}\backslash\{0,1\}$ is a universal covering of $\mathbb{C}\backslash\{0,1\}$. This map is invariant under the action of the subgroup $\Gamma_2 \subset PSL(2,\mathbb{Z})$*

$$\Gamma_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) \,|\, a \equiv d \equiv 1 \ (mod\, 2), \ \ b \equiv c \equiv 0 \ (mod\, 2) \right\}.$$

Proof. Let us consider $\lambda(\tau')$ and use the relation (4.51)

$$\lambda(\tau') = \frac{\wp(\frac{\tau'+1}{2};\tau') - \wp(\frac{\tau'}{2};\tau')}{\wp(\frac{1}{2};\tau') - \wp(\frac{\tau'}{2};\tau')} = \frac{\wp(\frac{1}{2}(b+d+(a+c)\tau);\tau) - \wp(\frac{1}{2}(a\tau+b);\tau)}{\wp(\frac{d+c\tau}{2};\tau) - \wp(\frac{1}{2}(a\tau+b);\tau)}.$$

It is straightforward to check that $\lambda(\tau') = \lambda(\tau)$ if and only if the modular transformation belongs to Γ_2. □

Remark 4.46. The group Γ_2 is the group of deck transformations of the covering $\lambda : \mathbb{H} \to \mathbb{C}\backslash\{0,1\}$, namely the set of homeomorphism $f : \mathbb{H} \to \mathbb{H}$ preserving the fibers of the covering. Such group is isomorphic to the fundamental group of $\mathbb{C}\backslash\{0,1\}$ and therefore [14]

$$\mathbb{H}/\Gamma_2 \simeq \mathbb{C}\backslash\{0,1\}.$$

Furthermore, the following identity is satisfied [12] $PSL(2,\mathbb{Z})/\Gamma_2 \simeq S_3$. Namely the quotient of the modular group under the group Γ_2 is isomorphic to the group of permutation S_3. The above identity and the lemma 4.45 explain the identification of the spaces $(\mathbb{C}\backslash\{0,1\})/S_3$ and $H/PSL(2,\mathbb{Z})$.

Exercise 4.47: Prove that any elliptic function with period lattice $\{2m\omega'' + 2n\omega'\}$ can be represented as a rational function of $\wp(z)$ and $\wp'(z)$.

Exercise 4.48: Show that if τ is pure imaginary then the branch points e_1, e_2 and e_3 are real.

Exercise 4.49: Consider the curve

$$\Gamma := \{(z,w) \in \mathbb{P}^2 \mid w^2 = z(z-1)(z-\lambda)\}$$

with $0 \leqslant \lambda \leqslant 1$ and consider the lattice $L = \{2m\omega' + 2n\omega'', \quad m,n \in \mathbb{Z}\}$ where

$$\int_\infty^0 \frac{dz}{w} = L + \omega'', \quad \int_\infty^1 \frac{dz}{w} = L + \omega' + \omega'', \quad \int_\infty^\lambda \frac{dz}{w} = L + \omega'.$$

Show that the curve Γ is isomorphic to the curve $w^2 = 4z^3 - g_2 z - g_3$ where g_2 and g_3 are defined in (4.46).

Exercise 4.50: Consider the Korteweg-de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx} \tag{4.58}$$

(here $u = u(x,t)$, and u_t stands for the derivative with respect to t, and u_x for derivative with respect to x. Show that any (complex) periodic solution of it with the form of a traveling wave has the form

$$u(x,t) = u(x-ct) = 2\wp(x-ct-x_0) - \frac{c}{6}, \tag{4.59}$$

where the Weierstrass function $\wp$ corresponds to some elliptic curve (4.49), and the velocity c and the phase x_0 are arbitrary.

Exercise 4.51: (see [8]). Look for a solution of the KdV equation in the form

$$u(x,t) = 2\wp(x-x_1(t)) + 2\wp(x-x_2(t)) + 2\wp(x-x_3(t)). \tag{4.60}$$

Derive for the functions $x_j(t)$ the system of differential equations

$$\ddot{x}_j = 12\sum_{k\neq j}\wp(x_j - x_k), \quad j = 1,2,3, \tag{4.61}$$

and its integrals

$$\sum_{k\neq j}\wp'(x_j - x_k) = 0, \quad j = 1,2,3. \tag{4.62}$$

Integrate this system in quadratures.

We define the Weierstrass ζ and σ functions (which are useful in the theory of elliptic functions) from the conditions

$$\zeta'(z) = -\wp(z), \quad \frac{\sigma'(z)}{\sigma(z)} = \zeta(z). \tag{4.63}$$

The series expansion of $\zeta(z)$ has the form

$$\zeta(z) = \frac{1}{z} + \sum_{\omega\in L\setminus\{0\}}\left[\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right]. \tag{4.64}$$

This function has simple poles at the nodes of the period lattice. The function $\sigma(z)$ is entire. It has simple zeros at the nodes of the period lattice and can be expanded in the infinite product

$$\sigma(z) = z\prod_{\omega\in L\setminus\{0\}}\left\{\left(1-\frac{z}{\omega}\right)\exp\left[\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right]\right\} \tag{4.65}$$

The functions $\zeta(z)$ and $\sigma(z)$ are not elliptic; under a translation of the argument by a vector of the period lattice they transform according to the law

$$\zeta(z+2m\omega'+2n\omega'') = \zeta(z)+2m\eta+2n\eta', \quad \eta=\zeta(\omega'),\ \eta'=\zeta(\omega''), \tag{4.66}$$
$$\sigma(z+2\omega') = \sigma(z)\exp[2\eta(z+\omega')],\ \ \sigma(z+2\omega'') = -\sigma(z)\exp[2\eta'(z+\omega'')] \tag{4.67}$$

where η and η' are constants depending on the period lattice.

Exercise 4.52: Prove the following identity:

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} = \wp(u)-\wp(v). \tag{4.68}$$

Other properties of the functions,$\wp$, ζ and σ and of other elliptic functions as well, can be found, for example, in the texts [2] and [7], or in the handbook [3].

4.1.4 The Jacobi variety, Abel's theorem

Let $e_1,\dots,e_g$ be the standard basis in the space $\mathbb{C}^g$, $e_j = (0,\dots,1,\dots,0)$, with one on the j-entry. Given $2g$ row vectors $\lambda_k \in \mathbb{C}^g$, $k = 1,\dots,2g$, with $\lambda_k = \sum_{j=1}^{g} \lambda_{kj} e_j$, we construct the $2g\times g$ matrix Λ having in the k-row the vector λ_k

$$\Lambda_{kj} = (\lambda_k)_j. \tag{4.69}$$

The matrix Λ generates a lattice in $\mathbb{C}^g$ of maximal rank, if its row vectors are linearly independent over the real numbers.

Consider in $\mathbb{C}^g$ the integer period lattice L generated by the vectors (4.69). The vectors in this lattice can be written in the form

$$L = \{v\in\mathbb{C}^g \mid v = \sum_{k=1}^{2g} m_k\lambda_k, \quad (m_1,\dots,m_{2g})\in\mathbb{Z}^{2g}\} \tag{4.70}$$

We assume that L generates a lattice of maximal rank in $\mathbb{C}^g$. Then the quotient of $\mathbb{C}^g$ by this lattice is the $2g$-dimensional torus

$$T^{2g} = \mathbb{C}^g/L \tag{4.71}$$

namely a g-dimensional complex manifold. Changing the basis in $\mathbb{C}^g$, namely $e_k \to e_k M$, with $M\in GL(g,\mathbb{C})$, the matrix $\Lambda \to \Lambda M$. Furthermore, the same lattice is given by vectors $(\tilde\lambda_1,\dots,\tilde\lambda_{2g})$ with

$$\tilde\lambda_k = \sum_{k=1}^{2g} n_{kj}\lambda_j$$

with $N = \{n_{kj}\}_{k,j=1}^{2g} \in SL(2g, \mathbb{Z})$. Therefore $\Lambda \to N\Lambda$. Summarizing, two matrices Λ and $\tilde{\Lambda}$ represent the same torus if

$$\tilde{\Lambda} = N\Lambda M, \quad M \in GL(g, \mathbb{C}), \ \ N \in SL(2g, \mathbb{Z}). \tag{4.72}$$

If we assume that the lattice generated by Λ has maximal rank, we can always choose Λ in such a way that $\Lambda = (2\pi i\Lambda_1, \Lambda_2)$ with $\Lambda_1 \in GL(g, \mathbb{C})$. Therefore, by (4.72) the two matrices Λ and $\Lambda\Lambda_1^{-1} = (2\pi i I_g, \Lambda_2\Lambda_1^{-1})$ with I_g the g-dimensional identity, represent the same torus.

Lemma 4.53. *The matrices*

$$\Lambda = (I_g, \Lambda_2), \quad \tilde{\Lambda} = (I_g, \tilde{\Lambda}_2)$$

represent the same torus if

$$\tilde{\Lambda}_2 = (cI_g + d\Lambda_2)(aI_g + b\Lambda_2)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2g, \mathbb{Z})$$

with a, b, c, d $g \times g$ matrices.

The proof of the lemma follows immediately from (4.72).

Let $B = (B_{jk})$ be an arbitrary symmetric $g \times g$ matrix with positive-definite imaginary part (as shown in Lecture 4.1.2, the period matrices of Riemann surfaces have this property). We consider the vectors

$$e_1, \ldots, e_g, \quad e_1B, \ldots, e_gB. \tag{4.73}$$

Lemma 4.54. *The vectors (4.73) are linearly independent over $\mathbb{R}$.*

Proof. Assume that these vectors are dependent over $\mathbb{R}$:

$$(\rho_1e_1 + \cdots + \rho_ge_g) + (\mu_1e_1 + \cdots + \mu_ge_g)B = 0, \ \ \rho_i, \ \mu_j \in \mathbb{R}.$$

Separating out the real part of this equality we get that $\Im((\mu_1e_1 + \cdots + \mu_ge_g)B) = 0$. But the matrix $\Im(B)$ is nonsingular, which implies $\mu_1 = \cdots = \mu_g = 0$. Hence also $\rho_1 = \cdots = \rho_g = 0$. The lemma is proved. □

Consider in $\mathbb{C}^g$ the integer period lattice generated by the vectors (4.73). The vectors in this lattice can be written in the form

$$m + nB, \quad m, n \in \mathbb{Z}^g. \tag{4.74}$$

By Lemma 4.54 the quotient of $\mathbb{C}^g$ by this lattice is a torus of maximal rank:

$$T^{2g} = T^{2g}(B) = \mathbb{C}^g/\{m + nB\}. \tag{4.75}$$

Definition 4.55. *Suppose that $B = (B_{jk})$ is a period matrix of a Riemann surface Γ of genus g. The torus $T^{2g}(B)$ in (4.75), constructed from this period matrix is called the Jacobi variety (or Jacobian) of the surface Γ and denoted by $J(\Gamma)$.*

Remark 4.56. What happens with the torus $J(\Gamma)$ when the canonical basis of cycles on Γ changes? Let $\alpha = (\alpha_1, \ldots, \alpha_g)^t$ and $\beta = (\beta_1, \ldots, \beta_g)^t$ be the column vectors of the canonical homology basis. Let α' and β' be a new canonical homology basis related to α and β by the symplectic transformation

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

Let $\omega = (\omega_1, \ldots, \omega_g)$ be the canonical homology basis of holomorphic differentials with respect to the basis α and β, namely

$$\int_\alpha \omega = I_g, \quad \int_\beta \omega = B$$

where I_g is the g dimensional identity matrix. Then

$$\int_{\alpha'} \omega = \int_{a\alpha + b\beta} \omega = aI_g + bB,$$

$$\int_{\beta'} \omega = \int_{c\alpha + d\beta} \omega = cI_g + dB.$$

So the canonical basis of holomorphic differentials $\omega' = (\omega'_1, \ldots, \omega'_g)$ with respect to the basis α' and β' is given by

$$\omega' = \omega(aI_g + bB)^{-1}$$

This implies that the corresponding period matrix

$$B' = \int_{\beta'} \omega' = (cI_g + dB)(aI_g + bB)^{-1}. \tag{4.76}$$

From lemma 4.53 it follows that the tori $T^{2g}(B)$ and $T^{2g}(B')$ are isomorphic. Accordingly, the Jacobian $J(\Gamma)$ changes up to isomorphism when the canonical basis changes.

We consider the primitives ("Abelian integrals") of the basis of holomorphic differentials:

$$u_k(P) = \int_{P_0}^{P} \omega_k, \quad k = 1, \ldots, g, \tag{4.77}$$

where P_0 is a fixed point of the Riemann surface. The vector-valued function

$$\mathcal{A}(P) = (u_1(P), \dots, u_g(P)) \tag{4.78}$$

is called the Abel mapping (the path of integration is chosen to be the same in all the integrals $u_1(P), \dots, u_g(P)$).

Lemma 4.57. *The Abel mapping is a well-defined holomorphic mapping*

$$\Gamma \to J(\Gamma). \tag{4.79}$$

Proof. (cf. Example 4.27). A change of the path of integration in the integrals (4.77) leads to a change in the values of these integrals according to the law

$$u_k(P) \to u_k(P) + \oint_\gamma \omega_k, \quad k = 1, \dots, g,$$

where γ is some cycle on Γ. Decomposing it with respect to the basis of cycles, $\gamma \simeq \sum m_j a_j + \sum n_j b_j$ we get that

$$u_k(P) \to u_k(P) + m_k + \sum_j B_{kj} n_j, \quad k = 1, \dots, g.$$

The increment on the right-hand side is the kth coordinate of the period lattice vector $2\pi i M + NB$ where $M = (m_1, \dots, m_g)$, $N = (n_1, \dots, n_g)$. The lemma is proved. □

The Jacobi variety together with the Abel mapping (4.79) is used for solving the following problem: what points of a Riemann surface can be the zeros and poles of meromorphic functions? We have the Abel's theorem.

