Toda Lattice

Tamara Grava

December 29, 2024

Contents

1 First integrals associated to a Lax pair

One of the most known method to construct first integrals of a Hamiltonian system is through symmetries of the space P. Another powerful method is due to Lax [**?**] and represents the starting point of the modern theory of integrable systems. Given an ODE

$$\dot{x} = f(x), \quad x = (x^1, \dots, x^N) \tag{1.1}$$

and two $m \times m$ matrices $L = (L_{ij}(x))$, $A = (A_{ij}(x))$, they constitute a *Lax pair* for the dynamical systems if for every solution $x = x(t)$ of (1.1) the matrices $L = (L_{ij}(x(t)))$ and $A = (A_{ij}(x(t)))$ satisfy the equation

$$\dot{L} = [A, L] := AL - LA \tag{1.2}$$

and the validity of (1.2) for $L = L(x)$, $A = A(x)$ implies (1.1).

Theorem 1.1 *Given a Lax pair for the dynamical system (1.1), then the eigenvalues $\lambda_1(x)$, ..., $\lambda_m(x)$ of $L(x)$ are integrals of motion for the dynamical system.*

Proof. The coefficients $a_1(x)$, ..., $a_m(x)$ of the characteristic polynomial

$$\det(L - \lambda I) = (-1)^m \left[\lambda^m - a_1(x)\lambda^{m-1} + a_2(x)\lambda^{m-2} + \cdots + (-1)^m a_m(x)\right] \tag{1.3}$$

of the matrix $L = L(x)$ are polynomials in tr L, tr L^2, ..., tr L^m:

$$a_1 = \mathrm{tr}\ L, \quad a_2 = \frac{1}{2}\left[(\mathrm{tr}\ L)^2 - \mathrm{tr}\ L^2\right],\ a_3 = \dots$$

Next we show that

$$\mathrm{tr}\ L^k, \quad k = 1,\, 2, \dots \tag{1.4}$$

are first integral of the dynamical system. Indeed for $k = 1$

$$\frac{d}{dt}\mathrm{tr}\ L = \mathrm{tr}\ \dot{L} = \mathrm{tr}\,(A\,L - L\,A) = 0.$$

more generally

$$\frac{d}{dt}\mathrm{tr}\, L^k = k\mathrm{tr}\,([A, L]\, L^{k-1}) = 0. \tag{1.5}$$

Since the coefficients of the characteristic polynomial $L(x)$ are constants of motion it follows that its eigenvalues are constants of motion. □

Another proof of the theorem, close to Lax's original proof, can be obtained observing that the solution of the equation $\dot{L} = [A, L]$ can be represented in the form

$$L(t) = Q(t)L(t_0)Q^{-1}(t) \tag{1.6}$$

where the evolution of $Q = Q(t)$ is determined from the equation

$$\dot{Q} = A(t)Q \tag{1.7}$$

with initial data

$$Q(t_0) = 1.$$

Then the characteristic polynomials of $L(t_0)$ e $Q(t)L(t_0)Q^{-1}(t)$ are the same and consequently the eigenvalues are the same.

2 The open Toda lattice

Let us consider the system of n points $q_1,\ q_2, \ldots, q_n$ on the real line interacting with nearest neighbour interaction potential

$$U(q_1, \ldots, q_n) = \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

the so called Toda lattice. The Hamiltonian $H(q, p) \in \mathcal{C}^\infty(T^*\mathbb{R}^n)$ takes the form

$$H(q, p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}. \tag{2.1}$$

There are two possible boundary conditions:

- open Toda with $q_0 = -\infty$ and $q_{n+1} = +\infty$
- closed Toda with $q_1 = q_{n+1}$.

Here we analyse the open Toda lattice. The Hamilton equations with respect to the canonical Poisson bracket

$$\{q_k, p_j\} = \delta_{kj}, \quad \{q_k, q_j\} = \{p_k, p_j\} = 0, jk = 1, \ldots, n \tag{2.2}$$

are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = p_k, \quad k = 1, \ldots, n$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = \begin{cases} -e^{q_1 - q_2} & \text{if } k = 1 \\ e^{q_{k-1} - q_k} - e^{q_k - q_{k+1}} & \text{if } 2 \leq k \leq n-1 \\ e^{q_{n-1} - q_n} & \text{if } k = n \end{cases}$$

Since the Hamiltonian is translation invariant, the total momentum is a conserved quantity together with the Hamiltonian.

Flaschka [5],[6] and Manakov [11] separetely showed that the Toda lattice is a completely integrable system. Let us introduce a new set of dependent variables

$$\begin{aligned} a_k &= \frac{1}{2}\, e^{\frac{q_k - q_{k+1}}{2}}, \quad k = 1, \dots, n-1 \\ b_k &= -\frac{1}{2}\, p_k, \quad k = 1, \dots, n, \end{aligned} \tag{2.3}$$

with evolution given by the equations

$$\begin{aligned} \dot{a}_k &= a_k(b_{k+1} - b_k), \quad k = 1, \dots, n-1 \\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2), \quad k = 1, \dots, n, \end{aligned} \tag{2.4}$$

where we use the convention that $a_0 = a_n = 0$. Observe that there are only $2n-1$ variables and this is due the translation invariance of the original system. The equations (4.3) have an Hamiltonian form with Hamiltonian

$$H(a,b) = 2\sum_{i=1}^{n} b_i^2 + 4\sum_{i=1}^{n-1} a_i^2$$

with Poisson bracket define on $(\mathbb{R}^*)^{n-1} \times \mathbb{R}^n$ given by

$$\{a_i, b_j\} = -\frac{1}{4}\delta_{ij} a_i + \frac{1}{4}\delta_{i,j-1} a_i, \quad i = 1, \dots, n-1, \;\; j = 1, \dots, n,$$

while all the other entries are equal to zero. We observe that the total momentum $\sum_{k=1}^{n} b_k$ is a Casimir of the above Poisson bracket

Next we introduce the tridiagonal $n \times n$ matrices:

$$
\begin{aligned}
L &= \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & 0 \\ a_1 & b_2 & a_2 & & 0 & 0 \\ 0 & a_2 & b_3 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & b_{n-1} & a_{n-1} \\ 0 & & & & a_{n-1} & b_n \end{pmatrix} \\
A &= \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & 0 \\ -a_1 & 0 & a_2 & & 0 & 0 \\ 0 & -a_2 & 0 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & 0 & a_{n-1} \\ 0 & & & & -a_{n-1} & 0 \end{pmatrix}
\end{aligned}
\tag{2.5}
$$

where $A = L_+ - L_-$ and we are using the following notation: for a square matrix X we call X_+ the upper triangolar part of X

$$(X_+)_{ij} = \begin{cases} X_{ij}, & i < j \\ 0, & \text{otherwise} \end{cases}$$

and in a similar way by X_- the lower triangular part of X

$$(X_-)_{ij} = \begin{cases} X_{ij}, & i < j \\ 0, & \text{otherwise.} \end{cases}$$

A straighforward calculation shows that

Lemma 2.1 *The Toda lattice equations (4.3) are equivalent to*

$$\frac{dL}{dt} = [A, L]. \tag{2.6}$$

Exercise 2.2 Determine the Lax pair for the closed Toda lattice.

The open Toda lattice equation is sometimes written in the literature in Hessebeg form. Conjugating the matrix L by a diagonal matrix $D = \text{diag}(1, a_1, a_1a_2, \dots, \prod_{j=1}^{n-1} a_j)$

yelds the matrix $\widehat{L} = DLD^{-1}$

$$\widehat{L} = \begin{pmatrix} b_1 & 1 & 0 & \dots & 0 & 0 \\ a_1^2 & b_2 & 1 & & 0 & 0 \\ 0 & a_2^2 & b_3 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & b_{n-1} & 1 \\ 0 & & & & a_{n-1}^2 & b_n \end{pmatrix} \tag{2.7}$$

The Toda equations (4.3) take the form

$$\frac{d\widehat{L}}{dt} = -2[\widehat{A}, \widehat{L}] \tag{2.8}$$

where the matrix $\widehat{A} = \widehat{L}_-$ namely

$$\widehat{A} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ a_1^2 & 0 & 0 & & 0 & 0 \\ 0 & a_2^2 & 0 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & 0 & 0 \\ 0 & & & & a_{n-1}^2 & 0 \end{pmatrix} \tag{2.9}$$

From the results of the previous section, the Lax formulation guarantees the existence of conserved quantities, namely the traces

$$F_j = \text{tr } L^{j+1}, \quad j = 0, \dots, n-1.$$

are conserved quantities. To show the independence of the integrals $F_0, \dots, F_{n-1}$ we observe that

$$F_{j-1} = \sum_{k=1}^{n} b_k^j + \text{lower order polynomials of } a_k \text{ and } b_k.$$

Since the polynomials $b_1^j + b_2^j + \dots + b_n^j$ for $j = 1, \dots, n$ are linearly independent with respect to the variables $b_1, \dots, b_n$, it follows that the integrals $F_0, \dots, F_{n-1}$ are functionally independent. Next we show that the integrals are in involution. For the purpose we need the following lemma.

Lemma 2.3 *(i) The spectrum of L consists of n distinct real numbers $\lambda_1 < \lambda_2 < \dots < \lambda_n$.*

(ii) Let $Lv = \lambda v$ with $v = (v_1, \dots v_n)^t$. Then $v_1 \neq 0$ and $v_n \neq 0$. Furthermore, $v_k = v_1 p_k(\lambda)$ where $p_k(\lambda)$ is a polynomial of degree k in λ.

Proof. We will first prove (ii). From the equation $Lv = \lambda v$ one obtains

$$(b_1 - \lambda)v_1 + a_1 v_2 = 0 \tag{2.10}$$

$$a_{k-1}v_{k-1} + (b_k - \lambda)v_k + a_k v_{k+1} = 0, \quad 2 \leq k < n. \tag{2.11}$$

Since $a_1 \neq 0$ clearly $v_1 = 0 \Longrightarrow v_2 = 0$, but then from (2.11) with $k = 2$, since $a_2 \neq 0$, then $v_1 = 0$ and $v_2 = 0$ implies $v_3 = 0$. Hence $v = 0$ if $v_1 = 0$. Therefore $v_1 \neq 0$. In the same way it can be proved that $v_n \neq 0$. From (2.10) and (2.11) it easily follows that v_k is a polynomial of degree k in λ. To prove (i), since L is symmetric, the eigenvalues are real. In order to show that the eigenvalues are distinct, let us suppose that v and $\tilde{v}$ are two eigenvalues corresponding to the same eigenvector λ. Then the linear combination $\alpha v + \beta \tilde{v}$ is also an eigenvector of L with eigenvalue λ. But then one can choose $\alpha \neq 0$ and $\beta \neq 0$ so that $\alpha v_1 + \beta \tilde{v}_1 = 0$ and by (ii) it follows that $\alpha v + \beta \tilde{v}$=0 implying that v and $\tilde{v}$ are dependent. □

Using the above lemma one has

$$\det \frac{\partial F_j}{\partial \lambda_k} = \begin{pmatrix} 1 & 1 & \dots 1 & \\ 2\lambda_1 & 2\lambda_2 & \dots & 2\lambda_n \\ \dots & \dots & \dots & \dots \\ n\lambda_1^{n-1} & n\lambda_2^{n-1} & \dots & n\lambda_n^{n-1} \end{pmatrix} = n! \prod_{i<j} (\lambda_i - \lambda_j) \neq 0,$$

because the eigenvalues are all distinct. This shows that we can take the eigenvalues $\lambda_1, \dots, \lambda_n$ as a new set of functionally independent variables. In order to show that the Toda lattice is an integrable system we also need to show that the functions $F_1, \dots, F_n$, or equivalently the eigenvalues $\lambda_1, \dots, \lambda_n$ commute with respect to the canonical Poisson bracket. For the purpose let us consider the equation

$$Lv = \lambda v, \tag{2.12}$$

where v is a normalised eigenvector, $v = (v_1, \dots, v_n)^t$ and $(v, v) = 1$. Then we introduce the discrete Wronkstian

$$W_i(v, w) = a_i(v_i w_{i+1} - v_{i+1} w_i) \tag{2.13}$$

where w is an eigenvector with respect to the eigenvalue μ. We use the convention that $W_0 = W_n = 0$. It is easy to see using the equation (2.12) that the Wronkstian satisfies

$$W_i = (\mu - \lambda)v_i w_i + W_{i-1}. \tag{2.14}$$

Indeed we have from (2.12)

$$(b_i-\lambda)v_i+a_{i-1}v_{i-1}+a_iv_{i+1}=0,\quad (b_i-\mu)w_i+a_{i-1}w_{i-1}+a_iw_{i+1}=0.$$

Multiplying the first equation by w_i and the second by v_i and subtracting them, one obtains the statement. We are ready to prove the following.

Proposition 2.4 *The eigenvalues of L commute with respect to the canonical Poisson bracket (2.2).*

Proof. Let us consider the equation (2.12) and its variational derivative

$$\delta L\, v + L\, \delta v = v\, \delta\lambda + \lambda\, \delta v$$

Taking the scalar product with respect to v and using $(v,v)=1$ one obtains

$$\delta\lambda = (v,\delta L\, v)+(v,(L-\lambda)\delta v) = (v,\delta L\, v)+((L-\lambda)v,\delta v)=(v,\delta L\, v) \tag{2.15}$$

where we use the fact that the operator L is symmetric.

Let λ and μ be two eigenvalues of L with normalized eigenvectors v an w respectively. Then from (2.15) one has

$$\begin{aligned}\frac{\partial\lambda}{\partial p_i} &= (v,\,\frac{\partial L}{\partial p_i}v) = -\frac{1}{2}v_i^2\\ \frac{\partial\lambda}{\partial q_i} &= (v,\,\frac{\partial L}{\partial q_i}v) = a_iv_iv_{i+1}-a_{i-1}v_iv_{i-1},\quad i=1,\ldots,n,\end{aligned} \tag{2.16}$$

where we use the fact that $(v,v)=1$ and we define $a_0=0=a_n$. The same relations hold for the eigenvalue μ. Then one has

$$\begin{aligned}\{\lambda,\,\mu\} &= \sum_{i=1}^{n}\left(\frac{\partial\lambda}{\partial q_i}\frac{\partial\mu}{\partial p_i}-\frac{\partial\lambda}{\partial p_i}\frac{\partial\mu}{\partial q_i}\right)\\ &= \frac{1}{2}\sum_{i=1}^{n}\left(v_iw_i(a_i(v_iw_{i+1}-v_{i+1}w_i)+a_{i-1}(w_iv_{i-1}-v_iw_{i-1})\right).\end{aligned} \tag{2.17}$$

Using the definition of Wronkstian in (2.13) and the identity (2.14) one can reduce the above relation to the form

$$\{\lambda,\,\mu\} = \frac{1}{2(\mu-\lambda)}\sum_{i=1}^{n}(W_i^2-W_{i-1}^2) = \frac{W_n^2-W_0^2}{2(\mu-\lambda)}=0.$$

□

Summarazing, we have proved the following theorem.

Theorem 2.5 *The Toda Lattice is a completely integrable Hamiltonian system.*

By Liouville theorem it follows that the Toda system can be integrated by quadratures. Let us show how to do this. By the lemma 2.3 it follows that

$$L = U\Lambda U^t \tag{2.18}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ with distinct eigenvalues and U is an orthogonal matrix $UU^t = 1$ with entries $U_{ij} = u_{ij}$ the normalized eigenvectors $u_i = (u_{1i}, \ldots u_{ni})^t$ of L. From $UU^t = U^tU = 1$ one has

$$(u_i, u_j) = \delta_{ij}, \quad \sum_{k=1}^{n} (u_{kj})^2 = 1, \quad i, j = 1, \ldots, n.$$

We know the eigenvalues of $L(t)$, since they are constants of motion. In order to know $L(t)$ at time t we need to know the orthogonal matrix $U = U(t)$, with entries $U_{ij} = u_{ij}$. From (2.6) and (2.18) one has that

$$\dot{U} = AU. \tag{2.19}$$

In particular, the dynamics implied by the above equation on the first row u_{1i}, $i = 1, \ldots, n$ of the matrix U is quite simple.

Lemma 2.6 *The time evolution of the first row of the matrix U, namely the entries u_{1i} $i = 1, \ldots, n$ are given by*

$$u_{1i}(t)^2 = \frac{e^{2\lambda_i t} u_{1i}(0)^2}{\sum_{k=1}^{n} e^{2\lambda_k t} u_{1k}(0)^2}, \quad i = 1, \ldots, n. \tag{2.20}$$

Proof. From (2.19) one has

$$\frac{du_{1i}}{dt} = (AU)_{1i} = a_1 u_{i2}$$

and from the relation $Lu_i = \lambda_i u_i$, with $u_i = (u_{1i}, \ldots, u_{ni})^t$, one reduces the above equation to the form

$$\frac{du_{1i}}{dt} = (\lambda_i - b_1) u_{1i}.$$

The solution is given by

$$u_{1i}(t) = E(t) e^{\lambda_i t} u_{1i}(0), \quad E(t) = exp\left(-\int_0^t b_1(\tau) d\tau\right)$$

Using the normalization conditions

$$1 = \sum_{i=1}^{n} u_{1i}(t)^2 = E(t)^2 \sum_{i=1}^{n} e^{2\lambda_i t} u_{1i}(0)^2$$

which implies

$$E(t)^2 = \left(\sum_{i=1}^{n} e^{2\lambda_i t} u_{1i}(0)^2 \right)^{-1}$$

one arrives to the statement of the lemma. □

2.1 Inverse spectral problem for the open Toda lattice

The goal of this section is to reconstruct the $2n-1$ variables $a_j(t)$ and $b_j(t)$ from the spectral data $\lambda_1, \dots, \lambda_n$ and the entries $u_{1i}(t)$, $i = 1, \dots, n$, of the matrix U with the constraint $\sum_{i=1}^{n} u_{1i}^2(t) = 1$. We are going to use three different procedure to solve inverse spectral problem and integrate the Toda lattice:

- the QR algorithm du to Symes;
- the continued fraction expansion due to Moser;
- orthogonal polynomials (follows from the first algorithm).

