KdV equation, direct and inverse scattering

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1 Introduction

The modern theory of integrable systems started with the discovery in 1967 that the Korteweg de Vries (KdV) equation is integrable [7]. Such equation is an evolutionary partial differential equation and corresponds to an integrable system with infinite degree of freedom. Before 1967 it was believed that integrability as opposed to chaotic behaviour was a rare phenomena, restricted to particular examples. Indeed there were few examples of known integrable systems and results concerning integrability:

- two-body problem in celestial mechanics (Kepler, Newton 1600-1687);
- geodesics on ellipsoids and separation of variables in Hamilton-Jacobi equation (Jacobi 1837);

- Liouville theorem about the integrability by quadratures of an integrable systems (Liouville 1838);
- harmonic oscillator on the unit sphere (Neumann 1859);
- Clebsh system (rigid body) 1871;
- Lagrange, Euler and Kovalevskaya (1888) tops;
- Noether theorem about the relation between symmetries and integrals of motion of a mechanical system (Emmy Noether 1915);
- global version of Liouville theorem (Arnold 1963).

In 1967 Gardner, Green, Kruskal and Miura realized that the spectrum of the Schrödinger equation $-\frac{d^2}{dx^2} + u(x,t)$ does not change with time if the potential $u(x,t)$ evolves according to the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0,$$

where $u = u(x,t)$ is a scalar function of $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$ and $u_t = \frac{\partial}{\partial t}u(x,t)$ and $u_x = \frac{\partial}{\partial x}u(x,t)$. With this observation it was realized that the KdV equation can be integrated by inverse scattering that can be thought of as a nonlinear analogue of the Fourier transform used to solve linear partial differential equations. Around 1974 there were finite-dimensional versions of the inverse scattering transform that were applied to solve finite dimensional integrable systems like the Toda lattice or the Calogero-Moser systems (Flaschka [6], Manakov [14], Moser [15]). The main goal of these notes is to study integrable sytems with finite and infinite degree of freedoms. We will first study the inverse scattering transform for the open finite Toda lattice. Then we will consider inverse scattering for the KdV equation with rapidly decreasing initial data and periodic initial data. In this latter case, when the periodic initial data is "finite gap", namely when the spectrum of the Hill's equation has only a finite number of open gaps, the evolution in time of the KdV solution $u(x,t)$ corresponds to a linear flow on a finite-dimensional tori.

2 A short review of the classical theory of finite-dimensional integrable systems

We review the basic definitions in the theory of finite-dimensional integrable systems.

2.1 Poisson manifolds

We start with the definition of Poisson bracket.

Definition 2.1. A manifold P is said to be a Poisson manifold if P is endowed with a Poisson bracket, that is a Lie algebra structure defined on the space $\mathcal{C}^\infty(P)$ of smooth functions over P

$$\begin{gathered}\mathcal{C}^\infty(P) \times \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P) \\ (f,g) \mapsto \{f,g\}\end{gathered} \tag{2.1}$$

so that $\forall f,g,h \in \mathcal{C}^\infty(P)$ the bracket $\{\,.\,,\,.\,\}$

- is antisymmetric:
$$\{g,f\} = -\{f,g\}, \tag{2.2}$$

- bilinear
$$\begin{aligned}\{af+bh,g\} &= a\{f,g\}+b\{h,g\},\\ \{f,ag+bh\} &= a\{f,g\}+b\{f,h\}, \quad a,\, b\in\mathbb{R}\end{aligned} \tag{2.3}$$

- satisfies Jacobi identity
$$\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}=0; \tag{2.4}$$

- it satisfies Leibnitz identity with respect to the product of function
$$\{f\,g,h\} = g\,\{f,h\}+f\,\{g,h\}. \tag{2.5}$$

A Poisson bracket defines an homomorphism from the space $\mathcal{C}^\infty(P)$ to the space of vector fields over P:

$$\begin{aligned}\mathcal{C}^\infty(P) &\to vect(P)\\ f &\to X_f = \{.\,,f\}\end{aligned}$$

so that
$$[X_f, X_g] = -X_{\{f,g\}},$$
where $[.,.]$ is the commutator of vector fields also known as Lie bracket: $L_X Y := [X,Y]$. In order to write the definition 2.1 in local coordinates $x=(x^1,\dots,x^N)$ let us introduce the matrix

$$\pi^{ij}(x) := \{x^i, x^j\}, \quad i,\, j=1,\dots,N=\dim P. \tag{2.6}$$

Theorem 2.2. *[5] 1) Given a Poisson manifold P, and a system of local coordinates over P, then the matrix $\pi^{ij}(x)$ defined in (2.6) is antisymmetric and satisfies*

$$\frac{\partial \pi^{ij}(x)}{\partial x^s}\pi^{sk}(x)+\frac{\partial \pi^{ki}(x)}{\partial x^s}\pi^{sj}(x)+\frac{\partial \pi^{jk}(x)}{\partial x^s}\pi^{si}(x)=0, \quad 1\le i<j<k\le N. \tag{2.7}$$

Furthermore the Poisson bracket of two smooth functions is calculated according to

$$\{f,g\} = \pi^{ij}(x)\frac{\partial f(x)}{\partial x^i}\frac{\partial g(x)}{\partial x^j}. \tag{2.8}$$

2) Given a change of coordinates

$$\tilde{x}^k = \tilde{x}^k(x), \quad k=1,\dots,N,$$

then the matrices $\pi^{ij}(x)=\{x^i,x^j\}$ e $\tilde{\pi}^{kl}(\tilde{x})=\{\tilde{x}^k,\tilde{x}^l\}$ satisfy the rule of transformation of a tensor of type (2,0):

$$\tilde{\pi}^{kl}(\tilde{x}) = \pi^{ij}(x)\frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial \tilde{x}^l}{\partial x^j}. \tag{2.9}$$

3) Viceversa, given a smooth manifold P and an antisymmetric tensor (2,0) $\pi^{ij}(x)$ such that (2.7) is satisfied, then (2.8) defines over P a Poisson bracket.

Proof. The matrix $\pi^{ij}(x)$ is clearly antisymmetric. In order to derive (2.8) one observe that for a fixed function f, the application

$$\begin{aligned} \mathcal{C}^\infty(P) &\to \mathcal{C}^\infty(P) \\ g &\to \{g, f\} \end{aligned}$$

is linear and satisfies Leibnitz rule (2.5), therefore it is a linear differential operator of first order, namely

$$\{g, f\} = X_f g,$$

for a vector field

$$X_f = X_f^j \frac{\partial}{\partial x^j},$$

where we are taking the sum over repeated indices. In order to determine the components of the vector field X_f one considers

$$X_f^j = X_f\, x^j = \{x^j, f\}.$$

Now let us fix x_j and consider the linear map

$$f \mapsto \{f, x^j\} = X_{x^j} f = X_{x^j}^k \frac{\partial}{\partial x^k} f.$$

Since $X_{x^j}^k = \pi^{kj}$ by (2.6), it follows from the above relations that

$$X_f^j = \pi^{jk} \frac{\partial}{\partial x^k} f,$$

so that

$$\{g, f\} = X_f g = X_f^j \frac{\partial g}{\partial x^j} = \pi^{jk} \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial x^j}.$$

Using the same rule for the change of coordinates $\tilde{x}^j = \tilde{x}^j(x)$ and $\tilde{x}^k = \tilde{x}^k(x)$ one obtains the tensor rule (2.9). Equation (2.7) follows from Jacobi identity.

To prove the sufficiency of the theorem one observe that given a $(2, 0)$ antisymmetric tensor $\pi^{ij}(x)$, the map

$$(f, g) \mapsto \{f, g\} := \pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

is bilinear, antisymmetric and satisfies Leibnitz rule. Furthermore it does not depend on the choice of local coordinates

$$\tilde{\pi}^{kl} \frac{\partial f}{\partial \tilde{x}^k} \frac{\partial g}{\partial \tilde{x}^l} = \pi^{st} \frac{\partial \tilde{x}^k}{\partial x^s} \frac{\partial \tilde{x}^l}{\partial x^t} \frac{\partial x^i}{\partial \tilde{x}^k} \frac{\partial x^j}{\partial \tilde{x}^l} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \pi^{st} \delta_s^i \delta_t^j \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}.$$

In order to show validity of the Jacobi identity it is sufficient to observe that for any functions f, g, h and any antisymmetric tensor $\pi^{ij}(x)$ the following identity is satisfied:

$$\begin{aligned} &\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} \\ &= \Big[\{\{x^i, x^j\}, x^k\} + \{\{x^k, x^i\}, x^j\} + \{\{x^j, x^k\}, x^i\} \Big] \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \frac{\partial h}{\partial x^k}. \end{aligned}$$

so that the Jacobi identity follows from (2.7). □

The Poisson bracket is said to be non degenerate if the rank $\pi = dim(P)$. Clearly the antisymmetry implies that only even dimensional manifolds can have a non-degenerate Poisson bracket.

Definition 2.3. Given a Poisson bracket the set of functions that commutes with any other functions of $\mathcal{C}^\infty(P)$, namely

$$\{h \in \mathcal{C}^\infty(P) \,|\, \{h, f\} = 0, \forall f \in \mathcal{C}^\infty(P)\}$$

are called *Casimirs* of the Poisson bracket.

For a nondegenerate Poisson bracket, the only Casimirs are the constant functions.

Definition 2.4. A $2n$-dimensional P manifold is called symplectic manifold if it is endowed with a close non degenerate 2-form ω.

In local coordinates one has

$$\omega = \sum_{i,j=1}^{n} \omega_{ij} dx^i \wedge dx^j$$

where $\wedge$ stands for the exterior product. We recall that the form ω is closed if $d\omega = \sum_{ijk=1}^{n} \frac{\partial}{\partial x^k} \omega_{ij} dx^k \wedge dx^i \wedge dx^j = 0$, which implies that

$$\frac{\partial}{\partial x^k}\omega_{ij} + \frac{\partial}{\partial x^i}\omega_{jk} + \frac{\partial}{\partial x^j}\omega_{ki} = 0, \quad i \neq j \neq k.$$

Lemma 2.5. *A Poisson manifold $\{P, \pi\}$ with non degenerate Poisson bracket π, is a symplectic manifold, with $\omega_{ij} = (\pi^{ij})^{-1}$.*

For a symplectic manifold (P, ω) one has the identies

$$\omega(X_f, .) = -df$$

and

$$\{f, g\} = \omega(X_g, X_f) = X_g(f) = \langle df, X_g \rangle$$

where $\langle ., . \rangle$ is the pairing between one form and vectors, i.e. for a one form $\alpha = \alpha_i dx^i$ and a vector $v = v^i \frac{\partial}{\partial x^i}$ then $\langle \alpha, v \rangle = \alpha_i v^i$. In order to verify the above second identity let X_f^i and X_g^j be the coordinates of the vector fields X_f and X_g respectively, then one has

$$\omega(X_g, X_f) = \sum_{i,j} \omega_{ij} X_g^i X_f^j = \sum_{i,j} \sum_{k,l} \omega_{ij} \pi^{il} \frac{\partial g}{\partial x^l} \pi^{jk} \frac{\partial f}{\partial x^k} = \sum_{il} \pi^{il} \frac{\partial g}{\partial x^l} \frac{\partial f}{\partial x^i} = \{f, g\}.$$

The *classical Darboux theorem* says that in the neighbourhood of every point of a symplectic manifold (P, ω), $\dim P = 2n$, there is a local systems of co-ordinates $(q^1, \ldots q^n, p_1, \ldots, p_n)$ called Darboux coordinates or canonical coordinates such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i \,. \tag{2.10}$$

In such coordinates the Poisson bracket takes the form

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right)$$

and the Hamiltonian vector field X_f takes the form

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i} \right)$$

and the Poisson tensor π is

$$\pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The existence of Darboux coordinates is related to the vanishing of the second group of the so called Poisson cohomology $H^*(P, \pi)$. If the Poisson bracket is non-degenerate, the Poisson cohomology coincides with the de-Rham cohomology and Darboux theorem is equivalent to the vanishing of second de-Rham cohomology group in an open set. There are many tools for computing de Rham cohomology groups, and these groups have probably been computed for most familiar manifolds. However, when π is not symplectic, then $H^*(P, \pi)$ does not vanish even locally [16] and it is is much more difficult to compute than the de Rham cohomology. There are few Poisson (non-symplectic) manifolds for which Poisson cohomology has been computed [8]. The Poisson cohomology $H^*(P, \pi)$ can have infinite dimension even when P is compact, and the problem of determining whether $H^*(P, \pi)$ is finite dimensional or not is already a difficult open problem for most Poisson structures that we know of. In the case of linear Poisson structures, Poisson cohomology is intimately related to Lie algebra cohomology, also known as Chevalley - Eilenberg cohomology, [18].

Given a Poisson manifold (P, π), $\dim P = N$, and a function $H \in \mathcal{C}^\infty(P)$, an Hamiltonian system in local coordinates $(x^1, \ldots, x^N)$ is a set of N first order ODEs defined by

$$\dot{x}^i = \{x^i, H\},$$

with initial condition $x^i(t = 0) = x^i_0$. For a symplectic manifold (P, ω), $\dim P = 2n$, the Hamilton equations in Darboux coordinates takes the form

$$\begin{aligned} \dot{q}^i &= \{q^i, H\} = \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= \{p_i, H\} = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n \end{aligned} \tag{2.11}$$

with initial conditions $q^i(t = 0) = q^i_0$, $p_i(t = 0) = p^0_i$.

Definition 2.6. A function $F \in \mathcal{C}^\infty(P)$ is said to be a conserved quantity for the Hamiltonian system (2.11) if

$$\frac{dF}{dt} = \{F, H\} = 0.$$

Namely conserved quantities Poisson commute with the Hamiltonian. We remark that if $F_1, \ldots, F_m$ are conserved quantities, then any function of $g = g(F_1, \ldots, F_m)$ is a conserved quantity.

Let Φ_F^s be the Hamiltonian flow associated with $F \in \mathcal{C}^\infty(P)$ and Φ_H^t the Hamiltonian flow associated with the Hamiltonian H. The flow $\Phi_F^t(\Phi_H^t(x))$ indicates the integral curve obtained by first applying the Hamiltonian vector field X_H and then the Hamiltonian vector field X_F. Viceversa $\Phi_H^t(\Phi_F^s(x))$ indicates the integral curve obtained by first applying the Hamiltonian vector field X_F and then the Hamiltonian vector field X_H. A natural question is to ask when $\Phi_F^s(\Phi_H^t(x)) = \Phi_H^t(\Phi_F^s(x))$.

Lemma 2.7. *Let $(P, \{\,.\,,\,.\,\})$ be a nondegenerate Poisson bracket. Consider the Hamiltonians $F, H \in \mathcal{C}^\infty(P)$, in involution $\{F, H\} = 0$ and the Hamiltonian flows $\Phi_H^t(x)$ and $\Phi_F^s(x)$ associated with the dynamical systems*

$$\frac{dx^i}{dt} = \{x^i, H\}, \quad i = 1, \ldots, N \tag{2.12}$$

and

$$\frac{dx^i}{ds} = \{x^i, F\}, \quad i = 1, \ldots, N. \tag{2.13}$$

Then the flows commute, namely

$$\Phi_H^t \circ \Phi_F^s = \Phi_F^s \circ \Phi_H^t.$$

Proof. It is sufficient to prove that the time derivative with respect to t and s commute. We have

$$\frac{d}{ds}\frac{dx^i}{dt} = \frac{d}{ds}\{x^i, H\} = \{\frac{dx^i}{ds}, H\} = \{\{x^i, F\}, H\}$$

and

$$\frac{d}{dt}\frac{dx^i}{ds} = \frac{d}{dt}\{x^i, F\} = \{\frac{dx^i}{dt}, F\} = \{\{x^i, H\}, F\}$$

subtracting the above two relations and using Jacobi identity

$$\frac{d}{ds}\frac{dx^i}{dt} - \frac{d}{dt}\frac{dx^i}{ds} = -\{\{F, H\}, x^i\} = 0$$

□

In order to introduce Liouville theorem, we first define the concept of Lagrangian sub manifold and integrable system.

Definition 2.8. Let P be a symplectic manifold of dimension $2n$. A a sub-manifold $G \subset P$ is called a Lagrangian submanifold if $\dim G = n$ and the symplectic form is identically zero on vectors tangent to G, namely

$$\omega(X, Y) = 0, \quad \forall X, Y \in TG.$$

Definition 2.9. A Hamiltonian system defined on a $2n$ dimensional Poisson manifold P with non degenerate Poisson bracket and with Hamiltonian $H \in \mathcal{C}^\infty(P)$ is called completely integrable if there are n independent conserved quantities $H = H_1, \ldots, H_n$ in involution, namely

$$\{H_j, H_k\} = 0, \quad j, k = 1, \ldots, n \tag{2.14}$$

and the gradients $\nabla H_1, \ldots \nabla H_n$ are linearly independent.

Let us consider the level surface

$$M_E = \{(p,q) \in P \,|\, H_1(p,q) = E_1, \quad H_2(p,q) = E_2, \quad H_n(p,q) = E_n\} \tag{2.15}$$

for some constants $E = (E_1, \dots, E_n)$.

We now introduce a special type of change of coordinates that leave the symplectic form invariant.

Definition 2.10. A change of coordinates $x \to \Phi(x)$ is defined by $2n$ functions. The change of coordinates is a *canonical transformation* if $\Phi^*\omega = \omega$ where Φ^* is the pullback of Φ.

Since $\omega = dW$ where the one-form W in canonical coordinates take the form

$$W = p_i dq^i$$

and the pullback commutes with differentiation, one has

$$0 = \omega - \Phi^*\omega = dW - \Phi^*(dW) = d(W - \Phi^*W) = 0.$$

Namely the form $d(W - \Phi^*W)$ is exact, so there is locally a function S such that $W - \Phi^*W = dS$ and S is called the generating function of the canonical transformation.