Theorem 4.58 (Abel's Theorem)**.** *The points $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ (some of the points can repeat) on a Riemann surface Γ are the respective zeros and poles of some function meromorphic on Γ if and only if the following relation holds on the Jacobian:*

$$\mathcal{A}(P_1) + \cdots + \mathcal{A}(P_n) \equiv \mathcal{A}(Q_1) + \cdots + \mathcal{A}(Q_n). \tag{4.80}$$

Here and below, the sign $\equiv$ will mean equality on the Jacobi variety (congruence modulo the period lattice (4.74)). We remark that the relation (4.80) does not depend on the choice of the initial point P_0 of the Abel map (4.77).

Proof. 1) Necessity. Suppose that a meromorphic function f has the respective points $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ as zeros and poles, where each zero and pole is written the number of times corresponding to its multiplicity. Consider the logarithmic differential

$\Omega = d(\log f)$. Since $f = \mathrm{const}\exp\int_{P_0}^{P}\Omega$, is a meromorphic function, the integral in the exponent does not depend on the path of integratio. It follows that all the periods of this differential Ω are integer multiples of $2\pi i$. On the other hand, we represent it in the form

$$\Omega = \sum_{j=1}^{n} \Omega_{P_jQ_j} + \sum_{s=1}^{g} c_s\omega_s, \tag{4.81}$$

where $\Omega_{P_jQ_j}$ are normalized differentials of the third kind (see Lecture 4.1.3) and $c_1, \dots, c_g$ are constant coefficients. Let us use the information about the periods of the differential. We have that

$$2\pi i n_k = \oint_{a_k} \Omega = c_k, \quad n_k \in \mathbb{Z},$$

which gives us $c_k = 2\pi i n_k$. Further,

$$2\pi i m_k = \oint_{b_k} \Omega = 2\pi i \sum_{j=1}^{n} \int_{Q_j}^{P_j} \omega_k + 2\pi i \sum_{s=1}^{g} n_s B_{sk}$$

(we used the formula (4.31)). From this,

$$u_k(P_1) + \cdots + u_k(P_n) - u_k(Q_1) - \cdots - u_k(Q_n) = \sum_{j=1}^{n} \int_{Q_j}^{P_j} \omega_k = m_k - \sum_{s=1}^{g} n_s B_{sk}. \tag{4.82}$$

The right-hand side is the kth coordinate of the vector $m + nB$ of the period lattice (4.74), where $m = (m_1, \dots, m_g)$, $n = (n_1, \dots, n_g)$. The necessity of the condition (4.80) is proved.

2) Sufficiency. Suppose that

$$u_k(P_1) + \cdots + u_k(P_n) - u_k(Q_1) - \cdots - u_k(Q_n) = m_k - \sum_{s=1}^{g} n_s B_{sk}. \tag{4.83}$$

Consider the function

$$f(P) = \exp\left[\sum_{j=1}^{g} \int_{P_0}^{P} \Omega_{P_jQ_j} + \sum_{j=1}^{g} c_j \int_{P_0}^{P} \omega_j\right]$$

where $\Omega_{P_jQ_j}$ are the normalised third kind differentials with poles in P_j and Q_j and c_j are constants. The function is a single valued meromorphic function if the integrals in the exponent do not depend on the path of integration. Let us study the behaviour of f when $P \to P + \alpha_k$:

$$f(P) \to f(P) \exp\left[\sum_{j=1}^{g} c_j \int_{\alpha_k} \omega_j\right].$$

In order to have a single valued function the constant $c_k = 2\pi n_k$, $n_k \in \mathbb{N}$. Next let us consider the behaviour of f when $P \to P + \beta_k$:

$$f(P) \to f(P) \exp\left[\sum_{j=1}^{g} \int_{\beta_k} \Omega_{P_j Q_j} + \sum_{j=1}^{g} n_j \int_{\beta_k} \omega_j\right] = f(P) \exp\left[\sum_{j=1}^{g} \int_{Q_j}^{P_j} \omega_k + 2\pi i \sum_{j=1}^{g} n_j \int_{\beta_k} \omega_j\right]$$

Using the relation (4.83) one obtains that $f(P) \to f(P) \exp[2\pi i m_k] = f(P)$ which shows that $f(P)$ is a meromorphic function on Γ. □

Example 4.59. We consider the elliptic curve

$$w^2 = 4z^3 - g_2 z - g_3. \tag{4.84}$$

For this curve the Jacobi variety $J(\Gamma)$ is a two-dimensional torus, and the Abel mapping (which coincides with (4.39)) is an isomorphism (see Example 4.21). Abel's theorem becomes the following assertion from the theory of elliptic functions: the sum of all the zeros of an elliptic function is equal to the sum of all its poles to within a vector of the period lattice.

Example 4.60. (also from the theory of elliptic functions). Consider an the elliptic function of the form $f(z, w) = az + bw + c$, where a, b, and c are constants. It has a pole of third order at infinity (for $b \neq 0$). Consequently, it has three zeros P_1, P_2, and P_3. In other words, the line $az + bw + c = 0$ intersects the elliptic curve (4.84) in three points (see the figure). We choose ∞ as the initial point for the Abel mapping, i.e., $u(\infty) = 0$. Let $u_i = u(P_i)$, $i = 1, 2, 3$. In other words,

$$P_i = (\wp(u_i), \wp'(u_i)), \quad i = 1, 2, 3,$$

where $\wp(u)$ is the Weierstrass function corresponding to the curve (4.84). Applying Abel's theorem to the zeros and poles of f, we get that

$$u_1 + u_2 + u_3 = 0.$$

Conversely, according to the same theorem, if $u_1 + u_2 + u_3 = 0$, i.e. $u_3 = -u_2 - u_1$ then the points P_1, P_2 and P_3 lie on a single line. Writing the condition of collinearity of these points and taking into account the evenness of $\wp$ and oddness of $\wp'$, we get the addition theorem for Weierstrass functions:

$$\det \begin{vmatrix} 1 & \wp(u_1) & \wp'(u_1) \\ 1 & \wp(u_2) & \wp'(u_2) \\ 1 & \wp(u_1+u_2) & -\wp'(u_1+u_2) \end{vmatrix} = 0. \tag{4.85}$$

4.1.5 Divisors on a Riemann surface. The canonical class. The Riemann-Roch theorem

Definition 4.61. *A divisor D on a Riemann surface is defined to be a (formal) integral linear combination of points on it:*

$$D = \sum_{i=1}^{n} n_i P_i, \quad P_i \in \Gamma, \quad n_i \in \mathbb{Z}. \tag{4.86}$$

For example, for any meromorphic function f the divisor (f) of its zeros $P_1, \dots, P_k$ and poles $Q_1, \dots, Q_l$ of multiplicities $m_1, \dots, m_k$, and $n_1, \dots, n_l$, respectively is defined

$$(f) = m_1 P_1 + \dots + m_k P_k - n_1 Q_1 - \dots - n_l Q_l. \tag{4.87}$$

Observe that given f and g two meromorphic functions

$$(fg) = (f) + (g), \quad (f/g) = (f) - (g).$$

Definition 4.62. *Divisors of meromorphic functions are also called principal divisors.*

Another useful notation for the divisor of a meromoprhic function is given by

$$(f) = \sum_P \mathrm{ord}_P(f) \cdot P$$

where we recall that the order of f in P is the minimum coefficient present in the Laurent expansion in a neighbourhood of the point P namely $\mathrm{ord}_P f = \min_{n \in \mathbb{Z}} \{n, |\alpha_n \neq 0\}$ where the Laurent expansion of f in P is $\sum_n \alpha_n z^n$. Such definition does not depend on the choice of the local coordinates. The set of all divisors on Γ, $\mathrm{Div}(\Gamma)$, obviously form an Abelian group (the zero is the empty divisor).

Definition 4.63. *The degree* $\deg D$ *of a divisor of the form (4.86) is defined to be the number*

$$\deg D = \sum_{i=1}^{N} n_i. \tag{4.88}$$

The degree is a linear function on the group of divisors. For instance,

$$\deg(f) = 0. \tag{4.89}$$

Two divisors D and D' are said to be linearly equivalent, $D \simeq D'$ if their difference is a principal divisor. Linearly equivalent divisors have the same degree in view of (4.89). For example, on $\mathbb{CP}^1$ any divisor of zero degree is principal, and two divisors of the same degree are always linearly equivalent.

Example 4.64. The divisor (ω) of any Abelian differential ω on a Riemann surface Γ is well-defined by analogy with (4.87). If ω' is another Abelian differential, then $(\omega) \simeq (\omega')$. Indeed, their ratio $f = \omega/\omega'$ is a meromorphic function on Γ, and $(\omega) - (\omega') = (f)$. We remark that any differential in a coordinate chart $\phi_\alpha : U_\alpha \to V_\alpha$, with $\phi_\alpha(P) = z_\alpha$ take the form

$$\omega = h_\alpha(z_\alpha)dz_\alpha, \quad \omega' = h'_\alpha(z_\alpha)dz_\alpha$$

where h_α and h'_α are meromorphic functions. The ratio $g_\alpha = h_\alpha/h'_\alpha$ is a meromorphic function of V_α. Now define $f := g_\alpha \circ \phi_\alpha$ which is a meromorphic function on U_α. It is easy to check that f is well defined and independent from the coordinate chart.

Definition 4.65. *The linear equivalence class of divisors of Abelian differentials is called the canonical class of the Riemann surface. We denote it by K_Γ.*

For example, the divisor $-2\infty = (dz)$ can be taken as a representative of the canonical class $K_{\mathbb{CP}^1}$.

We reformulate Abel's theorem in the language of divisors. Note that the Abel map extends linearly to the whole group of divisors. Abel's theorem obviously means that a divisor D is principal if and only if the following two conditions hold:

1. $\deg D = 0$;
2. $\mathcal{A}(D) \equiv 0$ on $J(\Gamma)$,

where

$$\mathcal{A}(D) = \sum_{j=1}^{M}(\mathcal{A}(P_j) - \mathcal{A}(Q_j)), \quad D = \sum_{j=1}^{M}(P_j - Q_j),$$

with $\mathcal{A}$ the Abel map defined in (4.78).

Let us return to the canonical class. We compute it for a hyperelliptic surface $w^2 = P_{2g+2}(z)$. Let $P_1, \ldots, P_{2g+2}$ be the branch points of the Riemann surface, and P_{∞^+} and P_{∞^-} its point at infinity. We have that

$$(dz) = P_1 + \cdots + P_{2g+2} - 2P_{\infty^+} - 2P_{\infty^-}.$$

Thus the degree of the canonical class on this surface is equal to $2g-2$. We prove an analogous assertion for an arbitrary Riemann surface.

Lemma 4.66. *Let $f : \Gamma \to X$ a holomorphic map between Riemann surfaces Γ and X and ω a meromorphic one form on X, then for any fixed point $P \in \Gamma$*

$$ord_P f^*\omega = (1 + ord_{f(P)}(\omega))mult_P(f) - 1 \tag{4.90}$$

where $f^*\omega$ denotes the pull back of ω via f. We recall that the multiplicity of f in P is the unique integer m such that there is local coordinatea near $P \in \Gamma$ and $f(P) \in X$ such that f takes the form $z \to z^m$.

Proof. Suppose that the map f can be represented near the point P and $f(P)$ with local coordinates τ and τ' as $\tau \to \tau' = \tau^m$. Suppose that near the point $f(P)$ the one form ω takes the form $\omega = g(\tau')d\tau'$ with $g(\tau') = \sum_{k\geqslant n}\alpha_k\tau'^k$. Then, the one form $f^*\omega$, near the point P, takes the form

$$f^*\omega = g(\tau^m)m\tau^{m-1}d\tau = \sum_{k\geqslant n}\alpha_k\tau^{mk+m-1}d\tau.$$

Looking at the coefficient in the exponent, one has the claim of the lemma. □

Definition 4.67. *Let $f : \Gamma \to X$ a holomorphic map between Riemann surfaces. The branch point divisor W_f is the divisor on Γ defined by*

$$W_f = \sum_{P\in\Gamma}[mult_P(f) - 1]P. \tag{4.91}$$

Definition 4.68. *Let $f : \Gamma \to X$ be a holomoprhic map between Riemann surfaces and let $Q \in X$. The inverse image of the divisor Q denoted $f^*(Q)$ is defined as*

$$f^*(Q) = \sum_{P\in f^{-1}(Q)} mult_p(f)\cdot P.$$

Applying (4.90) and (4.91) we arrive to the relation between divisors

$$(f^*\omega) = W_f + f^*(\omega), \tag{4.92}$$

where $f^*(\omega)$ is the inverse image of the divisor (ω) of the one form ω.

Suppose that the Riemann surface Γ is given by the equation $F(z,w) = 0$. Further, let $P_1, \ldots, P_N$ be the branch points of this surface with respective multiplicities $f_1, \ldots, f_N$ with respect to the meromorphic function $z : \Gamma \to \mathbb{CP}^1$. (see Lecture 1). The branch point divisor $W_z = f_1P_1 + \ldots f_NP_N$.

Lemma 4.69. *The canonical class of the surface* Γ *has the form*

$$K_\Gamma = W_z + z^*(K_{\mathbb{CP}^1}). \tag{4.93}$$

Here z^* *denotes the inverse image of a divisor in the class* $K_{\mathbb{CP}^1}$ *with respect to the meromorphic function* $z : \Gamma \to \mathbb{CP}^1$.

Proof. This follows immediately from (4.92). □

Corollary 4.70. *The degree of the canonical class* K_Γ *of a Riemann surface* Γ *of genus* g *is equal to* $2g - 2$.

Proof. We have from (4.93) that $\deg K_\Gamma = \deg W_z - 2\deg z$, where $\deg W_z$ is the total multiplicity of the branch points of the map z. By the Riemann-Hurwitz formula (2.4), $\deg W_z = f = 2g + 2\deg z - 2$. The corollary is proved. □

The divisor (4.86) is *positive* if all multiplicities n are positive. An *effective* divisor is a divisor linearly equivalent to a positive divisor. Divisors D and D' are connected by the inequality $D > D'$ if their difference $D - D'$ is a positive divisor.

With each divisor D we associate the linear space of meromorphic functions

$$L(D) = \{f \mid (f) \geqslant -D\}. \tag{4.94}$$

If D is a positive divisor, then this space consists of functions f having poles only at points of D, with multiplicities not greater than the multiplicities of these points in D. If $D = D_+ - D_-$, where D_+ and D_- are positive divisors, then the space $L(D)$ consists of the meromorphic functions with poles possible only at points of D_+, with multiplicities not greater than the multiplicities of these points in D , and with zeros at all points of D_- (at least), with multiplicities not less than the multiplicities of these points in D.