Introducing the notation

$$\xi_i(t) = u_{1i}(t), \quad i = 1, \dots, n \tag{2.21}$$

one can see from lemma 2.3 that the orthogonal matrix U can be written in the form

$$U = \begin{pmatrix} \xi_1(t)p_0(\lambda_1, t) & \xi_2(t)p_0(\lambda_2, t) & \dots & \xi_n(t)p_0(\lambda_n, t) \\ \xi_1(t)p_1(\lambda_1, t) & \xi_2(t)p_1(\lambda_2, t) & \dots & \xi_n(t)p_1(\lambda_n, t) \\ \vdots & \vdots & & \vdots \\ \xi_1(t)p_{n-1}(\lambda_1, t) & \xi_2(t)p_{n-1}(\lambda_2, t) & \dots & \xi_n(t)p_{n-1}(\lambda_n, t) \end{pmatrix}$$

where $p_k(\lambda, t)$ is a polynomial of degree k in λ. Since U is an orthogonal matrix, the orthogonality relations on the rows of U take the form

$$\sum_{k=1}^{n} \xi_k^2 p_m(\lambda_k) p_j(\lambda_k) = \delta_{mj}. \tag{2.22}$$

In other words, the polynomials $p_j(\lambda, t)$ are normalized orthogonal polynomials with respect to the discrete weights ξ_k^2 at the points λ_k. To find the orthogonal polynomials from the weights, is a standard procedure. We will use QR factorisation, which is a decomposition of a matrix $A = QR$ into an orthogonal matrix Q and an upper triangular matrix R. If A is invertible, then the decomposition is unique if the diagonal entries of R are positive.

Let us consider the non degenerate matrix V in the form

$$V = \begin{pmatrix} \sqrt{w_1(t)}p_0(\lambda_1, 0) & \sqrt{w_1(t)}p_1(\lambda_1, 0) & \dots & \sqrt{w_1(t)}p_{n-1}(\lambda_1, 0) \\ \sqrt{w_2(t)}p_0(\lambda_2, 0) & \sqrt{w_2(t)}p_1(\lambda_2, 0) & \dots & \sqrt{w_2(t)}p_{n-1}(\lambda_2, 0) \\ \vdots & \vdots & & \vdots \\ \sqrt{w_n(t)}p_0(\lambda_n, 0) & \sqrt{w_n(t)}p_1(\lambda_n, 0) & \dots & \sqrt{w_n(t)}p_{n-1}(\lambda_n, 0) \end{pmatrix}$$
$$= \operatorname{diag}\left(\sqrt{\frac{w_1(t)}{w_1(0)}}, \sqrt{\frac{w_2(t)}{w_2(0)}}, \dots, \sqrt{\frac{w_n(t)}{w_n(0)}}\right) U(0)^t$$

By the QR algorithm one has

$$V(t) = \operatorname{diag}\left(\sqrt{\frac{w_1(t)}{w_1(0)}}, \sqrt{\frac{w_2(t)}{w_2(0)}}, \dots, \sqrt{\frac{w_n(t)}{w_n(0)}}\right) U(0)^t = U(t)^t R.$$

It is easy to check that R is indeed upper-triangular by observing that the entry (j, l) of the product $U(t)V$ takes the form

$$\sum_{k=1}^{n} w_k(t) p_j(\lambda_k, t) p_l(\lambda_k, 0) = 0, \qquad \text{for } l < j,$$

due to the orthogonality of the polynomials $p_j(\lambda_k, t)$ with respect to the weights $w_1(t), \dots w_n(t)$. Multiplyingthe matrix V from the left by $U(0)$ and observing that

$$\sqrt{\frac{w_j(t)}{w_j(0)}} = E(t)e^{\lambda_j t},$$

one arrives to the relation

$$E(t)U(0)e^{t\Lambda}U(0)^t = E(t)e^{tL(0)} = U(0)U(t)^t R$$

or equivalently

$$e^{tL(0)} = U(0)U(t)^t R, \tag{2.23}$$

where the scalar term $E(t)$ has been absorbed in R. From the relation

$$L(t) = U(t)\Lambda U(t)^t = U(t)U(0)^t L(0)U(0)U(t)^t \tag{2.24}$$

one realizes that $L(t)$ can be obtained from $L(0)$ by knowing $U(0)U(t)^t$. Therefore, the solution of the Toda lattice equations can be obtained by the following steps

- from $L(0)$ determine $e^{tL(0)}$;
- apply Gram-Schmidt orthogonalization procedure to $e^{tL(0)} = U(0)U(t)^tR$ so that one obtains $U(0)U(t)^t$
- determine $L(t)$ from the identity $L(t) = U(t)U(0)^tL(0)U(0)U(t)^t$.

We are going to derive a different procedure to integrate the Toda lattice due to Moser [12]. Recall that we have denoted by $(\xi_1, \dots \xi_n)$ the first row of the matrix U. Consider the set

$$\text{Spec} = \{\lambda_1 < \lambda_2 < \cdots < \lambda_n,\ (\xi_1, \dots, \xi_n),\ \xi_i > 0,\ \sum_{i=1}^{n} \xi_i^2 = 1\}, \tag{2.25}$$

which is the spectral data associated to the matrix $L = L(a, b)$. The matrix L is a Jacobi matrix, namely a tridiagonal symmetric matrix where the lower and upper diagonal entries are positive.

Theorem 2.7 (Moser) *The spectral map*

$$S : L(a, b) \to Spec$$

is a bijection between Jacobi matrices and the set Spec.

Proof. We need to show that for a given set $(\lambda, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ where $\lambda = (\lambda_1, \dots, \lambda_n)$, with $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\xi = (\xi_1, \dots, \xi_n)$ with $(\xi, \xi) = 1$ and $\xi_i > 0$ there is a unique Jacobi matrix with such spectral data. For the purpose define for $j = 0, \dots, n-1$ the $(n-j) \times (n-j)$ matrices

$$\Delta_j(z) = \det \begin{pmatrix} z - b_{j+1} & -a_{j+1} & 0 & 0 & \dots & \dots \\ -a_{j+1} & z - b_{j+2} & -a_{j+2} & 0 & \dots & \dots \\ 0 & -a_{j+2} & z - b_{j+3} & -a_{j+3} & & \\ & \dots & \dots & & & \dots \\ \dots & & 0 & 0 & -a_{n-1} & z - b_n \end{pmatrix}$$

with $\Delta_n(z) := 1$ and $\Delta_{n+1}(z) = 0$ and $\Delta_0(z) = \det(zI - L)$. It is easy to see that Δ_j is a polynomial of degree $n - j$. Furthermore, expanding the determinant along the first column, one obtains the recursion relation

$$\Delta_j(z) = (z - b_{j+1})\Delta_{j+1} - a_{j+1}^2\Delta_{j+2}. \tag{2.26}$$

Now let us consider the entry $(1, 1)$ of the resolvent $R := (zI - L)^{-1}$. Such entry turns out the be equal to

$$R(z)_{11} = \frac{\Delta_1(z)}{\Delta_0(z)}$$

by the form of the inverse of a matrix. On the other hand, one also has

$$R(z)_{11} = (zI - L)^{-1}_{11} = (U(zI - \Lambda)U^t)^{-1}_{11} = (U(zI - \Lambda)^{-1}U^t)_{11} = \sum_{i=1}^{n} \frac{\xi_i^2}{z - \lambda_i}$$

Combining the above two relations and the recursive formula (2.26) one arrives to the continued fraction expansion

$$\sum_{i=1}^{n} \frac{\xi_i^2}{z - \lambda_i} = \frac{1}{\frac{\Delta_0}{\Delta_1}} = \frac{1}{z - b_1 - \frac{a_1^2}{\frac{\Delta_1}{\Delta_2}}} = \frac{1}{z - b_1 - \frac{a_1^2}{z - b_2 - \frac{a_2^2}{\frac{\vdots}{\cdots - \frac{a_{n-1}^2}{z - b_n}}}}} \tag{2.27}$$

□

For example from the continued fraction expansion one has

$$b_1 = \sum_{i=1}^{n} \lambda_i \xi_i^2, \quad a_{n-1} = \sum_{i<j} (\lambda_i - \lambda_j)\xi_i \xi_j.$$

So the integration of the Toda lattice is obtained by the following diagram:

$$\begin{array}{ccc} \{a_i(0),\, b_i(0)\} & \overset{\text{direct spectral problem}}{\Longrightarrow} & \{\lambda_1, \dots, \lambda_n,\, \xi_1(0) \dots, \xi_n(0)\} \\ & & \Downarrow \\ \{a_i(t),\, b_i(t)\} & \overset{\text{inverse spectral problem}}{\Longleftarrow} & \{\lambda_1, \dots, \lambda_n,\, \xi_1(t) \dots, \xi_n(t)\}. \end{array}$$

Such procedure is called inverse scattering.

Example 2.8 In the particular case $n = 2$ from the continued fraction expansion

$$\frac{\xi_1^2}{z - \lambda_1} + \frac{\xi_2^2}{z - \lambda_2} = \frac{1}{z - b_1 - \frac{a_1^2}{z - b_2}}$$

one can get easily the explicit formulas of the solution

$$b_1(t) = -\frac{1}{2}p_1 = \lambda_1 \xi_1(t)^2 + \lambda_2 \xi_2(t)^2 = \frac{\lambda_1 \xi_1(0)^2 e^{2\lambda_1 t} + \lambda_2 \xi_2(0)^2 e^{2\lambda_2 t}}{\xi_1(0)^2 e^{2\lambda_1 t} + \xi_2(0)^2 e^{2\lambda_2 t}}$$

$$b_2(t) = -\frac{1}{2}p_2 = \lambda_2 \xi_1(t)^2 + \lambda_1 \xi_2(t)^2 = \frac{\lambda_2 \xi_1(0)^2 e^{2\lambda_1 t} + \lambda_1 \xi_2(0)^2 e^{2\lambda_2 t}}{\xi_1(0)^2 e^{2\lambda_1 t} + \xi_2(0)^2 e^{2\lambda_2 t}}$$

$$a_1 = \frac{1}{2} e^{\frac{q_1 - q_2}{2}} = (\lambda_2 - \lambda_1)\xi_1(t)\xi_2(t) = \frac{(\lambda_2 - \lambda_1)\xi_1(0)\xi_2(0)e^{(\lambda_1+\lambda_2)t}}{\xi_1(0)^2 e^{2\lambda_1 t} + \xi_2(0)^2 e^{2\lambda_2 t}}$$

or equivalently

$$q_1 = -\log\left(\xi_1(0)^2 e^{2\lambda_1 t} + \xi_2(0)^2 e^{2\lambda_2 t}\right)$$

$$q_2 = -2(\lambda_1 + \lambda_2)t - 2\log(2(\lambda_1 - \lambda_2)\xi_1(0)\xi_2(0)) + \log\left(\xi_1(0)^2 e^{2\lambda_1 t} + \xi_2(0)^2 e^{2\lambda_2 t}\right).$$

Observe that for $t \to +\infty$ one has

$$a_1(t) \to 0, \quad b_1(t) \to \lambda_2, \quad b_2(t) \to \lambda_1.$$

Exercise 2.9 Prove that for $t \to +\infty$ the Lax matrix becomes diagonal with entries

$$L \to \mathrm{drag}(\lambda_n, \lambda_{n-1}, \ldots, \lambda_2, \lambda_1).$$

2.2 Toda flows and orthogonal polynomials

It is instructive to relate the integration of the Toda flows to orthogonal polynomials. Let $d\mu(\lambda)$ be a positive measure on the real line such that

$$\int_{\mathbb{R}} \lambda^k d\mu(\lambda) < \infty, \quad k \geq 0.$$

Consider the $(n+1) \times (n+1)$ Hankel matrix M_n with entries

$$(M_n)_{ij} = \int_{\mathbb{R}} \lambda^{i+j-2} d\mu(\lambda), \quad i, j = 1, \ldots, n+1.$$

Lemma 2.10 *The matrix M_n is positive definite.*

Proof. It is sufficient to consider the positive integral

$$0 < \int_{\mathbb{R}} (\sum_{k=0}^{n} t_k \lambda^k)^2 d\mu(\lambda) = \int_{\mathbb{R}} \sum_{j,k=0}^{n} t_k t_j \lambda^{k+j} d\mu(\lambda) = < t, M_n t >$$

where $t = (t_0, \ldots, t_n)$. For the arbitrariness of t it follows that M_n is a positive definite matrix. □

We define the determinant

$$D_n = \det M_n \tag{2.28}$$

which is by lemma 2.10 positive. For convenience we are setting $D_{-1} = 1$.

Let us now consider the polynomial of degree n

$$\pi_n(\lambda) = \det \begin{pmatrix} & & M_{n-1} & & \int \lambda^n d\mu(\lambda) \\ & & & & \dots \\ & & & & \int \lambda^{2n-1} d\mu(\lambda) \\ & & & & \\ \lambda^0 & \lambda^1 & \dots & \lambda^{n-1} & \lambda^n \end{pmatrix} \tag{2.29}$$

Lemma 2.11 *The polynomials*

$$\begin{aligned} p_0(\lambda) &= \frac{1}{\sqrt{D_0}} \\ p_n(\lambda) &= \frac{\pi_n(\lambda)}{\sqrt{D_n D_{n-1}}} = \sqrt{\frac{D_{n-1}}{D_n}} \left(\lambda^n + O(\lambda^{n-1})\right), \quad n > 0, \end{aligned} \tag{2.30}$$

are orthonormal polynomials with respect to the measure $d\mu(\lambda)$, namely

$$\int_{\mathbb{R}} p_n(\lambda) p_m(\lambda) d\mu(\lambda) = \delta_{nm}. \tag{2.31}$$

Proof. The orthonormality condition (2.31) is equivalent to the conditions $\int_{\mathbb{R}} p_n(\lambda)\lambda^m d\mu(\lambda) = 0$ for $m < n$ and $\int_{\mathbb{R}} p_n(\lambda)^2 d\mu(\lambda) = 1$ Using the fact that the determinant is a multilinear map one has

$$\int_{\mathbb{R}} p_n(\lambda)\lambda^m d\mu(\lambda) = \det \begin{pmatrix} & & M_{n-1} & & \int \lambda^n d\mu(\lambda) \\ & & & & \dots \\ & & & & \int \lambda^{2n-1} d\mu(\lambda) \\ & & & & \\ \int \lambda^m d\mu(\lambda) & \int \lambda^{m+1} d\mu(\lambda) & \dots & \int \lambda^{m+n-1} d\mu(\lambda) & \int \lambda^{m+n} d\mu(\lambda) \end{pmatrix} = 0, \quad m < n.$$

The above determinant is equal to zero because the last row of the above matrix is equal to the $(m+1)$th row. Regarding the normalising condition one has

$$\int_{\mathbb{R}} p_n(\lambda)^2 d\mu(\lambda) = \frac{1}{D_n D_{n-1}} \int_{\mathbb{R}} D_{n-1} \lambda^n \pi_n(\lambda) d\mu(\lambda) = 1.$$

□

Lemma 2.12 *The orthogonal polynomials (2.30) satisfy a 3-term recurrence relations*

$$\begin{aligned}\lambda p_0(\lambda) &= a_1 p_1(\lambda) + b_1 p_0(\lambda)\\ \lambda p_n(\lambda) &= a_{n+1} p_{n+1}(\lambda) + b_{n+1} p_n(\lambda) + a_n p_{n-1}(\lambda),\end{aligned} \tag{2.32}$$

with

$$a_{n+1} = \sqrt{\frac{D_{n+1} D_{n-1}}{D_n^2}} \tag{2.33}$$

$$b_{n+1} = \frac{G_n}{D_n} - \frac{G_{n-1}}{D_{n-1}}. \tag{2.34}$$

where G_{n-1} is the determinant of the minor of $D_n(\lambda)$ that is obtained by erasing the $(n+1)$ row and the n column,

Proof. The polynomial $\lambda p_n(\lambda)$ is of degree $n+1$ so one has

$$\lambda p_n(\lambda) = \sum_{k=0}^{n+1} \gamma_k^n p_k(\lambda),$$

for some constants γ_k^n. Multiplying both sides of the above identity by $p_j(\lambda)$, $0 \le j < n-1$ and integrating over $d\mu(\lambda)$ one has, using orthogonality

$$0 = \int_{\mathbb{R}} \lambda p_n(\lambda) p_j(\lambda) d\mu(\lambda) = \gamma_j^n, \quad 0 \le j < n-1.$$

because $\lambda p_j(\lambda)$ is a polynomial of degree at most $j+1$ and $\lambda p_n(\lambda)$ is at most of degree $n+1$. Therefore only $\gamma_{n+1}^n, \gamma_n^n$ and γ_{n-1}^n are different from zero. In order to determine the coefficient γ_{n+1}^n let us observe that

$$p_n(\lambda) = \sqrt{\frac{D_{n-1}}{D_n}} \lambda^n + O(\lambda^{n-1}),$$

and comparing the right and left-handside of (2.32) one has

$$\gamma_{n+1}^n = \sqrt{\frac{D_{n+1} D_{n-1}}{D_n^2}} := a_{n+1} \tag{2.35}$$

Regarding γ_{n-1}^n one has

$$\gamma_{n-1}^n = \int_{\mathbb{R}} \lambda p_n(\lambda) p_{n-1}(\lambda) d\mu(\lambda) = \sqrt{\frac{D_n D_{n-2}}{D_{n-1}^2}}$$

so that $\gamma_{n-1}^n = a_n$. Defining G_{n-1} the determinant of the minor of $D_n(\lambda)$ that is obtained by erasing the $(n+1)$ row and the n column, one has that

$$p_n(\lambda) = \sqrt{\frac{D_{n-1}}{D_n}}\lambda^n - \frac{G_{n-1}}{\sqrt{D_n D_{n-1}}}\lambda^{n-1} + O(\lambda^{n-2})$$

so that comparing the left and righthandside of (2.32) one obtains

$$b_{n+1} = \frac{G_n}{D_n} - \frac{G_{n-1}}{D_{n-1}}. \tag{2.36}$$

□

2.3 Integration of Toda lattice

Now let us consider the measure associated to the Toda lattice

$$d\tilde{\mu}(\lambda) = E^2(t)\sum_{j=1}^{n} e^{2\lambda_i t}\delta(\lambda-\lambda_i)u_{1,i}(0)^2 d\lambda,$$

with $E(t)$ a function of time as in (**??**). Then it is easy to check that the ratios G_n/D_n in (2.36) are independent from $E(t)$ as well as the ratios $\sqrt{\frac{D_{n+1}D_{n-1}}{D_n^2}}$ in the definition of a_n. Therefore we can set $E(t) = 1$ without loss of generality. It in an easy calculation to derive the identity

$$\frac{\partial D_n}{\partial t} = 2G_n.$$

So using the above identity one can write the coefficient b_{n+1} in the form

$$b_{n+1} = \frac{1}{2}\frac{\partial}{\partial t}\log\frac{D_n}{D_{n-1}}. \tag{2.37}$$