Theorem 2.11. *[Liouville, see e.g. [5]] Consider a completely integrable Hamiltonian system on a non degenerate Poisson manifold P of dimension $2n$ and with canonical coordinates (q,p). Let us suppose that the Hamiltonians $H_1(p,q)$, ..., $H_n(p,q)$ are linearly independent on the level surface M_E (2.15) for a given $E = (E_1, \dots, E_n)$. The Hamiltonian flows on M_E are integrable by quadratures.*

Proof. By definition the system posses n independent conserved quantities $H_1 = H$, $H_2, \dots H_n$. Without loosing generality, we assume that (q,p) are canonical coordinates with respect to the symplectic form ω and the Poisson bracket $\{.,.\}$.

The gradients

$$\nabla H_j = \left(\frac{\partial H_j}{\partial q^1}, \dots, \frac{\partial H_j}{\partial q^n}, \frac{\partial H_j}{\partial p_1}, \dots, \frac{\partial H_j}{\partial p_n}\right)$$

are orthogonal to the surface M_E. Since the vector fields X_{H_j} are orthogonal to ∇H_k because $\{H_j, H_k\} = 0$, it follows that the vector fields X_{H_j} are tangent to the level surface M_E. Furthermore, since the Hamiltonian H_j are linearly independent, it follows that the vector fields X_{H_j}, $j = 1, \dots, n$ generate all the tangent space TM_E. Therefore the symplectic form is identically zero on the tangent space to M_E, namely $\omega|_{TM_E} \equiv 0$ because

$$\omega(X_{H_j}, X_{H_k}) = \{H_k, H_j\} = 0.$$

This is equivalent to say that M_E is a Lagrangian submanifold. We also observe that since ∇H_j, $j = 1, \dots, n$ are linearly independent, it is possible to assume, without loosing in generality that

$$\det \frac{\partial H_j}{\partial p_k} \neq 0.$$

Then by the implicit function theorem we can define

$$p_k = p_k(q, E).$$

Putting together the last two observations, we have for fixed $E = (E_1, \ldots, E_n)$

$$0 = \omega|_{TM_E} = \sum_i dp_i(q, E) \wedge dq^i = \sum_{ij} \frac{\partial p_i}{\partial q^j} dq^j \wedge dq^i$$

which implies

$$\frac{\partial p_i}{\partial q^j} - \frac{\partial p_j}{\partial q^i} = 0, \quad i \neq j.$$

The above identity implies that the one form $W = p_i dq^i$ is exact, and therefore there exists a function $S = S(q, E)$ so that $W|_{M_E} = dS|_{M_E}$. The function S is the generating function of a canonical transformation. $(q, p) \to (\psi, E)$ where

$$\Phi^* W = -\sum \psi^i dE_i$$

and $W - \Phi^* W = dS$ implies

$$\sum p_i dq^i - \frac{\partial S}{\partial q^i} dq^i - \frac{\partial S}{\partial E_i} dE_i = -\sum \psi^i dE_i$$

so that

$$p_i = \frac{\partial S}{\partial q^i}, \quad \psi_i = \frac{\partial S}{\partial E_i}.$$

In the canonical coordinates (ψ, E) the Hamiltonian flow with respect to the Hamiltonian $H_1 = H$ takes the form

$$\dot{\psi}_i = \{\psi_i, H_1\} = \frac{\partial H_1}{\partial E_i} = \delta_{1i}$$
$$\dot{E}_i = \{E_i, H_1\} = -\frac{\partial H_1}{\partial \psi_i} = 0.$$

So the above equations can be integrated in a trivial way:

$$\psi_1 = t + \psi_1^0, \;\; \psi_i = \psi_i^0, \quad i = 2, \ldots, n \qquad\qquad E_i = E_i^0, \;\; i = 1, \ldots, n$$

where $\psi_i^0$and E_i^0 are constants. Therefore we have shown that the Hamiltonian flow can be integrated by quadratures. Furthermore

$$q = q(t + \psi_1^0, \psi_2^0, \ldots, \psi_n^0, E), \quad p = p(t + \psi_1^0, \psi_2^0, \ldots, \psi_n^0, E).$$

□

In 1968 Arnold observed that if the level surface M_E is compact, the motion takes place on a torus and is quasi-periodic.

Theorem 2.12 (Arnold)**.** *If the level surface M_{E^0} defined in (2.15) is compact and connected then the level surfaces M_E for $|E - E^0|$ sufficiently small are diffeomorphic to a torus*

$$M_E \simeq T^n = \{(\phi_1, \ldots, \phi_n) \in \mathbb{R}^n \,|\, \phi_i \sim \phi_i + 2\pi, \, i = 1, \ldots, n\}, \tag{2.16}$$

and the motion on M_E is quasi-periodic, namely

$$\phi_1(t) = \omega_1(E)\, t + \phi_1^0, \ldots, \phi_n(t) = \omega_n(E)\, t + \phi_n^0 \tag{2.17}$$

where $\omega_1(E)$, …, $\omega_n(E)$ depends on E and the phases ϕ_1^0, …, ϕ_n^0 are arbitrary.

Proof. To prove the theorem we use a standard lemma (see [5]).

Lemma 2.13. *Let M be a compact connected n-dimensional manifold. If on M there are n linearly independ vector fields $X_1, \dots, X_n$ such that*

$$[X_i, X_j] = 0, \quad i, j = 1 \dots, n$$

then $M \simeq T^N$, the n-dimensional torus.

In our case the vector field $X_{H_1}, \dots, X_{H_n}$ are linearly independent and commuting, so, in the case M_{E^0} is compact and connected, it is also isomorphic to a n-dimensional torus. By continuity, for small values of $|E - E^0|$ the surface M_E is also isomorphic to a torus. The coordinates $\psi = (\psi_1, \psi_2, \dots, \psi_n)$ introduced in the proof of Liouville theorem 2.11 are not angles on the torus. In the canonical variables (ψ, E) the vector fields

$$X_{H_m} = \{\, .\, , H_m\} = \sum_{j=1}^{n} \frac{\partial H_m}{\partial E_j} \frac{\partial}{\partial \psi_j} = \frac{\partial}{\partial \psi_m}.$$

Let us make a change of variable $\phi = \phi(\psi)$ so that the coordinates $\phi = (\phi_1, \dots, \phi_n)$ are angles on the torus and let $I_1(E), \dots I_n(E)$ be the canonical variables associated to the angles $(\phi_1, \dots, \phi_n)$. By construction one has for any Hamiltonian H_m, $m = 1, \dots, n$

$$X_{H_m} = \{\, ., H_m\} = \frac{\partial}{\partial \psi_m} = \sum_{j=1}^{n} \frac{\partial H_m}{\partial I_j} \frac{\partial}{\partial \phi_j},$$

since H_m depends only on E_m and ϕ depends only on ψ. It follows that ϕ_j and ψ_k are related by a linear transformation

$$\phi_j = \sum_m \sigma_{jm} \psi_m, \quad \sigma_{jm} = \sigma_{jm}(E), \quad \det \sigma_{jm} \neq 0.$$

Comparing the above two relations one arrives to

$$\sigma_{jm} = \frac{\partial H_m}{\partial I_j}.$$

Let us verify that (ϕ, I) are indeed canonical variables:

$$\{\phi_j, I_k\} = \{\sum_k \sigma_{jm} \psi_m, I_k\} = \sum_m \sigma_{jm} \{\psi_m, I_k\} = \sum_m \sigma_{jm} \frac{\partial I_k}{\partial E_m} = \sum_m \frac{\partial H_m}{\partial I_j} \frac{\partial I_k}{\partial E_m} = \delta_{jk}.$$

The equation of motions in the variables (ϕ, I) are given by

$$\dot{\phi}_k = \frac{\partial H_1}{\partial I_k} =: \omega_k(E)$$

$$\dot{I}_k = \frac{\partial H_1}{\partial \phi_k} = 0$$

therefore the motion is quasi periodic on the tori. In the variable (p, q), with $p = p(\phi, I)$, $q = q(\phi, I)$, the evolution is given as

$$q = q(\omega_1 t + \phi_1^0, \dots, \omega_n t + \phi_n^0, I)$$
$$p = p(\omega_1 t + \phi_1^0, \dots, \omega_n t + \phi_n^0, I),$$

where $(\phi_1^0, \dots, \phi_n^0)$ are constant phases. □

3 Korteweg de Vries equation and direct/inverse scattering

The Korteweg de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0$$

is a nonlinear evolutionary partial differential equation for the function $u = u(x,t)$, $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$. Here u_t stands for the partial derivative with respect to t and u_x for the partial derivative with respect to x. This equation was discovered in 1877 by Boussinesq and in 1895 by D. Korteweg ad G. de Vries. By the scaling $x \to \alpha x$, $t \to \beta t$ and $u \to \gamma u$ one can normalised the equation to the form

$$u_t - 6uu_x + u_{xxx} = 0. \tag{3.1}$$

This equation has an exact travelling wave solution, namely a solution of the form $u(x,t) = f(x-ct)$. Plugging the ansatz into the KdV equation and putting the integration constants equal to zero, one arrives to the ODE

$$\frac{df}{f\sqrt{2f+c}} = d\theta, \quad \theta = x - ct$$

$$u(x,t) = -\frac{1}{2}c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct + x_0)\right).$$

Such solution was called soliton by Zabusky and Kruskal (1965) and they observed quite remarkable properties of the KdV solitons. In 1967 Gardner, Green, Kruskal and Miura discovered that the KdV equation can be solved via inverse scattering transform method (ISTM). The rest of this section will be devoted to explain this method. The key observation is the following result. Let us introduce the operator

$$L := -\partial_x^2 + u(x,t) \tag{3.2}$$

$$A := 4\partial_x^3 - 6u\partial_x - 3u_x, \tag{3.3}$$

and define $L_t := u_t$.

Theorem 3.1. *The equations*

$$\dot{L} = [L, A] \tag{3.4}$$

is equivalent to the KdV equation (3.1).

Proof. Let us compute the commutator $[L, A]$ and show that is a multiplication operator.

$$[L, A] = [-\partial_x^2 + u, 4\partial_x^3 - 6u\partial_x - 3u_x] = 6[\partial_x^2, u\partial_x] + 3[\partial_x^2, u_x] + 4[u, \partial_x^3] - 6[u, u\partial_x].$$

Calculating each single term one has

$$[\partial_x^2, u\partial_x]f = uf_{xxx} + 2u_xf_{xx} + u_{xx}f_x - uf_{xxx}$$
$$[\partial_x^2, u_x]f = u_xf_{xx} + 2u_{xx}f_x + u_{xxx}f - u_xf_{xx}$$
$$[u, \partial_x^3]f = uf_{xxx} - 3u_xf_{xx} - 3u_{xx}f_x - u_{xxx}f - uf_{xxx}$$
$$[u, u\partial_x]f = u^2f_x - u^2f_x - uu_xf$$

so that

$$[L,A]f = 12u_x f_{xx} + 6u_{xx}f_x + 6u_{xx}f_x + 3u_{xxxx} - 12u_x f_{xx} - 12u_{xx}f_x - 4u_{xxx}f + 6uu_x f$$
$$= -u_{xxx}f + 6uu_x f$$

□

The operator L is an operator acting on functions defined on $\mathbb{R}$. Here we assume that $u(x,0)$ is such that

$$\int_{-\infty}^{\infty} |u(x)|(1+x^2)dx < \infty.$$

For functions ψ in the Hilbert space $L^2(\mathbb{R})$ with scalar product $\langle .\,,.\rangle$ one has

$$\langle \psi, L\psi\rangle = \int_{-\infty}^{+\infty} \bar{\psi}(x)(-\partial_x^2 + u)\psi(x)dx = \int_{-\infty}^{+\infty} [(-\partial_x^2 + u(x,t))\bar{\psi}(x)]\psi(x)dx$$

where we have integrated by parts. It follows that the operator L is Hermitian or self-adjoint, namely

$$\langle \psi, L\psi\rangle = \langle L\psi, \psi\rangle.$$

As a consequence of being Hermitian, the eigenvalues of L are real, since

$$\langle \psi, L\psi\rangle = \langle \psi, \lambda\psi\rangle = \lambda\langle \psi, \psi\rangle = \langle L\psi, \psi\rangle = \langle \lambda\psi, \psi\rangle = \bar{\lambda}\langle \psi, \psi\rangle$$

so that $\bar{\lambda} = \lambda$.

Lemma 3.2. *Let λ be an eigenvalue of L, then* $\dfrac{d\lambda}{dt} = 0$.

Proof.

$$\dot{L}\psi + L\dot{\psi} = \dot{\lambda}\psi + \lambda\dot{\psi},$$

so that

$$LA\psi - AL\psi + (L-\lambda)\dot{\psi} = \dot{\lambda}\psi.$$

Next taking the scalar product with respect to ψ one has

$$\begin{aligned}
\langle \psi, LA\psi\rangle - \langle \psi, AL\psi\rangle + \langle \psi, (L-\lambda)\dot{\psi}\rangle &= \dot{\lambda}\langle \psi,\, \psi\rangle \\
\langle L\psi, A\psi\rangle - \langle \psi, A\lambda\psi\rangle + \langle (L-\lambda)\psi, \dot{\psi}\rangle &= \dot{\lambda}\langle \psi,\, \psi\rangle \\
\lambda\langle \psi, A\psi\rangle - \lambda\langle \psi, A\psi\rangle + \langle (L-\lambda)\psi, \dot{\psi}\rangle &= \dot{\lambda}\langle \psi,\, \psi\rangle \\
0 &= \dot{\lambda}\langle \psi,\, \psi\rangle,
\end{aligned}$$

where we use the reality of the eigenvalues and the symmetry of the operator L. □

The solution of KdV will be obtained by first study the Schrödinger operator $L(t=0)$. To such operator we will associate the scattering data. This is called direct scattering problem. In the scattering coordinates the time evolution is trivial if the potential $u = u(x,t)$ evolves according to the KdV equation and it is straightforward to obtain the scattering data at time t. The inverse scattering aims at reconstructing the potential $u(x,t)$ from the scattering data at time t. The diagram below summarise the procedure just described.

$$\begin{array}{ccc} \{u(x,t=0)\} & \overset{\text{direct scattering}}{\longrightarrow} & \text{study } -\partial_x^2 + u(x,0)\text{: scattering data at } t=0 \\ & & \Big\downarrow \, KdV\,evolution \\ \{u(x,t)\} & \overset{\text{inverse scattering}}{\longleftarrow} & \text{scattering data at } t>0. \end{array}$$

Before starting the theory of scattering we will say few more words about integrability

3.1 Algebra of pseudo-differential operators

In the theory of evolutionary systems one of the most important issues is a systematic method for construction of integrable systems. The crucial point of the formalism is the observation that integrable dynamical systems can be obtained from the Lax equations on appropriate Lie algebras. The greatest advantage of this formalism, besides the possibility of systematic construction of the integrable systems, is the construction of bi-Hamiltonian structures and infinite hierarchies of symmetries and conserved quantities.

The classical R-matrix formalism that originated from the pioneering article [23] by Gelfand and Dickey, where they presented the construction of Hamiltonian soliton systems of KdV type by means of pseudo- differential operators.

Let $\mathfrak{g}$ be an algebra with respect to some multiplication, over $\mathbb{R}$ or $\mathbb{C}$ and let us assume that $\mathfrak{g}$ is endowed with a bilinear product given by a Lie bracket $[,]:\mathfrak{g}\to\mathfrak{g}$, which is skew-symmetric and satisfies the Jacobi identity.

Definition 3.3. A linear map $R:\mathfrak{g}\to\mathfrak{g}$ such that the bracket

$$[a,b]_R = [Ra,b] + [a,Rb], \quad \forall a,b\in\mathfrak{g} \tag{3.5}$$

is another Lie bracket on $\mathfrak{g}$ is called the classical R-matrix.

The map (3.5) defines a Lie bracket if it is skew symmetric and satisfies the Jacobi identity. The skew symmetry of (3.5) is obvious. The fact that (3.5) satisfies the Jacobi identity is not given for free, but some conditions on the map R have to be imposed. These conditions are called Yang-Baxter equations that we promptly derive. To impose the Jacobi identity on (3.5) one finds that

$$\begin{aligned} 0 &= [a,[b,c]_R]_R + \text{ cyclic permutation} \\ &= [Ra,[Rb,c]] + [Ra,[b,Rc]] + [a,R[b,c]_R] + \text{ cyclic permutation} \\ &= [Rb,[Rc,a]] + [Rc,[a,Rb]] + [a,R[b,c]_R] + \text{ cyclic permutation} \\ &= [a,R[b,c]_R - [Rb,Rc]] + \text{ cyclic permutation} \end{aligned}$$

where the last equality follows from the Jacobi identity for $[\cdot,\cdot]$. Hence, a sufficient condition for R to be a classical R-matrix is to satisfy the following so-called Yang-Baxter equation

$$[Ra, Rb] - R[a,b]_R + c[a,b] = 0 \tag{3.3}$$

where $c \in \mathbb{C}$.

3.2 An example of R-matrix

Assume that the Lie algebra $\mathfrak{g}$ can be split into a direct sum of two Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, namely

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \quad [\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm \quad \mathfrak{g}_+ \cap \mathfrak{g}_- = \emptyset$$

We denote the projections onto these subalgebras by $P_\pm$, where

$$P_+ + P_- = 1.$$

Next we define a linear map $R : \mathfrak{g} \to \mathfrak{g}$ as

$$R = \frac{1}{2}(P_+ - P_-) = P_+ - \frac{1}{2} = \frac{1}{2} - P_- \tag{3.13}$$

Let $a_\pm := P_\pm(a)$ for $a \in \mathfrak{g}$. Then it is immediate to verify that the map R defines a R-matrix. Indeed

$$[a,b]_R = [Ra, b] + [a, Rb] = [a_+, b_+] - [a_-, b_-]$$

Further

$$R[a,b]_R = \frac{1}{2}[a_+, b_+] + \frac{1}{2}[a_-, b_-]$$
$$[Ra, Rb] = \frac{1}{4}[a_+, b_+] - \frac{1}{4}[a_+, b_-] - \frac{1}{4}[a_-, b_+] + \frac{1}{4}[a_-, b_-]$$

so that

$$[Ra, Rb] - R[a,b]_R = -\frac{1}{4}[a_+, b_+] - \frac{1}{4}[a_+, b_-] - \frac{1}{4}[a_-, b_+] - \frac{1}{4}[a_-, b_-] = -\frac{1}{4}[a,b]$$

namely the Yang-Baxter equation (3.3) are satisfied for $c = \frac{1}{4}$.