Lemma 4.71. *If the divisors* D *and* D' *are linearly equivalent, then the spaces* $L(D)$ *and* $L(D')$ *are isomorphic.*

Proof. Let $D - D' = (g)$, where g is a meromorphic function. If $f \in L(D)$, then $f' = fg \in L(D')$. Indeed,

$$(f') + D' = (f) + (g) + D' = (f) + D > 0.$$

Conversely, if $f' \in L(D')$, then $f = g^{-1}f' \in L(D)$. The lemma is proved. □

We denote the dimension of the space $L(D)$ by

$$l(D) = \dim L(D). \tag{4.95}$$

By Lemma 4.71, the function $l(D)$ (as well as the degree $\deg D$) is constant on linear equivalence classes of divisors. We make some simple remarks about the properties of this important function.

Remark 4.72. A divisor D is effective if and only if $l(D) > 0$. Indeed, replacing D by a positive divisor D' linearly equivalent to it, we see that the space $L(D')$ contains the constants. Conversely, if $l(D) > 0$, then D is effective. Indeed, if the meromorphic function f is such that $D' = (f) + D > 0$, then the divisor D', which is linearly equivalent to D is positive.

Remark 4.73. For the zero (empty) divisor, $l(0) = 1$. If $\deg D < 0$, then $l(D) = 0$.

Remark 4.74. The number $l(D)-1$ is often denoted by $|D|$. According to Remark 4.72 $|D| \geqslant 0$ for effective divisors. The number $|D|$ admits the following intuitive interpretation. We show that $|D| \geqslant k$ if and only if for any points $P_1, \ldots, P_k$ there is a divisor $D' \simeq D$ containing the points $P_1, \ldots, P_k$ (the presence of coinciding points among $P_1, \ldots, P_k$ is taken into account by their multiple occurrence in D'). If $l(D) \geqslant k+1$, then there are linearly independent functions $f_1, \ldots, f_k \in L(D)$ such that the function $f = \sum_{i=1}^k c_i f_i - c_0$, where c_i, $i = 1, \ldots, k$ are arbitrary constants, has zeros in $P_1, \ldots, P_k$, namely

$$f(P_j) = 0, j = 1, \ldots, k.$$

This is a system of inhomogeneous linear equation for the constants $c_1, \ldots, c_k$ which has a solution for any choice of the points $P_1, \ldots, P_k$. So it follows that the divisor D' of zeros of f contains the point $P_1, \ldots, P_k$, which implies that $D + (f) = D'$, or equivalently $D' \simeq D$ and D' contains the points $P_1, \ldots, P_k$.

Viceversa suppose that there is a positive divisor D' containing the arbitrary points $P_1, \ldots, P_k$ and such that $D' \simeq D$. Then there is a meromorphic function f such that $(f) = D' - D$, or $(f) + D = D' > 0$. It follows that $f \in L(D)$ and f has zeros in arbitrary points $P_1, \ldots, P_k$. We write f is the form $f = \sum_{j=1}^k c_k f_k - c_0$ where $f_j \in L(D)$. If the function f has zeros in arbitrary points $P_1, \ldots, P_k$ it follows that the system of equations

$$f(P_j) = 0, j = 1, \ldots, k,$$

must be solvable for any set of points $P_1, \ldots, P_k$, but this is possible only if the functions $f_1, \ldots, f_k$ are linearly independent and different from the constant, which means that $l(D) \geqslant k+1$. One therefore says that $|D|$ is the number of mobile points in the divisor D.

Remark 4.75. Let $K = K_\Gamma$, be the canonical class of a Riemann surface. We mention an interpretation that will be important later for the space $L(K-D)$ for an arbitrary divisor D. First, if $D = 0$, then the space $L(K)$ is isomorphic to the space of holomorphic differentials on Γ. Indeed, choose a representative $K_0 > 0$ in the canonical class, taking K_0 to be the zero divisor of some holomorphic differential ω_0, $K_0 = (\omega_0)$. If $f \in L(K_0)$, i.e. $(f) + (\omega_0) \geqslant 0$, then the divisor $(f\omega_0)$ is positive, i.e., the differential $f\omega_0$ is holomorphic. Conversely, if ω is any holomorphic differential, then the meromorphic function $f = \omega/\omega_0$ lies in $L(K_0)$.

It follows from the foregoing and Theorem 4.12 that

$$l(K) = g.$$

Lemma 4.76. *For a positive divisor D the space $L(K-D)$ is isomorphic to the space*

$$\Omega(D) = \{\omega \in H^1(\Gamma) \mid (\omega) - D \geqslant 0\}$$

Proof. We choose a representative $K_0 > 0$ in the canonical class, taking K_0 to be the zero divisor of some holomorphic differential ω_0, $K_0 = (\omega_0)$. If $f \in L(K_0 - D)$, then the differential $f\omega_0$ is holomorphic and has zeros at the points of D, i.e., $f\omega_0 \in \Omega(D)$. Conversely, if $\omega \in \Omega(D)$, then $f = \omega/\omega_0 \in L(K_0 - D)$. The assertion is proved. □

The main way of getting information about the numbers $l(D)$ is the Riemann-Roch Theorem.

Theorem 4.77 (Riemann Roch Theorem)**.** *For any divisor D*

$$l(D) = 1 + \deg D - g + l(K-D). \tag{4.96}$$

Proof. For surfaces Γ of genus 0 (which are isomorphic to $\mathbb{CP}^1$ in view of Problem 6.1) the Riemann-Roch theorem is a simple assertion about rational functions (verify!). By Remarks 4.73 and 4.75 (above) the Riemann-Roch theorem is valid for $D = \varnothing$.

We first prove (4.96) for positive divisors $D > 0$. Let $D = \sum_{k=1}^m n_k P_k$ where all the $n_k > 0$. We first verify the arguments when all the n_k are $= 1$, i.e., $m = \deg D$. Let $f \in L(D)$ be a nonconstant function.

We consider the Abelian differential $\omega = df$. It has double poles and zero residues at the points $P_1, \ldots, P_m$ and does not have other singularities. Therefore, it is representable in the form

$$\Omega = df = \sum_{k+1}^{m} c_k \Omega^{(1)}_{P_k} + \psi$$

where $\Omega^{(1)}_{P_k}$ are normalized differentials of the second kind (see Lecture 4.1.3), $c_1, \ldots, c_m$ are constants, and the differential ψ is holomorphic. Since the function $f(P) = \int_{P_0}^P \Omega$ is single-valued on Γ, the integral $\int_{P_0}^P \Omega$is independent from the path of integration. This implies that

$$\oint_{\alpha_i} \Omega = 0, \quad \oint_{b_i} \Omega = 0, \quad i = 1, \ldots, g. \tag{4.97}$$

From the vanishing of the α-periods of the meromorphic differentials $\Omega^{(1)}_{P_k}$ we get that $\psi = 0$ (see Corollary 4.17). From the vanishing of the β-period we get, by (4.30) with $n = 1$, that

$$0 = \oint_{\beta_i} \Omega = \sum_{k=1}^{m} c_k \psi_{ik}(z_k)|_{z_k=0}, \quad i = 1, \ldots, g, \tag{4.98}$$

where z_k is a local parameter in a neighborhood of P_k, $z_k(P_k) = 0$, $k = 1, \dots, m$, and the basis of holomorphic differentials are written in a neighborhood of P_k in the form $\omega_i = \psi_{ik}(z)dz_k$. Defining $\omega_i(P_k) := \psi_{ik}(0)$, we write the system (4.98) in the form

$$\begin{pmatrix} \omega_1(P_1) & \omega_1(P_2) & \dots & \omega_1(P_m) \\ \omega_2(P_1) & \omega_2(P_2) & \dots & \omega_2(P_m) \\ \dots & \dots & \dots & \dots \\ \omega_g(P_1) & \omega_g(P_2) & \dots & \omega_g(P_m) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_m \end{pmatrix} = 0, \tag{4.99}$$

We have obtained a homogeneous linear system of $m = \deg D$ equations in the coefficients $c_1, \dots, c_m$. The nonzero solutions of this systems are in a one-to-one correspondence with the nonconstant functions f in $L(D)$, where f can be reproduced from a solution $c_1, \dots, c_m$ of the system (4.98) in the form

$$f(P) = \sum_{k=1}^{m} c_k \int_{P_0}^{P} \Omega_{P_k}^{(1)}.$$

Thus $l(D) = 1 + \deg D - \mathrm{rank}\mathcal{A}$ where $\mathcal{A}$ is the matrix of holomorphic differentials in (4.99) (the 1 is added because the constant function belong to the space $L(D)$). On the other hand the rank of the matrix $\mathcal{A}$ has another interpretation. Consider the holomorphic differential $\omega = \sum_{j=1}^{g} a_j \omega_j$. Such differential ω belongs to the space $\Omega(D)$ if

$$\omega(P_k) = 0, \quad k = 1, \dots, m.$$

The above system of equations can be written in the equivalent form

$$\begin{pmatrix} a_1 & a_2 & \dots & a_g \end{pmatrix} \begin{pmatrix} \omega_1(P_1) & \dots & \omega_1(P_m) \\ \dots & \dots & \dots \\ \omega_g(P_1) & \dots & \omega_g(P_m) \end{pmatrix} = 0. \tag{4.100}$$

The number of solutions of this system is equal to $g - \mathrm{rank}\mathcal{A}$ and it is in one to one correspondence with the linearly independent holomorphic differentials in $\Omega(D)$. Therefore $\dim\Omega(D) = g - \mathrm{rank}\mathcal{A}$. On the other hand we have obtained that

$$l(D) = 1 + \deg D - \mathrm{rank}\mathcal{A}$$

so that combining the two equations we obtain

$$l(D) = 1 + \deg D - g + \dim\Omega(D) = 1 + \deg D - g + l(K - D)$$

where the second identity is due to the fact that the space $\Omega(D)$ and $L(K-D)$ are isomorphic for positive divisors. Accordingly the Riemann-Roch theorem has been proved in this case.

We explain what happens when the positive divisor D has multiple points. For example suppose that $D = n_1 P_1 + \dots$. Then $\omega = df = \sum_{j=1}^{n_1} c_1^j \Omega_{P_1}^{(j)} + \dots$ and the system (4.98) can be written in the form

$$\sum_{j=1}^{n_1} c_1^j \, \frac{1}{j!} \frac{d^{j-1}\psi_{i1}}{dz_1^{j-1}}\bigg|_{z_1=0} + \dots = 0$$

If the rank of the coefficient matrix of this system is denoted (as above) by $\operatorname{rank}\mathcal{A}$, the dimension of the space $L(D)$ is equal to $l(D) = 1 + \deg D - \operatorname{rank}\mathcal{A}$ while the dimension of the space $\Omega(D)$ is equal to $g - \operatorname{rank}\mathcal{A}$. We have proved the Riemann-Roch theorem for all positive divisors and hence for all effective divisors, which (accordingly to Remark 4.72) are distinguished by the condition $l(D) > 0$. Next we note that the relation in this theorem can be written in the form

$$l(D) - \frac{1}{2}\deg D = l(K - D) - \frac{1}{2}\deg(K - D), \tag{4.101}$$

which is symmetric with respect to the substitution $D \to K - D$. Therefore the theorem is proved for all divisors D such that D or $K - D$ is equivalent to an integral divisor. If neither D nor $K - D$ are equivalent to an integral divisor, then $l(D) = 0$ and $l(K - D) = 0$ and the Riemann-Roch theorem reduces in this case to the equality

$$\deg D = g - 1. \tag{4.102}$$

Let us prove this equality. We represent D in the form $D = D_+ - D_-$, where D_+ and D_- are positive divisors and $\deg D_- > 0$. It follows from the validity of the Riemann-Roch theorem for D_+ that $l(D_+) \geqslant \deg D_+ - g + 1 = \deg D + \deg D_- - g + 1$. Therefore if $\deg D \geqslant g$, then $l(D_+) \geqslant 1 + \deg D_-$. Then the space $L(D_+)$ contains a nonzero function vanishing on D_-, i.e. belonging to the space $L(D_+ - D_-) = L(D)$. This contradicts the condition $l(D) = 0$. Similarly, the assumption $\deg(K - D) \leqslant g$ leads to a contradiction. This implies (4.102). The theorem is proved. □

4.1.6 Some consequences of the Riemann-Roch theorem. The structure of surfaces of genus 1. Weierstrass points. The canonical embedding

Corollary 4.78. *If* $\deg D \geqslant g$, *then the divisor* D *is effective.*

Corollary 4.79. *The Riemann inequality*

$$l(D) \geqslant 1 + \deg D - g, \tag{4.103}$$

holds for $\deg D \geqslant g$.

Definition 4.80. *A positive divisor D is called special if*

$$dim\Omega(D) > 0.$$

We remark that any effective divisor of degree less then g is special since $l(D) > 0$ and by Riemann-Roch theorem this implies $\dim\Omega(D) > 0$.

Corollary 4.81. *If* $\deg D > 2g - 2$, *then D is nonspecial.*

Proof. For $\deg D > 2g - 2$ we have that $\deg(K - D) < 0$, hence $l(K - D) = 0$ (see Remark 4.73). The corollary is proved. □

Exercise 4.82: Suppose that $k \geqslant g$; let the Abel mapping $A : \Gamma \to J(\Gamma)$ (see Lecture 4.1.4) be extended to the kth-power mapping

$$A^k : \underbrace{\Gamma \times \cdots \times \Gamma}_{k \text{ times}} \to J(\Gamma)$$

by setting $A^k(P_1, \dots, P_k) = A(P_1)+\cdots+A(P_k)$ (it can actually be assumed that A^k maps into $J(\Gamma)$ the kth symmetric power $S^k\Gamma$, whose points are the unordered collections $(P_1, \dots, P_k)$ of points of Γ). Prove that the special divisors of degree k are precisely the critical points of the Abel mapping A^k. Deduce from this that a divisor D with $\deg D \geqslant g$ in general position is nonspecial.