We conclude that the integration of the Toda lattice equation is given by the relation (2.37) and (2.33) with respect to the measure

$$d\mu(\lambda,t) = \sum_{j=1}^{n} u_{1,i}(0)^2 e^{2\lambda_i t}\delta(\lambda-\lambda_i)d\lambda.$$

We are now interested in determining the evolution of the coefficients a_n and b_n as a function of the parameter t. To operate in a more general setting let us introduce the modified weight

$$d\mu(\lambda) = e^{2\sum_{k=1}^{s}\lambda^k t_k} d\tilde{\mu}(\lambda),$$

with $d\tilde{\mu}(\lambda)$ independent from the times $t_k, k = 1, \ldots, s$ and with $t_1 = t$. Consider the tridiagonal seminfinte matrix L

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \ldots & 0 & 0 & \ldots \\ a_1 & b_2 & a_2 & & 0 & 0 & \ldots \\ 0 & a_2 & b_3 & & & 0 & \ldots \\ & & & & & & \\ \ldots & & & \ldots & & \ldots & \ldots \\ & & & & & & \\ 0 & & & & b_{n-1} & a_{n-1} & \ldots \\ 0 & & & & a_{n-1} & b_n & \ldots \\ \ldots & & & \ldots & & \ldots & \ldots \end{pmatrix} \tag{2.38}$$

and the infinite vector

$$p(\lambda) = \begin{pmatrix} p_0(\lambda) \\ p_1(\lambda) \\ p_2(\lambda) \\ \ldots \\ p_n(\lambda) \\ \ldots . \end{pmatrix}$$

Then the 3-term recurrence relation can be written in the compact form

$$\lambda p(\lambda) = L p(\lambda). \tag{2.39}$$

Now let us introduce the quasi-polynomials

$$\psi_k(\lambda) = p_k(\lambda) e^{\sum_{r=1}^{s} \lambda^r t_r}.$$

Clearly from the orthonormality of the polynomials $p_k(\lambda)$ it follows that

$$\int_{\mathbb{R}} \psi_k(\lambda)\psi_j(\lambda) d\tilde{\mu}(\lambda) = \delta_{kj}. \tag{2.40}$$

Now we are going to investigate the dependence of ψ_k on the times $t_1, \ldots, t_s$.

Lemma 2.13 *The following relation is satisfied:*

$$\frac{\partial \psi_j(\lambda)}{\partial t_\alpha} = \sum_{m=0}^{\infty} (A_\alpha)_{jm} \psi_m(\lambda), \quad \alpha = 1, \ldots, s, \tag{2.41}$$

with A_α antisymmetric matrix.

Proof. Let us differentiate with respect to t_α the orthonormality relations (2.40)

$$\int_{\mathbb{R}} \frac{\partial \psi_j(\lambda)}{\partial t_\alpha} \psi_k(\lambda) d\tilde{\mu}(\lambda) + \int_{\mathbb{R}} \psi_j(\lambda) \frac{\partial \psi_k(\lambda)}{\partial t_\alpha} d\tilde{\mu}(\lambda) = 0$$

so that

$$\begin{aligned}&\int_{\mathbb{R}} \sum_m (A_\alpha)_{jm} \psi_m(\lambda) \psi_k(\lambda) d\tilde{\mu}(\lambda) + \int_{\mathbb{R}} \psi_j(\lambda) \sum_m (A_\alpha)_{km} \psi_m(\lambda) d\tilde{\mu}(\lambda) \\ &= (A_\alpha)_{jk} + (A_\alpha)_{kj} = 0\end{aligned}$$

□

Lemma 2.14 *The following relation is satisfied*

$$A_\alpha = (L^\alpha)_+ - (L^\alpha)_-, \quad \alpha = 1, \ldots, s, \tag{2.42}$$

where $(L^\alpha)_\pm$ *is the projection of* L^α *to the upper/lowe triangular part of* L^α*.*

Proof. We observe that

$$\psi_k(\lambda) = \left(\sqrt{\frac{D_{k-1}}{D_k}} \lambda^k + O(\lambda^{k-1}) \right) e^{\sum_{\beta=1}^s \lambda^\beta t_\beta},$$

so that

$$\frac{\partial \psi_k(\lambda)}{\partial t_\alpha} = \psi_k(\lambda) \frac{\partial}{\partial t_\alpha} \left(\log \sqrt{\frac{D_{k-1}}{D_k}} \right) + \lambda^\alpha \psi_k(\lambda) + O(\lambda^{k-1}) e^{\sum_{\beta=1}^s \lambda^\beta t_\beta},$$

so that for $j > k$

$$\begin{aligned}A_{kj} &= \int_{\mathbb{R}} \frac{\partial \psi_k(\lambda)}{\partial t_\alpha} \psi_j(\lambda) d\tilde{\mu}(\lambda) = \int_{\mathbb{R}} \lambda^\alpha \psi_k(\lambda) \psi_j(\lambda) d\tilde{\mu}(\lambda) = \int_{\mathbb{R}} \sum_m (L^\alpha)_{km} \psi_m(\lambda) \psi_j(\lambda) d\tilde{\mu} \\ &= (L^\alpha)_{kj}.\end{aligned}$$

Using the antisymmetry of A_α, (2.42) follows. □

Lemma 2.15 *The semiinfinite matrix* L *satisfies the Lax equation*

$$\frac{dL}{dt_\alpha} = [A_\alpha, L], \quad \alpha = 1, \ldots, s. \tag{2.43}$$

Proof. We differentiate with respect to t_α the 3-term recurrence relation (2.39) to obtain

$$\frac{dL}{dt_\alpha}\psi + (L-\lambda)\frac{d\psi}{dt_\alpha} = 0 \tag{2.44}$$

where $\psi(\lambda) = p(\lambda)e^{\sum_{k=1}^{s} t_k\lambda^k}$. Using (2.41) one obtains

$$\frac{dL}{dt_\alpha}\psi + (L-\lambda)A_\alpha\psi = \left(\frac{dL}{dt_\alpha} - [A_\alpha, L]\right)\psi = 0$$

so that by the completeness of ψ one has (2.43). $\square$

Remark 2.16 Let $(\lambda_1, \dots, \lambda_n)$ be the zeros of the polynomial $p_n(\lambda)$, then the relation (2.39) takes the form

$$\begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & 0 \\ a_1 & b_2 & a_2 & & 0 & 0 \\ 0 & a_2 & b_3 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & b_{n-1} & a_{n-1} \\ 0 & & & & a_{n-1} & b_n \end{pmatrix} \begin{pmatrix} p_0(\lambda_j) \\ p_1(\lambda_j) \\ p_2(\lambda_j) \\ \\ \dots \\ \\ p_{n-2}(\lambda_j) \\ p_{n-1}(\lambda_j) \end{pmatrix} = \lambda_j \begin{pmatrix} p_0(\lambda_j) \\ p_1(\lambda_j) \\ p_2(\lambda_j) \\ \\ \dots \\ \\ p_{n-2}(\lambda_j) \\ p_{n-1}(\lambda_j) \end{pmatrix}.$$

The above equality says that the zeros of $p_n(\lambda)$ are the eigenvalues of L defined in (4.4) and therefore, by lemma 2.3, its eigenvalues are distinct and real. The eigenvector relative to the eigenvalue λ_j is given by $(p_0(\lambda_j), p_1(\lambda_j), \dots, p_{n-1}(\lambda_j))^t$.

Remark 2.17 From the construction of this section and the relation (2.22), in order to solve the Toda lattice equations, given the Lax matrix $L(0)$ at time $t = 0$, it is sufficient to determine its eigenvalues $\lambda_1, \dots, \lambda_n$ and the first entry of the eigenvectors $u_{1j}(0)$, $j = 1, \dots, n$ and then construct the measure

$$d\mu(\lambda) = \sum_{j=1}^{n} u_{1j}(0)^2 e^{2\lambda_j t}\delta(\lambda - \lambda_j)d\lambda,$$

where $\delta(\lambda)$ is the Dirac delta function. Given the measure $d\mu(\lambda)$ the solution of the Toda lattice equation is obtained from (2.28), (2.37) and (2.35).

3 Periodic Toda

We consider the Toda lattice for b_n, a_n with $n \in \mathbb{Z}$ and the periodicity condition in N: $(b_n, a_n) = (a_{n+N}, b_{n+N})$. The phase space becomes $\mathcal{M} = \mathbf{R^N} \times \mathbf{R^N_{>0}}$. The Hamiltonian takes the form

$$H = \frac{1}{2}\sum_{n=1}^{N} b_n^2 + \sum_{n=1}^{N} a_n^2.$$

Now the equations of motions are:

$$\begin{cases} \dot{b_n} = a_n^2 - a_{n-1}^2 \\ \dot{a_n} = \frac{1}{2} a_n (b_{n+1} - b_n) \end{cases} . \tag{3.1}$$

These equations have an Hamiltonian structure. Indeed, the canonical Poisson bracket transforms to

$$\{a_n, a_m\} = 0 = \{b_n, b_m\},$$
$$\{b_n, a_m\} = \frac{1}{2}(a_m \delta_{n,m} - a_m \delta_{n,m+1})$$

for every $1 \leq n, m \leq N$. Such bracket can be written in the compact form using a $2N \times 2N$ matrix:

$$J = \begin{pmatrix} 0 & \mathcal{A} \\ -\mathcal{A}^t & 0 \end{pmatrix}, \tag{3.2}$$

where $\mathcal{A}$ is a $N \times N$ matrix defined by:

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} a_1 & 0 & \dots & 0 & -a_N \\ -a_1 & a_2 & 0 & \dots & 0 \\ 0 & -a_2 & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -a_{N-1} & a_N \end{pmatrix}. \tag{3.3}$$

For every point of the phase space $(b, a) = (b_n, a_n)_{1 \leq n \leq N}$ the Poisson brackets between two generic $\mathcal{C}^\infty(\mathcal{M})$ functions F, G, is:

$$\begin{aligned} \{F, G\}_J(b, a) &= \langle (\nabla_b F, \nabla_a F), J(\nabla_b G, \nabla_a G) \rangle_{\mathbf{R^{2N}}} \\ &= \langle \nabla_b F, \mathcal{A} \nabla_a G \rangle_{\mathbf{R^N}} - \langle \nabla_a F, \mathcal{A}^t \nabla_b G \rangle_{\mathbf{R^N}}. \end{aligned} \tag{3.4}$$

Then we have that the equations given in 3.1 are exactly:

$$\dot{b}_n = \{b_n, H\}_J, \quad \dot{a}_n = \{a_n, H\}_J, \quad 1 \leq n \leq N.$$

From now on, we will consider the periodic Toda lattice in Flaschka coordinates.

Remark 3.1 Because of the old coordinates' periodicity, we have that $\prod_{n=1}^{N} a_n = 1$. Indeed:

$$\prod_{n=1}^{N} a_n = \prod_{n=1}^{N} e^{\frac{1}{2}(q_n - q_{n+1})} = e^{\frac{1}{2}\sum_{n=1}^{N}(q_n - q_{n+1})} = 1.$$

So we have immediatly found another constant of motion, together with $\sum_{n=1}^{N} b_n$.

The last thing to observe about the Poisson structure $\{\,,\,\}_J$, is that it is degenerate and it admits some Casimir functions. Indeed, one can observe that the matrix $\mathcal{A}$ has not maximum rank, since:

$$\det(\mathcal{A}) = a_1 \begin{vmatrix} a_2 & 0 & \dots & 0 \\ -a_2 & a_3 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & -a_{N-1} & a_N \end{vmatrix} - a_N(-1)^{N-1} \begin{vmatrix} -a_2 & a_3 & \dots & 0 \\ 0 & -a_3 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & -a_{N-1} \end{vmatrix} = 0,$$

but $\operatorname{rank}(\mathcal{A}) = N-1$: one can find a nonzero minor of this dimension, for example the one obatined removing the first column and the first row of $\mathcal{A}$ has the value $\prod_{i=2}^{N} a_i = 1/a_1 \neq 0$.

Then, from the definition of J, $\operatorname{rank}(J) = 2N-2$, in every point of the phase space.

Definition 3.2 A Casimir function for a Poisson structure $\{\,,\,\}$ is a smooth function $f : \mathcal{C}^\infty(\mathcal{M})$ with the property that $\{f,h\} \equiv 0$, $\forall h \in \mathcal{C}^\infty(\mathcal{M})$.

In our case, the two Casimir functions associated to the Poisson bracket $\{\,,\,\}_J$ are the conserved quantities:

$$C_1 = -\frac{1}{N}\sum_{n=1}^{N} b_n, \quad C_2 = \left(\prod_{n=1}^{N} a_n\right)^{\frac{1}{N}}. \tag{3.5}$$

Calculating the gradients in a certain point $(b,a) \in \mathcal{M}$, we find:

$$\nabla_{b,a}C_1 = (\nabla_b C_1, \nabla_a C_1) = (-\frac{1}{N}, \dots, -\frac{1}{N}, 0, \dots, 0), \tag{3.6}$$

$$\nabla_{b,a}C_2 = (\nabla_b C_2, \nabla_a C_2) = (0, \dots, 0, \frac{C_2}{Na_1}, \dots, \frac{C_2}{Na_N}), \tag{3.7}$$

namely they are linearly independent vectors for every point (b,a). Then, one can consider for every $(\beta, \alpha) \in \mathbf{R} \times \mathbf{R}_{>0}$ the level set given by:

$$\mathcal{M}_{\beta,\alpha} := \{(b,a) \in \mathcal{M} : (C_1, C_2) = (\beta, \alpha)\},$$

and since the gradients $\nabla_{b,a}C_1$ and $\nabla_{b,a}C_2$ are linearly independent on each point of $\mathcal{M}$, $\mathcal{M}_{\beta,\alpha}$ is smooth submanifold of $\mathcal{M}$ of codimension two. Every $\mathcal{M}_{\beta,\alpha}$ has also the property

to have the restricted Poisson structure (induced by J) nondegenerate, and then it has a symplectic structure.
In other words $\mathcal{M}_{\beta,\alpha}$ is the symplectic foliation of $\mathcal{M}$.
Furthermore in our specific case $C_2 = 1$, so α must be 1 and then the foliation is of the type $\mathcal{M}_{\beta,1}$.

The Lax pair for the periodic Toda lattice is given by two $N \times N$ matrices $L^{\pm}, A$:

$$L^{\pm}(b,a) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & \pm a_N \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ \pm a_N & \dots & 0 & a_{N-1} & b_N \end{pmatrix} \tag{3.8}$$

that is a Jacobi matrix with extra entry in the right-upper and left-lower corner (that leave it symmetric), and the skew-symmetric one:

$$A = \frac{1}{2} \begin{pmatrix} 0 & a_1 & 0 & \dots & -a_N \\ -a_1 & 0 & a_2 & \ddots & \vdots \\ 0 & -a_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \dots & 0 & -a_{N-1} & 0 \end{pmatrix}, \tag{3.9}$$

one can prove, through straightforward calculation, that the system given by

$$\dot{L}^{+} = \left[A, L^{+}\right]$$

is equivalent to the Toda lattice equations 3.1, i.e. L^{+}, A form a Lax pair for the periodic Toda lattice. (The same is true for the matrix L^{-} but changing the sign of the terms a_N in the matrix A).

Now,using the property of Lax pair formulation, recalled above, one finds the following proposition.

Proposition 3.3 *For every solution $(b(t), a(t))$ of the periodic Toda lattice 3.1, the eigenvalues $\left\{\lambda_j^{\pm}\right\}_{1 \le j \le N}$ of $L^{\pm}(b(t), a(t))$ are conserved quantities.*

This result explains why it is fundamental for us, to study the spectrum of such *periodic* Jacobi matrices: it gives us the constants of motion that we need in order to say that the Toda lattice is an integrable system.

4 Jacobi operator and its spectra

The spectral theory of the matrices $L^{\pm}$ defined in (3.8) is strictly connected with the spectral theory of Jacobi operators. So we dedicate a section entirely about this topic. We start from the Jacobi operator and we introduce some particular spectra for this operator, namely the periodic/antiperiodic spectrum and the Dirichlet spectrum. Then we show how these two spectra are associated to the Lax matrices $L^{\pm}$ of the periodic Toda lattice.

Definition 4.1 We call Jacobi operator, an operator acting on the space $l^2(\mathbf{Z})$ as the following combination of shift operators:

$$(\mathcal{L}_{b,a}y)(n) = a_{n-1}(S^{-1}y)(n) + b_n(S^0y)(n) + a_n(S^1y)(n), \tag{4.1}$$

for every $y \in l^2(\mathbf{Z})$, where $(S^m y)(n) = y(n+m), \forall m \in \mathbf{Z}$ is the $m-$shift operator, and $b = \{b_n\}_{n\in\mathbf{Z}}$ is a sequence of real numbers and $a = \{a_n\}_{n\in\mathbf{Z}}$ is a sequence of real and positive numbers.

On each sequence of $l^2(\mathbf{Z})$, the Jacobi operator is realised as an infinite dimensional symmetric tridiagonal matrix such that: the b_n stay on the diagonal and the a_n stay on the upper and lower diagonals.

Definition 4.2 A real Jacobi matrix of size N is the finite dimensional analogue of the Jacobi operator. So it is a symmetric, tridiagonal matrix:

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & 0 & \dots & a_{N-1} & b_N \end{pmatrix}, \tag{4.2}$$

where the entries $a_n \in \mathbf{R}_{>0}$ and $b_n \in \mathbf{R}$ for $1 \leq n \leq N$. A *periodic* Jacobi matrix is a matrix of the same kind, but it has also two nonzero positive elements: one in the right upper corner and another in the left lower corner, that leave it symmetric but no more tridiagonal. For example the matrix $L^{\pm}$ defined in 3.8 are periodic Jacobi matrix.