3.3 Algebra of pseudo-differential operators

The algebra of pseudo-differential operator (PDO) is the set of operators

$$\mathfrak{g} = \left\{ L = \sum_{i \geqslant -\infty}^{N} u_i(x)\partial_x^i \right\} \tag{3.6}$$

where the smooth functions $u_i(x)$ depend also on the times $\boldsymbol{t} = (t_1, t_2, \dots)$. The ∂_x operator is related to the total derivative with respect to x. Thus, the multiplication in $\mathfrak{g}$ is defined through the so-called generalized Leibniz rule

$$\partial^m u(x) = \sum_{n \geqslant 0} \binom{m}{n} u^{(n)}(x)\partial^{m-n} \tag{3.7}$$

where $u^{(n)}(x) := (\partial_x)^n u(x)$ and $\begin{pmatrix} m \\ n \end{pmatrix}$ stands for the standard binomial coefficient. For $m < 0$ one has the relation

$$\begin{pmatrix} m \\ n \end{pmatrix} = (-1)^n \begin{pmatrix} -m+n-1 \\ n \end{pmatrix}$$

From the above relation it follows that

$$\begin{aligned} \partial_x u &= u\partial_x + u_x \\ \partial^{-1} u &= u\partial_x^{-1} - \partial_x^{-1} u_x \partial_x^{-1} \\ &= u\partial_x^{-1} - u_x\partial_x^{-2} + u_{2x}\partial_x^{-3} - \ldots, \end{aligned}$$

The algebra (3.6) with the multiplication defined through (3.7) is an associative and noncommutative algebra. Therefore, we have a well-defined Lie algebra structure on $\mathfrak{g}$ with the natural commutator of operators, namely

$$[A, B] = AB - BA \quad A, B \in \mathfrak{g}$$

Consider the following decomposition of $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where

$$\mathfrak{g}_+ := \left\{ \sum_{i \geqslant 0} u_i \partial_x^i \right\}, \quad \mathfrak{g}_- := \left\{ \sum_{i<0} a_i \partial_x^i \right\}. \tag{3.25}$$

Then, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$ and $P_\pm : \mathfrak{g} \to \mathfrak{g}_\pm$ the projector operators. Consider an element L from $\mathfrak{g}$ of the form

$$L = \partial_x^N + u_{N-1}\partial_x^{N-1} + u_{N-2}\partial_x^{N-2} + \ldots u_0 \tag{3.27}$$

where $N > 0$. Then its N-th root

$$L^{\frac{1}{N}} = \partial_x + a_0 + a_{-1}\partial_x^{-1} + a_{-2}\partial_x^{-2} + \ldots, \tag{3.8}$$

where coefficients a_i are differential function of u_i, that are constructed by solving recursively the identity

$$\left(L^{\frac{1}{N}}\right)^N = \overbrace{L^{\frac{1}{N}} \cdot \ldots \cdot L^{\frac{1}{N}}}^{N} = L.$$

In general one can take the fractional powers of (3.8) to obtain

$$L^{\frac{n}{N}} = \overbrace{L^{\frac{1}{N}} \cdot \ldots \cdot L^{\frac{1}{N}}}^{n}.$$

Using the distributive property of the commutator one has that

$$0 = [L, L] = [\left(L^{\frac{1}{N}}\right)^N, L] = \left(L^{\frac{1}{N}}\right)^{N-1} [L^{\frac{1}{N}}, L] + \left(L^{\frac{1}{N}}\right)^{N-2} [L^{\frac{1}{N}}, L] L^{\frac{1}{N}} + \cdots + [L^{\frac{1}{N}}, L] \left(L^{\frac{1}{N}}\right)^{N-1}$$

so that one can conclude that

$$[L^{\frac{1}{N}}, L] = 0$$

and in general

$$[L^{\frac{k}{N}}, L] = 0 \quad k \geq 1.$$

Definition 3.4. The fractional powers of L generate the following Lax hierarchies

$$L_{t_k} = \left[\left(L^{\frac{k}{N}}\right)_+, L\right], \quad k = 1, \ldots \tag{3.9}$$

In order to investigate the type of equations obtained from (3.9) we analyse the properties of the above commutator.

Lemma 3.5. *The differential operator* $\left[\left(L^{\frac{k}{N}}\right)_+, L\right]$ *has degree* $N-2$.

Proof. We have that

$$0 = \left[\left(L^{\frac{k}{N}}\right)_+ + \left(L^{\frac{k}{N}}\right)_-, L\right]$$

so that

$$\left[\left(L^{\frac{k}{N}}\right)_+, L\right] = -\left[\left(L^{\frac{k}{N}}\right)_-, L\right].$$

On the other hand $\left(L^{\frac{k}{N}}\right)_- = a_{-1}\partial_x^{-1} + \text{l.o.t}$ where l.o.t stands for lower order terms, so that

$$[a_{-1}\partial_x^{-1}, \partial_x^N] = a_{-1}\partial_x^{N-1} - \partial_x^N(a_{-1}\partial_x^{-1}) = -N(\partial_x a_{-1})\partial_x^{N-2} + l.o.t$$

and the statement follows. □

Remark 3.6. From lemma 3.5 it follows that $\partial_{t_k} u_{N-1} = 0$ in (3.9). For this reason, we consider only operators where $u_{N-1} = 0$ since this term does not contribute to the dynamic, namely operators of the form

$$L = \partial_x^N + u_{N-2}\partial_x^{N-2} + \ldots + u_1\partial_x + u_0\,.$$

Example 3.7. [**KdV hierarchy**] Consider the case $N = 2$, then

$$L = -\partial_x^2 + u,$$

where for normalization reasons we have inserted the minus sign in front of the second derivative. One finds that

$$\begin{aligned}(-L)^{\frac{1}{2}} =& \partial_x - \frac{1}{2}u\partial_x^{-1} + \frac{1}{4}u_x\partial_x^{-2} - \frac{1}{8}\left(u_{2x} - u\right)\partial_x^{-3} + \frac{1}{16}\left(u_{3x} - 6uu_x\right)\partial_x^{-4} \\ &- \frac{1}{32}\left(u_{4x} - 14uu_{2x} - 11u_x^2 + 2u^3\right)\partial_x^{-5} + \ldots\end{aligned}$$

The fractional power of L gives for example

$$(-L)^{\frac{3}{2}} = -L \cdot (-L)^{\frac{1}{2}} = \partial^3 - \frac{3}{2}u\partial_x - \frac{3}{4}u_x + (\ldots)\partial_x^{-1} + \ldots$$

The Korteweg de Vries equation is recovered from

$$L_{t_1} = u_{t_1} = \left[4\,(-L)_+^{\frac{3}{2}}, L\right] \Longleftrightarrow u_{t_3} - 6uu_x + u_{xxx} = 0$$

In general, from Lemma 3.5 the commutator $\left[(-L)_+^{\frac{2k+1}{2}}, L\right]$ is a differential operator of degree zero, namely a differential polynomial in $u, u_x, u_{xx}, \dots$ and it defines the KdV hierarchy

$$L_{t_k} = u_{t_k} = 2^{2k}\left[(-L)_+^{\frac{3}{2}}, L\right], \quad k = 1, 2, \dots \tag{3.10}$$

where we have changed slightly the numbering of the times with respect to the Definition 3.4. For example

$$u_{t_2} = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x,$$

and in general the equation $u_{t_k} = u^{(2k+1)} + \dots + cu^k u_x$, where c is a constant. Namely the highest derivative has order $2k+1$ and the highest nonlinearity is of order $k+1$.

Example 3.8. [**Boussinesq equation**] When $N = 3$ the differential operator L takes the form

$$L = \partial_x^3 + u\partial_x + v$$

with $u = u(x, \boldsymbol{t})$ and $v = v(x, \boldsymbol{t})$. We have

$$\begin{aligned} L^{\frac{1}{3}} =& \partial_x + \frac{1}{3}u\partial_x^{-1} - \frac{1}{3}\left(u_x - v\right)\partial_x^{-2} + \frac{1}{9}\left(2u_{2x} - 3v_x - u^2\right)\partial_x^{-3} \\ &- \frac{1}{9}\left(u_{3x} - 2v_{2x} - 4uu_x + 2uv\right)\partial_x^{-4} \\ &+ \frac{1}{81}\left(3u_{4x} - 9v_{3x} - 45uu_{2x} + 36uv_x - 45u_x^2 + 45u_x v - 9v^2 + 5u^3\right)\partial_x^{-5} + \dots \end{aligned}$$

Then, for

$$\left(L^{\frac{2}{3}}\right)_+ = \partial_x^2 + \frac{2}{3}u,$$

one finds the Lax equation for the Boussinesq system

$$L_{t_2} = \left[\left(L^{\frac{2}{3}}\right)_+, L\right] \Longleftrightarrow \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} -u_{2x} + 2v_x \\ -\frac{2}{3}u_{3x} + v_{2x} - \frac{2}{3}uu_x \end{pmatrix}$$

or equivalently

$$u_{t_2 t_2} = (3u_{xxx} - \frac{4}{3}uu_x)_x,$$

that is called the Boussinesq equation.

Remark 3.9. The KdV equation and the Boussinesq equation are reductions (up to normalization) of a 2-dimensional integrable equation called the Kadomtsev-Petviashvili equation

$$(u_t + uu_x + u_{xxx})_x = u_{yy}\,.$$

t-independent solutions satisfy the Boussinesq equation, while y-independent solutions satisfies the KdV equation.

The very important fact that determines integrability is that the hierarchy of equations (3.9) commute, namely we have the following lemma.

Lemma 3.10. *The following relations are satified*

$$\frac{\partial}{\partial t_j} L_{t_k} = \frac{\partial}{\partial t_k} L_{t_j}, \quad j \neq k = 1, 2, \dots$$

Proof. Let $A_k := (L^{\frac{k}{N}})_+$. From definition 3.4 we have

$$\begin{aligned}
\frac{\partial}{\partial t_j} L_{t_k} - \frac{\partial}{\partial t_k} L_{t_j} &= \frac{\partial}{\partial t_j}[A_k, L] - \frac{\partial}{\partial t_k}[A_j, L] = [\frac{\partial}{\partial t_j} A_k - \frac{\partial}{\partial t_k} A_j, L] + [A_k, \frac{\partial}{\partial t_j} L] - [A_j, \frac{\partial}{\partial t_k} L] \\
&= [\frac{\partial}{\partial t_j} A_k - \frac{\partial}{\partial t_k} A_j, L] + [A_k, [A_j, L]] - [A_j, [A_k, L]] \\
&= \left[\frac{\partial}{\partial t_j} A_k - \frac{\partial}{\partial t_k} A_j, L\right] + [L, [A_k, A_j]]
\end{aligned}$$

that implies

$$\frac{\partial}{\partial t_j} A_k - \frac{\partial}{\partial t_k} A_j - [A_j, A_k] = 0 \,. \tag{3.11}$$

The above equation takes also the name of zero curvature equation. To show that such equation is satisfied we first show the following identity:

$$\frac{\partial}{\partial t_j} L^{\frac{k}{N}} = [A_j, L^{\frac{k}{N}}] \,.$$

To prove it let us start for $k = 1$,

$$\begin{aligned}
\frac{\partial}{\partial t_j} (L^{\frac{1}{N}})^N &= \sum_{i=0}^{N-1} (L^{\frac{1}{N}})^i \frac{\partial}{\partial t_j} L^{\frac{1}{N}} (L^{\frac{1}{N}})^{N-i-1} \\
&= \sum_{i=0}^{N-1} (L^{\frac{1}{N}})^i [A_j, L^{\frac{1}{N}}] (L^{\frac{1}{N}})^{N-i-1} = [A_j, L]
\end{aligned}$$

as required. The equality for $k > 1$ can be proved by the same computations. Next we have

$$(\frac{\partial}{\partial t_j} L^{\frac{k}{N}})_+ = [A_j, L^{\frac{k}{N}}]_+$$

where the underscore + stands for the projection on the subspace of positive and constant differential operators. We are ready to prove the zero curvature equation (3.11):

$$\begin{aligned}
\frac{\partial}{\partial t_j} A_k - \frac{\partial}{\partial t_k} A_j &= [(L^{\frac{j}{N}})_+, L^{\frac{k}{N}}]_+ - [(L^{\frac{k}{N}})_+, L^{\frac{j}{N}}]_+ \\
&= [(L^{\frac{j}{N}})_+, (L^{\frac{k}{N}})_+]_+ + [(L^{\frac{j}{N}})_+, (L^{\frac{k}{N}})_-]_+ - [(L^{\frac{k}{N}})_+, L^{\frac{j}{N}}]_+ \\
&= [(L^{\frac{j}{N}})_+, (L^{\frac{k}{N}})_+]_+ + [(L^{\frac{j}{N}}), (L^{\frac{k}{N}})_-]_+ - [(L^{\frac{k}{N}})_+, L^{\frac{j}{N}}]_+ \\
&= [(L^{\frac{j}{N}})_+, (L^{\frac{k}{N}})_+]_+ + [(L^{\frac{j}{N}}), (L^{\frac{k}{N}})]_+ \\
&= [(L^{\frac{j}{N}})_+, (L^{\frac{k}{N}})_+]_+ = [A_j, A_k].
\end{aligned}$$

□

We can conclude that the Lax equations (3.4) and in particular the KdV hierarchy (3.10) are an infinite dimensional set of commuting equations.

3.4 Direct scattering

We look for solution of the Schrödinger equation for potentials $u(x, t = 0)$ such that

$$\int_{-\infty}^{\infty} (1 + x^2)|u(x, t = 0)|dx < \infty \tag{3.12}$$

The first step is to determine the Jost solutions of the equation $L\psi = k^2\psi$, $k \in \mathbb{R}$ namely two sets of independent and bounded solutions such that

$$\phi(x, k) \to e^{-ikx}, \;\; \overline{\phi}(x, k) \to e^{ikx}, \quad x \to -\infty \tag{3.13}$$

and

$$\psi(x, k) \to e^{ikx}, \;\; \overline{\psi}(x, k) \to e^{-ikx}, \quad x \to +\infty. \tag{3.14}$$

Since the potential $u(x)$ is real, if $\phi(x, k)$ is a solution of the Schrödinger equation then also $\phi^*(x, k^*)$ is a solution, where * means complex conjugation. For this reason $\bar{\phi}(x, k) = \phi^*(x, k^*)$. We observe that for k real we also have $\bar{\phi}(x, k) = \phi(x, -k)$ and $\bar{\psi}(x, k) = \psi(x, -k)$. Let us define $w(x, k) := \phi(x, k)e^{ikx}$, and $\bar{w}(x, k) := \overline{\phi}(x, k)e^{-ikx}$. We observe that from (3.13) we have that

$$w(x, k) \to 1, \;\; \bar{w}(x, k) \to 1, \quad \text{as } x \to -\infty.$$

Lemma 3.11. *The following relation holds*

$$\left(w'e^{-2ikx}\right)' = uwe^{-2ikx} \tag{3.15}$$

where $'$ denotes the derivative with respect to x.