Remark 4.83. Let $\deg D = 0$, then if D is equivalent to a divisor of a meromorphic function, then $L(D) = 1$ otherwise $L(D) = 0$. Let $\deg D = 2g - 2$, then if D is equivalent to the canonical divisor, then $l(D) = g$ otherwise $l(D) = g - 1$. Furthermore if $\deg D > 2g - 2$, then by Riemann Roch theorem one has $l(D) = 1 + \deg D - g$. If $0 \leqslant \deg D \leqslant g - 1$ the minimum value of $l(D)$ is zero while for $g \leqslant \deg D \leqslant 2g - 2$, $\min(l(D)) = 1 - g + \deg D$.

The values of $l(D)$ for $0 \leqslant \deg D \leqslant 2g - 2$ are estimated by the Clifford theorem.

Theorem 4.84. *If* $0 \leqslant \deg D \leqslant 2g - 2$, *then*

$$l(D) \leqslant 1 + \frac{1}{2} \deg D. \tag{4.104}$$

Proof. If $l(D) = 0$ or $l(K - D) = 0$, the proof of the theorem is straightforward. Let us assume that $l(D) > 0$ and $l(K - D) > 0$ and consider the map $L((D) \times L(K - D) \to L(K)$ given by $(f, h) \to fh$ where $(f, h) \in L((D) \times L(K - D)$. Let V be the subspace in $L(K)$ which is the image of this map. Then one has

$$g = l(K) \geqslant \dim V = l(D)l(K - D) \geqslant l(D) + l(K - D) - 1$$

where in the last equality we use the identity which holds for real numbers a and b bigger then one: $(a-1)(b-1) \geqslant 0$ and so $ab \geqslant a+b-1$.

Therefore

$$g \geqslant l(D) + l(K-D) - 1 = 2l(D) + g - 2 - \deg D,$$

which implies (4.104). □

Let us make a plot of the possible values of $l(D)$ using Clifford theorem and the above observations.

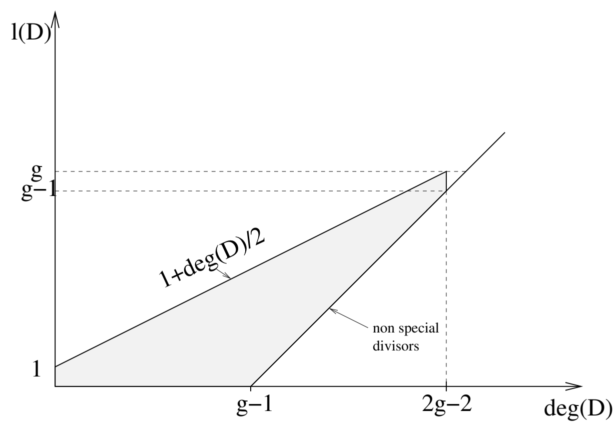


Figure 4.3: The values of $l(D)$ as a function of $\deg D$. One can see that the value of $l(D)$ of a special divisors is located between the two lines.

We now present examples of the use of the Riemann-Roch theorem in the study of Riemann surfaces.

Example 4.85. Let us show that any Riemann surface Γ of genus $g = 1$ is isomorphic to an elliptic surface $w^2 = P_3(z)$. Let P_0 be an arbitrary point of Γ. Here $2g-2 = 0$, therefore, any positive divisor is nonspecial. We have that $A(2P_0) = 2$, hence there is a nonconstant function z in $l(2P_0)$, i.e., a function having a double pole at P_0. Further $l(3P_0) = 3$, hence there is a function $w \in l(3P_0)$ that cannot be represented in the form $w = az + b$. This function has a pole of order three at P_0. Finally, since $l(6P_0) = 6$, the functions $1, z, z^2, z^3, w, w^2, wz$ which lie in $l(6P_0)$ are linearly independent. We have that

$$a_1 w^2 + a_2 wz + a_3 w + a_4 z^3 + a_5 z^2 + a_6 z + a_7 = 0. \tag{4.105}$$

The coefficient a_1 is nonzero (verify). Making the substitution

$$w \to w - \left(\frac{a_2}{2a_1} z + \frac{a_3}{2a_1}\right)$$

we get the equation of an elliptic curve from (4.105).

Example 4.86 (Riemann count of the moduli space of Riemann surface)**.** Consider a Riemann surface Γ of genus g and a meromorphic function of degree $n > g$. Such function represents Γ as a n-sheeted covering of the complex plane, branched over a number of points with total branching number b_f equal to

$$b_f = 2n + 2g - 2$$

where the Riemann-Hurwitz formula has been used. Generically the branch points have branching number equal to one so that b_f is also equal to the branch points of the Riemann surface. From the Riemann existence theorem, given the branch points and a permutation associated to each branch point such that the corresponding monodromy group is a transitive sub-group of S_n, then one can construct a Riemann surface Γ. Let $f : \Gamma \to \mathbb{P}^1$ be the obvious projection map. Such map To any set of branch points it correspond a finite number of Riemann surface of genus g together with a meromorphic function of degree n. Riemann surface is determined uniquely up to isomorphism.

Any meromorphic function of degree n on Γ will represent Γ as a n-sheeted covering of the complex plane. Let D_∞ be the divisor of poles of f. Since the degree of f is equal to n then $\deg D_\infty = n$. Furthermore from Riemann-Roch theorem

$$l(D_\infty) = n + 1 - g.$$

So the freedom of choosing the function f is given by the position of the poles, and this gives n parameters, and the number of functions having poles in D_∞, which is equal to $n+1-g$. The total number of parameters in choosing the meromorphic function of degree n is $2n - 1 - g$. So the total number of parameters for describing a curve of genus g is

$$2n + 2g - 2 - (2n - 1 - g) = 3g - 3.$$

Definition 4.87 (Weierstrass points)**.** *A point P_0 of a Riemann surface Γ of genus g is called a Weierstrass point if $l(kP_0) > 1$ for some $k \leqslant g$.*

It is clear that in the definition of a Weierstrass point it suffices to require that $l(gP_0) > 1$ when $g \geqslant 2$. There are no Weierstrass points on a surface of genus $g = 1$. On hyperelliptic Riemann surfaces of genus $g > 1$ all branch points are Weierstrass points, since there exist functions with second-order poles at the branch points (see Lecture 3).

Definition 4.88. *A Riemann surface is called hyperelliptic if and only if it admits a non constant meromorphic function of degree 2.*

The use of Weierstrass points can be illustrated in the next exercise.

Exercise 4.89: Let Γ be a Riemann surface of genus $g > 1$, and P_0 a Weierstrass point of it, with $l(2P_0) > 1$. Prove that Γ is hyperelliptic. Prove that the surface is also hyperelliptic if $l(P+Q) > 1$ for two points P and Q.

Exercise 4.90: Let Γ be a hyperellitpic Rieamnn surface and z a function of degree two. Prove that any other function f of degree two is a Moebius transformation of z.

We show that there exist Weierstrass points on any Riemann surface Γ of genus $g > 1$.

Lemma 4.91. *Suppose that z is a local parameter in a neighborhood P_0, $z(P_0) = 0$; assume that locally the basis of holomorphic differentials has the form $\omega_i = \psi_i(z)dz$, $i = 1, \ldots, g$. Consider the determinant*

$$W(z) = \det \begin{pmatrix} \psi_1(z) & \psi_1'(z) & \ldots & \psi_1^{(g-1)}(z) \\ \ldots & \ldots & & \ldots \\ \psi_g(z) & \psi_g'(z) & \ldots & \psi_g^{(g-1)}(z) \end{pmatrix}. \tag{4.106}$$

The point P_0 is a Weierstrass point if and only if $W(0) = 0$.

Proof. If P_0 is a Weierstrass point, i.e., $l(gP_0) > 1$, then $l(K - gP_0) > 0$ by the Riemann-Roch theorem. Hence, there is a holomorphic differential with a g-fold zero at P_0 on Γ. The condition that there be such a differential can be written in the form $W(0) = 0$ (cf. the proof of the Riemann-Roch theorem). The lemma is proved. ◻

Lemma 4.92. *Under a local change of parameter $z = z(w)$ the quantity W transforms according to the rule $\tilde{W}(w) = \left(\frac{dz}{dw}\right)^{\frac{1}{2}g(g+1)} W(z)$.*

Proof. Suppose that $\omega_i = \psi_i(z)dz = \tilde{\psi}_i(w)dw$. Then each $\tilde{\psi}_i = \psi_i \frac{dz}{dw}$, $i = 1, \ldots, g$. This implies that the derivatives $d^k\tilde{\psi}_i/dw^k$ can be expressed for each i in terms of the derivatives $d^l\psi_i/dz^l$ by means of a triangular transformation of the form

$$\frac{d^k\tilde{\psi}_i}{dw^k} = \left(\frac{dz}{dw}\right)^{k+1} \frac{d^k\psi_i}{dz^k} + \sum_{j=1}^{k-1} c_j \frac{d^j\psi_i}{dz^j}, \quad i = 1, \ldots g$$

(the coefficients c_s in this formula are certain differential polynomials in $z(w)$). The statement of the Lemma readily follows from the transformation rule. ◻

Let us define the *weight* of a Weierstrass point P_0 as the multiplicity of zero of $W(z)$ at this point. According to the previous Lemma the definition of weight does not depend on the choice of the local parameter.

The proof of existence of Weierstrass points for $g > 1$ can be easily obtained from the following statement.

Lemma 4.93. *The total weight of all Weierstrass points on the Riemann surface* Γ *of genus* g *is equal to* $(g-1)\,g\,(g+1)$.

Proof. Let us consider the ratio

$$W(z)/\psi_1^N(z).$$

Here $N = \frac{1}{2}\,g(g+1)$. According to lemma (4.92), the above ratio does not depend on the choice of the local parameter and, hence, it is a meromorphic function on Γ. This function has poles of multiplicity N at the zeroes of the differential ω_1 (the total number of all poles is equal to $2g-2$). Therefore this function must have $N\,(2g-2) = (g-1)\,g\,(g+1)$ zeroes (as usual, counted with their multiplicities). These zeroes are the Weierstrass points. $\square$

Let us do few more remarks about the Weierstrass points. Given a point $P_0 \in \Gamma$, let us consider the dimension $l(k\,P_0)$ as a function of the integer argument k. This function has the following properties. According to figure (4.3) we have

$$1 \leqslant l(k\,P_0) \leqslant g, \quad 1 \leqslant k \leqslant 2g-1.$$

In particular $l\,((2g-1)P_0) = g$. It follows that while k increases $2g-2$ times the function $l(k\,P_0)$ increases only $g-1$ times. The next lemma shows that the function $l(k\,P_0)$ is a piece-wise constant function where each step has size equal to one.

Lemma 4.94.

$$l(k\,P_0) = \begin{cases} l\,((k-1)P_0) + 1, & \text{if there exists a function with a pole of order } k \text{ at } P_0 \\ l\,((k-1)P_0)\,, & \text{if such a function does not exist} \end{cases}$$

Proof. The space $L(k\,P_0)$ is larger then the space $L((k-1)P_0)$ therefore $l(k\,P_0) \geqslant l((k-1)P_0)$. On the other hand, $\dim\Omega(kP_0) \leqslant \dim\Omega((k-1)P_0)$. From the Riemann Roch theorem one has

$$l(k\,P_0) - l((k-1)P_0) = 1 + \dim\Omega(kP_0) - \dim\Omega((k-1)P_0)$$

which, when combined with the above two inequalities, gives the statement. $\square$

When $l(k\,P_0) = l((k-1)P_0)$ we will say that the number k is a *gap* at the point P_0. From the previous remarks it follows the following *Weierstrass gap theorem*:

Theorem 4.95. *There are exactly* g *gaps* $1 = a_1 < ... < a_g < 2g$ *at any point* P_0 *of a Riemann surface of genus* g.

The gaps have the form $a_i = i$, $i = 1, \ldots, g$, for a point P_0 in general position (which is not a Weierstrass point). Namely for a non Weierstrass point the function $l(kP_0)$ is non zero only for $k > g$ and one has $l(kP_0) = 1 + k - g$ for $k > g$. A Weierstrass point P_0 is called normal if the Weierstrass gap sequence takes the form $1, 2, \ldots, g-1, g+1$ where g is the genus of the surface. Namely a meromorphic function with only a pole in P_0 has order at least equal to g. Normal Weierstrass points are generic. A Weierstrass point P_0 is called hyperelliptical is the Weierstrass gap sequence takes the form $1, 3, 5, \ldots, 2g-1$. In this case a meromorphic function with only a pole in P_0 has order equal to two.

Exercise 4.96: Show that every compact Riemann surface of genus g is conformally equivalent to a $(g+1)-$sheeted covering surface of the complex plane.

Exercise 4.97: Prove that for branch points of a hyperelliptic Riemann surface of genus g the gaps have the form $a_i = 2i-1$, $i = 1, \ldots, g$. Prove that a hyperelliptic surface does not have other Weierstrass points. Next suppose that the hyperelliptic Riemann surface has genus 2 and let P_0 be a Weierstrass point. Show that there exist meromorphic functions z and w with only a pole in P_0 and such that

$$w^2 + a_1wz + a_2wz^2 + a_3z^5 + a_4z^4 + a_5z^3 + a_6z^2 + a_7z + a_8 = 0.$$

Exercise 4.98: Prove that any Riemann surface of genus 2 is hyperelliptic.

Exercise 4.99: Let Γ be a hyperelliptic Riemann surface of the form $w^2 = P_{2g+l}(z)$. Prove that any birational (biholomorphic) automorphism $\Gamma \to \Gamma$ has the form $(z, w) \to (\frac{az+b}{cz+d}, \pm w)$, where the linear fractional transformation leaves the collection of zeros of $P_{2g+2}(z)$ invariant.