Remark 4.3 The open Toda lattice, is defined by the equations

$$\begin{aligned} \dot{a}_k &= a_k(b_{k+1} - b_k), \quad k = 1, \dots, N-1 \\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2), \quad k = 1, \dots, N, \end{aligned} \tag{4.3}$$

where we have $a_0 = a_N = 0$. Its Lax pair is given by the matrix (4.2) and the matrix

$$A = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & 0 \\ -a_1 & 0 & a_2 & & 0 & 0 \\ 0 & -a_2 & 0 & & & 0 \\ & & & & & \\ \dots & & & \dots & & \dots \\ & & & & & \\ 0 & & & & 0 & a_{n-1} \\ 0 & & & & -a_{n-1} & 0 \end{pmatrix} \tag{4.4}$$

so that the equations (4.3) are equivalent to the Lax equations

$$\frac{dL}{dt} = [A, L].$$

Now we look at the difference equation for the Jacobi operator:

$$(\mathcal{L}_{b,a} y)(n) = \lambda y(n), \ n \in \mathbf{Z} \tag{4.5}$$

in our case, when b_n, a_n are periodic, and we define two different spectra associated to this equation.

Remark 4.4 Solutions $\{y(n)\}_{n\in\mathbf{Z}}$ of the equation 4.5, are functions of λ, so we denote $y(\cdot) = y(\cdot, \lambda)$ for every λ satysfing the equation.

Definition 4.5 The periodic/antiperiodic spectrum of the Jacobi operator $\mathcal{L}_{b,a}$ (of period N), is formed by the $\lambda \in \mathbf{R}$ such that the eigenfunctions of 4.5 satisfy the periodic/antiperiodic condition $y(n+N) = \pm y(n)$, for every $n \in \mathbf{Z}$.

We now show that this spectrum exactly corresponds to the set of eigenvalues of the matrix $L^{\pm}$. Indeed: we know from the properties of Flaschka-Manakov coordinates that our sequences of b_n, a_n are periodic of period N. Then using the periodic condition on the eigenfunction, we obtain that the the difference equation 4.5 is reduced to N equations:

$$\begin{cases} a_0 y(0) + b_1 y(1) + a_1 y(2) = \lambda y(1) \Leftrightarrow a_N y(N) + b_1 y(1) + a_1 y(2) = \lambda y(1) \\ a_{k-1} y(k-1) + b_k y(k) + a_k y(k+1) = \lambda y(k), 2 \le k \le N-1 \\ a_{N-1} y(N-1) + b_N y(N) + a_N y(N+1) = \lambda y(N) \Leftrightarrow \\ \qquad \Leftrightarrow a_{N-1} y(N-1) + b_N y(N) + a_N y(1) = \lambda y(N), \end{cases}$$

that are equivalent to the eigenvalues equation for L^+:

$$(L^+ - \lambda I)(y(1), \dots, y(N))^t = 0,$$

and similarly for the anti-periodic spectrum.

Definition 4.6 The Dirichlet spectrum of the Jacobi operator $\mathcal{L}_{b,a}$, is formed by $\lambda \in \mathbf{R}$ such that they are solution of 4.5 and the correspondent eigenfunction respects the zero boundary conditions, i.e. $y(1) = 0 = y(N+1)$.

Also in this case, we can find a relation between this spectrum and the eigenvalues of a matrix related to the Lax matrix for Toda periodic L^+. In particular the Dirichlet spectrum corresponds to the eigenvalues of the matrix L_2^N obtained removing the first column and the first row of L^+. Indeed using the periodicity of b_n, a_n and the zero boundary conditions, the equation 4.5 is reduced to $N-1$ equations:

$$\begin{cases} a_1 y(1) + b_2 y(2) + a_2 y(3) = \lambda y(2) \Leftrightarrow b_2 y(2) + a_2 y(3) = \lambda y(2) \\ a_{k-1} y(k-1) + b_k y(k) + a_k y(k+1) = \lambda y(k), 3 \leq k \leq N-1 \\ a_{N-1} y(N-1) + b_N y(N) + a_N y(N+1) = \lambda y(N) \Leftrightarrow a_{N-1} y(N-1) + b_N y(N) = \lambda y(N) \end{cases},$$

that are equivalent to the eigenvalues equation for L_2^N:

$$(L_2^N - \lambda I)(y(2), \dots, y(N))^t = 0.$$

We recall that from Lemma 2.3 the eigenvalues $\mu_1, \dots, \mu_{N-1}$ of the matrix L_2^N are all distinct.

Remark 4.7 For the periodic Toda lattice, the matrices $L^\pm$, do not have necessarily distinct eigenvalues. The Dirichlet spectrum $\mu_1(0), \dots, \mu_{N-1}(0)$ that are the eigenvalues of the matrix L_2^N are distinct, and this enable us to use the orthogonal decomposition, namely $L_2^N(0) = U(0)\text{diag}(\mu_1, \dots, \mu_{N-1})U^t(0)$ where $U(0)$ is an orthogonal matrix whose column are orthogonal polynomials with respect to an orthogonal measure $\rho(0)$. Such measure is determined from the eigenvalues $\lambda_1^\pm, \dots, \lambda_N^\pm$ of $L^\pm$ and the Dirchlet spectrum. However in this case, the eigenvalues μ_j are not constant of motion but evolve according to so called Dubrovin type equation [**?**]. The integration is much more complicated, however once we have the measure $\rho(t)$ and the Dirichlet spectrum $\mu_1(t), \dots, \mu_{N-1}(t)$, we can obtain the orthogonal matrix $U(t)$ by Gram-Schmidt procedure and then recover the matrix $L_2^N(t)$ as

$$L_2^N = U(t)\text{diag}(\mu_1(t), \dots, \mu_{N-1}(t))U^t(t), \quad t \geq 0.$$

This enable to obtain $a_2, \dots, a_{N-1}$ and $b_2, \dots, b_N$. The evolution of $a_1(t)$, $b_1(t)$ and $a_N(t)$ is obtained using the two conserved quantities of the Toda lattice, namely C_1, C_2, and by fixing the normalization of the measure $\rho(t)$.

We summarise the integration procedure with the following diagram:

$$\begin{array}{ccc} \{b_k(t=0), a_k(t=0)\} & \overset{\text{direct spectral problem}}{\longrightarrow} & \text{Eigenvalues of } L^{\pm}(0),\ L_2^N(0) \\ & & \text{and spectral measure } \rho(0) \\ & & \Big\downarrow \ \text{Toda evolution} \\ \{b_k(t), a_k(t)\} & \overset{\text{inverse spectral problem}}{\longleftarrow} & \text{Eigenvalues of } L^{\pm}(t),\ L_2^N(t) \\ & & \text{and spectral measure } \rho(t) \end{array}$$

The key step is to observe that the Dirichlet sprectrum, after a nonlinear change of coordinate, evolves linearly on the Jacobi variety of the Riemann surface

$$S := \{(w, \lambda) \in \mathbf{C}^2 \mid w^2 = \prod_{j=1}^{N} (\lambda - \lambda_j^+)(\lambda - \lambda_j^-)\},$$

where we assume that generically the $\lambda_j^{\pm}$ are distinct. Such coordinates correspond to the angle variable of the Toda lattice.

5 Direct spectral problem

In this section we want to describe the periodic/antiperiodic spectrum and the Dirichlet spectrum for the operator $\mathcal{L}_{b,a}$. As a first step we construct a basis of normalised eigenvectors.

Definition 5.1 We call fundamental solutions $c(k, \lambda), s(k, \lambda)$ the solutions of 4.5 with initial conditions:

$$c(0, \lambda) = 1, \quad s(0, \lambda) = 0,$$
$$c(1, \lambda) = 0, \quad s(1, \lambda) = 1.$$

Then given any arbitrary initial conditions, the solution $y(k, \lambda)$ will be a linear combination of the two fundamental solutions:

$$y(k, \lambda) = y(0, \lambda)c(k, \lambda) + y(1, \lambda)s(k, \lambda).$$

Definition 5.2 We call Wronskian, for every $k \in \mathbf{Z}$, the quantity:

$$W(k) = a_k(c(k)s(k+1) - c(k+1)s(k)). \tag{5.1}$$

The Wronkstian does not depend on k.

Proposition 5.3 (Wronskian Identity) *The following is satisfied:*

$$W(k) = a_0, \quad \forall k \in \mathbf{N}. \tag{5.2}$$

Proof. Using the relation (4.5) we have

$$\begin{aligned} W(k) &= a_k(c(k)s(k+1) - c(k+1)s(k)) \\ &= a_k \left[\frac{c(k)}{a_k}((\lambda - b_k)s(k) - a_{k-1}s(k-1)) - \frac{s(k)}{a_k}((\lambda - b_k)c(k) - a_{k-1}c(k-1)) \right] \\ &= a_{k-1}(c(k-1)s(k) - s(k-1)c(k)) = W(k-1). \end{aligned} \tag{5.3}$$

Using the definition 5.1 we have that $W(0) = a_0$. □

Now we characterise the fundamental solutions $c(k, \lambda), s(k, \lambda)$.

Proposition 5.4 *For each $k \in \mathbf{N}$, the eigenfunctions $c(k, \lambda)$ and $s(k, \lambda)$ are polynomials in λ of degree at $k-2$ and $k-1$ respectively. In particular we have:*

$$\begin{aligned} c(N+1, \lambda) &= -(a_1 a_2 \dots a_{N-1})^{-1} \lambda^{N-1} + \dots, \\ s(N+1, \lambda) &= (a_1 a_2 \dots a_N)^{-1} \lambda^N + \dots \end{aligned} \tag{5.4}$$

Proof. We first introduce the following notation: L_i^j is the the quadratic matrix obtained from L^+ without first $i-1$ rows and columns and without last $N-j$ rows and columns. Then we can take the determinant of this quadratic matrix:

$$\Delta_i^j = \det(\lambda I - L_i^j) \tag{5.5}$$

for $1 \le i \le j \le N$. We also put $\Delta_i^{i-1} \equiv 1$ and $\Delta_i^{i-2} \equiv 0$.
We are going to prove that, $\forall k \in \mathbf{Z}$, the fundamental solutions can be expressed with the following formulas:

$$c(k) = -a_0(a_1 \dots a_{k-1})^{-1} \Delta_2^{k-1}, \quad s(k) = (a_1 \dots a_{k-1})^{-1} \Delta_1^{k-1}, \tag{5.6}$$

and since $\Delta_2^{k-1}, \Delta_1^{k-1}$ are monic polynomials in λ of degree $k-2$ and $k-1$ respectively we have the first statement of the proposition. Then, in the case $k = N+1$ we obtain exactly the relation (5.4).
We have to prove the relation (5.6). We prove it by induction on k. We restrict to the eigenfunction $c(k, \lambda)$, since the case for $s(k, \lambda)$ can be obtained in a similar way. If $k = 1$, from the equation 4.5 we have that:

$$\lambda c(1, \lambda) = a_0 c(0, \lambda) + b_1 c(1, \lambda) + a_1 c(2, \lambda),$$

and substituting initial conditions, one obtains $c(2,\lambda) = -a_0 a_1^{-1}$: then recalling that we put $\Delta_i^{i-1} \equiv 1$ we are done.
Now, supposing that the formula is valid for $k-1$, we prove that it is also valid for k. Recalling the same eigenvalue equation:

$$\lambda c(k-1,\lambda) = a_{k-2} c(k-2,\lambda) + b_{k-1} c(k-1,\lambda) + a_{k-1} c(k,\lambda),$$

and using the induction hypothesis, we find:

$$\begin{aligned} c(k,\lambda) &= a_{k-1}^{-1}(-a_{k-2}c(k-2,\lambda) + (\lambda - b_{k-1})c(k-1,\lambda)) \\ &= a_{k-1}^{-1}(-a_{k-2}(-a_0(a_1 \dots a_{k-3})^{-1}\Delta_2^{k-3}) + (\lambda - b_{k-1})(-a_0(a_1 \dots a_{k-2})^{-1}\Delta_2^{k-2}) \\ &= -a_0(a_1 \dots a_{k-1})^{-1}(-a_{k-2}^2\Delta_2^{k-3} + (\lambda - b_{k-1})\Delta_2^{k-2}) \\ &= -a_0(a_1 \dots a_{k-1})^{-1}\Delta_2^{k-1}. \end{aligned}$$

□

For every eigenfunction $f(k)$ of the operator $\mathcal{L}_{b,a}$ we have

$$f(k,\lambda) = f(0,\lambda)c(k,\lambda) + f(1,\lambda)s(k,\lambda)$$

and in particular

$$\begin{pmatrix} f(N) \\ f(N+1) \end{pmatrix} = \begin{pmatrix} c(N) & s(N) \\ c(N+1) & s(N+1) \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}.$$

Using the Wronkstian identity 5.2, the matrix

$$M_N = \begin{pmatrix} c(N) & s(N) \\ c(N+1) & s(N+1) \end{pmatrix} \tag{5.7}$$

has $\det(M_N) = 1$. Therefore its eigenvalues are

$$det(M_N - \xi I) = \xi^2 - \Delta\xi + 1 = 0 \quad \Leftrightarrow \quad \xi_\pm = \frac{\Delta \pm \sqrt{\Delta^2 - 4}}{2},$$

where $\Delta(\lambda) = Tr(M_N(\lambda)) = c(N,\lambda) + s(N+1,\lambda)$.
The quantity $\Delta(\lambda)$, is called discriminant. We observe that the eigenvalues ξ of the matrix M_N have the following properties

- if $\Delta(\lambda) = \pm 2$, then $\xi_\pm = \pm 1$ is a double eigenvalue of the matrix M_N;
- if $|\Delta| < 2$, then the eigenvalues $\xi_\pm$ are complex conjugate and $|\xi_\pm| = 1$;
- if $|\Delta| > 2$, then the eigenvalues are real and $\xi_+ > 1$ and $\xi_- < 1$.

The eigenvectors of the matrix M_N can be parematrized as

$$\begin{pmatrix} \chi_\pm \\ 1 \end{pmatrix}, \quad \chi_\pm = \frac{\xi_\pm - s(N+1)}{c(N+1)}.$$

The Bloch eigenfunctions are defined by

$$\psi_\pm(k+1) = s(k+1) + \chi_\pm c(k+1), \quad k \in \mathbf{Z}.$$

We observe that $\psi_\pm(1) = 1$. One can immediately verify that

$$\psi_\pm(N+1) = \xi_\pm \psi_\pm(1) = \xi_\pm$$

so that

$$\psi_\pm(m(N+1)) = \xi^m \psi_\pm(1), \quad m \in \mathbf{Z}.$$

Therefore when the eigenvalues $\xi_\pm$ are real, the Bloch eigenfunction $\psi_+(m(N+1))$ grows exponentially for m positive and $\psi_-(m(N+1))$ grows exponentially for m negative. When the eigenvalues $\xi_\pm$ are complex we have

$$|\psi_\pm(m(N+1))| = |\xi^m||\psi_\pm(1)| = 1, \quad m \in \mathbf{Z}.$$

The values of λ for which

$$\Delta(\lambda) = \pm 2,$$

correspond to the periodic and anti-periodic spectrum of the operator $\mathcal{L}_{b,a}$. This spectrum corresponds to the eigenvalues of the matrices $L^\pm$ respectively. Since the matrices $L^\pm$ are symmetric, it follows that the equations $\Delta(\lambda) = \pm 2$ have N real solutions each.

The set of values $\lambda \in \mathbf{R}$ for which

$$|\Delta(\lambda)| < 2$$

corresponds to the stability zones of the spectrum of $\mathcal{L}_{b,a}$.
The Dirichlet spectrum corresponds to the values of $\lambda \in \mathbf{R}$ for which the polynomial $c(N+1)$ of degree $N-1$ is equal to zero. Indeed, from the relation 5.6, we have:

$$c(1,\lambda) = 0, \quad c(N+1,\lambda) = 0.$$

Such spectrum corresponds to the eigenvalues of the matrix L_2^N, which is a tri-diagonal matrix. The next result shows that the eigenvalues of tridiagonal matrices are all distinct.

Proposition 5.5 *Every Jacobi matrix L of size N has exactly N different real eigenvalues.*

Proof. The idea of the proof is first to show that every eigenvector $v = (v(1), \ldots, v(N))^t$ of L for a certain eigenvalue λ, i.e. $(L - \lambda I)v = 0$, has the property that the first and the last entry must be different from zero. Indeed, if for example $v(1)$ were zero, then writing explicitly the eigenvalues equation one has:

$$(b_1 - \lambda)v(1) + a_1 v(2) = 0,$$

that implies $v(2) = 0$. Then:

$$a_1 v(1) + (b_2 - \lambda)v(2) + a_2 v(3) = 0,$$

that implies $v(3) = 0$. Going on in this way one obtains that all the components of v are zero, in contraddiction with the fact that v was eigenvector. The same proof is valid supposing that $v(N) = 0$.
Now we can prove our thesis. Suppose that a certain λ, eigenvalue of L, we have two different eigenvectors v, w: we are going to show that they are linearly dependent. Indeed: from the property above we know that $v(1) \neq 0$ and also $w(1) \neq 0$. Then we can choose a couple of real numbers c, d such that $(c, d) \neq (0, 0)$ and $cv(1) + dw(1) = 0$. And now we can consider the vector $cv + dw$: this is still an eigenvector for L with respect to λ, but its first component is zero, and so we have that $cv + dw$ is the null vector for c, d coefficients nonzero both. This exactly means that v, w are linearly dependent and so every eigenvalue of L has multeplicity one. □

Remark 5.6 In this chapter we introduced all the tools to prove the integrability of the periodic Toda lattice. Indeed we introduce

- the Flaschka-Manakov coordinates $\{a_k, b_k\}_{k=1}^N$ and we show that there are two constants of motion:

$$\sum_{n=1}^N b_n, \qquad \prod_{n=1}^N a_n.$$

- We show that the Hamilton equation of motion for the periodic Toda lattice are equivalent to the Lax equation

$$\dot{L}^{\pm} = \left[L^{\pm}, A\right],$$

  where $L^{\pm}$ are the periodic Jacobi matrices defined in (3.8) and A is defined in (3.9). This allowed us to show that all the eigenvalues of the Lax matrix $L^{\pm}$ are constant of motions. Clearly these quantities cannot be all independent.