Proof. Substituting ϕ into the Schrödinger equation we have

$$-\left(we^{-ikx}\right)'' + uve^{-ikx} = k^2we^{-ikx}.$$

Taking the second derivative and re-arraging the terms one arrives to the statement of the lemma. □

Lemma 3.12. *The function $w(x, k)$ satisfies the following integral equations*

$$w(x, k) = 1 - \int_{-\infty}^{x} u(\xi)w(\xi, k)\frac{1 - e^{2ik(x-\xi)}}{2ik}d\xi. \tag{3.16}$$

Proof. We first integrate the equation (3.15) once obtaining

$$w'e^{-2ikx} = \int_{-\infty}^{x} u(\xi)w(\xi, k)e^{-2ik\xi}d\xi$$

Next we integrate another time from $-\infty$ to x and use the fact that $w(x, k) \to 1$ as $x \to -\infty$ obtaining

$$w(x, k) = 1 + \int_{-\infty}^{x} dx'e^{2ikx'} \int_{-\infty}^{x'} u(\xi)w(\xi, k)e^{-2ik\xi}d\xi.$$

Exchanging the order of integration we arrive to

$$w(x, k) = 1 + \int_{-\infty}^{x} d\xi u(\xi)w(\xi, k)e^{-2ik\xi} \int_{\xi}^{x} dx'e^{2ikx'} = 1 + \int_{-\infty}^{x} d\xi u(\xi)w(\xi, k)e^{-2ik\xi}\frac{e^{2ikx} - e^{2ik\xi}}{2ik}$$

which is equivalent to the statement of the lemma. □

In a similar way one has

$$\bar{w}(x,k) = 1 + \int_{-\infty}^{x} u(\xi)]\bar{w}(\xi,k)\frac{1-e^{-2ik(x-\xi)}}{2ik}d\xi. \tag{3.17}$$

Lemma 3.13. *The integral equation (3.16) has a solution.*

Proof. We give a sketch of the proof (see Lemma 1, Sect. 2.1 in [?]). We consider an iteration scheme with $w_0(x,k) = 0$, $w_1(x,k) = 1$ and

$$w_{j+1}(x,k) = 1 + \int_{-\infty}^{x} u(\xi)w_j(\xi,k)D_k(x-\xi)d\xi$$

where $D_k(x) = \dfrac{1-e^{2ikx}}{2ik}$. For $|k| > 0$ one has the estimate $|D_k(x-\xi)| < \dfrac{1}{|k|}$ uniformly in $x, \xi \in \mathbb{R}$. Here $|\,.\,|$ stands for the modulus of a complex number. We claim that for all $j \geq 0$ and $|k| > 0$ one has the estimate

$$|w_{j+1}(x,k) - w_j(x,k)| \leq \frac{1}{j!}\left(\frac{1}{|k|}\int_{-\infty}^{x} d\xi|u(\xi)|\right)^j < \frac{1}{j!}\left(\frac{1}{|k|}\int_{-\infty}^{+\infty} d\xi|u(\xi)|\right)^j. \tag{3.18}$$

Such estimate can be easily proved by induction. From the above estimate it follows that $w_j(x,k)$ is a Cauchy sequence and therefore there exists the limit $w(x,k)$ in $L^\infty(\mathbb{R})$. Using the above estimate we also have that

$$|w_N(x,k)| = \left|\sum_{j=0}^{N-1} w_{j+1} - w_j\right| \leq \sum_{j=0}^{N-1}\frac{1}{j!}\left(\frac{1}{|k|}\int_{-\infty}^{+\infty} d\xi|u(\xi)|\right)^j$$

that shows that the limit $w(x,k)$ is uniformly bounded by

$$|w(x,k)| \leq \exp\left(\frac{1}{|k|}\int_{-\infty}^{+\infty} d\xi|u(\xi)|\right), \quad |k| > 0. \tag{3.19}$$

In order to prove the existence of a solution of the integral equation (3.16) for k near zero, we need the estimate $|D_k(x-\xi)| < |x-\xi|$ for all real x, ξ. In this case we can prove that $w_j(x,k)/(1+x)$ converges for all real x provided that $\int_{-\infty}^{+\infty}|u_0(\xi)||c_1\xi + c_2|d\xi < \infty$ for some constants c_1 and c_2. Namely a stronger decay at infinity of the function u is required for the case k near the origin. □

Remark 3.14. For proving the existence of the first derivative with respect to k of the function $w(x,k)$ one has to prove the existence of the solution of the integral equation

$$F(x,k) = 1 - \int_{-\infty}^{x} u(\xi)F(\xi,k)D_k(x-\xi)d\xi + \int_{-\infty}^{x} u(\xi)w(\xi,k)\frac{\partial}{\partial k}D_k(x-\xi)d\xi.$$

and show that the function $F(x,k)$ coincides with $\partial_k w(x,k)$. For proving the existence of $F(x,k)$ one must have that the function $u_0(x)$ decays at infinity in such a way that the condition (3.12) is satisfied. In general the smoothness of the function $w(x,k)$ with respect to the parameter k depends on how fast the function u decays at infinity, [19].

Regarding the Jost function normalised at $+\infty$ we define

$$p(x,k) := \psi(x,k)e^{-ikx}, \quad \bar{p}(x,k) := \overline{\psi}(x,k)e^{ikx}$$

In a similar way as done before, one can obtain an integral equation for the function p and q, that is

$$\begin{aligned} p(x,k) &= 1 - \int_x^{+\infty} d\xi u(\xi)p(\xi,k)\frac{1-e^{-2ik(x-\xi)}}{2ik}, \\ \bar{p}(x,k) &= 1 + \int_x^{+\infty} d\xi u(\xi)\bar{p}(\xi,k)\frac{1-e^{2ik(x-\xi)}}{2ik}. \end{aligned} \tag{3.20}$$

Remark 3.15. From the formula (3.15) and (3.17) it is immediate to see that the function $D_k(x-\xi) = \dfrac{1-e^{2ik(x-\xi)}}{2ik}$ for $x > \xi$ is exponentially small for $\operatorname{Im} k > 0$ while $\overline{D_k(x-\xi)}$ for $x > \xi$ is exponentially small for $\operatorname{Im} k < 0$. On the other hand $D_k(x-\xi)$ for $x < \xi$ is exponentially small for $\operatorname{Im} k < 0$ while $\overline{D_k(x-\xi)}$ for $x < \xi$ is exponentially small for $\operatorname{Im} k > 0$. From this simple observation it follows from (3.15) and (3.17) and (3.37) that

- the functions $\phi(x,k)$ and $\psi(x,k)$ or $p(x,k)$ and $w(x,k)$ have an analytic continuation in the upper half space $\operatorname{Im} k \geq 0$.
- the functions $\overline{\phi}(x,k)$ and $\overline{\psi}(x,k)$ or $\bar{p}(x,k)$ and $\bar{w}(x,k)$ have an analytic continuation in the lower half space $\operatorname{Im} k \leq 0$.

Next we consider the Wronkstian $W(f,g) = f_x g - g_x f$.

Lemma 3.16. *The following relation is satisfied*

$$W(\phi,\bar{\phi}) = W(\bar{\psi},\psi) = -2ik.$$

Proof. First of all let us observe that

$$\frac{d}{dx}W(\phi\,,\bar{\phi}) = \phi_{xx}\bar{\phi} + \phi_x\bar{\phi}_x - \phi\bar{\phi}_{xx} - \phi_x\bar{\phi}_x = 0,$$

because ϕ and $\bar{\phi}$ satisfy the Schrödinger equation. Namely the Wronkstian of two independent solutions of the Schrödinger equation is equal to a constant. In order to evaluate the constant, we evaluate the Wronkstian with respect to ϕ and $\bar{\phi}$ at $x=-\infty$ and the Wronkstian with respect to $\bar{\psi}$ and ψ at $x=+\infty$ obtaining the statement of the lemma.

□

Since the Jost functions ϕ and $\bar{\phi}$ and ψ and $\bar{\psi}$ are two independent sets of solutions of the Schrödinger equation that is an ODE of second degree it follows that they are related by a linear transformation

$$(\phi,\bar{\phi}) = (\bar{\psi},\psi)\begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \tag{3.21}$$

for some constants $a = a(k)$ and $b = b(k)$. The matrix

$$S = \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \tag{3.22}$$

is called scattering matrix.

Lemma 3.17. *The matrix S has determinant equal to one.*

Proof. It is enough to evaluate the Wronkstian

$$-2ik = W(\phi, \bar{\phi}) = W(a\bar{\psi} + b\psi, \bar{a}\psi + \bar{b}\bar{\phi}) = -2ik(a\bar{a} - b\bar{b}).$$

□

Lemma 3.18. *The following identities are satisfied*

$$\begin{aligned} a(k) &= 1 - \frac{1}{2ik}\int_{-\infty}^{+\infty} u(\xi)p(\xi,k)d\xi, \\ b(k) &= \frac{1}{2ik}\int_{-\infty}^{+\infty} u(\xi)\bar{p}(\xi,k)e^{-2ik\xi}d\xi. \end{aligned} \tag{3.23}$$

Proof. For the first relation we evaluate $W(\psi, \phi)$ obtaining

$$W(\psi, \phi) = W(\psi, a\bar{\psi} + b\psi) = aW(\psi, \bar{\psi}) = 2iak.$$

On the other hand one also has that

$$W(\psi, \phi) = W(pe^{ikx}, we^{-ikx}) = ikpw + p'w - pw' + ikpw = (p' + 2ikp)w - pw'.$$

Evaluating the last relation at $x = -\infty$ one obtains

$$W(\psi, \phi) = (p' + 2ikp)w - pw' = 2ik - \int_{-\infty}^{+\infty} u(\xi)p(\xi,k)d\xi$$

which combined with the first identity gives the statement of the lemma. Regarding the derivation of b, we consider the Wronkstian

$$W(\bar{\psi}, \phi) = W(\bar{\psi}, a\bar{\psi} + b\psi) = bW(\bar{\psi}, \psi) = -2ikb.$$

On the other hand we also have

$$W(\bar{\psi}, \phi) = W(\bar{p}e^{-ikx}, we^{-ikx}) = (\bar{p}'w - \bar{p}w')e^{-2ikx} = -\int_{-\infty}^{+\infty} u(\xi)\bar{p}(\xi,k)e^{-2ik\xi}d\xi,$$

where the last identity has been obtained by evaluating the Wronkstian at $x = -\infty$. Combining the last two equations one obtains the statement of the lemma. □

Lemma 3.19. *The function $a(k)$ is analytic on the upper half plane and it has at most a finite number of zeros on the positive imaginary axis. The zeros are all simple and*

$$i\frac{da}{dk}|_{k=i\kappa_j} = \frac{1}{\beta_j}\int_{-\infty}^{+\infty} \phi^2(x, i\kappa_j)dx = \beta_j\int_{-\infty}^{+\infty} \psi^2(x, i\kappa_j)dx. \tag{3.24}$$

Remark 3.20. The linear spectral problem can have a point spectrum where the function $a(k)$ has higher order zeros, but the corresponding potential $u(x)$ is not in $L^2(\mathbb{R})$.

Proof. The function $a(k)$ has an analytic extension for $\operatorname{Im} k > 0$ because of formula (3.23) and the fact that the function $p(x,k)$ has an analytic continuation on the upper half space. Since the function $p(x,k)$ is uniformly bounded for $|k| > 0$, (see relation (3.19) for the equivalent function $w(x,k)$), one also has that

$$a(k) \to 1, \quad |k| \to \infty, \quad \operatorname{Im} k > 0.$$

Therefore $a(k)$ is analytic on the upper half space and bounded at infinity. It follows that $a(k)$ has at most a finite number of zeros. To prove that the zeros lie on the complex imaginary axis, let us suppose that $k_j = \xi_j + i\kappa_j$, $\kappa_j > 0$ is a zero of a, namely $a(k_j) = 0$. Then $W(\psi(x,k_j), \phi(x,k_j)) = 0$, which implies that the functions are proportional to each other

$$\phi(x,k_j) = \beta_j \psi(x,k_j). \tag{3.25}$$

We also observe that

$$\psi(x,k_j) \simeq e^{ik_jx} = e^{i\xi_jx}e^{-\kappa_jx}, \quad \text{as } x \to +\infty$$

and

$$\phi(x,k_j) \simeq e^{-ik_jx} = e^{-i\xi_jx}e^{\kappa_jx}, \quad \text{as } x \to -\infty.$$

It follows that the function $f(x,k_j) = \phi(x,k_j) = \beta\psi(x,k_j)$ is in $L^2(\mathbb{R})$ and $Lf = k_0^2 f$ where k_j^2 is an eigenvalue of the operator L. Since the eigenvalues of L are all real, it follows that k_0 is equal to ξ_j or to $i\kappa_j$. The first possibility has to be excluded because otherwise the functions ψ and ϕ cannot be proportional. Hence $k_j = i\kappa_j$, $\kappa_j > 0$. In order to show that the zeros of $a(k)$ are all simple we consider the $W(\psi,\phi) = 2ika$ and differentiate with respect to k. Under the assumption (**??**) and remark 3.14, the Jost functions ψ and ϕ are differentiable and we obtain

$$W(\psi_k, \phi) + W(\psi, \phi_k) = 2ia + 2ik\frac{da}{dk},$$

so that evaluating the above relation at $i\kappa_j$ we obtain

$$2\kappa_j \frac{da}{dk}|_{k=i\kappa_j} = W(\phi, \psi_k)|_{i\kappa_j} + W(\phi_k, \psi)|_{i\kappa_j}.$$

Using the relation $\phi(x, i\kappa_j) = \beta_j\psi(x, i\kappa_j)$ we can reduce the above relation to the form

$$2\kappa_j\beta_j \frac{da}{dk}|_{k=i\kappa_j} = \beta_j^2 W(\psi, \psi_k)|_{i\kappa_j} + W(\phi_k, \phi)|_{i\kappa_j}. \tag{3.26}$$

In order to evaluate the above Wronkstians we consider the sum of the equations $(\phi_{xx} - u\phi + k^2\phi)\phi_k = 0$ and $(-\phi_{xxk} + u\phi_k - 2k\phi - k^2\phi_k)\phi$ that gives

$$(\phi_{xk}\phi - \phi_x\phi_k)_x = -2k\phi^2$$

so that

$$\frac{d}{dx}W(\phi_k, \phi) = -2k\phi^2$$

or

$$W(\phi_k(+\infty,k), \phi(+\infty,k)) - W(\phi_k(-\infty,k), \phi(-\infty,k)) = -2k\int_{-\infty}^{+\infty}\phi(x,k)^2dx.$$

Evaluating the above relation at $i\kappa_j$ and subtracting equation (3.26) evaluated at $x = +\infty$ we obtain

$$[W(\phi_k(-\infty,k),\phi(-\infty,k)) + \beta_j^2 W(\psi(+\infty,k),\psi_k(+\infty,k))]|_{i\kappa_j} \\ = 2i\kappa_j \int_{-\infty}^{+\infty} \phi(x,i\kappa_j)^2 dx + 2\kappa_j\beta_j \frac{da}{dk}|_{k=i\kappa_j}. \quad (3.27)$$

Since the left hand size is equal to zero, because $\phi(x,k) \simeq e^{-ikx}$ as $x \to -\infty$ and $\psi(x,k) \simeq e^{ikx}$ as $x \to +\infty$, we have that

$$\frac{da}{dk}|_{k=i\kappa_j} = -\frac{i}{\beta_j}\int_{-\infty}^{+\infty} \phi(x,i\kappa_j)^2 dx. \quad (3.28)$$

Setting $k = i\kappa_j$ in the integral equation (3.16) shows that $w(x,i\kappa_j)$ is real which implies that $\phi(x,i\kappa_j)$ is real, therefore, the integral in the r.h.s. of the above equation is real and non zero. This shows that the zeros of $a(k)$ are simple at $i\kappa_j$. □

Now let us consider the function $\phi(x,k)$

$$\phi(x,k) = a(k)\bar{\psi}(x,k) + b(k)\psi(x,k) = \begin{cases} a(k)e^{-ikx} + b(k)e^{ikx}, & x \to \infty \\ e^{-ikx} & x \to -\infty \end{cases} \quad (3.29)$$

then

$$\frac{1}{a}\phi(x,k) = \begin{cases} e^{-ikx} + \frac{b(k)}{a(k)}e^{ikx}, & x \to +\infty \\ \frac{1}{a(k)}e^{-ikx} & x \to -\infty \end{cases} \quad (3.30)$$

Looking at the first row of the above equation, we can consider e^{-ikx} as incident wave arriving from $+\infty$ and $\frac{b(k)}{a(k)}e^{ikx}$ as reflected wave, while in the second row we can consider $\frac{1}{a(k)}e^{-ikx}$ as transmitted wave.

Definition 3.21. The quantities

$$r(k) := \frac{b(k)}{a(k)}, \quad t(k) := \frac{1}{a(k)} \quad (3.31)$$

are called reflection and transmission coefficients from the right of the Schrödinger equation.

We observe that since $|a(k)|^2 - |b(k)|^2 = 1$ for k real one has

$$|r(k)|^2 + |t(k)|^2 = 1, \quad k \in \mathbb{R} \quad (3.32)$$

which expresses the conservation of the wave flux. One can also define the reflection coefficient from the left by considering

$$\frac{1}{a}\psi(x,k) = \begin{cases} e^{ikx} - \frac{\bar{b}(k)}{a(k)}e^{ikx}, & x \to -\infty \\ \frac{1}{a(k)}e^{ikx} & x \to +\infty \end{cases} \quad (3.33)$$

and defining $r_L(x) = -\frac{\bar{b}(k)}{a(k)}$ and $t_L(k) = \frac{1}{a(k)}$ the reflection and transmission coefficients from the left.

Lemma 3.22. *The reflection coefficient has the following properties*

1. $|r(k)| \leq 1$ *for all* $k \in \mathbb{R}$*;*
2. $r^*(k) = r(-k)$*, for all* $k \in \mathbb{R}$*;*
3. $r(k) \simeq \frac{1}{|k|}$ *as* $|k| \to \infty$*;*
4. $r(k) \in L^2(\mathbb{R})$ *and* $r'(k) \in L^2(\mathbb{R})$ *if the initial data satisfies (3.12).*

Proof. The relation 1. follows from (3.32). The relation 2. follows from the symmetry property of the Jost solutions. Indeed for real k one has $\psi^*(x,k) = \psi(x,-k)$ so that by (3.23) one has that $a^*(k) = a(-k)$ and $b^*(k) = b(-k)$, k real.

Regarding relation 3. since $p(x,k)$ is uniformily bounded for $|k| > 0$ it follows from the definition of $a(k)$ and $b(k)$ in (3.23) that $|a(k)| \to 1$ and $|b(k)| = O(1/|k|)$ as $|k| \to \infty$. The relation 4. follows from the remark 3.14 and some norm estimates of the derivative of $p(x,k)$ with respect to k. □

Remark 3.23. It is proved by Beals and Coifmann [3] that if $u(x)x^j \in L^1(\mathbb{R})$ then $r \in C^j(\mathbb{R})$ and $r^{(\ell)}(k) \to 0$ as $|k| \to \infty$ with $\ell = 1, \ldots, j$ and $j \geq 2$. Here $r^{(\ell)}(k)$ is the partial derivative with respect to k. Further, if $u(x)x^\ell \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then $r^{(\ell)} \in L^2(\mathbb{R})$ $\ell = 1 \ldots, j$.

The scattering data for the potential $u(x,0)$ is the following set

$$\mathcal{S}(0) = \{r(k), \{\kappa_j,\ \beta_j\}_{j=1}^n\}$$

where $r(k) = r(k, t=0)$ and $\beta_j = \beta_j(t=0)$. The eigenfunction f_j with respect to the eigenvalue $-\kappa_j^2$ is defined as

$$f_j(x, \kappa_j) = \begin{cases} e^{\kappa_j x}, & \text{as } x \to -\infty \\ \beta_j e^{-\kappa_j x}, & \text{as } x \to \infty. \end{cases}$$

If the function $u(x,t)$ evolve according to the KdV equation the scattering data at time t can be obtained in a straightforward way.