Example 4.100 (The canonical embedding)**.** . Let Γ be an arbitrary Riemann surface of genus $g \geqslant 2$. We fix on Γ a canonical basis of cycles $a_1, \ldots, a_g$, $b_1, \ldots, b_g$; let $\omega_1, \ldots, \omega_g$ be the corresponding normal basis of holomorphic differentials. This basis gives a canonical mapping $\Gamma \to \mathbb{CP}^{g-1}$ according to the rule

$$P \to (\omega_1(P) : \omega_2(P) : \cdots : \omega_g(P)). \tag{4.107}$$

Indeed, it suffices to see that all the differentials $\omega_1, \ldots, \omega_g$ cannot simultaneously vanish at some point of the surface. If P were a point at which any holomorphic differential vanished, i.e., $l(K-P) = g$, (see Remark 4.75), then $l(P)$ would be $= 2$ in view of the Riemann-Roch theorem, and this means that the surface Γ is rational (verify!). Accordingly (4.107) really is a mapping $\Gamma \to \mathbb{CP}^{g-1}$; it is obviously well-defined.

Lemma 4.101. *If Γ is a nonhyperelliptic surface of genus $g \geqslant 3$, then the canonical mapping (4.107) is a smooth embedding. If Γ is a hyperelliptic surface of genus $g \geqslant 2$, then the image of the canonical embedding is a rational curve, and the mapping itself is a two-sheeted covering.*

Proof. We prove that the mapping (4.107) is an embedding. Assume not: assume that the points P_1 and P_2 are merged into a single point by this mapping. This means that the rank of the matrix

$$\begin{pmatrix} \omega_1(P_1) & \omega_1(P_2) \\ \dots & \dots \\ \omega_g(P_1) & \omega_g(P_2) \end{pmatrix}$$

is equal to 1. But then $l(P_1 + P_2) > 1$ (see the proof of the Riemann-Roch theorem). Hence, there exists on Γ a nonconstant function with two simple poles at P_1 and P_2 i.e., the surface Γ is hyperelliptic. The smoothness is proved similarly: if it fails to hold at a point P, then the rank of the matrix

$$\begin{pmatrix} \omega_1(P) & \omega_1'(P) \\ \dots & \dots \\ \omega_g(P) & \omega_g'(P) \end{pmatrix}$$

is equal to 1. Then $l(2P) > 1$, and the surface is hyperelliptic. Finally, suppose that Γ is hyperelliptic. Then it can be assumed of the form $w^2 = P_{2g+1}(z)$. Its canonical mapping is determined by the differentials (5.42). Performing a projective transformation of the space $\mathbb{CP}^{g-1}$ with the matrix (c_{jk}) (see the formula (5.42)), we get the following form for the canonical mapping:

$$P = (z, w) \to (1 : z : \dots : z^{g-1}) \tag{4.108}$$

Its properties are just as indicated in the statement of the lemma. The lemma is proved. □

We remark that the canonical mapping smoothly embeds a nonhyperelliptic Riemann surface of genus g in $\mathbb{P}^{g-1}$. It is proved in [16] that the image of such embedding can be smoothly project to $\mathbb{P}^3$. Namely every smooth Riemann surface has an holomorphic embedding in $\mathbb{P}^3$.

Exercise 4.102: Suppose that the Riemann surface Γ is given in $\mathbb{CP}^2$ by the equation

$$\sum_{i+j=4} a_{ij} \xi^i \eta^j \zeta^{4-i-j} = 0, \tag{4.109}$$

and this curve is nonsingular in $\mathbb{CP}^2$ (construct an example of such a nonsingular curve). Prove that the genus of this surface is equal to 3 and the canonical mapping is the identity up to a projective transformation of $\mathbb{CP}^2$. Prove that Γ is a non hyperelliptic surface. Prove that any non hyperelliptic surface of genus 3 can be obtained in this way.

The range $\Gamma' \subset \mathbb{CP}^{g-1}$ of the canonical mapping is called the canonical curve.

Exercise 4.103: Prove that any hyperplane in $\mathbb{CP}^{g-1}$ intersects the canonical curve Γ' in $2g - 2$ points (counting multiplicity).

Chapter 5

Jacobi inversion problem and theta-functions

5.1 Statement of the Jacobi inversion problem. Definition and simplest properties of general theta functions

In Lecture 4.1.2 we saw that inversion of an elliptic integral leads to elliptic functions. For a surface of genus $g > 1$ the Inversion of integrals of Abelian differentials is not possible since any such differential has zeros (at least $2g - 2$zeros). Instead of the problem of inverting a single Abelian integral, Jacobi proposed for hyperelliptic surfaces $w^2 = P_5(z)$ the problem of solving the system

$$\begin{aligned} \int_{P_0}^{P_1} \frac{dz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{dz}{\sqrt{P_5(z)}} &= \eta_1 \\ \int_{P_0}^{P_1} \frac{zdz}{\sqrt{P_5(z)}} + \int_{P_0}^{P_2} \frac{zdz}{\sqrt{P_5(z)}} &= \eta_2 \end{aligned} \tag{5.1}$$

where η_1, η_2 are given numbers from which the location of the points $P_1 = (z_1, w_1)$, $P_2 = (z_2, w_2)$ is to be determined. It is clear, moreover, that P_1 and P_2 are determined from (5.1) only up to permutation. Jacobi's idea was to express the symmetric functions of P_1 and P_2 as functions of η_1 and η_2. He noted also that this will give meromorphic functions of η_1 and η_2 whose period lattice is generated by the periods of the basis of holomorphic differentials $dz/\sqrt{P_5(z)}$ and $zdz/\sqrt{P_5(z)}$. This Jacobi inversion problem was solved by Göepel and Rosenhain by means of the apparatus of theta functions of two variables. The generalization of the Jacobi inversion problem to arbitrary Riemann surfaces and its solution are due to Riemann. We give a precise statement of the Jacobi inversion problem. Let Γ be an arbitrary Riemann surface of genus g, and fix a canonical basis of cycles $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ on Γ; as above let $\omega_1, \ldots, \omega_g$ be be the corresponding basis of normalized holomorphic differentials. Recall (see Lecture 4.1.4) that the Abel mapping has the form

$$A : \Gamma \to J(\Gamma), \quad A(P) = (u_1(P), \ldots, u_g(P)), \tag{5.2}$$

where $J(\Gamma)$ is the Jacobi variety,

$$u_i(P) = \int_{P_0}^{P} \omega_i, \tag{5.3}$$

P_0 is a particular point of Γ, and the path of integration from P_0 to P is the same for all $i = 1, \ldots, g$. Consider the gth symmetric power $S^g\Gamma$ of Γ. The unordered collections $(P_1, \ldots, P_g)$ of g points of Γ are the points of the manifold $S^g\Gamma$. The meromorphic functions on $S^g\Gamma$ are the meromorphic symmetric functions of g variables $P_1, \ldots, P_g$, $P_j \in \Gamma$. The Abel mapping (5.2) determines a mapping

$$A^{(g)} : \ S^g\Gamma \to J(\Gamma), \quad A^g(P_1, \ldots, P_g) = A(P_1) + \cdots + A(P_g), \tag{5.4}$$

which we also call the Abel mapping.

Lemma 5.1. *If the divisor $D = P_1 + \cdots + P_g$ is nonspecial, then in a neighborhood of a point $A^{(g)}(P_1, ..., P_g) \in J(\Gamma)$ the mapping $A^{(g)}$ has a single-valued inverse.*

Proof. Suppose that all the points are distinct; let $z_1, \ldots, z_g$ be local parameters in neighborhoods of the respective points $P_1, \ldots, P_g$ with $z_k(P_k) = 0$ and $\omega_i = \psi_{ik}(z_k)dz_k$ the normalized holomorphic differentials in a neighborhood of P_k. The Jacobi matrix of the mapping (5.4) has the following form at the points $(P_1, \ldots, P_g)$

$$\begin{pmatrix} \psi_{11}(z_1 = 0) & \ldots & \psi_{1g}(z_g = 0) \\ \ldots & \ldots & \ldots \\ \psi_{g1}(z_1 = 0) & \ldots & \psi_{gg}(z_g = 0) \end{pmatrix}.$$

If the rank of this matrix is less than g, then $l(K - D) > 0$, i.e., the divisor D is special by the Riemann-Roch theorem. The case when not all the points $P_1, \ldots, P_g$ are distinct is treated similarly. We now prove that the inverse mapping is single-valued. Assume that the collection of points $(P'_1, \ldots, P'_g)$ is also carried into $A^{(g)}(P_1, \ldots, P_g)$. Then the divisor $D' = P'_1 + \cdots + P'_g$ is linearly equivalent to D by Abel's theorem. If $D' \neq D$, then there would be a meromorphic function with poles at points of D and with zeros at points of D'. This would contradict the fact that D is nonspecial. Hence, $D' = D$, and the points $P'_1, \ldots, P'_g$ differ from $P_1, \ldots, P_g$ only in order. The lemma is proved. □

Since a divisor $P_1 + ... + P_g$ in general position is nonspecial (see Problem 4.82), the Abel mapping (5.4) is invertible almost everywhere. The problem of inversion of this mapping in the large is the Jacobi inversion problem. Thus, the Jacobi inversion problem can be written in coordinate notation in the form

$$\begin{cases} u_1(P_1) + \cdots + u_1(P_g) &= \eta_1 \\ \cdots\cdots\cdots \\ u_g(P_1) + \cdots + u_g(P_g) &= \eta_g \end{cases} \tag{5.5}$$

which generalizes (5.1). To solve this problem we need the apparatus of multi-dimensional theta functions.

5.2 Theta-functions

The g-dimensional theta-functions are defined by their Fourier serie. Let $B = (B_{jk})$ be a symmetric $g \times g$ matrix with negative-definite real part and let $z = (z_1, \dots, z_g)$ and $N = (N_1, \dots, n_g)$ be g-dimensional column vectors. The Riemann theta function is defined by its multiple Fourier series,

$$\theta(z) = \theta(z; B) = \sum_{N \in \mathbb{Z}}^{g} \exp\left(\pi i \langle NB, N\rangle + \langle N, z\rangle\right), \tag{5.6}$$

where the angle brackets denote the Euclidean inner product:

$$\langle N, z\rangle = \sum_{k=1}^{g} N_k z_k, \quad \langle NB, N\rangle = \sum_{j,k=1}^{g} B_{kj} N_j N_k.$$

The summation in (5.6) is over the lattice of integer vectors $N = (N_1, \dots, N_g)$. The obvious estimate $\Re(i\langle NB, N\rangle) \leqslant -b\langle N, N\rangle$, where $b > 0$ is the smallest eigenvalue of the matrix $\Im(B)$, implies that the series (5.6) defines an entire function of the variables $z_1, \dots, z_g$.

Proposition 5.2. *The theta-function has the following properties.*

1. $\theta(-z\,;\,B) = \theta(z\,;\,B)$.

2. *For any integer vectors* $M, K \in \mathbb{Z}^g$,

$$\theta(z + K + MB; B) = \exp\left(-\pi i\langle MB, M\rangle - 2\pi i\langle M, z\rangle\right)\theta(z; B). \tag{5.7}$$

3. *It satisfies the heat equation*

$$\begin{aligned} \frac{\partial}{\partial B_{ij}}\theta(z\,;\,B) &= \frac{1}{2\pi i}\frac{\partial^2}{\partial z_i z_j}\theta(z\,;\,B), \quad i \neq j \\ \frac{\partial}{\partial B_{ii}}\theta(z\,;\,B) &= \frac{1}{4\pi i}\frac{\partial^2}{\partial z_i^2}\theta(z\,;\,B). \end{aligned} \tag{5.8}$$

Proof. The proof of properties 1. and 3. is straightforward. Let us prove property 2. In the series for $\theta(z+K+MB)$ we make the change of summation index $N \to N-M$. The relation (5.7) is obtained after this transformation. □

The integer lattice $\{N+MB\}$ is called the period lattice.

Remark 5.3. It is possible to define the function $\theta(z)$ as an entire function of $z_1, \ldots, z_g$ satisfying the transformation law (5.7) (this condition determines $\theta(z)$ uniquely to within a factor).

The theta-function is an analytic multivalued function on the g-dimensional torus $T^g = \mathbb{C}^g/\{N+MB\}$. In order to construct single valued functions, i.e. meromorphic functions on the torus, one can take for example, for any two vectors $e_1, e_2 \in \mathbb{C}^g$ the product

$$\frac{\theta(z+e_1)\theta(z-e_1)}{\theta(z+e_2)\theta(z-e_2)}.$$

Indeed the above expression is by (5.7) a single valued function on the g-dimensional torus. In general for any two sets of g vectors $e_1, \ldots e_g \in \mathbb{C}^g$, $v_1, \ldots v_g \in \mathbb{C}^g$ satisfying the constraint

$$e_1 + \ldots e_g = 0, \quad v_1 + \ldots v_g = 0$$

the product

$$\prod_{j=1}^{g} \frac{\theta(z+e_j)}{\theta(z+v_j)},$$

is a meromorphic function on the torus (verify this!).

Let p and q be arbitrary real g-dimensional row vectors. We define the theta function with characteristics p and q:

$$\begin{aligned}\theta[p,q](z) &= \exp\left(\pi i\langle pB, p\rangle + 2\pi i\langle z+q, p\rangle\right)\theta(z+q+pB) \\ &= \sum_{N\in\mathbb{Z}^g} \exp\left(\pi i\langle (N+p)B, N+p\rangle + 2\pi i\langle z+q, N+p\rangle\right).\end{aligned} \tag{5.9}$$

For $p=0$ and $q=0$ we get the function $\theta(z)$. The analogue of the law (5.7) for the functions $\theta[p,q](z)$ has the form

$$\theta[p,q](z+K+MB) = \theta[p,q](z)\exp\left[-\pi i\langle MB, M\rangle - 2\pi i\langle M, z+q\rangle + 2\pi i\langle K, p\rangle\right]. \tag{5.10}$$

Observe that all the coordinates of the characteristics p and q are determined modulo 1.

Definition 5.4. *The characteristics p and q with all coordinates equal to* 0 *or* 1/2 *are called half periods. A half period* $[p, q]$ *is said to be even if* $4\langle p, q\rangle \equiv 0 \pmod 2$ *and odd if* $4\langle p, q\rangle \equiv 1 \pmod 2$.

Exercise 5.5: Prove that the function $\theta[p, q](z)$ is even if $[p, q]$ is an even half period and odd if $[p, q]$ is an odd half period.