- Then we related this set of eigenvalues to the periodic, anti-periodic spectrum of the Jacobi operator 4.1, and we defined also its Dirichlet spectrum, showing that it coincides with the set of eigenvalues of the matrix L_2^N, submatrix of $L^{\pm}$ obtained by erasing the first row and column of $L^{\pm}$.

In the next chapter we are going to solve the inverse spectral problem for the periodic Jacobi matrix, namely we will show how to reconstruct the periodic Jacobi matrix associated to Toda lattice from its spectral data.

6 Integrability of Toda periodic

In this chapter we are going to show a way to integrate the equations in 3.1.
The main idea is to think at the problem from another point of view: we want to resolve the inverse problem for the spectrum of periodic Jacobi matrix. This means that we suppose to know the eigenvalues of the matrix L^+, and we want to reconstruct, starting from these, the entries of L^+. Unfortunately the only spectrum of the matrix it is not enough, but we have to use the already known Dirichlet spectrum. This information, together with the construction of a basis of eigenvectors with respect to every Dirichlet eigenvalue, permit us to reconstruct the periodic Jacobi matrix L^+, that has the initial fixed spectrum and Dirichlet spectrum, and $\prod_{n=1}^{N} a_n$ constant.
Actually, this can be done with a certain degree of non uniqueness, that depends essentially on the choice of a sign in front of some square roots needed in the construction of the eigenvectors cited above. In particular, one can see that the space of isospectral periodic Jacobi matrices with constant $\prod_{n=1}^{N} a_n$ is a torus of dimension at most $N-1$. This is all for the inverse spectral problem for periodic Jacobi matrix.
So in the case of periodic Toda equations, fixing the initial conditions $a_n(0), b_n(0)$, we can compute the eigenvalues of $L^\pm$, that are constants, as we already know, and we have that $\prod_{n=1}^{N} a_n = 1$. Then applying the result for the inverse spectral problem, if we are able to calculate also the time evolution for the Dirichlet spectrum, that it is no more constant, then we are able to describe the temporal evolution of the coordinates a_n, b_n .
For this reason, in the last part of the chapter we will show how to find the system of differential equations for the Dirichlet eigenvalues.

7 The union of periodic and anti-periodic spectra

This section is kind of technical. We are going to introduce a new matrix Q of dimension $2N$, that will be useful for other computations.
First of all we want to show how the spectrum of this matrix is related to the periodic/anti-periodic spectra of 4.1, already defined in the section 4.

Definition 7.1 . We define a new periodic Jacobi matrix Q, of size $2N$, constructed by taking two copies of L^+:

$$Q = \begin{pmatrix} b_1 & a_1 & 0 & \dots & \dots & \dots & \dots & a_N \\ a_1 & b_2 & \ddots & & & & & \\ & \ddots & \ddots & a_{N-1} & & & & \\ & & a_{N-1} & b_N & a_N & & & \\ \vdots & & & a_N & b_1 & a_1 & & \vdots \\ & & & & a_1 & \ddots & \ddots & \\ & & & & & \ddots & & a_{N-1} \\ a_N & \dots & \dots & \dots & \dots & \dots & a_{N-1} & b_N \end{pmatrix}.$$

The spectrum of Q is described by the following result. Note that since Q is a real symmetric matrix, it has $2N$ real eignevalues, but we do not know anything about the multeplicity of each one.

Theorem 7.2 (Q spectrum) *The spectrum of Q*

$$\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \dots < \lambda_{2N-2} \leq \lambda_{2N-1} < \lambda_{2N}$$

is such that: if N is even, then $\lambda_1, \lambda_4, \lambda_5, \dots, \lambda_{2N-4}, \lambda_{2N-3}, \lambda_{2N}$ are the roots of $\Delta(\lambda)-2 = 0$ and they have periodic eigenvectors of period N; all the others are the roots of $\Delta(\lambda)+2 = 0$ and they have eigenvectors of period $2N$. If N is odd, then $\lambda_2, \lambda_3, \lambda_6, \dots, \lambda_{2N-4}, \lambda_{2N-3}, \lambda_{2N}$ are the roots of $\Delta(\lambda) - 2 = 0$, with eignvectors of period N and the others are the roots of $\Delta(\lambda)+2 = 0$, with eigenvectors of period $2N$. Only the intervals $[\lambda_{2i}, \lambda_{2i+1}]$ may degenerate to one point.

Before to start the proof we need the following lemma, that describes the derivative of $\Delta(\lambda)$.

Lemma 7.3 *We have that:*

$$\frac{d\Delta}{d\lambda} = \frac{1}{a_N} \sum_{k=1}^{N} (s(N)c^2(k) - (c(N) - s(N+1))c(k)s(k) - c(N+1)s^2(k)). \tag{7.1}$$

Proof. We take $y(k)$ and $z(k)$ generic solutions of eigenvalues equation 4.5, respectivly for two different λ and μ. Then:

$$\begin{aligned}(\lambda-\mu)\sum_{k=1}^{N} y(k)z(k) &= \sum_{k=1}^{N}(\lambda y(k)z(k)-\mu z(k)y(k)) \\ &= \sum_{k=1}^{N}(z(k)(a_{k-1}S^{-1}+b_kS^0+a_kS^1)y(k)-y(k)(a_{k-1}S^{-1}+b_kS^0+a_kS^1)z(k)) \\ &= \sum_{k=1}^{N}(a_{k-1}(z(k)y(k-1)-z(k-1)y(k))+a_k(z(k)y(k+1)-y(k)z(k+1))) \\ &= a_0(y(0)z(1)-z(0)y(1))+a_N(z(N)y(N+1)-y(N)z(N+1)).\end{aligned}$$

Choosing $y(k)=c(k,\lambda)$ and $z(k)=c(k,\mu)$ and using initial conditions, remains:

$$(\lambda-\mu)\sum_{k=1}^{N} c(k,\lambda)c(k,\mu) = a_N(c(N,\mu)c(N+1,\lambda)-c(N,\lambda)c(N+1,\mu)),$$

and now dividing by $\lambda-\mu$ and for $\lambda \longrightarrow \mu$, we finally have an expression for the norm of c:

$$\|c\|^2 = \sum_{k=1}^{N} c^2(k) = a_N(c(N)c'(N+1)-c(N+1)c'(N)). \tag{7.2}$$

The same is valid for the norm of s and something similar is true for the scalar product (c,s). Then substituting these formulas in $\frac{d\Delta}{d\lambda} = c'(N,\lambda)+s'(N+1,\lambda)$ one obtains the thesis. □

It follows the proof of the previous theorem on the spectrum of Q.

Proof. (Q spectrum) Now we are looking for f solution of the eigenvalues equation, so $f = f_0c + f_1s$ a linear combination of the fundamental solutions, where $f_0 = f(0,\lambda), f_1 = f(1,\lambda)$. Then we also impose that it is periodic (or antiperiodic) of period N, i.e. $f(k+N) = \pm f(k)$. These two things together are equivalent to the following condition (in the periodic case):

$$\begin{pmatrix} f(0) \\ f(1) \end{pmatrix} = \begin{pmatrix} f(N) \\ f(N+1) \end{pmatrix} = \begin{pmatrix} c(N) & s(N) \\ c(N+1) & s(N+1) \end{pmatrix} \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}.$$

But this means that the matrix:

$$M_N = \begin{pmatrix} c(N) & s(N) \\ c(N+1) & s(N+1) \end{pmatrix} \tag{7.3}$$

has only eigenvalues 1. Now observe that these eigenvalues come from:

$$det(M_N - \xi I) = \xi^2 - \Delta\xi + 1 = 0 \quad \Leftrightarrow \quad \xi = \frac{\Delta \pm \sqrt{\Delta^2 - 4}}{2},$$

where $\Delta = Tr(M_N) = c(N) + s(N+1)$.
So, we have that f is a periodic solution of period N of the eigenvalues equations for Q, if and only if $\Delta - 2 = 0$. We can conclude that the eigenvalues of Q that admits periodic eigenvector comes from the zeroes of the polynomial $\Delta(\lambda) - 2$ (that is of degree N). Repeating the same procedure with antiperiodic condition, one finds that the eigenvalues satysfing this condition are exactly the roots of $\Delta(\lambda) + 2$. The eigenvectors then can be extended to periodic solution of period $2N$.
To show the alternetion of the eigenvalues, we are going to use the formula from the past lemma:

$$\frac{d\Delta}{d\lambda} = \frac{1}{a_N} \sum_{k=1}^{N} (s(N)c^2(k) - (c(N) - s(N+1))c(k)s(k) - c(N+1)s^2(k)).$$

We observe that here appears a quadratic form, whose discriminant is exactly $\Delta^2 - 4$. Indeed:

$$\begin{aligned}((c(N) - s(N+1))^2 + 4s(N)c(N+1) \\ &= c^2(N) + s^2(N+1) - 2c(N)s(N+1) + 4s(N)c(N+1) \\ &= c^2(N) + s^2(N+1) + 2c(N)s(N+1) - 4 \\ &= \Delta^2 - 4.\end{aligned}$$

Then we can say that, as long as $\Delta^2 - 4 < 0$, $\frac{d\Delta}{d\lambda}$ will have the sign of $c(N+1)$. Moreover this polynomial cannot vanish untill $\Delta^2 - 4 < 0$, because otherwise from the Wronskian relation we would have $c(N)s(N+1) = 1$, and then:

$$|\Delta| = |c(N) + \frac{1}{c(N)}| \geq 2,$$

that is in contraddiction with the fact that $\Delta^2 - 4 < 0$. So $c(N+1)$ vanishes or changes of sign only when $\Delta^2 - 4 > 0$. □

Corollary 7.4 *The eigenvalues of Q are the union of the eigenvalues of $L^{\pm}$.*

Remark 7.5 [Desplacements of Dirichlet eigenvalues, respect to Q eigenvalues]We already observed in the section 4, that the definition of Dirichlet spectrum for the operator $\mathcal{L}_{b,a}$, for which are requested zero boundary conditions, coincides exactly with the spectrum of the matrix L_2^N and also with the roots of the polynomial $c(N+1)$. Then, from

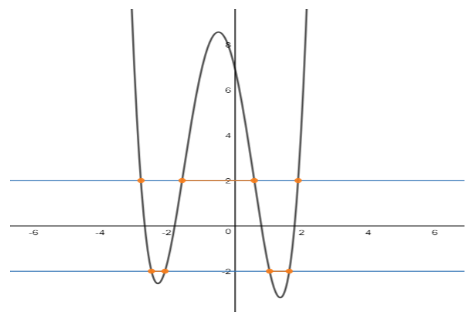

Figure 1: This is the graphic of a generic polynomial $\Delta(\lambda)$ for $N = 4$. All the eigenvalues for the matrix Q are identified by the abcsissas of the orange points, that describes the intersections between $y = \Delta(\lambda)$ and the lines $y = \pm 2$. In this case all the gaps are open, and we find 8 different eigenvalues of Q, $\lambda_1 < \lambda_2 \leq \lambda_3 < \cdots < \lambda_8$.

the proposition 5.5, we have exactly $N - 1$ distinct Dirichlet eigenvalues that we call $\mu_1 < \cdots < \mu_{N-1}$.
One can then observe that everyone of these eigenvalues stay in an interval of the type $[\lambda_{2i}, \lambda_{2i+1}]$, already called *interval of instability*. Indeed, evaluating the Wronskian relation for $k = N$, for $\lambda = \mu_s, s = 1, \dots N - 1$ one obtains:

$$c(N, \mu_s) = \frac{1}{s(N+1, \mu_s)} \tag{7.4}$$

and then the discriminant in μ_s becomes:

$$\Delta(\mu_s) = \frac{1}{s(N+1, \mu_s)} + s(N+1, \mu_s). \tag{7.5}$$

Therefore $|\Delta(\mu_s)| \geq 2$. In this way, we find that $\mu_s \in [\lambda_{2s}, \lambda_{2s+1}]$, and by the formula proved for $\frac{d\Delta}{d\lambda}$ we have that there is only one Dirichlet eigenvalue for each of these intervals.

For example, in the Figure 1, for $N = 4$, we find the 3 Dirichlet eigenvalues that lay one on each of the orange segments, that are exactly the ones for which $|\Delta(\lambda)| \geq 2$.

8 Inverse spectral problem

Suppose we have fixed the spectrum of the matrix Q (and so of $L^{\pm}$), and also the $\prod_{n=1}^{N} a_n = A$ and we want to describe the entries of these matrices. We now enunciate the main result for the inverse spectral problem.
Its proof explains in detail why the matrix Q, of given spectrum and Dirichlet spectrum, is not univocally defined. The proof is constructive, so one can also find in which way the entries of the matrix are reconstructed from these spectra.

Theorem 8.1 *If* $\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \cdots < \lambda_{2N-2} \leq \lambda_{2N-1} < \lambda_{2N}$ *is the spectrum of a matrix* Q*, then for every sequence* $\{\mu_s\}_{s=1}^{s=N-1}$ *such that* $\lambda_{2s} \leq \mu_s \leq \lambda_{2s+1}$*, there are exactly* 2^r *matrices* Q *with that periodic spectrum and* $\Delta_2^N(\mu_s) = 0$*,* $\forall s$*.*
Note that r *is the number of* μ_s *that are nondegenerating in* $\lambda_{2s,2s+1}$*.*

Proof. We follow the proof in [20].
For every Dirichlet eigenvalues we define $\Phi(k, \mu_s)$ for $k = 1, \ldots, N-1$ the corrispondent eigenvector for L_2^N, with the condition $\Phi(1, \mu_s) = 1$. Then we have:

$$\Phi(k, \mu_s) = -\frac{a_1}{a_N} c(k+1, \mu_s), \forall k.$$

Indeed, $\Phi(k, \mu_s)$ must solve the equations:

$$\begin{cases} (b_2 - \lambda)\Phi(1) + a_2\Phi(2) = 0 \\ a_{k-1}\Phi(k-2) + (b_k - \lambda)\Phi(k-1) + a_k\Phi(k) = 0, k = 3, \ldots, N-1 \\ a_{N-1}\Phi(N-2) + (b_N - \lambda)\Phi(N-1) = 0 \end{cases},$$

that are exactly satisfied from $-a_1 \setminus a_N c(k+1, \mu_s)$, for $k = 1, \ldots, N-1$.
Recalling the formula 7.2 for the norm of c and evaluating it in $\lambda = \mu_s$ remains:

$$\|c(\mu_s)\|^2 = \sum_{k=1}^{N} c^2(k, \mu_s) = a_N c(N, \mu_s) c'(N+1, \mu_s).$$

Then we also have that:

$$\sum_{k=1}^{N} \Phi^2(k, \mu_s) = \frac{a_1^2}{a_N} \frac{c'(N+1, \mu_s)}{s(N+1, \mu_s)},$$

where in the last equality we used the Wronskian relation 7.4 for $k = N$ already used. The latter formula also give us a new way to write $s(N+1, \mu_s)$:

$$s(N+1, \mu_s) = \frac{\Delta(\mu_s) \pm \sqrt{\Delta^2(\mu_s) - 4}}{2}, \tag{8.1}$$

where we know from before that the radicand is positive or zero, so the squareroot is a real number.
Moreover observing that we can rewrite the polynomial $c(N+1, \lambda)$ as:

$$c(N+1, \lambda) = -(a_1 \ldots a_{N-1})^{-1} \prod_{s=1}^{N-1} (\lambda - \mu_s) = -(a_1 \ldots a_{N-1})^{-1} P(\lambda),$$

we find that the square of the norm of the eigenvector correspondent to μ_s, r_s^2, is exactly:

$$r_s^2 = -a_1^2 A P'(\mu_s) \left(\frac{\Delta(\mu_s) \pm \sqrt{\Delta^2(\mu_s) - 4}}{2} \right)^{-1}. \tag{8.2}$$

Now, if we consider the normalized eigenvector for μ_s, its components are:

$$\frac{\Phi(k, \mu_s)}{r_s^2}, k = 1, \dots N-1.$$

But then, thinking about the orthogonal matrix formed by the basis of normalized eigenvectors for L_2^N, we also have that:

$$\sum_{s=1}^{N-1} \frac{\Phi^2(k, \mu_s)}{r_s^2} = 1, k = 1, \dots N-1$$

and in particular for $k = 1$, using the condition imposed above:

$$\sum_{s=1}^{N-1} \frac{1}{r_s^2} = 1.$$

This means that:

$$a_1^2 = -A \sum_{s=1}^{N-1} \frac{\sigma(\mu_s)}{P'(\mu_s)}, \tag{8.3}$$

where we set

$$\sigma(\mu_s) = \frac{\Delta(\mu_s) \pm \sqrt{\Delta^2(\mu_s) - 4}}{2}.$$

With this last formula we can now describe the entry a_1, knowing the values of A and of all μ_s. It's important to note that in this reconstruction of a_1, we have to choose a sign in the factor $\sigma(\mu_s)$ every time that $\Delta^2(\mu_s) - 4 \neq 0$, i.e. for every μ_s that not coincides to $\lambda_{2s,2s+1}$, beacuse we know that the zeroes of this polynomial are the eigenvalues of Q. This is the reason from which comes the nonuniqueness of the matrix Q.
Once one has a_1, one can use the trace formula in order to determine b_1:

$$b_1 = \frac{1}{2}(Tr(L^+) + Tr(L^-)) - Tr(L_2^N) = \frac{1}{2}(\lambda_1 + \lambda_N) + \frac{1}{2} \sum_{i=1}^{N-1} (\lambda_{2i} + \lambda_{2i+1} - 2\mu_i). \tag{8.4}$$

Then, for what concernes the entries $a_2, \dots, a_{N-1}$ and $b_2, \dots b_N$ we use some results about orthogonal polynomials with respect to a given measure.