Theorem 3.24.

$$\frac{dr}{dt} = 8ik^3 r, \quad \text{or} \quad r(k,t) = r(k,0)e^{8ik^3 t} \tag{3.34}$$

$$\frac{d\beta_j}{dt} = 8\kappa_j \beta_j \quad \text{or} \quad \beta_j(t) = \beta_j(0)e^{8\kappa_j^3 t}. \tag{3.35}$$

Proof. Let us consider the equation $L\psi = \lambda\psi$ where ψ is either a Jost function or an eigenfunction. Taking the derivative with respect to time of this equation and using the Lax equation $\dot{L} = [L, A]$ we obtain (here dot denotes time derivation)

$$L(A\psi + \dot{\psi}) = \lambda(A\psi + \dot{\psi}).$$

It follows that there are constants $c_1 = c_1(k)$ and $c_2 = c_2(k)$ such that

$$A\psi + \dot{\psi} = c_1\psi + c_2\bar{\psi}. \tag{3.36}$$

In order to evaluate these constants, we evaluate the above relation at $+\infty$. We assume that the KdV dynamics does not change the boundary conditions of $u(x,t)$ at infinity. Recall that $\psi(k;x,t) \simeq e^{ikx}$ and $\bar{\psi}(k;x,t) \simeq e^{-ikx}$ as $x \to +\infty$ and using (3.37) we obtain

$$\psi(k;x,t) = e^{ikx} - \int_x^{+\infty} d\xi u(\xi,t)\psi(\xi,t;k)\frac{e^{ik(x-\xi)} - e^{-ik(x-\xi)}}{2ik}. \tag{3.37}$$

Therefore taking a derivative with respect to x we obtain

$$\psi'(k;x,t) = ike^{ikx} - \int_x^{+\infty} d\xi u(\xi,t)\psi(\xi,t;k)\cos(k(x-\xi)) = ike^{ikx} + o(1), \quad \text{as } x \to +\infty$$

In a similar way

$$\psi'''(k;x,t) = (ik)^3 e^{ikx} + k^2\int_x^{+\infty} d\xi u(\xi,t)\psi(\xi,t;k)\cos(k(x-\xi)) = (ik)^3 e^{ikx} + o(1), \quad \text{as } x \to +\infty$$

and

$$\dot{\psi}(k;x,t) = \int_x^{+\infty} d\xi[\dot{u}(\xi,t)\psi(\xi,t;k) + u(\xi)\dot{\psi}(\xi,t;k)]\sin(k(x-\xi)) = o(1), \quad \text{as } x \to +\infty$$

Substituting the above relations in (3.36) one obtains

$$4(ik)^3 e^{ikx} = c_1 e^{ikx} + c_2 e^{-ikx} + o(1), \quad \text{as } x \to +\infty,$$

that gives $c_2 = 0$ and $c_1 = -4ik^3$ so that

$$A\psi + \dot{\psi} = -4ik^3\psi. \tag{3.38}$$

In a similar way we obtain

$$A\phi + \dot{\phi} = 4ik^3\phi$$

and analogous relations hold for the Jost function $\bar{\phi}$ and $\bar{\psi}$. Since $\phi = a\bar{\psi} + b\psi$ we obtain that

$$\begin{aligned} 4ik^3(a\bar{\psi} + b\psi) &= A(a\bar{\psi} + b\psi) + \frac{d}{dt}(a\bar{\psi} + b\psi) \\ &= a(A\bar{\psi} + \dot{\bar{\psi}}) + b(A\psi + \dot{\psi}) + \dot{a}\bar{\psi} + \dot{b}\psi \\ &= 4ik^3 a\bar{\psi} - 4ik^3 b\psi + \dot{a}\bar{\psi} + \dot{b}\psi \end{aligned} \tag{3.39}$$

which gives

$$\dot{b} = 8ik^3 b, \quad \dot{a} = 0.$$

Since the reflection coefficient is $r = \frac{b}{a}$ it follows from the above relation that

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{b}{a}\right) = 8ik^3\frac{b}{a} = 8ik^3 r.$$

In order to obtain the evolution of the norming constants we recall the notation

$$\phi(x,t;i\kappa_j) = \beta_j(t)\psi(x,t;i\kappa_j).$$

Using the relation $A\phi + \dot{\phi} = 4ik^3\phi$ we obtain

$$A(\beta_j(t)\psi(x,t;i\kappa_j) + \frac{d}{dt}(\beta_j(t)\psi(x,t;i\kappa_j)) = 4i(i\kappa_j)^3\psi(x,t;i\kappa_j)$$

which is equivalent to

$$\beta_j(t)(A\psi(x,t;i\kappa_j) + \dot{\psi}(x,t;i\kappa_j)) + \dot{\beta}(t)_j\psi(x,t;i\kappa_j) = 4\kappa_j^3\beta_j(t)\psi(x,t;i\kappa_j).$$

Using the relation (3.38) evaluated at $i\kappa_j$ the above relation gives $\dot{\beta}_j = 8\kappa_j^3\beta_j$. □

So the integration of the KdV equation is obtained by the following diagram

$$\{u(x,t=0)\} \quad \overset{\text{direct scattering}}{\longrightarrow} \quad \mathcal{S}(0) = \{r(k), \{\beta_j, \kappa_j\}_{j=1}^N\}$$

$\downarrow$ *KdV evolution*

$$\{u(x,t)\} \quad \overset{\text{inverse scattering}}{\longleftarrow} \quad \mathcal{S}(t) = \{r(k)e^{8ik^3t}, \{\beta_j e^{8\kappa_j t}, \kappa_j\}_{j=1}^N\}$$

The inverse scattering is the problem to determine the potential $u(x,t)$ from the spectral data $\mathcal{S}(t)$.

Remark 3.25. For the inverse problem Beals and Coifmann [3] proved that if $r(k)k^n \in L^2(\mathbb{R})$ then the distributional derivative of $u^{(\ell)} \in L^2(\mathbb{R}) \cup L^\infty(\mathbb{R})$, $\ell = 0\dots,n$. If further $r^{(\ell)} \in L^2(\mathbb{R})$ for $\ell = 0,\dots,n+1$ then also

$$[u(x)x^{n+1}] \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

Comparing with remak 3.23 one can see that there is no bijective map between the physical space, where $u(x)$ is defined and the scattering space. In order to obtain such a bijective map a substantial analytical work has been done by X. Zhou [19] that proved the bijectivity between

$$u(x) \in L^2((1+x^{2n})dx) \cap H^j(\mathbb{R})$$

and

$$r(k) \in H^n(\mathbb{R}) \cap L^2((1+k^{2j})dk).$$

3.5 Inverse spectral problem as a Riemann-Hilbert problem

We define the vector function

$$m(k;x,t) := \begin{cases} \left(\frac{1}{a(k)}\phi(k;x,t)e^{ikx}, \psi(k;x,t)e^{-ikx}\right) & \operatorname{Im} k \geq 0 \\ \left(\bar{\psi}(k;x,t)e^{ikx}, \frac{1}{\bar{a}(k)}\bar{\phi}(k;x,t)e^{-ikx}\right) & \operatorname{Im} k \leq 0 \end{cases} \tag{3.40}$$

Observe that $m(k;x,t)$ is analytic for $k \in \mathbb{C}\backslash(\mathbb{R} \cup \{\pm i\kappa_j\}_{j=1}^n)$; Furthermore let us define $m_\pm = \lim_{\epsilon\to 0} m(x, k \pm i\epsilon)$. Then we have the following lemma

Lemma 3.26. *The following relation is satisfied*

$$m_+(k;x,t) = m_-(k;x,t)v(k;x,t), \quad k \in \mathbb{R}$$

where

$$v(k;x,t) = \begin{pmatrix} 1-|r(k)|^2 & -\bar{r}(k)e^{-\theta(k;x,t)} \\ r(k)e^{\theta(k;x,t)} & 1 \end{pmatrix}, \quad \theta = 2ikx + 8ik^3t \tag{3.41}$$

and $r(k)$ is calculated at $t = 0$.

Proof. We consider the r.h.s. of the above relation and using (3.21)

$$\begin{aligned}\left(\bar{\psi}e^{ikx}, \frac{1}{\bar{a}(k)}\bar{\phi}e^{-ikx}\right)v(k) &= \left(((1-|r|^2)\bar{\psi} + \frac{r}{\bar{a}}\bar{\phi})e^{ikx}, (\frac{\bar{\phi}}{\bar{a}(k)} - \bar{r}\bar{\psi})e^{-ikx}\right) \\ &= \left(((1-|r|^2)\bar{\psi} + \frac{r}{\bar{a}}(\bar{b}\bar{\psi} + \bar{a}\psi))e^{ikx}, (\frac{\bar{b}}{\bar{a}(k)}\bar{\psi} + \psi - \bar{r}\bar{\psi})e^{-ikx}\right).\end{aligned}$$

Using the time evolution of the reflection coefficient $r(k)$ in (3.34), one obtains the statement of the Lemma.

□

Further as $k \to i\kappa_j$ using the fact that $a(k)$ has simple zeros at κ_j

$$\mathrm{Res}_{k=i\kappa_j} m(k;x,t) = \left(\frac{\phi(x,t;i\kappa_j)}{a'(i\kappa_j)}e^{-\kappa_j x}, 0\right) = \left(\frac{\beta_j(t)\psi(x,t;i\kappa_j)}{a'(i\kappa_j)}e^{-\kappa_j x}, 0\right) \tag{3.42}$$

where $a'(i\kappa_j) = \frac{da}{dk}|_{k=i\kappa_j}$. Using (3.24) and definining

$$ic_j := \frac{\beta_j(t=0)}{a'(i\kappa_j)} = \frac{i}{\int_{-\infty}^{+\infty}\psi^2(x,t=0;i\kappa_j)dx} \in i\mathbb{R}^+$$

we can recast the condition (3.42) as a limit, namely

$$\mathrm{Res}_{k=i\kappa_j} m(k;x,t) = \lim_{k\to i\kappa_j} m(k;x,t)\begin{pmatrix} 0 & 0 \\ ic_j e^{-2\kappa_j x + 8\kappa_j^3 t} & 0 \end{pmatrix}$$

and similarly

$$\begin{aligned}\mathrm{Res}_{k=-i\kappa_j} m(k;x,t) &= \mathrm{Res}_{k=-i\kappa_j}\left(\bar{\psi}(k;x,t)e^{ikx}, \frac{1}{\bar{a}(k)}\bar{\phi}(k;x,t)e^{-ikx}\right) \\ &= \left(0, \frac{1}{\bar{a}'(-i\kappa_j)}\bar{\phi}(x,t;-i\kappa_j)e^{-\kappa_j x}\right) = \lim_{k\to -i\kappa_j} m(k;x,t)\begin{pmatrix} 0 & -ic_j e^{-2\kappa_j x + 8\kappa_j^3 t} \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Summarizing we arrive to the following **Riemann-Hilbert boundary value problem** for the vector $m(k;x,t)$

Theorem 3.27. *Let $\mathcal{S} = \{r(k);\ (\kappa_j, \beta_j)_{j=1}^N\}$ be the (right) scattering data of the Lax operator $L(0)$. Then $m = m(k;x,t)$ defined in (4.7) is the solution of the following Riemann-Hilbert problem.*

- *$m(k;x,t)$ is analytic for $k \in \mathbb{C}\backslash(\mathbb{R} \cup \{\pm i\kappa_j\}_{j=1}^n)$ with simple poles at $\pm i\kappa_j$, with $\kappa_j > 0$, $j = 1, \ldots, n$;*
- *the boundary values $m_\pm = \lim_{\epsilon\to 0} m(x, k \pm i\epsilon)$ satisfy the jump condition*

$$m_+(k;x,t) = m_-(k;x,t)v(k;x,t), \quad k \in \mathbb{R}\,; \tag{3.43}$$

- *the residus conditions at the poles $\pm i\kappa_j$, $j = 1, \ldots, n$,*

$$\mathrm{Res}_{k=i\kappa_j} m(k;x,t) = \lim_{k\to i\kappa_j} m(k;x,t) \begin{pmatrix} 0 & 0 \\ ic_j e^{\theta(i\kappa_j,x,t)} & 0 \end{pmatrix} \tag{3.44}$$

and

$$\mathrm{Res}_{k=-i\kappa_j} m(k;x,t) = \lim_{k\to i\kappa_j} m(k;x,t) \begin{pmatrix} 0 & -ic_j e^{-\theta(-i\kappa_j,x,t)} \\ 0 & 0 \end{pmatrix}$$

where $c_j \in \mathbb{R}^+$;

- *for $|k| \to \infty$*

$$m(x,k) \to (1,1) + O(\frac{1}{|k|}).$$

- *Further the symmetry of the Jost solutions implies the symmetry*

$$m(-k;x,t) = m(k;x,t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Analyzing the integral equations (3.16) and (3.37) for the functions $w(k;x,t)$ and $p(k;x,t)$ we see that expanding for large k we have

$$\begin{aligned} w(x,k) &= 1 - \int_{-\infty}^{x} u(\xi)w(\xi,k)\frac{1-e^{2ik(x-\xi)}}{2ik}d\xi = 1 - \frac{1}{2ik}\int_{-\infty}^{x} u(\xi)d\xi \\ p(x,k) &= 1 - \int_{x}^{+\infty} d\xi u(\xi)p(\xi,k)\frac{1-e^{-2ik(x-\xi)}}{2ik} = 1 - \frac{1}{2ik}\int_{x}^{+\infty} u(\xi)d\xi \end{aligned} \tag{3.45}$$

we deduce that

$$u(x,t) = -2i\partial_x \lim_{k\to\infty} k(m_1(k;x,t) - 1)\,. \tag{3.46}$$

3.6 Reflectionless potentials and N-soliton solutions

When $r(k) = 0$, the Riemann-Hilbert problem for $m(k)$ is solved by a vector $m(k)$ rational in k.

Indeed in this case $m(k) = (m_1(k), m_2(k))$ is a meromorphic function of $\mathbb{C}$ and $m_1(k)$ has a simple pole at $i\kappa_j$ while $m_2(k)$ has simle poles at $-i\kappa_j$. Keeping track of the symmetry of the solution in (4.7) we look for a solution that has the form

$$m(k;x,t) = \left(1 + \sum_{j=1}^{N} \frac{i\alpha_j}{k - i\kappa_j}, 1 - \sum_{j=1}^{N} \frac{i\alpha_j}{k + i\kappa_j}\right)$$

where the constants (in k) $\alpha_j = \alpha_j(x,t) \in \mathbb{R}$ [1] are to be determined from the residue conditions of the Riemann-Hilbert problem. Notice that $m_2(k) = m_1^*(k^*)$. From the constants α_j the solution of KdV is obtained from (3.46)

$$u(x,t) = -2\partial_x \sum_{j=1}^{N} \alpha_j(x,t).$$

Plugging the ansatz for $m(k)$ into the residue conditions (3.43) gives the equations

$$\alpha_\ell = \left(1 - \sum_{j=1}^{N} \frac{\alpha_j}{k_\ell + k_j}\right) c_\ell e^{\theta(i\kappa_\ell)} \quad \Leftrightarrow \quad \frac{\alpha_\ell e^{-\frac{1}{2}\theta(i\kappa_\ell)}}{\sqrt{c_\ell}} = \left(1 - \sum_{j=1}^{N} \frac{\sqrt{c_j} e^{\frac{1}{2}\theta(i\kappa_j)}}{k_\ell + k_j} \frac{\alpha_j e^{-\frac{1}{2}\theta(i\kappa_j)}}{\sqrt{c_j}}\right) \sqrt{c_\ell} e^{\frac{1}{2}\theta(i\kappa_\ell)}$$

$\forall\ k = 1, \ldots N$. Let $\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{c}} \in \mathbb{R}^N$ and $(\tilde{\boldsymbol{\alpha}})_\ell = \frac{\alpha_\ell}{\sqrt{c_\ell}} e^{-\frac{1}{2}\theta(i\kappa_\ell)}$, $(\tilde{\boldsymbol{c}})_k = \sqrt{c_k} e^{\frac{1}{2}\theta(i\kappa_\ell)}$, and

$$(\boldsymbol{K})_{j\ell} = \frac{\sqrt{c_j}\sqrt{c_\ell} e^{\frac{1}{2}\theta(i\kappa_\ell) + \frac{1}{2}\theta(i\kappa_j)}}{k_\ell + k_j}.$$

Then the linear system of equations for α_j can be recast in the form

$$(\boldsymbol{I}_N + \boldsymbol{K})\, \tilde{\boldsymbol{\alpha}} = \tilde{\boldsymbol{c}}, \qquad \tilde{\boldsymbol{\alpha}} = (\boldsymbol{I}_N + \boldsymbol{K})^{-1} \tilde{\boldsymbol{c}}.$$

Proposition 3.28. *The matrix $\boldsymbol{I}_N + \boldsymbol{K}$ is invertible and the N soliton solution takes the form*

$$u(x,t) = -2 \frac{\partial^2}{\partial x^2} \ln \det (\boldsymbol{I}_N + \boldsymbol{K})\ . \tag{3.47}$$

Proof. We follow the same argument as in [**?**, Proposition B1].