In particular the function $\theta(z)$ is even and for $e = q + Bp$ with $4\langle p, q\rangle \equiv 1 \pmod 2$ one has

$$\theta(e) = 0.$$

Example 5.6. For $g = 1$ the theta-function reduces to the Jacobi theta-function. Let τ be an arbitrary number with $\Im\tau > 0$. The Jacobi theta function is defined by the series

$$\theta(z;\tau) = \sum_{-\infty<n<\infty} \exp\left(\pi i\tau n^2 + 2\pi i n z\right). \tag{5.11}$$

Since

$$\left|\exp\left(\pi i\tau n^2 + 2\pi i n z\right)\right| = \exp\left(-\pi\Im\tau n^2 - 2\pi n\Im z\right)$$

the series (5.11) converges absolutely and uniformly in the strips $|\Im(z)| \leqslant \text{const}$ and defines an entire function of z.

The series (5.11) can be rewritten in the form common in the theory of Fourier series:

$$\theta(z) = \sum_{-\infty<n<\infty} \exp(\pi i\tau n^2) e^{2\pi i z n} \tag{5.12}$$

(the function $\vartheta_3(z\,;\,\tau)$) in the standard notation; see [[3]). The function $\theta(z)$ has the following periodicity properties:

$$\theta(z+1) = \theta(z) \tag{5.13}$$

$$\theta(z+\tau) = \exp(-\pi i\tau - 2\pi i z)\theta(z) \tag{5.14}$$

The integer lattice with basis 1 and τ is called the period lattice of the theta function. The remaining Jacobi theta-functions are defined with respect to the lattice $1, \tau = b/2\pi i$ as

$$\vartheta_1(z\,;\,\tau) := \theta[\frac{1}{2}, \frac{1}{2}](z) = \sum_{-\infty<n<\infty} \exp\left[\pi i\tau\left(n+\frac{1}{2}\right)^2 + 2\pi i\left(z+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\right]$$

$$\vartheta_2(z\,;\,\tau) := \theta[\frac{1}{2}, 0](z) = \sum_{-\infty<n<\infty} \exp\left[\pi i\tau\left(n+\frac{1}{2}\right)^2 + 2\pi i z\left(n+\frac{1}{2}\right)\right]$$

$$\vartheta_4(z\,;\,\tau) := \theta[0, \frac{1}{2}](z) = \sum_{-\infty<n<\infty} \exp\left[\pi i \tau n^2 + 2\pi i \left(z + \frac{1}{2}\right) n\right].$$

The functions $\vartheta_2(z\,;\,\tau)$, $\vartheta_3(z\,;\,\tau)$ and $\vartheta_4(z\,;\,\tau)$ are even functions of z while $\vartheta_1(z\,;\,\tau)$ is odd. So for $g = 1$, the theta-function $\theta(z\,;\,\tau) = \vartheta_3(z\,;\,\tau) = 0$ for $z = \dfrac{1+\tau}{2}$.

Exercise 5.7: Prove that the zeros of the function $\theta(z)$ form an integer lattice with the same basis 1, τ and with origin at the point $z_0 = \dfrac{1+\tau}{2}$.

By multiplying theta function (5.9) we obtain higher order theta functions. The function $f(z)$ is said to be a nth order theta function with characteristics p and q if it is an entire function of $z_1, \ldots, z_g$ and transforms according to the following law under translation of the argument by a vector of the period lattice

$$f(z + N + MB) = \exp\left[-\pi i n \langle MB, M\rangle - 2\pi i n \langle M, z+q\rangle + 2\pi i \langle p, N\rangle\right] f(z). \tag{5.15}$$

Exercise 5.8: Prove that the nth order theta functions with given characteristics q, p form a linear space of dimension n^g. Prove that a basis in this space is formed by the functions

$$\theta[\frac{p+\gamma}{n}, q](nz\,;\, nB), \tag{5.16}$$

where the coordinates of the vector γ run independently through all values from 0 to $n-1$.

Under a change of the homology basis $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$ under a symplectic transformation

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2g, \mathbb{Z}).$$

The period matrix transforms as (see 4.76)

$$B' = \int_{\beta'} \omega' = (cI_g + dB)(aI_g + bB)^{-1}.$$

Denote by R the matrix

$$R = aI_g + bB \tag{5.17}$$

The transformed values of the argument of the theta-function and of the characteristics are determined by

$$z = z'R$$
$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \frac{1}{2}\operatorname{diag}\begin{pmatrix} cd^t \\ ab^t \end{pmatrix}. \tag{5.18}$$

Here the symbol diag means the vectors of diagonal elements of the matrices ab^t and cd^t. We have the equality

$$\theta[p', q'](z'\,;\,B') = \chi\,\sqrt{\det R}\exp\left\{\frac{1}{2}\sum_{i\leqslant j} z_i z_j \frac{\partial \log\det R}{\partial B_{ij}}\right\}\theta[p, q](z\,;\,B), \tag{5.19}$$

where χ is a constant independent from z and B. See [19] for a proof.

Exercise 5.9: Prove the formula (5.19) for $g = 1$. *Hint*. Use the Poisson summation formula (see [20],[19]: if

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx$$

is the Fourier transform of a sufficiently nice function $f(x)$, then

$$\sum_{n=-\infty}^{\infty} f(2\pi n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

Theta function are connected by a complicated system of algebraic relations, which are called addition theorems. These are basically relations between formal Fourier series (see [19]). We present one of these relations. Let

$$\hat{\theta}[n](z;B) = \theta[\frac{n}{2}, 0](2z\,;\,2B),$$

according to (5.16) this is a basis of second order theta functions.

Lemma 5.10. *The following identity holds:*

$$\theta(z+w)\theta(z-w) = \sum_{n\in(\mathbb{Z}_2)^g} \hat{\theta}[n](z)\hat{\theta}[n](w). \tag{5.20}$$

The expression $n \in (Z_2)^g$ means that the summation is over the g-dimensional vectors n whose coordinates all take values in 0 or 1.

Proof. Let us first analyze the case $g = 1$. The formula (5.20) can be written as

$$\theta(z+w)\theta(z-w) = \hat{\theta}(z)\hat{\theta}(w) + \hat{\theta}[1](z)\hat{\theta}[1](w) \tag{5.21}$$

where

$$\theta(z) = \sum_k \exp(\pi i b k^2 + 2\pi i k z), \quad \hat{\theta}(z) = \sum_k \exp(2\pi i b k^2 + 4\pi i k z),$$

$$\hat{\theta}[1](z) = \sum_k \exp(\left[2\pi i b(\frac{1}{2}+k)^2 + 4\pi i(k+1/2)z\right], \quad \Im(b) > 0.$$

The left-hand side of (5.21) has then the form

$$\sum_{k,l} \exp\left[\pi i b(k^2+l^2) + 2\pi i k(z+w) + 2\pi i l(z-w)\right]. \tag{5.22}$$

We introduce new summation indices m and n by setting $m = (k+l)/2$ and $n = (k-l)/2$. The numbers m and n simultaneously are integers or half integers. In these variables the sum (5.22) takes the form

$$\sum \exp[2\pi i b m^2 + 4\pi i m z + 2\pi i b n^2 + 4\pi i n w]. \tag{5.23}$$

We break up this sum into two parts. The first part will contain the terms with integers m and n, while in the second part m and n are both half-integers. In the second part we change the notation from m to $m+\frac{1}{2}$ and from n to $n+\frac{1}{2}$. Then m and n are integers, and the expression (5.19) can be written in the form

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}} \exp[2\pi i b m^2 + 4\pi i m z]\exp[2\pi i b n^2 + 4\pi i n w] + \\
&\sum_{m,n\in\mathbb{Z}} \exp[2\pi i b(m+\frac{1}{2})^2 + 4\pi i(m+\frac{1}{2})z]\exp[2\pi i b(n+\frac{1}{2})^2 + 4\pi i(n+\frac{1}{2})w] = \\
&\hat{\theta}(z)\hat{\theta}(w) + \hat{\theta}[1](z)\hat{\theta}[1](w).
\end{aligned}$$

The lemma is proved for $g = 1$. In the general case $g > 1$ it is necessary to repeat the arguments given for each coordinate separately. The lemma is proved. □

Exercise 5.11: Suppose that the Riemann matrix B has a block-diagonal form $B = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}$, where B' and B'' are $k \times k$ and $l \times l$ Riemann matrices, respectively with $k + l = g$. Prove that the corresponding theta function factors into the product of two theta function

$$\begin{aligned}
\theta(z\,; B) &= \theta(z'\,; B')\theta(z''\,; B''), \\
z = (z_1, \ldots, z_g),\ \ z' &= (z_1, \ldots, z_k),\ \ z'' = (z_{k+1}, \ldots, z_g).
\end{aligned} \tag{5.24}$$

Notte that the period matrix of a Riemann surface never has a block diagonal structure.

5.2.1 The Riemann theorem on zeros of theta functions and its applications

To solve the Jacobi inversion problem we use the Riemann θ-function $\theta(z) = \theta(z\,;B)$ on the Riemann surface Γ. As usual we assume that $\alpha_1, \dots \alpha_g$ and $\beta_1, \dots, \beta_g$ is a canonical homology basis. The basis of holomorphic differentials $\omega_1, \dots, \omega_g$ is normalized

$$\int_{\alpha_j} \omega_k = \delta_{jk}, \quad \int_{\beta_j} \omega_k = B_{jk}.$$

Even though $\theta(z \,|\, B)$ is not single-valued on $J(\Gamma)$, the set of zeros is well defined because of (5.7). The set of zeros of $\theta(z \,|\, B)$ is an analytic set of codimension one in $J(\Gamma)$. Let $e = (e_1, \dots, e_g) \in \mathbb{C}^g$ be a given vector. We consider the function $F : \Gamma \to \mathbb{C}$ defined as

$$F(P) = \theta(A(P) - e), \tag{5.25}$$

where the Abel map A

$$A(P) = \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g \right),$$

is a holomorphic map of maximal rank of Γ into $J(\Gamma)$. Because of the periodicity properties of the theta-function (5.7), the function $F(P)$ transforms in the following way:

- $F(P + \alpha_j) = F(P)$ $$\tag{5.26}$$
- $F(P + \beta_j) = F(P) \exp \left[-\pi i B_{jj} - 2\pi i \int_{P_0}^{P} \omega_j + 2\pi i e_j \right].$ $$\tag{5.27}$$

The study of the zeros of $F(P)$ is thus the study of the intersection of $A(\Gamma) \subset J(\Gamma)$ with the set of zeros of $\theta(z\,;B)$ which form a well defined compact analytic sub-variety of the torus $J(\Gamma)$. Since Γ is compact, there are only two possibilities. Either $F(P)$ is identically zero on Γ or else $F(P)$ has only a finite number of zeros. The function $F(P)$ is single-valued and analytic on the cut surface $\tilde{\Gamma}$ (the Poincaré polygon). Assume that it is not identically zero. This will be the case if, for example $\theta(e) \neq 0$.

Lemma 5.12. *If $F(P) \not\equiv 0$, then the function $F(P)$ has g zeros on $\tilde{\Gamma}$ (counting multiplicity).*

Proof. To compute the number of zeros it is necessary to compute the logarithmic residue

$$\frac{1}{2\pi i} \oint_{\partial \tilde{\Gamma}} d \log F(P) \tag{5.28}$$

(assume that the zeros of $F(P)$ do not lie on the boundary of $\partial\tilde{\Gamma}$). We sketch a fragment of $\partial\tilde{\Gamma}$ (cf. the proof of lemma 4.15). The following notation is introduced for brevity and

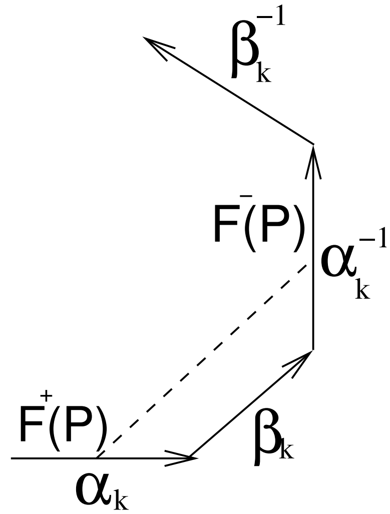


Figure 5.1: A fragment of $\tilde{\Gamma}$.

used below: F^+ denotes the value taken by F at a point on $\partial\tilde{\Gamma}$ lying on the segment α_k or β_k and F^- the value of F at the corresponding point α_k^{-1} or β_k^{-1} (see the figure 5.1).

The notation u^+ and u^- has an analogous meaning. In this notation the integral (5.28) can be written in the form

$$\frac{1}{2\pi i}\oint_{\partial\tilde{\Gamma}} d\log F(P) = \frac{1}{2\pi i}\sum_{k=1}^{g}\left(\int_{\alpha_k} + \int_{\beta_k}\right)[d\log F^+ - d\log F^-]. \tag{5.29}$$

Note that if P is a point on α_k then

$$u_j^-(P) = u_j^+(P) + \int_{\beta_k}\omega_j = u_j^+(P) + B_{jk}, \quad j = 1,\ldots,g, \tag{5.30}$$

(cf. (4.10)), while if P lies on β_k, then

$$u_j^+(P) = u_j^-(P) + \int_{\alpha_k}\omega_j = u_j^-(P) + \delta_{jk}, \quad j = 1,\ldots,g, \tag{5.31}$$

(cfr. (4.11)). We get from the law of transformation (5.7) of the theta function or from (5.27), that for P on the cycle α_k one has

$$\log F^-(P) = -\pi i B_{kk} - 2\pi i u_k^+(P) + 2\pi i e_k + \log F^+(P); \tag{5.32}$$

while on the cycle β_k from (5.26) one has

$$\log F^+ = \log F^-. \tag{5.33}$$

From this on α_k

$$d\log F^-(P) = d\log F^+(P) - frm{-}e\pi iomega_k(P), \tag{5.34}$$

and on β_k

$$d\log F^-(P) = d\log F^+(P). \tag{5.35}$$

Accordingly, from (5.34) and (5.34) the sum (5.29) can be written in the form

$$\frac{1}{2\pi i}\oint\limits_{\partial\tilde{\Gamma}} d\log F = \sum_k \oint\limits_{\alpha_k} \omega_k = g,$$

where we have used the normalization condition $\oint_{\alpha_k}\omega_k = 2\pi i$. The lemma is proved □

Note that although the function $F(P)$ is not a single-valued function on Γ, its zeros $P_1,\ldots,P_g$ do not depend on the location of the cuts along the canonical basis of cycles. Indeed, if this basis cycles is deformed then the path of integration from P_0 to P can change in the formulas for the Abel map. A vector of the form $(\oint_\gamma \omega_1,\ldots,\oint_\gamma \omega_g)$ is added to the argument of the theta-function $\theta(z)$ in (5.25). This is a vector of period lattice $\{N + MB\}$. As a result of this the function $F(P)$ can only be multiplied by a non zero factor in view of (5.7).