First, we observe that since $(\Phi(1,\mu_s),\dots,\Phi(N-1,\mu_s))^t, s=1,\dots,N-1$ are eigenvectors of a real standard Jacobi matrix L_2^N, they correspond to a family of orthogonal polynomials with respect to the spectral measure $d\rho(\lambda)$ associated to L_2^N. Moreover, being real orthogonal polynomials, they satiesfies a three terms recurrence relation (given exactly from the eigenvalues equation for L_2^N), where it is well known that the coefficients are determined by special formulas:

$$\begin{aligned} b_k &= \int \lambda\Phi^2(k-1,\lambda)d\rho(\lambda), k=2,\dots,N \\ a_k &= \int \lambda\Phi(k,\lambda)\Phi(k-1,\lambda)d\rho(\lambda), k=2,\dots,N-1 \end{aligned}.$$

Defining the spectral measure as the following discreate measure:

$$d\rho(\lambda) = \frac{1}{a_1^2}\sum_{s=1}^{N-1}\frac{1}{r_s^2}\delta(\lambda-\mu_s)d\lambda, \tag{8.5}$$

we finally find:

$$\begin{aligned} b_k &= \frac{1}{a_1^2}\sum_{s=1}^{N-1}\mu_s\frac{\Phi^2(k-1,\mu_s)}{P'(\mu_s)}\delta(\mu_s), \quad k=2,\dots,N \\ a_k &= \frac{1}{a_1^2}\sum_{s=1}^{N-1}\mu_s\frac{\Phi(k-1,\mu_s)\Phi(k,\mu_s)}{P'(\mu_s)}\delta(\mu_s), \quad k=2,\dots,N-1 \end{aligned}. \tag{8.6}$$

At the end, one can obtain the last term a_N, dividing the costant A by the product $a_1\cdot\dots\cdot a_{N-1}$.

□

With this theorem we are able to write 2^r different matrices Q with same eigenvalues, Dirichlet eigenvalues and fixed A, just choosing different signs for every $\sigma(\mu_s)$ everytime that μ_s is not degenerate.

9 Time evolution of Dirichlet eigenvalues

The main result of this section will be the following formula for the derivative of each Dirichlet eigenvalue, that is the analogue of the formula of the Dirichlet eigenvalues for periodic problem of the KdV equation.

Proposition 9.1 *For every $s=1,\dots N-1$ is true that:*

$$\left(\frac{1}{2}\frac{d\mu_s}{dt}P'(\mu_s)\right)^2 = \Delta^2(\mu_s)-4. \tag{9.1}$$

In the proof of this proposition, we need some algebraic properties of the matrices L_2^{N-1} and L_3^N, that we state in few tecnical lemmas.

Lemma 9.2 *The time derivative of the quantity $\Delta_2^j(\lambda)$, for every value of λ, is:*

$$\frac{d}{dt}\Delta_2^j = 2\left(a_j^2\Delta_2^{j-1} - a_1^2\Delta_3^j\right). \tag{9.2}$$

Proof. We procede by induction over j. Before to start we recall that from the equations of motion we have:

$$\frac{d}{dt}(\lambda - b_j) = 2(a_{j-1}^2 - a_j^2)$$

and

$$\frac{d}{dt}(-a_{j+1}^2) = -2a_{j+1}\frac{d}{dt}a_{j+1} = 2(-a_{j+1}^2)\left((\lambda - b_{j+1}) - (\lambda - b_{j+2})\right).$$

Then for $j = 3$ the equation is satisfied since:

$$\begin{aligned}
\frac{d}{dt}\Delta_2^3 &= \frac{d}{dt}\left((\lambda - b_2)(\lambda - b_3) - a_2^2\right) \\
&= 2(a_1^2 - a_2^2)(\lambda - b_3) + 2(\lambda - b_2)(a_2^2 - a_3^2) - 2a_2^2\left((\lambda - b_2) - (\lambda - b_3)\right). \\
&= a_1^2(\lambda - b_3) - a_3^2(\lambda - b_2) \\
&= a_1^2\Delta_3^3 - a_3^2\Delta_2^2.
\end{aligned}$$

Then supposing that the formula is true for every $j < N-1$, we prove it for $i = j+1$, indeed:

$$\begin{aligned}
\frac{d}{dt}\Delta_2^{j+1} &= \frac{d}{dt}\left((\lambda - b_{j+1})\Delta_2^j - a_j^2\Delta_2^{j-1}\right) \\
&= 2(a_j^2 - a_{j+1}^2)\Delta_2^j + (\lambda - b_{j+1})2(a_1^2\Delta_3^j - a_j^2\Delta_2^{j-1}) + \\
&\quad - 2a_j^2((\lambda - b_j) - (\lambda - b_{j+1}))\Delta_2^{j-1} - 2a_j^2(a_1^2\Delta_3^{j-1} - a_{j-1}^2\Delta_2^{j-2}) \\
&= -2a_{j+1}^2\Delta_2^j + 2a_1^2((\lambda - b_{j+1})\Delta_3^j - a_j^2\Delta_3^{j-1}) - 2a_j^2(-\Delta_2^j + (\lambda - b_j)\Delta_2^{j-1} - a_{j-1}^2\Delta_2^{j-2}) \\
&= 2(a_1^2\Delta_3^{j+1} - a_{j+1}\Delta_2^j),
\end{aligned}$$

where we used the induction hypothesis for $j, j-1$. So the formula is true for every value of $j \leq N$. □

Lemma 9.3 *For every $s = 1, \ldots, N-1$ the following formula is true:*

$$\Delta_2^{N-1}(\mu_s)\Delta_3^N(\mu_s) = \prod_{n=2}^{N-1} a_n^2. \tag{9.3}$$

Proof. In order to prove the relation stated in the lemma, we prove the following one, for every i, j and for each value of λ:

$$\Delta_i^j \Delta_{i+1}^{j+1} - \Delta_i^{j+1} \Delta_{i+1}^j = a_i^2 \dots a_j^2. \tag{9.4}$$

Then for $i = 2, j = N-1$, evaluating the above formula in one of the μ_s, we obtain the thesis:

$$\Delta_2^{N-1} \Delta_3^N(\mu_s) - \underbrace{\Delta_2^N \Delta_3^{N-1}(\mu_s)}_{=0} = a_2^2 \dots a_{N-1}^2.$$

Now, for induction over i, j, one proves 9.4, simple expressing the determinants $\Delta_{i+1}^{j+1}, \Delta_{i+1}^j$ respect to the last row:

$$\begin{aligned}\Delta_i^j \Delta_{i+1}^{j+1} - \Delta_i^{j+1} \Delta_{i+1}^j &= \Delta_i^j ((\lambda - b_{j+1}) \Delta_{i+1}^j - a_j^2 \Delta_{i+1}^{j-1}) - ((\lambda - b_{j+1}) \Delta_i^j - a_j^2 \Delta_i^{j-1}) \Delta_{i+1}^j \\ &= a_j^2 \underbrace{(\Delta_i^{j-1} \Delta_{i+1}^j - \Delta_i^j \Delta_{i+1}^{j-1})}_{=a_i^2 \dots a_{j-1}^2},\end{aligned}$$

where in the last equality we used the induction hypothesis. □

Now, using these two relations, we prove the proposition stated at the begenning of the section.

Proof. First of all, looking for the proof of the proposition above, we observe that the determinant of the matrix $L^{\pm} - \lambda I$ explicited respect to the first row is:

$$\begin{aligned}\det(L^{\pm} - \lambda I) &= (b_1 - \lambda)\Delta_2^N - a_1^2 \Delta_3^N \pm (-1)^{N-1} a_1 a_N \begin{vmatrix} a_2 & 0 & & & 0 \\ b_3 - \lambda & a_3 & 0 & & 0 \\ a_3 & b_4 - \lambda & a_4 & & 0 \\ \vdots & & \ddots \ddots & & \vdots \\ 0 & & & b_{N-1} - \lambda & a_{N-1} \end{vmatrix} \\ &\pm (-1)^{N-1} a_N a_1 \begin{vmatrix} a_2 & b_3 - \lambda & a_3 & & 0 \\ 0 & a_3 & b_4 - \lambda & a_4 & 0 \\ 0 & 0 & a_4 & b_5 - \lambda & 0 \\ \vdots & & & \ddots \quad \ddots & \vdots \\ 0 & & & & a_{N-1} \end{vmatrix} - a_N^2 \Delta_2^{N-1} \\ &= (b_1 - \lambda)\Delta_2^N - a_1^2 \Delta_3^N - a_N^2 \Delta_2^{N-1} \pm 2(-1)^{N-1} \prod_{n=1}^N a_n.\end{aligned} \tag{9.5}$$

If we valuate this formula in $\lambda = \mu_s, \forall s$ we have:

$$det(L^{\pm} - \mu_s I) = -a_1^2 \Delta_3^N(\mu_s) - a_N^2 \Delta_2^{N-1}(\mu_s) \pm 2(-1)^{N-1}. \tag{9.6}$$

Now we recall what we have proved in the section 4 for the spectrum of periodic Jacobi matrices: the polynomials $\Delta(\lambda) \pm 2$ correspond respectevely to the caratteristic polynomials of $L^{\pm}$. In this way, we can say that:

$$\begin{aligned} \Delta^2(\mu_s) - 4 &= \det(L^+ - \mu_s I)\det(L^- - \mu_s I) \\ &= \left(a_1^2 \Delta_3^N(\mu_s) + a_N^2 \Delta_2^{N-1}(\mu_s)\right)^2 - 4 \\ &= \left(a_1^2 \Delta_3^N(\mu_s) - a_N^2 \Delta_2^{N-1}(\mu_s)\right)^2, \end{aligned} \tag{9.7}$$

where in the last equality we used the formula 9.3 in such a way that

$$a_1^2 a_N^2 \Delta_3^N(\mu_s) \Delta_2^{N-1}(\mu_s) = 1.$$

At the end, combining the last formula with 9.2 one obtains the thesis:

$$\Delta^2(\mu_s) - 4 = \left(\frac{1}{2}\frac{d}{dt}\Delta_2^N\right)^2 = \left(\frac{1}{2}\frac{d\mu_s}{dt}P'(\mu_s)\right)^2.$$

□

9.1 Integration of the equations of the Dirichlet spectrum

We are finally going to show how to determine the time dependence of the Dirichlet eigenvalues. Remembering that these are the the roots of the polynomial $P(\lambda)$, one can show the following formulas:

$$\sum_{s=1}^{N-1} \frac{\mu_s^r}{P'(\mu_s)} = 0, r = 0, \ldots, N-3 \tag{9.8}$$

and

$$\sum_{s=1}^{N-1} \frac{\mu_s^{N-2}}{P'(\mu_s)} = 1. \tag{9.9}$$

Indeed, the function $G(\lambda) = \frac{\lambda^r}{P(\lambda)}$, for every $r \geq 0$, has $\mu_1, \ldots, \mu_{N-1}$ as poles of first order. Then taking γ any contour around all these poles, using the residual calculation rules, one has:

$$\frac{1}{2\pi i}\int_\gamma \frac{\lambda^r}{P(\lambda)} d\lambda = \sum_{s=1}^{N-1} \frac{\mu_s^r}{P'(\mu_s)}.$$

But, on the other side:

$$\frac{1}{2\pi i}\int_{\gamma}\frac{\lambda^r}{P(\lambda)}d\lambda = -\mathrm{Res}_{\lambda=\infty}\frac{\lambda^r}{P(\lambda)} = \begin{cases} 0, r = 0,\dots,N-3 \\ 1, r = N-2 \end{cases} \quad .$$

Now, substituting the formula for $P'(\mu_s)$ from 9.1 in these polynomials properties, one finds the following system of differential equations for the Dirichlet eigenvalues:

$$\sum_{s=1}^{N-1}\mu_s^r\frac{d\mu_s/dt}{\pm\sqrt{\Delta^2(\mu_s)-4}} = 0, r = 0,\dots,N-3 \tag{9.10}$$

and

$$\sum_{s=1}^{N-1}\mu_s^{N-2}\frac{d\mu_s/dt}{\pm\sqrt{\Delta^2(\mu_s)-4}} = 2. \tag{9.11}$$

Note that these equations make sense if and only if we suppose the nondegenerancy of all the intervals of instability (otherwise some denominator in the above formula goes to zero). On the other side, for every interval that degenerates, we know that the corrispondent μ_s will be constant $\mu_s = \lambda_{2s,2s+1}$, so we do not need any differential equations to describe its temporal evolution. In a certain way, the number of equations of the above system decrease in a proportional way respect to the degenerating μ_s. For simplicity we assume that the instability zones are all open.

We introduce the change of variables $(\mu_1,\dots,\mu_{N-1}) \to (\xi_1,\dots,\xi_{N-1})$

$$\xi_l = \sum_{j=1}^{N-1}\int_{\lambda_{2j}}^{\mu_j}\frac{\lambda^{l-1}}{\sqrt{R(\lambda)}}d\lambda, \tag{9.12}$$

for every $l,\dots,N-1$.
We check that the trasformation that sends $(\mu_1,\dots,\mu_{N-1})$ into $(\xi_1,\dots,\xi_{N-1})$ is really a change of variables, verifying that its Jacobian is nondegenerate.
Calculating its determinant one finds that it can be expressed through a multiple of the determinant of Vandermonde matrix in $\mu_1,\dots,\mu_{N-1}$.
Then, since the Dirichlet eigenvalues are all dinsticts, the determinant is nonzero and the trasformation above is a change of variables.

Indeed:

$$det\left(\frac{\partial\xi_l}{\partial\mu_j}\right)_{l,j=1}^{N-1}=\begin{vmatrix}\frac{1}{\sqrt{R(\mu_1)}} & \frac{1}{\sqrt{R(\mu_2)}} & \dots & \frac{1}{\sqrt{R(\mu_{N-1})}}\\ \frac{\mu_1}{\sqrt{R(\mu_1)}} & \frac{\mu_2}{\sqrt{R(\mu_2)}} & \dots & \frac{\mu_{N-1}}{\sqrt{R(\mu_{N-1})}}\\ \vdots & & & \vdots\\ \frac{\mu_1^{N-2}}{\sqrt{R(\mu_1)}} & \frac{\mu_2^{N-2}}{\sqrt{R(\mu_{N-2})}} & \dots & \frac{\mu_{N-}^{N-2}}{\sqrt{R(\mu_{N-1})}}\end{vmatrix}$$

$$=\frac{1}{\sqrt{\prod_{i=1}^{N-1}R(\mu_i)}}\begin{vmatrix}1 & 1 & \dots & 1\\ \mu_1 & \mu_2 & \dots & \mu_{N-1}\\ \vdots & & & \vdots\\ \mu_1^{N-2} & \mu_2^{N-2} & \dots & \mu_{N-1}^{N-2}\end{vmatrix}$$

$$=\frac{1}{\sqrt{\prod_{i=1}^{N-1}R(\mu_i)}}\underbrace{\prod_{1\le j<l\le N-1}(\mu_j-\mu_l)}_{\neq 0}\neq 0.$$

The temporal evolution of these new variables takes the following form:

$$\frac{d\xi_l}{dt}=\sum_{j=1}^{N-1}\frac{\mu_j^{l-1}}{\sqrt{R(\mu_j)}}\frac{d\mu_j}{dt}=2\sum_{j=1}^{N-1}\frac{\mu_j^{l-1}}{P'(\mu_j)},$$

where in the last equality we used the already proved formula (9.1) that describes the derivative of μ_j. Now using the property of polynomials given by (9.8),(9.9), we finally obtain:

$$\frac{d\xi_l}{dt}=\begin{cases}0, l=1,\dots N-2\\ 2, l=N-1\end{cases}.$$

Then:

$$\xi_l=\begin{cases}c_l, l=1,\dots,N-2\\ 2t+c_{N-1}, l=N-1\end{cases},$$

and setting all the integration constants to zero, one has:

$$\begin{pmatrix}0\\0\\ \vdots\\0\\2t\end{pmatrix}=\begin{pmatrix}\sum_{j=1}^{N-1}\int_{\lambda_{2j}}^{\mu_j}\frac{1}{\sqrt{R(\lambda)}}d\lambda\\ \sum_{j=1}^{N-1}\int_{\lambda_{2j}}^{\mu_j}\frac{\lambda}{\sqrt{R(\lambda)}}d\lambda\\ \vdots\\ \sum_{i=1}^{N-1}\int_{\lambda_{2j}}^{\mu_j}\frac{\lambda^{N-2}}{\sqrt{R(\lambda)}}d\lambda\end{pmatrix}.$$

For every non degenerate μ_s, the sign of its time derivative at the time zero must be chosen in such a way that it is consistent with initial data $a_k(0), b_k(0)$. The main issue with the explicit integration (a concrete useful formula) is the explicit expression of the inversion of the above integrals, namely the expression of $\mu_j = \mu_j(t)$. Such problem is a classical problem in the theory of Riemann surface and it is called Jacobi inversion problem.

10 Action-angle variables

In order to define the action-angle variables for the periodic Toda lattice, we have to define another pair of variables $\mu_i, \nu_i, i = 1, \dots N-1$, where the μ_i are exactly the Dirichlet eigenvalues (and we are assuming that they are all non degenerate). We will show that these new variables are canonical, and from that we will construct actions and angles [20].

11 Translated coordinates

We consider the space Ω formed by:

$$\Omega = \left\{ \lambda_1^+ < \lambda_2^+ \leq \lambda_3^+ < \dots < \lambda_N^+, (\mu_i, \pm\sqrt{\Delta^2(\mu_i) - 4}), \begin{array}{r} s.\ t.\ every\ \mu_i\ stays\ in\ appropriate\ interval, \\ \sum \lambda_i^+ = 0,\ and\ \Delta(\lambda) + 2\ has\ N\ roots. \end{array} \right\}$$

For what we proved in previous chapters, we know that this set is in corrispondence to the space that describes the periodic Toda lattice in Flasckha coordinates, i.e:

$$D = \left\{ a_i, b_i, i = 1, \dots, N\ s.\ t. \sum b_i = 0, \prod a_i = 1, a_i > 0 \right\}.$$

In Ω we introduce a two form ω that is globally defined and closed (so it is a symplectic form):

$$\omega = \sum_{i=1}^{N-1} \frac{d\mu_i \wedge d\Delta(\mu_i)}{\sqrt{\Delta^2(\mu_i) - 4}},$$

where Δ is the discriminant.

With next theorem we are going to show that it admits a global canonical form, choosing appropriate coordinates on Ω that we will call *translated* coordinates (for reasons that will be clear after the proof).