The matrix $\boldsymbol{K}$ is symmetric and (even better) it is positive definite, as it can be seen as an inner-product type of matrix:

$$\boldsymbol{K}_{kj} = \int_x^\infty \sqrt{c_j}\sqrt{c_\ell} e^{-s(k_\ell + k_j)} e^{8t(k_\ell^3 + k_j^3)} \mathrm{d}s = \left\langle e^{8tk_\ell^3} \sqrt{c_\ell} e^{-sk_\ell}, \sqrt{c_j} e^{-sk_j} e^{8tk_j^3} \right\rangle \tag{3.48}$$

and the functions $f_k(s) = \sqrt{c_k} e^{-sk_\ell}$ are linearly independent in $L^2(x+\infty)$ ($\forall\ x$), since $k_\ell \neq k_j$ by assumption. Therefore for any verctor $\boldsymbol{v} = (v_1, \ldots, v_N) \in \mathbb{R}^N$ we have

$$\langle \boldsymbol{v}, K\boldsymbol{v} \rangle = \sum_{j,\ell} K_{j\ell} v_\ell v_j = \sum_{j,\ell} \left\langle v_\ell \sqrt{c_\ell} e^{8tk_\ell^3} e^{-sk_\ell}, v_j \sqrt{c_j} e^{-sk_j} e^{8tk_j^3} \right\rangle = \langle \boldsymbol{w}, \boldsymbol{w} \rangle > 0,$$

where $\boldsymbol{w} = (w_1, \ldots, w_N)$ and $w_j = v_j \sqrt{c_j} e^{-sk_j} e^{8tk_j^3}$. We conclude that the matrix $\boldsymbol{K}$ is positive definite.

[1] In principle the constants α_j should be taken complex, but to simplify the presentation we take them real, as this fact is derived from solving the linear system obtained from the residue conditions.

Notice now that

$$\alpha_j = \sum_{\ell=1}^{N} (\boldsymbol{I}_N + \boldsymbol{K})^{-1}_{j\ell} \sqrt{c_j}\sqrt{c_\ell} e^{\frac{1}{2}\theta(i\kappa_\ell)+\frac{1}{2}\theta(i\kappa_j)}, \tag{3.49}$$

therefore

$$\sum_{k=1}^{N} \alpha_\ell = -\operatorname{Tr}\left((\boldsymbol{I}_N + \boldsymbol{K})^{-1} \frac{\partial}{\partial x} \boldsymbol{K} \right) = -\frac{\partial}{\partial x} \ln\det(\boldsymbol{I}_N + \boldsymbol{K}) \tag{3.50}$$

where we used the fact that $\frac{\partial}{\partial x}\boldsymbol{K}_{k\ell} = -\sqrt{c_j}\sqrt{c_\ell}e^{\frac{1}{2}\theta(i\kappa_\ell)+\frac{1}{2}\theta(i\kappa_j)}$ and the formula $\frac{\partial}{\partial x}\ln\det \boldsymbol{A} = \operatorname{Tr}\left(\boldsymbol{A}^{-1}\frac{\partial}{\partial x}\boldsymbol{A}\right)$ for a generic matrix-valued function $\boldsymbol{A}(x)$. Therefore, our KdV N-soliton solution looks like (3.51). □

Example 3.29. In the case $N = 1$ we obtain the 1-soliton solution with spectral data (κ_0, c_0) in the form

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\ln\det\left(1 + \frac{c}{2\kappa_0}e^{-2\kappa_0 x + 8\kappa_0^2 t}\right). \tag{3.51}$$

which gives

$$u(x,t) = -2\kappa_0^2 \mathrm{sech}^2(\kappa_0(x - 4\kappa_0^2 t - x_0), \quad x_0 = \frac{1}{2\kappa_0}\log\frac{c_0}{2\kappa_0}.$$

4 Cauchy Operators

In this section we will show the existence of solution of the Riemann-Hilbert problem 3.27 for all $t \geq 0$. For the purpose we need to introduce the concept of Cauchy operator.

4.1 Cauchy operator

Let Γ be a close contour oriented anticlockwise in $\mathbb{C}$ and D_+ the interior of Γ and D_- is the complement to D_+ and Γ in $\mathbb{C}$. We have the following result, using the Cauchy theorem:

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{\mu - z} d\mu = \begin{cases} 1 \quad , & \text{if } z \in D_+ \\ 0 & \text{if } z \in D_-. \end{cases}$$

When $\lambda \in \Gamma$, the integral is defined as a principal value by the limit

$$p.v.\int_\Gamma \frac{d\mu}{\mu - \lambda} d\mu := \lim_{\epsilon \to 0}\int_{\Gamma_\epsilon} \frac{d\mu}{\mu - \lambda} d\mu$$

where $\Gamma_\epsilon := \Gamma \backslash \gamma_\epsilon$ and $\gamma_\epsilon = \{\mu \in \Gamma \,|\, |\mu - \lambda| \leq \epsilon\}$ for some $\epsilon > 0$. Performing the integral and choosing the phase of the logarithmic to be equal to zero on the orthogonal direction with respect to the contour at the point λ we obtain

$$\lim_{\epsilon \to 0}\frac{1}{2\pi i}\int_{\Gamma_\epsilon} \frac{d\mu}{\mu - \lambda} d\mu = \log(\epsilon e^{\frac{i\pi}{2}}) - \log(\epsilon e^{\frac{-i\pi}{2}}) = \frac{1}{2}.$$

Next, for a function $f(z)$ defined on Γ we aim to study the Cauchy integral

$$\Phi(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\mu)}{\mu - z} d\mu \tag{4.1}$$

which is analytic in $\mathbb{C}\backslash\Gamma$ and it has a jump discontinuity across Γ. To study such a jump discontinuity we start by making the assumption that that $f(z)$ is Hölder continuous on Γ, namely

$$|f(z_1) - f(z_2)| \leq c|z_1 - z_2|^\alpha, \quad 0 \leq \alpha \leq 1, \quad c > 0.$$

The function

$$\psi(z) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\mu) - f(z)}{\mu - z} d\mu$$

is clearly continuous across Γ and therefore

$$\lim_{z\to\lambda_+} \psi(z) = \psi_+(\lambda) = \lim_{z\to\lambda_-} \psi(z) = \psi_-(\lambda) = \psi(\lambda)$$

where $\lambda_\pm$ denote the limit to Γ from the left/right with respect to the oriented contour Γ in non tangential directions with respect to Γ. On the other hand we have

$$\psi_+(\lambda) = \lim_{z\to\lambda_+}\left[\frac{1}{2\pi i}\int_\Gamma \frac{f(\mu)}{\mu - z} d\mu - \frac{f(z)}{2\pi i}\int_\Gamma \frac{1}{\mu - z} d\mu\right] = \Phi_+(\lambda) - f(\lambda)$$

$$\psi_-(\lambda) = \lim_{z\to\lambda_-}\left[\frac{1}{2\pi i}\int_\Gamma \frac{f(\mu)}{\mu - z} d\mu - \frac{f(z)}{2\pi i}\int_\Gamma \frac{1}{\mu - z} d\mu\right] = \Phi_-(\lambda)$$

and for $\lambda \in \Gamma$

$$\psi(\lambda) = \frac{1}{2\pi i}\left[v.p.\int_\Gamma \frac{f(\mu)}{\mu - \lambda} d\mu - \frac{f(t)}{2\pi i} v.p.\int_\Gamma \frac{1}{\mu - \lambda} d\mu\right] = \Phi(\lambda) - \frac{1}{2} f(\lambda)$$

Since the function $\psi(z)$ is continuous for $z \in \mathbb{C}$ it follows that $\psi_+(\lambda) = \psi_-(\lambda) = \psi(\lambda)$ for $\lambda \in \Gamma$ and therefore we have

$$\Phi_+(\lambda) - \Phi_-(\lambda) = f(\lambda), \quad \Phi_\pm(\lambda) = \pm\frac{1}{2} f(\lambda) + \Phi(\lambda)$$

or equivalently

$$\Phi_\pm(\lambda) = \pm\frac{1}{2} f(\lambda) + \frac{v.p.}{2\pi i}\int_\Gamma \frac{f(\mu)}{\mu - \lambda} d\mu, \quad \lambda \in \Gamma$$

The above relation is called the Plemelj-Sokhotskij formula. The second term of the r.h.s. coincides with the the Hilbert transform

$$(Hf)(\lambda) := \frac{v.p.}{2\pi i}\int_\Gamma \frac{f(\mu)}{\lambda - \mu} d\mu$$

so that we can re-write it

$$\Phi_\pm(\lambda) = \pm\frac{1}{2} f(\lambda) - (Hf)(\lambda), \quad \lambda \in \Gamma \tag{4.2}$$

It can be proved that if $f(\lambda)$ is Hölder continuous with coefficient $0 < \alpha < 1$, then functions $\Phi_{\pm}(\lambda)$, $\lambda \in \Gamma$, are Holder continuous with the same index α. In general, given a function or a $n \times n$ matrix f defined on Γ, the map $f \to \pm\frac{1}{2}f(\lambda) - (Hf)(\lambda)$ defines the Cauchy integral operators $C_{\pm}$

$$[C_{\pm}f](\lambda) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\mu)}{\mu - \lambda_{\pm}}d\mu, \tag{4.3}$$

where as before $\lambda_{\pm}$ denote the limit to Γ from the left/right with respect to an oriented contour Γ in non tangential directions with respect to Γ.

The operators $\mathbb{C}_{\pm}$ enjoy several important properties, among which we will use the following:

- If $f \in L^p(\Gamma, |dz|, Mat(\mathbb{C}^n)), 1 \leq p < \infty$, then $[C_{\pm}f](\lambda)$ exists for $\lambda \in \Gamma$ almost everywhere.
- Let $f \in L^p(\Gamma, |dz|, Mat(\mathbb{C}^n)), 1 < p < \infty$.

  Then $C_{\pm}$ are bounded operators in $L^p(\Gamma, |dz| Mat(\mathbb{C}^n))$, i.e. there exists such a constant c_p that
  $$\|C_{\pm}f\|_{L^p(\Gamma)} \leq c_p\|f\|_{L^p(\Gamma)} \tag{4.4}$$
  where $\|f\|_{L^p(\Gamma)}$ is the norm of the matrix whose entries are $|f_{ij}|_{L^p(\Gamma)}$
- As operators in $L^p(\Gamma), 1 < p < \infty$, the Cauchy operators satisfy the Plemelj-Sokhotskij formula
  $$C_{\pm} = \pm\frac{1}{2}\mathrm{Id} - \frac{1}{2}H,$$
  where Id is the identity operator in $L^p(\Gamma)$ and H is the Hilbert transform,
  $$[Hf](\lambda) = \frac{1}{\pi i}v.p.\int_{\Gamma}\frac{f(\mu)}{\lambda - \mu}d\mu. \tag{3.27}$$
  Note that the map $f \to Hf$ is not bounded in L^1. The formula (3.26) implies that
  $$C_+ - C_- = \mathrm{Id}, \quad C_+ + C_- = -H. \tag{3.28}$$
- One has $H^2 = Id$ so that $C_{\pm}$ are orthogonal projectors, namely
  $$C_+^2 = C_+, \quad C_-^2 = C_-, \quad C_+C_- = 0.$$

The proof of these properties can be found in [1]. We describe an example that is sufficient for our purposes.

Example 4.1. Let $\Gamma = \mathbb{R}$ and $f \in L^2(\mathbb{R})$. In this case the Fourier transform $\widehat{f}(s) = \int_{\mathbb{R}} f(\mu)e^{-2\pi i\mu s}d\mu$ is well defined and we have the Parseval identity.

$$\int_{\mathbb{R}}|f(\mu)|^2 d\mu = \int_{\mathbb{R}}|\widehat{f}(s)|^2 ds.$$

We show that $[C_{\pm}f](\lambda) \in L^2(\mathbb{R})$. This is obtained by using the residue theoerm.

$$\begin{aligned} [C_+ f](\lambda) &= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_\Gamma \frac{f(\mu)}{\mu - (\lambda + i\epsilon)} d\mu \\ &= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{2\pi i s\mu} \widehat{f}(s) ds}{\mu - (\lambda + i\epsilon)} d\mu \\ &= \int_0^{+\infty} e^{2\pi i \lambda s} \widehat{f}(s) ds \in L^2(\mathbb{R}) \end{aligned} \tag{4.5}$$

and similarly

$$\begin{aligned} [C_- f](\lambda) &= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_\Gamma \frac{f(\mu)}{\mu - (\lambda - i\epsilon)} d\mu \\ &= \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{2\pi i s\mu} \widehat{f}(s) ds}{\mu - (\lambda - i\epsilon)} d\mu \\ &= -\int_{-\infty}^{0} e^{2\pi i \lambda s} \widehat{f}(s) ds \in L^2(\mathbb{R}) \end{aligned} \tag{4.6}$$

On the other hand the Hilbert transform

$$[Hf](\lambda) = \frac{1}{\pi i} v.p. \int_{\mathbb{R}} \frac{f(\mu)}{\lambda - \mu} d\mu$$

is the convolution of f with the kernel $K(\mu) = \frac{1}{\pi i \mu}$ and the Fourier transform

$$\widehat{K}(s) = \frac{1}{\pi i} v.p. \int_{-\infty}^{\infty} \frac{e^{2\pi i s\mu}}{\mu} d\mu = \operatorname{sign}(s)$$

so that

$$[Hf](\lambda) = -\int_{\mathbb{R}} \operatorname{sign}(s) e^{2\pi i \lambda s} \widehat{f}(s) ds$$

that combined with (4.5) and (4.6) gives (3.28).

Furthermore we observe that

$$H^2 = Id$$

so that the Cauchy operators $C_{\pm}$ are orthogonal projectors, namely

$$C_+^2 = C_+, \quad C_-^2 = C_-, \quad C_+C_- = C_-C_+ = 0.$$

4.2 The Riemann-Hilbert problem as a Fredholm integral equation

In order to show existence of solutions to the Riemann-Hilbert problem (3.27) we need first to write it for a matrix function by taking the first derivates with respect to x of the Jost functions. Namely we define

$$M(k;x,t) := \begin{cases} \begin{bmatrix} \frac{1}{a(k)}\phi(k;x,t)e^{ikx} & \psi(k;x,t)e^{-ikx} \\ \frac{1}{a(k)}\phi'(k;x,t)e^{ikx} & \psi'(k;x,t)e^{-ikx} \end{bmatrix} & \operatorname{Im} k \geq 0 \\ \\ \begin{bmatrix} \bar{\psi}(k;x,t)e^{ikx} & \frac{1}{\bar{a}(k)}\bar{\phi}(k;x,t)e^{-ikx} \\ \bar{\psi}'(k;x,t)e^{ikx} & \frac{1}{\bar{a}(k)}\bar{\phi}'(k;x,t)e^{-ikx} \end{bmatrix}, & \operatorname{Im} k \leq 0. \end{cases} \tag{4.7}$$

For simplifying the derivation we also assume that there is no discrete spectrum. Then the matrix $M(k)$ is analytic in $\mathbb{C}\backslash\mathbb{R}$ and satisfies the boundary problem

$$M_+(k) = M_-(k)v(k), \quad k \in \mathbb{R}$$

and

$$M(x) = \begin{bmatrix} 1 & 1 \\ -ik & ik \end{bmatrix} + O(k^{-1}), \quad \text{as } k \to \infty.$$

We define

$$N(k) := \begin{bmatrix} 1 & 1 \\ -ik & ik \end{bmatrix}^{-1} M(k)$$

then

$$\boxed{\begin{cases} N_+ = N_- v(k) & k \in \mathbb{R} \\ N \xrightarrow{k\to\infty} I + O(k^{-1}) \end{cases}} \tag{4.8}$$

Now the solution of the KdV equation is recovered from

$$u(x,t) = -2i\partial_x \lim_{k\to\infty} (N_{11} + N_{21} - 1)k.$$

Lemma 4.2. *If the solution of the Riemann-Hilbert problem (4.8) exists, it is unique.*

Proof. Since $\det v = 1$, we have that $(\det N)_+ = (\det N)_-$ so that $\det N$ is analytic in $\mathbb{C}$ and $\det N(k) \to 1$ as $k \to \infty$. By Liouville theorem $\det N = 1$. □

The Riemann Hilbert problem (4.8) can be reduced to an integral equation observing that

$$N_+(k) - N_-(k) = N_-(k)(v(k) - I)$$

and using the Cauchy integral we obtain

$$N(k) = I + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{N_-(s)(v(ks - I)}{s-k} ds$$

In particular the $-$ boundary value gives

$$N_-(k) = I + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{N_-(s)(v(ks - I)}{s-k_-} ds \tag{4.9}$$

Introducing the Cauchy operator

$$[C_vF](k) := [C_-(F(v-I))](k) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F(s)\,(v(s)-I)}{s-k_-}ds, \tag{4.10}$$

the equation (4.9) becomes

$$[(\mathrm{Id}-C_v)N_-](k) = I. \tag{4.11}$$

We have thus associated to the di RH problem (4.8) the operator $\mathrm{Id}-C_v$. The goal is to show that such operator is invertible. For the purpose we show that such operators is a Fredholm operator with trivial kernel and co-kernel. A Fredholm operator $K : X_1 \to X_2$ between two Banach spaces X_1, X_2 is a linear bounded operator with finite dimensional kernel and co-kernel. The index of the operator

$$\mathrm{ind}\, K = \dim\ker K - \dim\mathrm{coker}\, K.$$

To show that $\mathrm{Id}-C_v$ is Fredholm we follow closely [19].