Now we will show now that the g zeros of $F(P)$ give a solution of the Jacobi inversion problem for a suitable choice of the vector e.

Theorem 5.13. *Let $e \in \mathbb{C}^g$, suppose that $F(P) = \theta(A(p) - e) \not\equiv 0$ and $P_1,\ldots,P_g$ are its zeros on Γ. Then on the Jacobi variety $J(\Gamma)$*

$$A^g(P_1,\ldots,P_g) = e + \mathcal{K}, \tag{5.36}$$

where $\mathcal{K} = (\mathcal{K}_1,\ldots,\mathcal{K}_g)$ is the vector of Riemann constants,

$$\mathcal{K}_j = -\frac{1+B_{jj}}{2} + \sum_{l\neq j}\left(\oint\limits_{\alpha_l}\omega_l(P)\int_{P_0}^{P}\omega_j\right), \quad j = 1,\ldots,g. \tag{5.37}$$

Proof. Consider the integral

$$\zeta_j = \frac{1}{2\pi i}\oint\limits_{\partial\tilde{\Gamma}} u_j(P)\,d\log F(P). \tag{5.38}$$

This integral is equal to the sum of the residues of the integrands i.e.,

$$\zeta_j = u_j(P_1) + \cdots + u_j(P_g), \tag{5.39}$$

where $P_1, \ldots, P_g$ are the zeros of $F(P)$ of interest to us. On the other hand, this integral can be represented by analogy with the proof of Lemma 5.12 in the form

$$\begin{aligned}
\zeta_j &= \frac{1}{2\pi i} \sum_{k=1}^{g} \left(\int_{\alpha_k} + \int_{\beta_k} \right) \left(u_j^+ d \log F^+ - u_j^- d \log F^-) \right) \\
&= \frac{1}{2\pi i} \sum_{k=1}^{g} \int_{\alpha_k} [u_j^+ d \log F^+ - (u_j^+ + B_{jk})(d \log F^+ - 2\pi i \omega_k)] \\
&+ \frac{1}{2\pi i} \sum_{k=1}^{g} \int_{\beta_k} u_j^+ d \log F^+ - (u_j^+ - \delta_{jk}) d \log F^+] \\
&= \frac{1}{2\pi i} \sum_{k=1}^{g} \left[\int_{\alpha_k} 2\pi i u_j^+ \omega_k - B_{jk} \int_{a_k} d \log F^+ + 2\pi i B_{jk} \right] + \frac{1}{2\pi i} \int_{b_j} d \log F^+,
\end{aligned}$$

in the course of computation we used formula (5.30)-(5.35). The function F takes the same values at the endpoints of α_k, therefore

$$\int_{\alpha_k} d \log F^+ = 2\pi i n_k,$$

where n_k is an integer. Further let Q_j and $\tilde{Q}_j$ be the initial and terminal point of β_j. Then

$$\begin{aligned}
&\int_{\beta_j} d \log F^+ = \log F^+(\tilde{Q}_j) - \log F^+(Q_j) = \\
&= \log \theta(A(Q_j) + f_j - e) - \log \theta(A(Q_j) - e) = -\pi i B_{jj} + 2\pi i e_j - 2\pi i u_j(Q_j),
\end{aligned}$$

where $f_j = (B_{1j}, \ldots, B_{gj})$ is a vector of the period lattice. The expression for ζ_j can now be written in the form

$$\begin{aligned}
\zeta_j &= u_j(P_1) + \cdots + u_j(P_j) = \\
&= e_j - \frac{1}{2} B_{jj} - u_j(Q_j) + \sum_k \int_{a_k} u_j \omega_k + \sum_k B_{jk}(-n_k + 1).
\end{aligned} \tag{5.40}$$

The last two terms can be thrown out, they correspond to the j-coordinate of some vector of the period lattice. Thus the relation (5.40) coincides with the desired relation (5.36) if it is proved that the constant in this equality reduces to (5.37), i.e.

$$-\frac{1}{2} B_{jj} - u_j(Q_j) + \sum_k \int_{\alpha_k} u_j \omega_k = \mathcal{K}_j, \quad j = 1, \ldots, g.$$

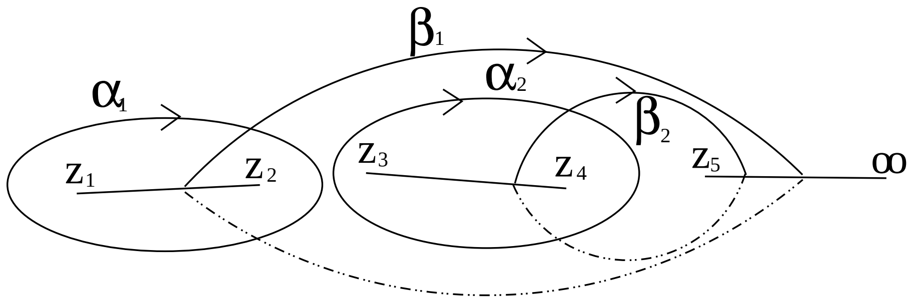


Figure 5.2: Homology basis.

To get rid of the term $u_j(Q_j)$ we transform the integral

$$\oint_{\alpha_j} u_j\omega_j = \frac{1}{2}[u_j^2(Q_j) - u_j^2(R_j)],$$

where R_j is the beginning of α_j and Q_j is its end (which is also the beginning of b_j). Further $u_j(Q_j) = u_j(R_j) + 1$. We obtain

$$\oint_{\alpha_j} u_j\omega_j = \frac{1}{2}[2u_j(Q_j) - 1],$$

hence

$$-u_j(Q_j) + \sum_{k=1}^{g}\int_{\alpha_k} u_j\omega_k = -\frac{1}{2} + \sum_{k\neq j,k=1}^{g}\int_{\alpha_k} u_j\omega_k.$$

The theorem is proved. □

Remark 5.14. We observe that the vector of Riemann constant depends on the choice of the base point P_0 of the Abel map. Indeed let $\mathcal{K}_{P_0}$ be the vector of Riemann constants with base point P_0. Then $\mathcal{K}_{Q_0}$ is related to $\mathcal{K}_{P_0}$ by

$$\mathcal{K}_{Q_0} = \mathcal{K}_{P_0} + (g-1)\int_{Q_0}^{P_0}\omega. \tag{5.41}$$

Example 5.15. The vector of Riemann constants can be easily calculated for hyperelliptic Riemann surfaces. In particular let us consider the curve $w^2 = \prod_{i=1}^{5}(z-z_i)$ of genus $g = 2$, and choose a basis of cycles as indicated in the figure 5.2. A normal basis of holomorphic differentials has the form

$$\omega_j = \frac{\prod_{k=1}^{2} c_{jk} z^{k-l} dz}{w}, \quad j = 1, 2, \tag{5.42}$$

where the constants c_{jk} are uniquely determined by

$$\int_{\alpha_k} \omega_j = \delta_{jk} .$$

We chose as base point of the Abel map the point $P_0 = (\infty, \infty)$. We need to compute

$$\left(\oint_{\alpha_2} \omega_2(P) \int_{P_0}^{P} \omega_1 \right), \quad \left(\oint_{\alpha_1} \omega_1(P) \int_{P_0}^{P} \omega_2 \right).$$

Using the fact that

$$\begin{aligned} \oint_{\alpha_2} \omega_2(P) \int_{P_0}^{P} \omega_1 &= \oint_{\alpha_2} \omega_2(P) \int_{P_0}^{z_4} \omega_1 + \int_{z_3}^{z_4} \omega_2(z,w) \int_{z_4}^{(z,w)} \omega_1 - \int_{z_3}^{z_4} \omega_2(z,-w) \int_{z_4}^{(z,-w)} \omega_1 \\ &= \oint_{\alpha_2} \omega_2(P) \int_{P_0}^{z_4} \omega_1 = \int_{P_0}^{z_4} \omega_1 = (-\frac{1}{2} - \frac{B_{12}}{2}) \end{aligned}$$

one obtains

$$\mathcal{K}_1 = -\frac{1 + B_{11}}{2} - \frac{1}{2} - \frac{B_{12}}{2} = -1 - \frac{B_{11} + B_{12}}{2}$$

In the same way calculating

$$\begin{aligned} \oint_{\alpha_1} \omega_1(P) \int_{P_0}^{P} \omega_2 &= \oint_{\alpha_1} \omega_1(P) \int_{P_0}^{z_2} \omega_2 + \int_{z_1}^{z_2} \omega_1(z,w) \int_{z_2}^{(z,w)} \omega_2 - \int_{z_1}^{z_2} \omega_1(z,-w) \int_{z_2}^{(z,-w)} \omega_2 \\ &= \oint_{\alpha_1} \omega_1(P) \int_{P_0}^{z_2} \omega_2 = -B_{21}/2 \end{aligned}$$

one obtains that

$$\mathcal{K}_2 = -\frac{1 + B_{22} + B_{21}}{2}$$

Observe that the vector $\mathcal{K}$ can be written in the form

$$\mathcal{K} = \left(0, \frac{1}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}\right) B$$

Namely, given the odd characteristic

$$p = \left(\frac{1}{2}, \frac{1}{2}\right), \quad q = \left(0, \frac{1}{2}\right),$$

one has that $\mathcal{K} = q + pB$. From this expression it follows that

$$\theta(\mathcal{K}) = 0.$$

It is a general result not restricted to this particular example that $\theta(z)|_{z=\mathcal{K}} = 0$.

Corollary 5.16. *Let D a positive divisor of degree g. If the function*

$$\theta(A(P) - A(D) + \mathcal{K})$$

does not vanish identically on Γ then its divisor of zeros coincides with D.

Accordingly, if the function $\theta(A(P) - e)$ is not identically equal to zero on Γ, then its zeros give a solution of the Jacobi inversion problem (5.5) for the vector $\eta = e+\mathcal{K}$. We have shown that the map (5.4) $A^g : S^g\Gamma \to J(\Gamma)$ is a local homeomorphism in a neighborhood of a non special positive divisor D of degree g. Since $\theta(z) \not\equiv 0$ for $z \in J(\Gamma)$, then $\theta(A^g(D))$ does not vanish identically on open subsets of $S^g\Gamma$. In the next subsection, we characterize the zero set of the θ-function. The zeros of the theta-function form an analytic subvariety of $J(\Gamma)$. The collection of these zeros forms the theta divisor in $J(\Gamma)$.

5.3 The Theta Divisor

In this section we study the set of zeros of the theta functions and in particular the Riemann vanishing theorem which prescribes in a rather detail manner the set of zeros of the theta-function on $\mathbb{C}^g$.

Theorem 5.17. *Let $e \in \mathbb{C}^g$, then $\theta(e) = 0$ if and only if $e = A(D_{g-1}) - \mathcal{K}$ where D_{g-1} is a positive divisor of degree $g - 1$ and $\mathcal{K}$ is the vector of Riemann constants (5.37).*

Remark 5.18. For $D \in S^{(g-1)}\Gamma$ the expression $A(D) - \mathcal{K}$ does not depend on the base point of the Abel map. The theorem 5.17 says that the theta-function vanishes on a $g - 1$-dimensional variety parametrized by $g - 1$ points of Γ. Defining $A(S^{g-1}\Gamma) = W_{g-1}$ the theta function vanishes on $W_{g-1} - \mathcal{K}$.

Proof. We first prove sufficiency. Let $P_1 + \cdots + P_g$ be a non special divisor and $v = A(P_1 + \cdots + P_g) - \mathcal{K}$. Let us consider $F(P) = \theta(A(P) - v)$. Either F is identically zero or not. In the former case for each $k = 1, \dots g$

$$F(P_k) = \theta(A(P_1 + \cdots + \hat{P}_k + \cdots + P_g) - \mathcal{K}) = 0,$$

where we use the symbol $\hat{P}_k$ to mean that P_k does not appear in the divisor. So for $e = A(P_1 + \cdots + \hat{P}_k + \cdots + P_g) - \mathcal{K}$ we have $\theta(e) = 0$.

In the latter case $F(P) \not\equiv 0$, we have that F has precisely g zeros on Γ due to lemma 5.12. Let $Q_1, \ldots Q_g$ be the zeros of F, then according to theorem 5.13 one has

$$A(Q_1 + \cdots + Q_g) = v + \mathcal{K} = A(P_1 + \cdots + P_g).$$

Since $P_1 + \cdots + P_g$ is not special, it follows from the Riemann-Roch and the Abel theorems that $Q_1 + \cdots + Q_g = P_1 + \cdots + P_g$. Therefore also in this case $F(P_k) = \theta(A(P_1 + \cdots + \hat{P}_k + \cdots + P_g) - \mathcal{K}) = 0$ for $k = 1, \ldots, g$. Since the set of non-special divisor of degree g is dense in $S^{(g)}\Gamma$, the divisors of the form $P_1 + \cdots + \hat{P}_k + \cdots + P_g$ form a dense subset of $S^{(g-1)}\Gamma$. Since the function $\theta(z)$ is continuous, it follows that $\theta(z)$ is identically zero on $W_{g-1} - \mathcal{K}$, where in general $W_n \subset J(\Gamma)$, is the Abel image of $S^{(n)}\Gamma$ for $n \geqslant 1$.