We first recall some properties of the discriminant Δ that will be used later. Combining 9.6, and the corollary 7.4 we have that (for example if N is even):

$$\Delta(\lambda) = (b_1 - \lambda)\Delta_2^N(\lambda) - a_1^2\Delta_3^N(\lambda) - a_N^2\Delta_2^{N-1}(\lambda),$$

and then for $\lambda = \mu_i$ for $i = 1, \dots, N-1$ it remains:

$$\Delta(\mu_i) = -a_1^2\Delta_3^N(\mu_i) - a_N^2\Delta_2^{N-1}(\mu_i), \tag{11.1}$$

because $\Delta_2^N(\mu_i) = 0$. Then in particular, as seen in 9.7, one has:

$$\sqrt{\Delta^2(\mu_i) - 4} = \pm(a_1^2 \Delta_3^N(\mu_s) - a_N^2 \Delta_2^{N-1}(\mu_s)).$$

Theorem 11.1 *On Ω there is a global system of coordinates q_i^0, p_i^0 for $2 \leq i \leq N$ such that ω can be rewritten as:*

$$\omega = \frac{1}{2} \sum_{j=2}^{N-1} dq_j^0 \wedge dp_j^0.$$

Proof. The main idea is to write ω using only a_i, b_i and then through Flasckha trasformation we will return to standard coordinates q_i, p_i.
First of all we define for every μ_i the corrispondent:

$$\nu_i = \pm \frac{1}{2} \log |c(N, \mu_i)|.$$

From the relations above and from the Wronskian relation 5.2, we express $c(N, \mu_i)$ in terms of Δ, and we obtain:

$$\begin{aligned} \nu_i &= \pm \frac{1}{2} \log |\frac{\Delta(\mu_i) \pm \sqrt{\Delta^2(\mu_i) - 4}}{2}| \\ &= \pm \frac{1}{2} \log |\frac{1}{2}((-a_1^2 \pm a_1^2)\Delta_3^N(\mu_i) + (-a_N^2 \mp a_N^2)\Delta_2^{N-1}(\mu_i))|, \end{aligned}$$

where in the last equality we used the relations recalled above.
Note that the sign corresponds to the sign chosen in Ω for $(\mu_i, \pm\sqrt{\Delta^2(\mu_i) - 4})$.
We can observe that, since:

$$d\nu_i = \frac{1}{2} \frac{2}{\Delta \pm \sqrt{\Delta^2 - 4}} \left(1 \pm \frac{\Delta}{\sqrt{\Delta^2 - 4}}\right) d\Delta(\mu_i) = \mp \frac{1}{2} \frac{1}{\sqrt{\Delta^2 - 4}} d\Delta(\mu_i),$$

we can write ω with the second definition of ν_i:

$$\omega = \sum_{i=1}^{N-1} d\mu_i \wedge d\nu_i.$$

Now from the formula 9.3:

$$\Delta_3^N \Delta_2^{N-1}(\mu_i) = \frac{1}{a_1^2 a_N^2},$$

we show that ν_i can be written as function of a_i, b_i, in the following way:

$$\nu_i = \pm \frac{1}{2} \log |\frac{1}{2}((-1 \pm 1)a_1^2 \Delta_3^N(\mu_i) + (-1 \mp 1)\frac{1}{a_1^2 \Delta_3^N(\mu_i)})| = -\frac{1}{2} \log |-a_1^2 \Delta_3^N(\mu_i)|.$$

Now we introduce the following notation: we call $\mu_k^{(l)}$ for $k = 1, \ldots, N-l+1$ the eigenvalues of L_l^N $(1 \leq l \leq N)$, so that our $\mu_k = \mu_k^{(2)}$. Recall that, expanding the determinant respect to the first row, one has:

$$\Delta_l^N(\lambda) = (b_l - \lambda)\Delta_{l+1}^N(\lambda) - a_l^2 \Delta_{l+2}^N(\lambda),$$

then

$$\Delta_l^N(\mu_k^{(l+1)}) = -a_l^2 \Delta_{l+2}^N(\mu_k^{(l+1)}). \tag{11.2}$$

It follows that:

$$\omega = -\frac{1}{2}\sum_{i=1}^{N-1} d\mu_i \wedge \frac{d\Delta_1^N(\mu_i)}{\Delta_1^N(\mu_i)}.$$

In order to manipulate this last expression of ω, we are going to show two recursive formulas, with the notation introduced above:

$$-\frac{1}{2}\sum_{i=1}^{N-l+1} d\mu_i^{(l)} \wedge \frac{d\Delta_{l+1}^N(\mu_i^{(l)})}{\Delta_{1+1}^N(\mu_i^{(l)})} = -\frac{1}{2}\sum_{j=1}^{N-l} d\mu_j^{(l+1)} \wedge \frac{d\Delta_l^N(\mu_j^{(l+1)})}{\Delta_1^N(\mu_j^{(l+1)})} \tag{11.3}$$

$$-\frac{1}{2}\sum_{j=1}^{N-l} d\mu_j^{(l+1)} \wedge \frac{d\Delta_l^N(\mu_j^{(l+1)})}{\Delta_1^N(\mu_j^{(l+1)})} = -\sum_{j=1}^{N-l} d\mu_j^{(l+1)} \wedge \frac{da_l}{a_l} - \frac{1}{2}\sum_{j=1}^{N-l} d\mu_j^{(l+1)} \wedge \frac{d\Delta_{l+2}^N(\mu_j^{(l+1)})}{\Delta_{l+2}^N(\mu_j^{(l+1)})} \tag{11.4}$$

(11.5)

.

and this is true for every $l = 1, \ldots, N$. Indeed:

$$-\frac{1}{2}\sum_{i=1}^{N-l+1} d\mu_i^{(l)} \wedge \frac{d\Delta_{l+1}^N(\mu_i^{(l)})}{\Delta_{1+1}^N(\mu_i^{(l)})} = -\frac{1}{2}\sum_{i=1}^{N-l+1} d\mu_i^{(l)} \wedge d(\log \Delta_{l+1}^N(\mu_i^{(l)}))$$

$$= -\frac{1}{2}\sum_{i=1}^{N-l+1} d\mu_i^{(l)} \wedge d(\sum_{j=1}^{N-l} \log(\mu_j^{(l)} - \mu_j^{(l+1)})$$

$$= \frac{1}{2}\sum_{i=1}^{N-l+1}\sum_{j=1}^{N-l} \frac{d\mu_i^{(l)} \wedge d\mu_j^{l+1}}{\mu_i^{(l)} - \mu_j^{(l+1)}}$$

$$= \frac{1}{2}\sum_{i=1}^{N-l+1}\sum_{j=1}^{N-l} \frac{d\mu_j^{(l+1)} \wedge d\mu_i^{l}}{\mu_j^{(l+1)} - \mu_i^{(l)}}$$

$$= -\frac{1}{2}\sum_{j=1}^{N-l} \mu_j^{(l+1)} \wedge d\log\left(\prod_{i=1}^{N-l+1}(\mu_j^{(l+1)} - \mu_i^{(l)})\right)$$

$$= -\frac{1}{2}\sum_{j=1}^{N-l} \mu_j^{(l+1)} \wedge \frac{d\Delta_l^N(\mu_j^{(l+1)})}{\Delta_l^N(\mu_j^{(l+1)})}.$$

Then, going on from this one can also obtain the second formula, applying 11.2 to the result above:

$$-\frac{1}{2}\sum_{j=1}^{N-l} \mu_j^{(l+1)} \wedge \frac{d\Delta_l^N(\mu_j^{(l+1)})}{\Delta_l^N(\mu_j^{(l+1)})} = -\frac{1}{2}\sum_{j=1}^{N-l} \mu_j^{(l+1)} \wedge \frac{da_l^2\Delta_{l+2}^N(\mu_j^{(l+1)})}{a_l^2\Delta_{l+2}^N(\mu_j^{(l+1)})}$$

$$= -\sum_{j=1}^{N-l} d\mu_j^{(l+1)} \wedge \frac{da_l}{a_l} - \frac{1}{2}\sum_{j=1}^{N-l} d\mu_j^{(l+1)} \wedge \frac{d\Delta_{l+2}^N(\mu_j^{(l+1)})}{\Delta_{l+2}^N(\mu_j^{(l+1)})}.$$

Taking the last expression for ω, we use this last formula for $l = 1$ and then the first for $l = 2$, it follows that:

$$\omega = -\sum_{j=1}^{N-1} d\mu_j^{(2)} \wedge \frac{da_1}{a_1} - \frac{1}{2}\sum_{j=1}^{N-2} d\mu_j^{(3)} \wedge \frac{d\Delta_2^N(\mu_j^{(3)})}{\Delta_2^N(\mu_j^{(3)})} = \ldots$$

and reapiting this procedure for every l one arrives to:

$$\cdots = -\sum_{j=1}^{N-1} d\mu_j^{(2)} \wedge \frac{da_1}{a_1} - -\sum_{j=1}^{N-2} d\mu_j^{(3)} \wedge \frac{da_2}{a_2} - \cdots - d\mu_1^{(N)} \wedge \frac{da_{N-1}}{a_{N-1}},$$

that is equivalent to:

$$\omega = -d\left(\sum_{j=1}^{N-1}\mu_j^{(2)} - \sum_{j=1}^{N-2}\mu_j^{(3)}\right)\wedge\frac{da_1}{a_1} - d\left(\sum_{j=1}^{N-2}\mu_j^{(3)} - \sum_{j=1}^{N-3}\mu_j^{(4)}\right)\wedge\left(\frac{da_1}{a_1}+\frac{da_2}{a_2}\right) - \dots$$
$$- d\mu_1^{(N)}\wedge\left(\frac{da_1}{a_1}+\frac{da_2}{a_2}+\dots+\frac{da_{N-1}}{a_{N-1}}\right).$$

Then remembering the fact that every L_l^N is a standard Jacobi matrix, so in particular it is diagonalizable (i.e. $L_l^N = U diag(\mu_1^{(l)}, \dots, \mu_{N-l+1}^{(l)})U^t$ with U orthogonal matrix), and using the invariance for similitude of the trace, one finds that

$$\sum_{k=1}^{N-l+1}\mu_k^{(l)} = \sum_{k=l}^{N} b_k,$$

and therefore:

$$\omega = -db_2\wedge\frac{da_1}{a_1} - db_3\wedge\left(\frac{da_1}{a_1}+\frac{da_2}{a_2}\right) - \dots - db_N\wedge\left(\frac{da_1}{a_1}+\frac{da_2}{a_2}+\dots+\frac{da_{N-1}}{a_{N-1}}\right).$$

Finally, we can write ω in a more compact form using that $\prod_i a_i = 1$. This implies that:

$$\log(\prod_i a_i) = \sum_i \log a_i = 0$$

and then:

$$d\left(\sum_i \log a_i\right) = \sum_i \frac{da_i}{a_i} = 0.$$

In this way:

$$\omega = \sum_{j=2}^{N} db_j \wedge \sum_{j\le i\le N}\frac{da_i}{a_i}. \qquad (11.6)$$

So ω is expressed only through a_i, b_i. Applying Flaschka transformation one can returns to the begenning coordinates:

$$2\frac{da_j}{a_j} = dq_j - dq_{j+1}$$
$$dp_j = -2db_j$$

and then

$$\omega = \frac{1}{2}\sum_{j=2}^{N}(dq_j - dq_1)\wedge dp_j.$$

Defining new coordinates $q_j^0 = q_j - q_1$ and $p_j^0 = p_j$ one then obtains the thesis. □

12 Construction of angles and actions

Finally we can start from these new canonical coordinates μ_i, ν_i to construct action-angle variables. First we define the actions: for every $l = 1, \dots, N-1$, choosing α_l a curve around the interval $[\lambda_{2l}, \lambda_{2l+1}]$

$$J_l = \oint_{\alpha_l} \sum_{i=1}^{N-1} \nu_i d\mu_i = 2\int_{\lambda_{2l}}^{\lambda_{2l+1}} \frac{1}{2} \log |\frac{\Delta(\mu_i) \pm \sqrt{\Delta^2(\mu_i) - 4}}{2}| d\mu_i. \tag{12.1}$$

For now on we will use the following notation $R(\lambda) = \Delta^2(\lambda) - 4$.
In order to construct the angles, we have to send the periodic variables μ_i into some new ones that will be linear in time. The intermediate step is to consider the variable ξ_j defined in (9.12). We define the $1-$ forms:

$$\eta_l = \frac{\lambda^{l-1}}{\sqrt{R(\lambda)}} d\lambda, \quad l = 1, \dots, N-1. \tag{12.2}$$

Then we call A, the $(N-1) \times (N-1)$ matrix given by:

$$A_{j,l} = 2\int_{\lambda_{2j}}^{\lambda_{2j+1}} \eta_l.$$

It is a standard result in the theory of Riemann surface that the determinant of A is nonzero, and so A is invertible . Then we can define the $N-1$ normalized forms:

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{N-1} \end{pmatrix} = A^{-1} \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{N-1} \end{pmatrix},$$

in the sense that $\int_{\lambda_{2j}}^{\lambda_{2j+1}} w_l = \delta_{j,l}$.
Finally we define the angle variables θ_l as:

$$\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_{N-1} \end{pmatrix} = \sum_{j=1}^{N-1} \begin{pmatrix} \int_{\lambda_{2j}}^{\mu_j} \omega_1 \\ \int_{\lambda_{2j}}^{\mu_j} \omega_2 \\ \vdots \\ \int_{\lambda_{2j}}^{\mu_j} \omega_{N-1} \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 2t \end{pmatrix} = 2t \begin{pmatrix} \tilde{A}_{1,N-1} \\ \tilde{A}_{2,N-1} \\ \vdots \\ \tilde{A}_{N-1,N-1} \end{pmatrix}, \tag{12.3}$$

where $\tilde{A}$ is the inverse of the matrix A. These functions θ_l are actually angles, indeed sending $\mu_j \to \mu_j + \alpha_j$ where α_j is a circle surrounding $[\lambda_{2j}, \lambda_{2j+1}]$, we have that:

$$
\begin{aligned}
\theta_l(\mu_j + \alpha_j) &= \sum_{k=1}^{N-1} A_{lk}^{-1} \xi_k(\mu_j + \alpha_j) \\
&= \sum_{k=1}^{N-1} A_{lk}^{-1} \left(\sum_{j=1}^{N-1} \int_{\lambda_{2j}}^{\mu_j} \frac{\lambda^{k-1}}{\sqrt{R(\lambda)}} d\lambda + \oint_{\alpha_j} \frac{\lambda^{k-1}}{\sqrt{R(\lambda)}} d\lambda \right) \\
&= \theta_l(\mu_j) + \sum_{k=1}^{N-1} A_{lk}^{-1} \left(\sum_{j=1}^{N-1} A_{jk} \right) \\
&= \theta_l(\mu_j) + \sum_{j=1}^{N-1} \delta_{j,l} = \theta_l(\mu_j) + 1.
\end{aligned}
$$

So for $\mu_j \to \mu_j + \alpha_j$, then $\theta_l \to \theta_l + 1$, and this is true for every $l = 1, \dots, N-1$.

Lemma 12.1 *The variables* (θ, J) *defined in* (12.1) *and* (12.3) *respectively, are canonically conjugate variables.*

The proof follows the classical lines of constructing a generating function of a canonical transformation between the (μ, ν) variables and the action angle variables.

13 Solution through Riemann Theta functions

In the work done before this chapter, we proved the integrability for the periodic Toda lattice and we constructed angle action variables to describe the motion of the system on the torus where it takes place.
Now we are looking for an explicit formulation for the a_n, b_n: in order to do this, we have to work on the Riemann surface associated to our problem. Here we can construct a homology basis of cycles and of holomorfic differential: through these then are defined the so called Theta functions, with which we are going to write our solution.
Once stated the main features of these objects, we have to define the second tool we need. We call the $k-$shifted solutions $c(k,n), s(k,n)$, the solutions of the spectral equation for the $k-$shifted operator $\mathcal{L}_{b,a}$, obtained replacing a_n, b_n with a_{n+k}, b_{n+k}. Then we are able to find an expression for every b_k, through the sum of all the $N-1$ zeros, $\mu_j(k)$, of $c(k, N+1)$ one of the two fundamental solutions for the shifted spectral equation. This formula will be used at the end to find the Theta functions formulation for b_k. To obtain it, we have to work on the sum of these $\mu_j(k)$.

Recalling some particular solutions for the spectral equation for L^+, called Bloch eigenfunctions and already defined in 4, and studying their analytical properties on the Riemann surface we will find a connection between them and the $\mu_j, \mu_j(k)$. Here comes the crucial point: we will define a $1-$ form on the Riemann surface as the differential of the logarithm of the Bloch function, and using some typical properties of the Riemann surfaces as the Riemann bilinear relations and the Riemann vanishing theorem, we will arrive to the expression through Theta functions for the sum of the $\mu_j(k)$.

14 Riemann surfaces

In this section we recall the basic ingredients of Riemann surfaces, see e.g. [**?**]. In order to define the Riemann surface associated to the periodic Toda lattice, we have to recall the eigenvalues for the matrix Q, given by the roots of the polynomial $\Delta^2 - 4$: $\lambda_1 < \lambda_2 \leq \lambda_3 < \lambda_4 \leq \cdots < \lambda_{2N-2} \leq \lambda_{2N-1} < \lambda_{2N}$.
Now we consider only the case in which there are exactly $2g+2$, listing eigenvalues and we rename them as:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{2g+2}.$$

Furthermore we have the inequality $\lambda_{2j} < \mu_j < \lambda_{2j+1},\ j = 1\ldots, g$.
We define the polynomial of degree $2g+2$:

$$R(\lambda) = \prod_{j=1}^{2g+2} (\lambda - \lambda_j).$$

Finally, we define the Riemannian surface S associated to the periodic Toda lattice as:

$$S = \left\{ (w, \lambda) \in \mathbf{C}^2 | F(\lambda, w) - w^2 - R(\lambda) = 0 \right\} \tag{14.1}$$

For $g = 1$, S is also called an elliptic curve and for $g > 1$ an hyperelliptic curve.