Lemma 4.3. *$Id - C_v$ is Fredholm operator in $L^2(\mathbb{R})$.*

Proof. In order to prove that such operator is Fredholm we show that the operator $\mathrm{Id}-C_v$ has a compact pseudo-inverse. Let us define the operator

$$[C_{v^{-1}}F](k) := [C_-(F(v^{-1}-I))](k) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F(s)\left(v^{-1}(s)-I\right)}{s-k_-}ds, \tag{4.12}$$

then we show that $\mathrm{Id}-C_{v^{-1}}$ is a pseudo-inverse of $\mathrm{Id}-C_v$

$$\begin{aligned}(\mathrm{Id}-C_{v^{-1}})(\mathrm{Id}-C_v)F &= F-[C_vF]-[C_{v^{-1}}F]+[C_{v^{-1}}[C_vF]] =\\ &= F-[C_vF]-[C_{v^{-1}}F]+[C_-([C_-F(v-I)])(v^{-1}-I)] =\\ &= F-[C_vF]-[C_{v^{-1}}F]+[C_-\left(([C_+F(v-I)]-F(v-I))\,(v^{-1}-I)\right)] =\\ &= F-[C_vF]-[C_{v^{-1}}F]-[C_-F(v-I)(v^{-1}-I)]+[C_-\left([C_+F(v-I)](v^{-1}-I)\right]\right) =\\ &= F+[C_-\left([C_+F(v-I)](v^{-1}-I)\right]\right) =: F+[\mathcal{K}F] = (\mathrm{Id}+\mathcal{K})F\end{aligned} \tag{4.13}$$

It remains to show the compactness of the operator $\mathcal{K}$ defined above. Let F_j be a sequence in $L^2(\mathbb{R})$ convergent weakly to zero. We have to show that

$$\|[\mathcal{K}F_j]\|_{L^2(\mathbb{R})} \xrightarrow{j\to+\infty} 0. \tag{4.14}$$

This is easily obtained by applying (4.4) where the L^2 norm is now the matrix norm. In the same way it can be checked that $(\mathrm{Id}-C_v)(\mathrm{Id}-C_v^{-1})$ is compact. It follows that $\mathrm{Id}-C_v$ is Fredholm operator. □

Lemma 4.4. *The Fredholm operator $Id - C_v$ has zero index.*

Proof. Let us consdier the family of operators

$$\mathrm{Id}-C_v^{(t)} \qquad t\in[0,1] \tag{4.15}$$

where $C_v^{(t)}$ is defined:

$$[C_v^{(t)}F] := C_+[F(v-I)t] \tag{4.16}$$

There is a continuous map $\mathcal{O} : [0,1] \to \mathcal{B}(L^2(\mathbb{R}), L^2(\mathbb{R}))$. Furthermore, from the previous lemma the operator $\mathrm{Id} - C_v^{(t)}$ is Fredholm for every $t \in [0,1]$. In particular $C_v^{(1)} = C_v$ e $C_v^{(0)} = 0$. Using the continuity property of the index of an operator and the continuity of the map $\mathcal{O}$ we obtain

$$\mathrm{ind}[\mathrm{Id} - C_v] = \mathrm{ind}[\mathrm{Id}] = 0, \tag{4.17}$$

which gives the result. □

We conclude that the operator associated to the problem (4.8) is a Fredholm operator with zero index. Next we want to show that the kernel is only the zero element. For the purpose we consider the same Riemann- Hilbert problem (4.8) with vanishing conditions at infinity, namely

$$\boxed{\begin{cases} N_+ = N_- V(k) & k \in \mathbb{R} \\ N \xrightarrow{k\to\infty} O(k^{-1}) \end{cases}} \tag{4.18}$$

Theorem 4.5. *The RH problem (4.18) has only the trivial solution.*

Proof. Let us define

$$H(k) := N(k)\bar{N}^t(\bar{k}). \tag{4.19}$$

Then clearly $\bar{H}^t(\bar{k}) = H(k)$. Further we observe that $H(k)$ is analytic for $\mathrm{Im}\, k > 0$ and

$$H(k) = O(k^{-2}) \qquad \text{for } k \to \infty \tag{4.20}$$

It follows by Cauchy theorem that

$$\int_{\mathbb{R}} H_+(s) ds = 0. \tag{4.21}$$

Hence

$$0 = \int_{\mathbb{R}} N_+(s)\bar{N}_-^t(s)ds = \int_{\mathbb{R}} N_-(s)V(s)\bar{N}_-^t(s)ds = \frac{1}{2}\int_{\mathbb{R}} N_-(s)\left(V(s) + \bar{V}^t(s)\right)\bar{N}_-^t(s)ds \tag{4.22}$$

Since

$$V(s) + \bar{V}^t(s) = 2\begin{pmatrix} 1 - |r(s)|^2 & 0 \\ 0 & 1 \end{pmatrix}$$

is positive definite $N_-(k)$ mus be identically zero. In the same one can show that $N_+(k) \equiv 0$, and it follows that $N(k) \equiv 0$. □

Lemma 4.6 (Vanishing lemma)**.** *The RH problem (4.8), admits a solution if and only if the solution to the homogeneous RH (4.18) has only the trivial solution.*

Proof. The RH (4.18) is equivalent to the integral equation

$$[(\mathrm{Id} - C_v)N_-](k) = 0 \tag{4.23}$$

We proof only $\Leftarrow$. By theorem 4.5, the only solution of (4.23) is the zero solution. It follows from lemma 4.3 and 4.4 that $\mathrm{Id} - C_v$ is a Ferdholm operator with zero index and trivial kernel, and therefore it is invertible. This guarantees the solvability of the RH problem (4.8). □

5 Fredholm Determinant

We start with the definition of a standard determinant, first in the conventional form and then in Fredholm's form which is specially adapted to the passage of the limit $d \to \infty$. Let K be $d \times d$ matrix with values in $\mathbb{C}$ and consider $\det(I - \lambda K)$ that is different from zero for $|\lambda| < |K|^{-1}$ and $\lambda \in \mathbb{C}$,

$$\det(I - \lambda K) = \sum_{\pi \in S_d} \text{sign}(\pi) \prod_{i=1}^{d} \left(I_{i\pi(i)} - \lambda K_{i\pi(i)} \right)$$

where S_d is the permutation group of d elements and $\text{sign}(\pi)$ is the sign of the permutation. We can expand in λ the above product and by denoting with $I_p = (i_1 < i_2 < \cdots < i_p)$, $i_1 < i_2 < \ldots, i_p$, a subsets of the d integers $(1, 2, \ldots, d)$ and J_p its complement we obtain

$$\det(I - \lambda K) = \sum_{p=1}^{d} (-\lambda)^p \sum_{I_p} \sum_{\pi \text{ fixing } J_p} \chi(\pi) \prod_{i \in I_p} K_{i\pi(i)} = \sum_{p=1}^{d} (-\lambda)^p \sum_{I_p} \det \left[K_{ij} \right]_{i,j \in I_p}$$

$$= 1 - \lambda \sum_{1 \le i \le d} K_{ij} + \lambda^2 \sum_{1 \le i < j \le d} \det \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix} - \lambda^3 \sum_{1 \le i < j < k \le d} \det \begin{bmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{bmatrix} + \ldots$$

The above expression can be slightly changed if we allowed an unordered set of indices in the sum, namely

$$\det(I - \lambda K) = \sum_{p=1}^{d} \frac{(-\lambda)^p}{p!} \sum_{|\widehat{I}_p| = p} \det \left[K_{ij} \right]_{i,j \in \widehat{I}_p}$$

$$= 1 - \lambda \sum_{1 \le i \le d} K_{ij} + \frac{\lambda^2}{2!} \sum_{1 \le i \le j \le d} \det \begin{bmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{bmatrix} - \frac{\lambda^3}{3!} \sum_{1 \le i \le j \le k \le d} \det \begin{bmatrix} K_{ii} & K_{ij} & K_{ik} \\ K_{ji} & K_{jj} & K_{jk} \\ K_{ki} & K_{kj} & K_{kk} \end{bmatrix} + \ldots +$$

where now $\widehat{I}_p$ is an unordered set of distinct p integers in the set $\{1, 2, \ldots d\}$. The goal is to give a sense to the limit $d \to \infty$ of the above series that takes the name of Fredholm series.

Properties of determinants.

Below we summarise the properties of determinants that will be useful to us.

- Let $\mathbf{k_1}, \ldots \mathbf{k_d}$ be the rows or columns of the matrix K. Then

  $$\det(K) = \text{signed volume of the parallelepiped spanned by the vectors } \mathbf{k_1}, \ldots \mathbf{k_d}$$

- The following relation is satisfied:

  $$\log \det(I - \lambda K) = \text{Tr} \log(I - \lambda K) \,. \tag{5.1}$$

  To prove the identity we consider first the matrix B defined as $I - \lambda K = e^B$. Then the following identities are satisfied:

  $$\det e^B = \det \left(e^{\frac{B}{N}} \ldots e^{\frac{B}{N}} \right) = \left(\det e^{\frac{B}{N}} \right)^N = \lim_{N \to \infty} \left(\det e^{\frac{B}{N}} \right)^N$$

  $$= \lim_{N \to \infty} \left(\det(1 + \frac{B}{N} + O(N^{-2})) \right)^N = \lim_{N \to \infty} \left(1 + \frac{\text{Tr} B}{N} + O(N^{-2}) \right)^N = e^{\text{Tr} B} \,.$$

  Taking the log of both sides one obtains the statement.

- Let A be a $m \times m$ matrix, D a $n \times n$ matrix, B a $m \times n$ matrix and C a $n \times m$ matrix. If $\det A \neq 0$ and $\det B \neq 0$ we have

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D)\det(A - BD^{-1}C) = \det A \det(D - CA^{-1}B)\,.$$

To prove the statement we first factor the determinant of A:

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \det \begin{bmatrix} A^{-1} & -A^{-1}B \\ 0 & I_n \end{bmatrix}$$

where we assume that $\det A \neq 0$. Taking the product of matrices we obtain

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \det \begin{bmatrix} I_m & 0 \\ CA^{-1} & -CA^{-1}B + D \end{bmatrix} = \det A \det(D - CA^{-1}B)$$

and similarly if $\det D \neq 0$ one obtains

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(D)\det(A - BD^{-1}C)\,.$$

When $A = I_m$ and $D = I_n$ we have the equality

$$\det(I_m - BC) = \det(I_n - CB) \tag{5.2}$$

- Derivative of a determinant. Let us supposed that the matrix K depends on a parameter x, then

$$\frac{\partial}{\partial x} \log\det(I - \lambda K) = -\lambda \mathrm{Tr}\left((I - \lambda K)^{-1} \frac{\partial}{\partial x} K\right). \tag{5.3}$$

- Matrix inverse of a $n \times n$ invertible matrix A:

$$(A)^{-1} = \frac{\mathrm{adj} A}{\det(A)} = \frac{((-1)^{i+j} M_{ji})_{i,j=1}^{n}}{\det(A)} \tag{5.4}$$

where adj stands for adjugate matrix and where M_{ij} is the ij minor of A, namely is the determinant of the matrix obtained from A by delating the $i-th$ row $j-th$ column.

5.1 Extension of the notion of determinant to operators

Let us consider a smooth function $K : [0,1] \times [0,1] \to \mathbb{R}$ and for $\phi \in C^1([0,1], \mathbb{R})$ let us consider the integral operator

$$(\mathcal{K}\phi)(y) = \int_0^1 K(y,w)\phi(w)dw$$

The Fredholm integral equation is defined as

$$\phi(y) - \lambda \int_0^1 K(y,w)\phi(w)dw = f(y), \tag{5.5}$$

or

$$(I - \lambda\mathcal{K})\phi = f$$

where I is the identity operator. If the operator $(I-\lambda K)$ is invertible the solution of the Fredholm integral equation is given by

$$\phi = (I-\lambda\mathcal{K})^{-1}f$$

As in the finite dimensional case (5.4) the inverse of the integral operator can be build from the ratio between its determinant called Fredholm determinant and another quantity, that takes the role of the adjugate matrix. The goal of the section is to define the Fredholm determinant of the operator $I-\lambda K$ and then construct the inverse of this operator.

We first start by discretising the function $\phi(y)$, $0 \le y \le 1$, by the d-dimensional vector $\phi(k/d), 1 \le k \le d$ and similarly the integral operator K becomes the $d\times d$ matrix

$$[K(i/d, j/d)/d]_{1\le i,j\le d}$$

so that the integral

$$\int_0^1 K(x,y)\phi(y)dy$$

is approximated by

$$\sum_{j=1}^{d} K\left(\frac{i}{d},\frac{j}{d}\right)\frac{1}{d}\phi\left(\frac{j}{d}\right)$$

The Fredholm's series for the determinant of $[I+K(i/d,j/d)/d]_{1\le i,j\le d}$ is

$$1+\sum_{p=1}^{d}\frac{\lambda^p}{p!}\sum_{|I_p|=p}\det\left[K\left(\frac{i}{d},\frac{j}{d}\right)\frac{1}{d}\right]_{i,j\in I_p}, \tag{5.6}$$

where I_p an unordered set of distinct p integers in the set $\{1,2,\dots d\}$.

Lemma 5.1. *The series (5.6) converges in the limit $d\to\infty$ for all $\lambda\in\mathbb{C}$.*

Proof. A finite-dimensional determinant is the signed volume of the parallelepiped spanned by the rows/columns of the matrix in hand so setting $m=\max\{|K(x,y)| : 0\le x,y\le 1\}$ we have

$$\det\left[K\left(\frac{i}{d},\frac{j}{d}\right)\frac{1}{d}\right]_{i,j\in I_p} \le \prod_{i\in\mathbf{k}}\sqrt{\sum_{j\in I_p}\left|K\left(\frac{i}{d},\frac{j}{d}\right)\right|^2\times d^{-2}} \le p^{p/2}\left(\frac{m}{d}\right)^p$$

and

$$\frac{1}{p!}\sum_{|I_p|=p}\det\left[K\left(\frac{i}{d},\frac{j}{d}\right)\frac{1}{d}\right]_{i,j\in I_p}$$

is dominated by

$$\begin{aligned}\frac{1}{p!}d(d-1)\dots(d-p+1)p^{p/2}\left(\frac{m}{d}\right)^p &\lesssim \frac{p^{p/2}m^p}{p^{p+\frac{1}{2}}e^{-p}\sqrt{2\pi}} \quad \text{by Stirling's approximation}\\ &< (me)^p p^{-p/2}\end{aligned}$$

independently of d. we conclude that

$$\left|\sum_{p=1}^{d} \frac{\lambda^p}{p!} \sum_{|I_p|=p} \det\left[K\left(\frac{i}{d},\frac{j}{d}\right)\frac{1}{d}\right]_{i,j\in I_p}\right| \leq \sum_{p=1}^{d} (\lambda m e)^p p^{-p/2}$$

and in the limit $d \to \infty$ it has radius of convergence R equal to

$$\frac{1}{R} = \lim_{p\to\infty} \left(me)^p p^{-p/2}\right)^{\frac{1}{p}} = 0,$$

that shows that the radius of convergence is infinity. □

The Fredholm determinant of the operator $\mathcal{K}$ is defined as

$$\det(I - \lambda\mathcal{K}) = 1 + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p!} \int_0^1 \dots \int_0^1 \det\left[K(x_i, x_j)\right]_{1\leq i,j\leq p} dx^p \tag{5.7}$$

and it is an entire function of λ. The second term of the above series coincides with the *trace* of the operator $\mathcal{K}$, namely

$$\mathrm{Tr}K = \int_0^1 K(x,x)dx\,. \tag{5.8}$$

5.2 Construction of the resolvent

The goal is to solve the Fredholm integral equation

$$\phi(y) - \lambda \int_\Gamma K(y,w)\phi(w)dw = f(y), \tag{5.9}$$

where Γ is in general some interval of the real line. If the modulus of λ is small enough one can consider the so called Neumann series

$$\phi(y) = f(y) + \lambda\psi_1(y) + \lambda^2\psi_2(y) + \dots \tag{5.10}$$

where

$$\begin{aligned}
\psi_1(y) &= \int_\Gamma K(y,w)f(w)dw \\
\psi_2(y) &= \int_\Gamma K(y,w)\psi_1(w)dw = \int_\Gamma K_2(y,w)f(w)dw \\
\psi_3(y) &= \int_\Gamma K(y,w)\psi_2(w)dw = \int_\Gamma K_3(y,w)f(w)dw \\
\dots &\quad \dots
\end{aligned}$$

and

$$\begin{aligned}
K_2(y,w) &= \int_\Gamma K(y,s_1)K(s_1,w)ds_1 \\
K_3(y,w) &= \int_\Gamma\int_\Gamma K(y,s_1)K(s_1,s_2)K(s_2,w)ds_1ds_2 \\
\dots &\quad \dots \\
K_n(y,w) &= \int_\Gamma K_j(y,s_1)K_{n-j}(s_1,w)ds_1,
\end{aligned} \tag{5.11}$$

with j any value from $1, \dots, n-1$ and $K_1 = K$. If we expand formally the inverse

$$(I - \lambda \mathcal{K})^{-1} = I + \lambda \mathcal{K} + \lambda^2 \mathcal{K} \circ \mathcal{K} + \lambda^3 \mathcal{K} \circ \mathcal{K} \circ \mathcal{K} + \dots$$

we see that the corresponding kernel of the opertator $\mathcal{K} + \lambda \mathcal{K} \circ \mathcal{K} + \lambda^2 \mathcal{K} \circ \mathcal{K} \circ \mathcal{K} + \dots$ takes the form

$$K + \lambda K_2 + \lambda^2 K_3 + \dots \tag{5.12}$$

where K_j are the kernels of the Neumann series (5.11). Introducing the L^2 norm of the kernel K as

$$||K||^2 = \int_\Gamma \int_\Gamma K^2(y, w) dy dw < m^2$$

for some constant m, it is quite straightforward to check that the series (5.12) converges for $|\lambda| < ||K||^{-1}$. The series (5.12) is exactly the kernel $R(y, w; \lambda)$ of the **resolvent** operator $\mathcal{R}$ defined as

$$(I + \lambda \mathcal{R})(I - \lambda \mathcal{K}) = (I - \lambda \mathcal{K})(I + \lambda \mathcal{R}) = I. \tag{5.13}$$

At the same time, using the above equation we can see that the resolvent satisfies

$$\mathcal{R} - \mathcal{K} - \lambda \mathcal{R} \circ \mathcal{K} = \mathcal{R} - \mathcal{K} - \lambda \mathcal{K} \circ \mathcal{R} = 0$$

that implies that the kernel R satisfies the integral equation

$$R(y, w; \lambda) - K(y, w) - \lambda \int_\Gamma R(y, s; \lambda) K(s, w) ds = 0 \tag{5.14}$$

or

$$R(y, w; \lambda) - K(y, w) - \lambda \int_\Gamma K(y, s) R(s, w; \lambda) ds = 0 \tag{5.15}$$