Conversely, let $\theta(e) = 0$. Then by Jacobi inversion theorem, since θ is not identically zero on $J(\Gamma)$. Then there exists an integer s, $1 \leqslant s \leqslant g$, so that

$$\theta(A(\tilde{D}_1 - \tilde{D}_2) - e) = 0, \quad \forall \tilde{D}_1, \tilde{D}_2 \in S^{(s-1)}\Gamma$$

but

$$\theta(A(D_1 - D_2) - e) \neq 0, \quad D_1, D_2 \in S^{(s)}\Gamma.$$

Let $D_1 = P_1 + \cdots + P_s$ and $D_2 = Q_1 + \cdots + Q_s$ where we assume that the points of the divisors are mutually distinct. Now let us consider the function

$$F(P) = \theta(A(P) + A(P_2 + \cdots + P_s) - A(Q_1 + \cdots + Q_s) - e)$$

Since $F(P_1) \neq 0$, this function is not identically zero on Γ. Therefore, by theorem 5.13 it has g zeros on Γ. These zeros are by construction $Q_1, \ldots, Q_s$ plus some other $g - s$ points $T_{s+1}, \ldots, T_g$. By theorem 5.13 one has

$$A(Q_1 + \cdots + Q_s + T_{s+1}, + \cdots + T_g) - \mathcal{K} = A(Q_1 + \cdots + Q_s) - A(P_2 + \cdots + P_s) + e$$

or equivalently

$$e = A(P_2 + \cdots + P_s + T_{s+1}, + \cdots + T_g) - \mathcal{K}$$

which is a point in $W_{g-1} - \mathcal{K}$. □

Regarding the zeros of the theta-function it is possible to prove a little bit more then stated in the previous theorems. Let $D \in S^{(g-1)}\Gamma$ and let $e = A(D) - \mathcal{K}$. Then

$$\mathrm{mult}_{z=e}\theta(z) = l(D).$$

where $l(D)$ is the dimension of the space $L(D)$. The proof of this identity can be found in [20].

Remark 5.19. The vector of Riemann constants has a characterisation in terms of divisors. Indeed there is a non positive divisor Δ of degree $g-1$ such that its Abel image coincides with $\mathcal{K}$, namely $A(\Delta) = \mathcal{K}$. Furthermore let D be a positive divisor of degree $g-1$, then the vector

$$e = A(D) - \mathcal{K}$$

is a zero of the theta-function, namely $\theta(e) = 0$. By the parity of the theta-function one has $\theta(-e) = 0$. It follows by theorem 5.17 that

$$-e = A(D') - \mathcal{K}$$

where D' is a positive divisor of degree $g-1$. Then summing up the two relations we obtain

$$2\mathcal{K} = A(D + D')$$

where $D + D'$ is a positive divisor of degree $2g-2$. It can be proved that the divisor $D + D$ is the divisor of a holomorphic differential, namely the vector $2\mathcal{K}$ is the Abel image of the divisor of a differential. More precisely a divisor D is canonical if and only if $A(D) = 2\mathcal{K}$ (see [19] for a proof of these results).

Using the characterization of the theta-divisor one can complete the description of the function $F(P)$.

Lemma 5.20. *Let $F(P) = \theta(A(P) - e)$ where $e = A(D) - \mathcal{K}$, $D \in S^{(g)}\Gamma$ and $\mathcal{K}$ the vector of Riemann constants defined in (5.37). Then*

1. *$F(P) \equiv 0$ iff the divisor D is special;*
2. *$F(P) \not\equiv 0$ iff $\dim\Omega(D) = 0$, i.e. the divisort D is not special. In this last case D is the divisor of zeros of $F(P)$.*

Proof. Let's prove part 1. of the lemma. Let $F(P) \equiv 0$, then by theorem 5.17 there is a positive divisor $\tilde{D}$ of degree $g-1$ so that

$$A(D) - \mathcal{K} - A(P) = A(\tilde{D}) - \mathcal{K}.$$

By Abel theorem, the identity holds if and only if D and $\tilde{D} + P$ are linearly equivalent, that is there is a meromorphic function in $L(D)$ with a zero in an arbitrary point $P \in \Gamma$. This is possible only if $l(D) > 1$ or equivalently $\dim\Omega(D) > 0$, namely D is special. Conversely, if $D \in S^g\Gamma$ is special then $l(D) > 1$ and therefore there is a function $f \in L(D)$ with an arbitrary zero in a point $P \in \Gamma$ so that $(f) = P + \tilde{D} - D$. where $\tilde{D} \in S^{(g-1)}\Gamma$. It follows by Abel theorem that $A(P) - A(D) + \mathcal{K} = -A(\tilde{D}) + \mathcal{K}$, then by theorem 5.17, one has $\theta(A(\tilde{D}) - \mathcal{K}) = 0$.

Now let us prove part 2. of the lemma. Suppose now that D is not special, then $F(P) \not\equiv 0$ and by theorem 5.13, the divisors of zeros of $F(P)$ coincides with D. $\square$

Corollary 5.21. *Let $e = A(D) - \mathcal{K}$ with $D \in S^{g-1}\Gamma$. Them the function $F(P) = \theta(A(P) - e)$ vanishes identically if and only if* $\mathbf{dim}\Omega(D + P_0) \geqslant 1$ *(Check!!) where P_0 is the base point of the Abel map.*

Proof. Let P_0 be the base point of the Abel map, then $A(P - P_0) = A(P)$. Suppose $F(P) \equiv 0$, then by theorem 5.17 there exists a positive divisor $\tilde{D}$ of degree $g - 1$ such that

$$A(P - P_0) - A(D) + \mathcal{K} = -A(\tilde{D}) + \mathcal{K}$$

which implies that $A(D + P_0) = A(\tilde{D} + P)$. By Abel theorem, there is a nontrivial meromorphic function h with divisor

$$(h) = \tilde{D} + P - D - P_0$$

for all $P \in \Gamma$. This implies that $l(D + P_0) \geqslant 2$ or equivalently, $D + P_0$ is a special divisor. Viceversa suppose that $\dim\Omega(D+P_0) \geqslant 1$, then $l(D+P_0) > 1$ so that $L(D+P_0)$ is generated by $\{1, h\}$ where h is a meromorphic function. So there is a nontrivial meromorphic function with poles in $D+P_0$ and having zero in an arbitrary point P (take for example the function $h - h(P)$) and some other $g - 1$ points given by the divisor $\tilde{D}$. It follows that

$$A(D + P_0) = A(\tilde{D} + P)$$

or equivalently

$$A(P - P_0) - A(D) + \mathcal{K} = -A(\tilde{D}) - \mathcal{K}$$

which implies by theorem 5.17 that $0 = \theta(-A(\tilde{D}) - K) = \theta(A(P - P_0) - A(D) - \mathcal{K}) = \theta(A(P) - A(D) - \mathcal{K})$ where we recall that P_0 is the base point of the Abel map. □

The zeros of the theta function (the points of the theta divisor) form a variety of dimension $2g - 2$ (for $g \geqslant 3$). If we delete from $J(\Gamma)$, the theta divisor, then we get a connected $2g$-dimensional domain. We get that the Jacobi inversion problem is solvable for all points of the Jacobian $J(\Gamma)$ and uniquely solvable for almost all points. Thus the collection $(P_1, \dots, P_g) = (A^{(g)})^{-1}(\eta)$ of points of the Riemann surface Γ (without consideration of order) is a single valued function of a point $\eta = (\eta_1, \dots \eta_g) \in J(\Gamma)$ (which has singularities at points of the theta divisor.) To find an analytic expression for this function we take an arbitrary meromorphic function $f(P)$ on Γ. Then the specification of the quantities $\eta_1, \dots, \eta_g$ uniquely determines the collection of values

$$f(P_1), \dots, f(P_g), \quad A^{(g)}(P_1, \dots, P_g) = \eta. \tag{5.3.43}$$

Therefore, any symmetric function of $f(P_1), \dots, f(P_g)$ is a single-valued meromorphic function of the g variables $\eta = (\eta_1, \dots, \eta_g)$, that is $2g$-fold periodic with period lattice

$\{2\pi i M + BN\}$. All these functions can be expressed in terms of a Riemann theta function. The following elementary symmetric functions has an especially simple expression:

$$\sigma_f(\eta) = \sum_{j=1}^{g} f(P_j). \tag{5.3.44}$$

From Theorem 5.36 and the residue formula we get for this function the representation

$$\begin{aligned}\sigma_f(\eta) = \frac{1}{2\pi i} \oint_{\partial\tilde{\Gamma}} f(P) d \log \theta(A(P) - \eta + \mathcal{K}) \\ - \sum_{f(Q_k)=\infty} \operatorname*{Res}_{P=Q_k} f(P) d \log \theta(A(P) - \eta + \mathcal{K}),\end{aligned} \tag{5.3.45}$$

the second term in the right hand side is the sum of the residue of the integrand over all poles if $f(P)$. As in the proof of Lemma 5.12 and Lemma 5.13, it is possible to transform the first term in (5.3.45) by using the formulas (5.34) and (5.35). The equality (5.3.45) can be written in the form

$$\sigma_f(\eta) = \frac{1}{2\pi i} \sum_k \int_{a_k} f(P)\omega_k - \sum_{f(a_k)=\infty} \operatorname*{Res}_{P=Q_k} f(P) d \log \theta(A(P) - \eta + \mathcal{K}). \tag{5.3.46}$$

Here the first term is a constant independent of η. We analyze the computation of the second term (the sum of residue) using an example.

Example 5.22. Γ is an hyperelliptic Riemann surface of genus g given by the equation $w^2 = P_{2g+1}(z)$, and the function f has the form $f(z, w) = z$, the projection on the z-plane. This function on Γ has a unique two-fold pole at ∞. We get an analytic expression for the function σ_f constructed according to the formula (5.3.44). In other words if $P_1 = (z_1, w_1)$, $\ldots, P_g = (z_g, w_g)$ is a solution of the inversion problem $A(P_1) + \cdots + A(P_g) = \eta$, then

$$\sigma_f(\eta) = z_1 + \cdots + z_g. \tag{5.3.47}$$

We take ∞ as the base point P_0 (the lower limit in the Abel mapping). According to (5.3.46) the function $\sigma_f(\eta)$ has the form

$$\sigma_f(\eta) = c - \operatorname*{Res}_{\infty} \left[z d \log \theta(A(P) - \eta + \mathcal{K}) \right].$$

Let us compute the residue. Take $\tau = z^{-\frac{1}{2}}$ as a local parameter in a neighborhood of ∞. Suppose that the holomorphic differentials ω_i have the form $\omega_i = \psi_i(\tau) d\tau$ in a neighborhood of ∞. Then

$$d\log\theta(A(P)-\eta+\mathcal{K}) = \sum_{i=1}^{g}[\log\theta(A(P)-\eta+\mathcal{K}]_i\omega_i(P) =$$
$$= \sum_{i=1}^{g}[\log\theta(A(P)-\eta+\mathcal{K})]_i\psi_i(\tau)d\tau$$

where $[\dots]_i$ denotes the partial derivative with respect to the ith variable. By the choice of the base point point $P_0 = \infty$, the decomposition of the vector-valued function $A(P)$ in a neighborhood of ∞ has the form

$$A(P) = \tau U + O(\tau^2),$$

where the vector $U = (U_1, \dots, U_g)$ has the form

$$U_j = \psi_j(0), \quad j = 1, \dots, g.$$

From these formulas we finally get

$$\sigma_f(\eta) = -(\log\theta(\eta-\mathcal{K}))_{i,j}U_iU_j + c = -\partial_x^2\log\theta(xU+\eta-\mathcal{K})|_{x=0} + c, \tag{5.3.48}$$

where $(\log\theta(\eta-\mathcal{K}))_{i,j}$ denotes derivative with respect to the $i-th$ and $j-th$ argument of the *theta*-function and c is a constant.

We shall show in the next Section that the function

$$u(x,t) = \frac{\partial^2}{\partial x^2}\log\theta(Ux+Wt-\eta+\mathcal{K}) + c$$

where $W_k = \dfrac{1}{3}\psi''(0)$ solves the Korteweg de Vries equation

$$u_t = \frac{1}{4}(6uu_x + u_{xxx}).$$

Exercise 5.23: Suppose that a hyperelliptic Riemann surface of genus g is given by the equation $w^2 = P_{2g+2}(z)$. Denotes its points at infinity by P_- and P_+. Chose P_- as the base point P_0 of the Abel mapping. Take $f(z,w) = z$ as the function f. Prove that the function $\sigma_f(\eta)$ has the form

$$\sigma_f(\eta) = \left(\log\frac{\theta(\eta-\mathcal{K}-A(P_+))}{\theta(\eta-\mathcal{K})}\right)_j U_j + c \tag{5.3.49}$$

where the vector $U = (U_1, \dots, U_g)$ has the form

$$U_j = \psi_j(0), \quad j = 1, \dots, g, \tag{5.3.50}$$

where the basis of holomorphic differentials have the form

$$\omega_j(P) = \psi_j(\tau) d\tau, \quad \tau = z^{-1}, \quad P \to \infty.$$

Exercise 5.24: Let Γ be a Riemann surface $w^2 = P_5(z)$ of genus 2. Consider the two systems of differential equations:

$$\frac{dz_1}{dx} = \frac{\sqrt{P_5(z_1)}}{z_1 - z_2}, \quad \frac{dz_2}{dx} = \frac{\sqrt{P_5(z_2)}}{z_2 - z_1} \tag{5.3.51}$$

$$\frac{dz_1}{dt} = \frac{z_2\sqrt{P_5(z_1)}}{z_1 - z_2}, \quad \frac{dz_2}{dt} = \frac{z_1\sqrt{P_5(z_2)}}{z_2 - z_1}. \tag{5.3.52}$$

Each of these systems determined a law of motion of the pair of points

$$P_1 = (z_1, \sqrt{P_5(z_1)}), \quad P_2 = (z_2, \sqrt{P_5(z_2)})$$

on the Riemann surface Γ. Prove that under the Abel mapping (5.1) these systems pass into the systems with constant coefficients

$$\frac{d\eta_1}{dx} = 0, \quad \frac{d\eta_2}{dt} = 1$$

$$\frac{d\eta_1}{dt} = -1, \quad \frac{d\eta_2}{dt} = 0.$$

In other words, the Abel mapping (5.1) is simply a substitution integrating the equations (5.3.51) and (5.3.52).

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