The goal is to show that the resolvent kernel $R(y, w; \lambda)$ has an expansion in ratio of determinants of the form

$$R(y, w; \lambda) = \frac{1}{\det(I - \lambda \mathcal{K})} \sum_{p=0}^{\infty} C_p(y, w; \lambda) \frac{(-\lambda)^p}{p!}$$

where the coefficients C_p are obtained recursively from the above integral equations (5.14)-(5.15). Plugging the abose ansatz into (5.14)-(5.15) gives

$$\sum_{p=0}^{\infty} C_p(y, w; \lambda) \frac{(-\lambda)^p}{p!} - \det(I - \lambda \mathcal{K}) K(y, w) = \begin{cases} \lambda \int_\Gamma ds K(y, s) \sum_{p=0}^{\infty} C_p(s, w; \lambda) \frac{(-\lambda)^p}{p!} \\ \lambda \int_\Gamma ds K(s, w) \sum_{p=0}^{\infty} C_p(y, s; \lambda) \frac{(-\lambda)^p}{p!} \end{cases}$$

or (assuming it is possible to exchange the sum with the integral)

$$C_p(y, w) - D_p K(y, w) = \begin{cases} -p \int_\Gamma ds K(y, s) C_{p-1}(s, w; \lambda) \\ -p \int_\Gamma ds K(s, w) C_{p-1}(y, s; \lambda) \end{cases} \quad p = 1, 2, \dots \tag{5.16}$$

where

$$D_p = \int_0^1 \dots \int_0^1 \det \left[K\left(x_i, x_j\right) \right]_{1 \le i, j \le p} dx^p,$$

and $C_0(y,w) = K(y,w)$. We make an ansatz for $C_p(y,w)$ by defining

$$C_p^*(y,w) = \int\limits_\Gamma \int\limits_\Gamma \cdots \int\limits_\Gamma K \begin{pmatrix} y, & s_1, & s_2, & \dots, & s_p \\ w, & s_1, & s_2, & \dots, & s_p \end{pmatrix} ds_1 ds_2, \dots ds_p$$

where

$$K \begin{pmatrix} s_0, & s_1, & s_2, & \dots, & s_p \\ y_0, & y_1, & y_2, & \dots, & y_p \end{pmatrix} = \det K(s_i, y_j)_{i,j=0}^p \tag{5.17}$$

Expanding the determinant along the first row we obtain

$$\begin{aligned} C_p^*(y,w) &= \int\limits_\Gamma \int\limits_\Gamma \cdots \int\limits_\Gamma K(y,w) K \begin{pmatrix} s_1, & s_2, & \dots, & s_p \\ s_1, & s_2, & \dots, & s_p \end{pmatrix} ds_1 ds_2, \dots ds_p \\ &+ \int\limits_\Gamma \int\limits_\Gamma \cdots \int\limits_\Gamma \sum_{j=1}^p (-1)^j K(y,s_j) K \begin{pmatrix} s_1, & s_2, & \dots, & s_j & \dots & s_p \\ w, & s_2, & \dots, & \widehat{s}_j & \dots & s_p \end{pmatrix} ds_1 ds_2, \dots ds_j \dots ds_p \\ &= K(y,w) D_p - \int\limits_\Gamma \int\limits_\Gamma \cdots \int\limits_\Gamma \sum_{j=1}^p K(y,s_j) K \begin{pmatrix} s_j, & s_1, & \dots, & \widehat{s}_j & \dots & s_p \\ w, & s_2, & \dots, & \widehat{s}_j & \dots & s_p \end{pmatrix} ds_1 ds_2, \dots ds_j \dots ds_p \\ &= K(y,w) D_p - \sum_{j=1}^p \int_\Gamma K(y,s_j) C_{p-1}^*(s_j,w) ds_j \end{aligned}$$

where $\widehat{s}_j$ means that the variable has been dropped. Since this last integral does not depends on s_j we can rewrite it in the form

$$C_p^*(y,w) = K(y,w) D_p - p \int_\Gamma K(y,s) C_{p-1}^*(s,w) ds \tag{5.18}$$

and $C_0^*(y,w) = K(y,w)$. In a similar way expanding $C_p^*(y,w)$ along the first column we obtain

$$C_p^*(y,w) = K(y,w) D_p - p \int_\Gamma C_{p-1}^*(y,s) K(s,w) ds, \tag{5.19}$$

Comparing (5.16) with (5.18) and (5.19) we obtain $C_p^*(y,w) = C_p(y,w)$ and we have the elegant power expansion of the resolvent

Theorem 5.2. *The resolvent kernel $R(y,w;\lambda)$ of the operator $I - \lambda\mathcal{K}$ takes the form*

$$R(y,w;\lambda) = \frac{1}{\det(1-\lambda\mathcal{K})} \sum_{p=0}^\infty \frac{(-\lambda)^p}{p!} \int\limits_\Gamma \int\limits_\Gamma \cdots \int\limits_\Gamma K \begin{pmatrix} y, & s_1, & s_2, & \dots, & s_p \\ w, & s_1, & s_2, & \dots, & s_p \end{pmatrix} ds_1 ds_2, \dots ds_p \tag{5.20}$$

where $K \begin{pmatrix} y, & s_1, & s_2, & \dots, & s_p \\ w, & s_1, & s_2, & \dots, & s_p \end{pmatrix}$ is defined in (5.17). The kernel $R(y,w;\lambda)$ is a meromorphic function of $\lambda \in \mathbb{C}$.

Proof. The only point that remains to be proved is the fact that $R(y,w;\lambda)$ is a meromorphic function of λ. This can be achieved recalling that $\det(1-\lambda\mathcal{K})$ is an entire function of λ by Lemma 5.1 and showing (along the same lines of the proof of Lemma 5.1) that the infinite sum in (5.20) is convergent for all λ in $\mathbb{C}$. □

Now we consider a very specific case of integral equation called Gelfand-Levitan-Marchenko equation that is derived in the context of inverse scattering (see below). It is an equation of the form

$$G(x,y) + F(x+y) + \int_x^{+\infty} G(x,z)F(z+y)dz = 0 \tag{5.21}$$

where the function F and G have some suitable decay at infinity. We introduce the integral operator of Hankel type $\mathcal{F} : L^2(x,\infty) \to L^2(x,\infty)$

$$\mathcal{F}f(y) = \int_x^{\infty} F(y+z)f(z)dz \tag{5.22}$$

and we set $\phi(y;x) = G(x,y)$ so that the Gelfand-Levitan-Marchenko equation takes the form

$$\phi(y;x) + F(x+y) + (\mathcal{F}\phi)(y) = 0 \tag{5.23}$$

or

$$\phi = -(I+\mathcal{F})^{-1}F = -(I+\mathcal{R})F.$$

Using the resolvent kernel $R(y,z;x)$ of the operator $\mathcal{F}$ we have

$$G(y;x) = \phi(y;x,t) = -F(x+y;t) + \int_x^{\infty} R(y,z;x)F(z+x)dz. \tag{5.24}$$

Further introducing the Fredholm determinant $\det(I+\mathcal{F})$ we have the following lemma.

Lemma 5.3. *Let $\mathcal{F}$ be the integral operator (5.22). Then the solution $G(x,x)$ of the Gelfand-Levitan-Marchenko equation (5.21) on the diagonal $x = y$ is given by*

$$G(x,x) = \frac{\partial}{\partial x}\log\det(I+\mathcal{F}). \tag{5.25}$$

Proof. We use the property (5.3) so that

$$\frac{\partial}{\partial x}\log\det(I+\mathcal{F}) = \mathrm{Tr}((I+\mathcal{F})^{-1}\mathcal{F}_x) = \mathrm{Tr}((I-\mathcal{R})\mathcal{F}_x)$$

and we observe from the definition (5.22) that

$$\frac{\partial}{\partial x}(\mathcal{F}f(y)) = -F(y+x;t)f(x)$$

namely the operator $\mathcal{F}_x$ is a multiplication operator. Further $\mathrm{Tr}(\mathcal{F}_x) = -F(2x)$ and

$$-\mathrm{Tr}(\mathcal{R}\mathcal{F}_x) = \int_x^{\infty} R(x,z)F(z+x)dz$$

so that one obtains

$$\frac{\partial}{\partial x}\log\det\det(I+\mathcal{F}) = -F(2x) + \int_x^{\infty} R(x,z;x)F(z+x)dz \tag{5.26}$$

that coincides with (5.24) when $y = x$.

□

In the next section we show how to relate the derivative of the above Fredholm determinant to the solution of the KdV equation.

5.3 τ-function of the solution of KdV as Fredholm determinant

The concept of τ-function in integrable systems has been introduced by Jimbo-Miwa-Ueno in the context of isomonodromic deformation equations. At the same time, Hirota [10] introduced a new formulation of KdV and in general of integrable systems. He introduced the following operator

$$D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x't')|_{x=x', t=t'} \tag{5.27}$$

Then by straighforward algebra it is immediate to check that if $\tau(x,t)$ is a nowhere vanishing function satisfying

$$(D_x D_t + D_t^4)\tau \cdot \tau = 0 \tag{5.28}$$

then $w = \frac{\tau_x}{\tau}$ solves the equation

$$w_t + 6w_x^2 + w_{xxx} = 0 \tag{5.29}$$

and $u(x,t) = -2\partial_x^2 \log(\tau(x,t))$ solves the KdV equation. The goal of this section is to show that the τ function of the KdV solution for initial data vanishing at infinity is a Fredholm determinant. This goal was first achieved by Dyson [2] for the specific case of data vanishing at infinity. Later Poppe [17] introduced a more general class of initial data.

Theorem 5.4. *Let F be a solution of the linearized KdV equation*

$$F_t + 8F_{xxx} = 0 \tag{3.8}$$

decaying sufficiently fast for $|x| \to \infty$ such that F and its derivatives up to order 4 in x and order 2 in t are are decreasing sufficiently fast.

Let $\mathcal{F}$ be the integral operator

$$\mathcal{F}f(y) = \int_x^\infty F(y+z;t) f(z) dz \tag{5.30}$$

Then

$$\tau(x,t) = \det(1 + \mathcal{F})$$

is a tau-function for the KdV equation, namely

$$u(x,t) := -2\frac{\partial^2}{\partial x^2} \log\det(1 + \mathcal{F}) \tag{3.10}$$

is a solution of the KdV equation.

The proof of this theorem was obtained in [17] by making x and t derivative of the Fredholm determinant. We follow Dyson's work that obtained the formula by solving the Gelfand-Levitan Marchenko integral equation (5.21) where the function F is related to the scattering data by

$$F(x;t) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} r(k) e^{ikx + 8ik^3 t} dk + \sum_{s=1}^{n} \frac{b_s e^{-k_s x + 8k_s^3 t}}{ia'(ik_s)}$$

and we assume that r satisfies the assumption of Lemma 3.22. Now let us recall the definition of Jost solutions

$$\phi(x,k) \to e^{-ikx}, \;\; \overline{\phi}(x,k) \to e^{ikx}, \quad x \to -\infty \tag{5.31}$$

and

$$\psi(x,k) \to e^{ikx}, \ \ \overline{\psi}(x,k) \to e^{-ikx}, \quad x \to +\infty \tag{5.32}$$

where $\phi(x,k)$ and $\psi(x,k)$ are analytic in the upper half plane and $\overline{\phi}$ and $\overline{\psi}$ are analytic in the lower half plane. The following general statement from the theory of Fourier integrals is useful for establishing the triangularity.

Lemma 5.5. *If $f(k)$ is analytic in the lower half plane and behaves like $O\left(\frac{1}{k}\right)$ for $|k| \to +\infty$, then the Fourier transform*

$$\widehat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(k)e^{ikx}dk$$

is zero for $x < 0$, and viceversa.

Proof. The shift $k \mapsto k - ia$ with $a > 0$ changes the exponential from e^{ikx} to e^{ikx+ax}. Such a shift does not change the integral. Therefore the modulus $|\widehat{f}(x)|$ for negative x admits an upper estimate as small as we want. □

The Jost solution ψ admits an analytic continuation into the upper half plane $\operatorname{Im} k > 0$.

$$p(x,k) := \psi(x,k)e^{-ikx}$$

an asymptotic expansion of the form

$$p(x,k) \sim 1 + O\left(\frac{1}{k}\right), \quad |k| \to \infty, \quad \operatorname{Im} k > 0$$

Denote by

$$A(x,y) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iky}\left(p(x,k) - 1\right)dk$$

the Fourier transform of $p(x,k) - 1$ with respect to k. Due to Lemma 5.5

$$A(x,y) = 0 \quad \text{for} \quad y < 0$$

Now taking the inverse transform we get

$$p(x,k) = 1 + \int_0^{+\infty} A(x,y)e^{ikx}dy$$

where the integral starts from 0 thanks to the lemma. Finally,

$$\begin{aligned}\psi(x,k) &= e^{ikx} + \int_0^{+\infty} A(x,y)e^{ik(x+y)}dy = \\ &= e^{ikx} + \int_x^{+\infty} A(x,\tilde{y} - x)e^{ik\tilde{y}}d\tilde{y}\end{aligned}$$

changing variable ($\tilde{y} := y + x$). Denoting $G(x,y) := A(x, y - x)$, we get

$$\psi(x,k) = e^{ikx} + \int_x^{+\infty} G(x,y)e^{iky}dy, \tag{5.33}$$

where $G(x,y) = 0$ for $y < x$.

Theorem 5.6. *The function $G(x,y)$ satisfies the Gelfand-Levitan-Marchenko equation (5.21). The solution of the KdV equation is recovered from the relation*

$$u(x,t) = -2\partial_x G(x,x) = -2\frac{\partial^2}{\partial x^2}\log\det(I+\mathcal{F}). \tag{5.34}$$

Proof. The second relation in (5.34) follows from Lemma 5.3. We first derive the Gelfand-Levitan-Marchenko equation for the kernel $G(x,y)$ and then we prove the first identity in (5.34).

From $\phi(x,k) = a(k)\bar{\psi}(x,k) + b(k)\psi(x,k)$, muliplying by $\frac{e^{iky}}{a(k)}$ and integrating with respect to k, we obtain

$$\int_{-\infty}^{+\infty}\frac{\phi(x,k)}{a(k)}e^{iky}dk = \int_{-\infty}^{+\infty}\left(\bar{\psi}(x,k) + r(k)\psi(x,k)\right)e^{iky}dk$$

Since these integrals will be not well defined, we need to subtract something:

$$\int_{-\infty}^{+\infty}\left(\frac{\phi(x,k)}{a(k)} - e^{-ikx}\right)e^{iky}dk = \int_{-\infty}^{+\infty}\left(\bar{\psi}(x,k) - e^{-ikx} + r(k)\psi(x,k)\right)e^{iky}dk \tag{5.35}$$

We perform the first integral, using contour deformation and Cauchy theorem and the fact that

$$\left(\frac{\phi(x,k)}{a(k)} - e^{-ikx}\right)$$

is $O\left(\frac{e^{-ikx}}{k}\right)$ when $\operatorname{Im} k > 0$ and we obtain when $y - x > 0$

$$\begin{aligned}\int_{-\infty}^{+\infty}\left(\frac{\phi(x,k)}{a(k)} - e^{ikx}\right)e^{iky}dk &= 2\pi i\sum_{j=1}^{N}\frac{\phi(x,i\kappa_j)}{a'(i\kappa_j)}e^{-\kappa_j y} = 2\pi i\sum_{j=1}^{N}\frac{\beta_j\psi(x,i\kappa_j)}{a'(i\kappa_j)}e^{-\kappa_j y}\\ &= 2\pi i\sum_{j=1}^{N}\frac{\beta_j(e^{-\kappa_j x} + \int_x^{+\infty}G(x,s)e^{-\kappa_j s}ds)}{a'(e\kappa_j)}e^{-\kappa_j y}\end{aligned} \tag{5.36}$$

where the constants β_j have been defined in (3.25) and in the last relation we used (5.33). The second integral

$$\int_{-\infty}^{+\infty}\left(\bar{\psi}(x,k) - e^{-ikx}\right)e^{iky}dk = 0$$

using contour deformation and Cauchy theorem. The third integral gives

$$\int_{-\infty}^{+\infty}r(k)\psi(x,k)e^{iky}dk = \int_{-\infty}^{+\infty}r(k)e^{ik(y+x)}dk + \int_{-\infty}^{+\infty}dke^{iky}\int_x^{+\infty}G(x,s)e^{iks}ds. \tag{5.37}$$

Substituting (5.36), (5.37) into (5.35) one obtains the Gelfand-Levitan-Marchenko equation (5.21). To show that the solution of KdV is recovered from the kernel $G(x,y)$ it is enough to observe that from the integral equation (3.37) for $p(x,k)$ we have

$$p(x,k) = 1 - \frac{1}{2ik}\int_x^{+\infty}d\xi u(\xi) + O(k^{-2}), \tag{5.38}$$

while from (5.33) we have

$$p(x,k) = 1 + \int_x^{+\infty} G(x,y)e^{ik(y-x)}dy = 1 - \frac{1}{ik}G(x,x) - \frac{1}{ik}\int_x^{+\infty} \partial_y G(x,y)e^{ik(y-x)}dy$$

where we assume that $G(x,y)$ is differentiable in y. Comparing the above two relation one obtains

$$G(x,x) = \frac{1}{2}\int_x^{+\infty} d\xi u(\xi)$$

which is equivalent to (5.34).

□

The proof of Theorem 5.4 follows from the proof of Theorem 5.6.

We have thus shown that the inverse spectral problem is reduced to a linear Fredholm integral equation with Hankel type kernel. The solution of this integral equation is obtained via Fredholm determinant. This is a very general feature in the theory of integral systems. The nonlinear problem is linearized in the scattering variables, the solution of the inverse problem is obtained via Fredholm determinant in the infinite-dimensional setting, or via the usual determinant in the finite-dimensional setting.

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