Remark 14.1 Note that the projection map $\pi : S \to \mathbf{C}, \pi(\lambda, w) = \lambda$ realizes S as a two-sheeted coverings of the $\lambda-$plane. The branch points of this covering are the points for which the pre-images merge in one point, i.e. the points determined by the system:

$$\begin{cases} F(\lambda, w) = 0 \\ F_w(\lambda, w) = 0 \end{cases} \Leftrightarrow \begin{cases} w^2 - R(\lambda) = 0 \\ w = 0 \end{cases},$$

namely $P_j = (\lambda_j, 0)$, $j = 1, \ldots, 2g+1$. All branch points have multeplicity one.

We choose the function $\sqrt{R(\lambda)}$ analytic in $\mathbf{C}\backslash[\lambda_1,\lambda_2]\cup[\lambda_3,\lambda_4]\cdots\cup[\lambda_{2g+1},\lambda_{2g+2}]$ and we assume that $\sqrt{R(\lambda)}$ is real and positive on $[\lambda_{2g+2},\infty)$. We define as the first sheet of the Riemann surface S and denote it by S^+, the sheet where $\sqrt{R(\lambda)}$ is positive on the interval $[\lambda_{2g+2},\infty)$. Clearly the second sheet S^- is identified with the sheet where $\sqrt{R(\lambda)}$ is negative on the interval $[\lambda_{2g+2},\infty)$.

The curve S can be compactified to a compact Rieamnn surface by adding the two points at infinity $\infty^{\pm}$ on the sheet $S^{\pm}$ and a complex structure. The complex structure is obtain as follows. In a neighborhood of any point of S that is not a branch point the projection λ is a local parameter, or local chart. Near such points the surface S has a local parametric representation $(\lambda,\sqrt{R(\lambda)})$ for a suitable choice of the sign of the square root.

Instead, in a neighborhood of a branch point P_j a local chart is given by

$$\tau=\sqrt{\lambda-\lambda_j}, \tag{14.2}$$

Then for the branch points of S we get the local parametric representation

$$\lambda=\lambda_j+\tau^2,\quad w=\tau\sqrt{\prod_{j\neq i}(\tau^2+\lambda_i-\lambda_j)} \tag{14.3}$$

where the radical is a single-valued holomorphic function for sufficiently small τ;(the expression under the root sign does not vanish), and $dw/d\tau\neq 0$ for $\tau=0$. The two points at infintiy have local chart

$$\lambda=\frac{1}{\tau},\quad \tau\in\mathbf{C}^*.$$

The genus of the surface S can be calculated using the Riemann-Hurwitz formula

$$\text{genus}=\frac{1}{2}\sum_{j=1}^{m}b_j-n+1. \tag{14.4}$$

where in our case, $b_j=1$ is the branching number of each branch point and $n=2$ is the degree of the covering map π. So we obtain

$$\text{genus}=\frac{1}{2}(2g+2)-2+1=g.$$

In the next three paragraphs we are going to summarise the most important features and properties of a Riemann surface, applied in particular on S . We limit to give some statements of theorems, and all the prooves can be find in [**?**].

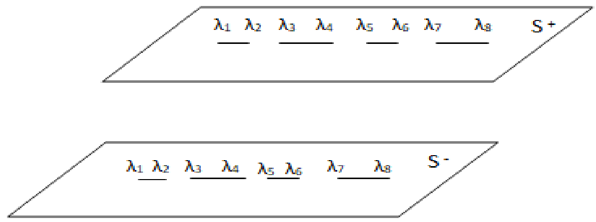

Figure 2: Representation of the Riemann surface S, in the case of genus $g = 3$, as two copies of the complex plane with the appropriate cuts.

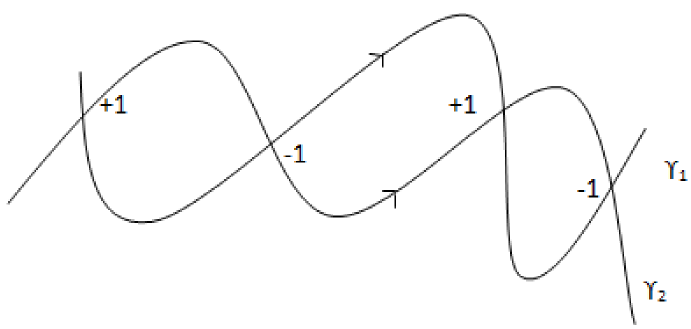

Figure 3: Two curves γ_1, γ_2 for which is calculated the value of the function $\nu(P)$, for every P in which they intersect.

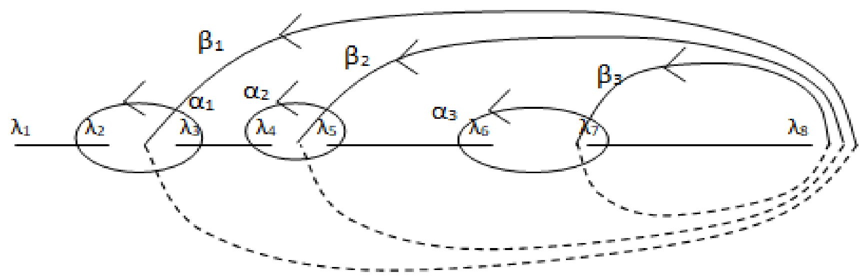

Figure 4: The canonical homology basis of cycles choosen for S, for genus $g = 3$. Observe that with the dashed line we denoted the passage of the curve on the lower sheet of S.

Canonical homology basis Now we want to define a basis for the for the first homology group of S, such that it is *canonical*.

Definition 14.2 We call a *canonical* basis for $H_1(S, \mathbf{Z})$, a basis of cycles $\alpha_1, \ldots, \alpha_g$ and $\beta_1, \ldots, \beta_g$ such that the intersection number satisfies the following conditions:

$$\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0, \quad \alpha_i \circ \beta_j = \delta_{ij}, \;\; i, j = 1 \ldots, g \tag{14.5}$$

Now, using the representation of S made by two copies of $\mathbf{C}$ with cuts along the real intervals $[\lambda_{2j-1}, \lambda_{2j}]$ for every $j = 1, \ldots, g+1$, we choose a canonical basis of cycles, as represented in the figure 4 in this way:

- every α_j is a counterclockwise cycle around the interval $[\lambda_{2j}, \lambda_{2j+1}]$ in the upper sheet, and two different α_j never intersect themselves;
- every β_j is a counterclockwise cycle that starts in the interval preceding λ_8, intersects only the α_j cycle (and the correspondent surrounded interval) in the upper sheet, then passes to the lower sheet returning to the point of begenning.

Holomorphic and meromorphic differentials We are going to introduce some properties of differentials on Riemann surfaces, that will be useful later on.

Definition 14.3 A differential ω is holomorphic/meromorphic if in local coordinates can be written as

$$\omega = h_\alpha(z_\alpha) dz_\alpha,$$

where $h_\alpha(z_\alpha)$ is a holomorphic/meromorphic function.

Note that holomorphic differentials are all closed differentials. Now we use the following result to construct a basis of exactly g holomorphic differentials.

Theorem 14.4 *The space of holomorphic differentials on a Riemann surface of genus g has dimension g (as a vectorial space).*

In particular, for our S, we can explicitly find a basis of holomorphic differentials, given by:

$$\eta_k = \frac{\lambda^{k-1} d\lambda}{w} = \frac{\lambda^{k-1} d\lambda}{\sqrt{R(\lambda)}}, \quad k = 1, \ldots, g. \tag{14.6}$$

We have to check that η_k are all holomorphic.
This is certainly true at any finite point that is not a branch point of S, so first we see what happens in one of the branch points $P_j = (\lambda_j, 0)$. We consider the local parameter $\tau = \sqrt{\lambda - \lambda_j}$, then we obtain $\eta_k = \zeta_k(\tau) d\tau$ with:

$$\zeta_k(\tau) = \frac{(\tau^2 + \lambda_j)^{k-1} 2\tau}{\sqrt{\prod_{i=1}^{2g+2} (\tau^2 + \lambda_j - \lambda_i)}} = \frac{2(\lambda_j + \tau^2)^{k-1}}{\sqrt{\prod_{j \neq i} (\tau^2 + \lambda_j - \lambda_i)}},$$

that is holomorphic for small τ.
Then at the infinity point we consider the local parameter $\tau = \frac{1}{\lambda}$, and in this case we obtain $\eta_k = \phi_k(\tau) d\tau$ with:

$$\phi_k(\tau) = \frac{-\left(\frac{1}{\tau}\right)^{k+1}}{\left(\frac{1}{\tau}\right)^{g+1} \prod_{i=1}^{2g+2} \sqrt{(1 - \lambda_i \tau)}} = \frac{-2\tau^{g-k}}{\prod_{i=1}^{2g+2} \sqrt{(1 - \lambda_i \tau)}},$$

that is also holomorphic for small τ.
So we can conclude that η_k form a basis of holomorphic differential of S.

Remark 14.5 We already met this holomorphic differentials during the construction of angle action variables in the previous section 12. Indeed, they are exactly the differential constructed in 12.2 to define the coordinates change from the Dirichlet eigenvalues to some new coordinates ξ_k linear in time.

Matching this last result with what we found in the previous paragraph 14, we can take a canonical homology basis for S, formed by $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, and we can normalize the basis of holomorfic differentials η_k in the sense that the normalized basis of holomorphic differential ω_k is such that:

$$\oint_{\alpha_j} \omega_k = \delta_{j,k}, \quad j, k = 1, \ldots, g. \tag{14.7}$$

Indeed, we set:

$$A_{jk} = \oint_{\alpha_j} \eta_k = 2\int_{\lambda_{2j}}^{\lambda_{2j+1}} \frac{\lambda^{k-1}d\lambda}{\sqrt{R(\lambda)}}, \ \ j,k=1,\dots,g. \tag{14.8}$$

This matrix is nonsingular. Indeed, otherwise there are constants $c_l,\dots,c_g$, not all zero, such that $\sum_k A_{jk}c_k = 0$. But then $\sum_k c_k\eta_k = 0$, since this differential has zero a-periods. This contradicts the independence of the differentials $\eta_1,\dots,\eta_g$.
So we can take $\tilde{A}_{k,j}$ the inverse matrix of $A_{j,k}$, and we define:

$$\omega_j = \sum_{k=1}^{g} \tilde{A}_{jk}\eta_k = \frac{\sum_{k=1}^{g}\tilde{A}_{jk}\lambda^{k-1}d\lambda}{\sqrt{R(\lambda)}}, \quad j=1,\dots,g. \tag{14.9}$$

Then we can consider the integral of the normalized basis on the β-cycles and the matrix we will obtain in this way has the following properties:

Proposition 14.6 *Let $\omega_1,\dots\omega_g$ be the normalized basis of holomorphic differentials as in (14.7) and* (14.9). *Let*

$$B_{jk} = \oint_{\beta_j} \omega_k, \quad j,k=1,\dots,g. \tag{14.10}$$

Then the matrix (B_{jk}) is symmetric and has positive-definite imaginary part.
The matrix (B_{jk}) is called a period matrix of the Riemann surface S.

Remark 14.7 The surface S has all the branch points real. Furthermore, with the choice of the homology basis given in fig. 4 and the choice of the branch cuts for $\sqrt{R(\lambda)}$, we can conclude that the matrix A is real and the matrix B is pure imaginary.

We will also need to consider meromorphic differentials on S. They take the form $\omega = H(\lambda,w)d\lambda$, where $H(\lambda,w)$ is a rational function of λ and w. If P_0 is a pole of multeplicity k for ω, in a local coordinate $z(P)$ centred in P_0, namely $z(P_0)=0$, it takes the form

$$\omega = \left(\frac{c_{-k}}{z^k} + \cdots + \frac{c_{-1}}{z} + O(1)\right)dz.$$

For the meromorphic differential there is the freedom to fix the normalisation to zero on the α cycles. Indeed it is possible to add the meromorphic differentials so that the differential

$$\tilde{\omega} = \omega + \sum_{j=1}^{g} c_j\omega_j$$

has zero α-periods, namely it is sufficient to chose $c_j = -\int_{\alpha_l}\omega$.

A property of the meromorphic differentials is given by the following theorem.

Theorem 14.8 (The Residue Theorem) *The sum of the residues of a meromorphic differential ω on a Riemann surface, taken over all poles of this differential, is equal to zero.*

Any meromorphic differential can be represented as the sum of a holomorphic differential and the following elementary meromorphic differentials.

1. Normalised abelian differential of the second kind Ω_P^n with a only a pole of multiplicity $n+1$ at P and principal part of the form

$$\Omega_P^n = \left(\frac{1}{z^{n+1}} + O(1)\right) dz \tag{14.11}$$

   with respect to some local parameter z, $z(P) = 0$, $n = 1, 2, \dots$.

2. Normalised Abelian differential of the third kind Ω_{PQ} with simple poles at the points P and Q with residues $+1$ and -1 respectively.

For the hyperelliptic Riemann surface S and given the points $P = (p, w_p)$ and $Q = (q, w_q)$ we have that

$$\Omega_{PQ} = \frac{d\lambda}{2w}\left(\frac{w + w_p}{\lambda - p} - \frac{w + w_q}{\lambda - q}\right) + \sum_{j=1}^{g} c_j \omega_j$$

where c_j are the normalising constants so that $\int_{\alpha_j} \Omega_{PQ} = 0$ for $j = 1, \dots, g$.

Regarding the second kind differentials we consider for future use the one with a pole at $\infty^{\pm}$ of order two which we denote as $\Omega^{(1)}_{\infty^{\pm}}$. It takes the form

$$\Omega^{(1)}_{\infty^{\pm}} = \left(\lambda^{g+1} - \lambda^{g}\sum_{j=1}^{2g+1} \lambda_j + s_{g-1}\lambda^{g-1} + \dots s_0\right)\frac{d\lambda}{w}, \tag{14.12}$$

where the constants $s_0, \dots, s_{g-1}$ are determined by $\int_{\alpha_j} \Omega^{(1)}_{\infty^{\pm}} = 0$, $j = 1, \dots, g$. The β periods of the above canonical meromorphic differentials can be expressed in terms of the holomorphic differential using the Riemann bilinear relation.

Lemma 14.9 *Let Ω_{PQ} a normalized Abelian differential of the third kind, and ω_k a normalized basis of holomorfic differentials with respect to the canonical homology basis chosen. Then:*

$$\oint_{\beta_k} \Omega_{PQ} = 2\pi i \int_Q^P \omega_k, \quad i = 1, \dots, g, \tag{14.13}$$

where the integration from Q to P in the integral does not intersect the cycles $\alpha_1, \dots, \alpha_g$, $\beta_1, \dots, \beta_g$.

As an example we consider the third kind differential $\Omega_{\infty^-,\infty^+}$. Its periods are

$$U_k := \frac{1}{2\pi i}\oint_{\beta_k}\Omega_{\infty^-,\infty^+} = \int_{\infty^+}^{\infty^-}\omega_k \in \mathbf{R} \;\; i=1,\dots,g, \tag{14.14}$$

where from our choice of homology basis, the periods U_k are all real. A smilar results holds for the second kind differential

Lemma 14.10 *Let $\Omega^{(1)}_{\infty^\pm}$ be the second kind normalised differential with second order pole at $\pm\infty$. Then*

$$\oint_{\beta_k}\Omega^{(1)}_{\pm\infty} = 4\pi i\psi_k(z)|_{z=0}, \;\; k=1,\dots,g, \tag{14.15}$$

where $\omega_k = \psi_k(z)dz$ is the holomorphic differential in the local coordinate $z = \dfrac{1}{\lambda}$ near the point ∞^+.

We observe that

$$\begin{pmatrix} \oint_{\beta_1}\Omega^{(1)}_{\pm\infty} \\ \oint_{\beta_2}\Omega^{(1)}_{\pm\infty} \\ \vdots \\ \oint_{\beta_g}\Omega^{(1)}_{\pm\infty}\end{pmatrix} = -4\pi i A^{-1}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix} \tag{14.16}$$

where A is the period matrix (14.8) of the non normalised holomorphic differentials. We define the vector V as

$$V_j = \frac{1}{2\pi i}\oint_{\beta_j}\Omega^{(1)}_{\pm\infty}, \quad j=1,\dots,g. \tag{14.17}$$

From (14.16) we have that the vector V is real and it is related to the angle variables defined in (**??**).

Jacobi variety, Abel map and Theta functions. With the same notations above for the canonical homology basis of cycles and the basis of holomorphic differentials on the Riemann surface S of genus g, we consider the period matrix of S, that we already called B. We recall that B is a $g \times g$ complex matrix, with the properties, stated in 14.6, to be symmetric and with imaginary part positive-definite.
Then, thanks to these properties, for any basis $e_1,\dots,e_g$ of $\mathbf{C}^g$ we have that the vectors:

$$e_1,\dots,e_g, Be_1,\dots,Be_g$$

are linearly independent on $\mathbf{R}$. Now consider in $\mathbf{C}^g$ the integer period lattice generated by the this family of independent vectors. Every vector in this lattice can be written in the form:

$$m + nB, \quad m,n \in \mathbf{Z}^g. \tag{14.18}$$

and by the independence of the generetors $e_1, \ldots, e_g, Be_1, \ldots, Be_g$, the quotient of $\mathbf{C}^g$ by this lattice is a torus of maximal rank, that we define as the Jacoby variety (or Jacobian) of S.

Definition 14.11 If B is the period matrix of S, then the torus:

$$T^{2g} = T^{2g}(B) = \mathbf{C}^g/\{m + nB\}. \tag{14.19}$$

denoted by $J(S)$ is the Jacobi variety of S.

Now we can define the Abel map. Fixing a point $P_0 \in S$, for every $P \in S$ we consider the integral:

$$u_k(P) = \int_{P_0}^{P} \omega_k, \quad k = 1, \ldots, g. \tag{14.20}$$

Then the vector-valued function

$$\mathcal{A}(P) = (u_1(P), \ldots, u_g(P)) \tag{14.21}$$

is called the Abel map (the path of integration is chosen to be the same in all the integrals $u_1(P), \ldots, u_g(P)$). This function is connected with the Jacobi variety of S in the sense that it takes values on it.

Lemma 14.12 *The Abel mapping is a well-defined holomorphic mapping*

$$\mathcal{A} : S \to J(S).$$

Proof. A change of the path of integration in the integrals $u_k(P)$ leads to a change in the values of these integrals according to the law

$$u_k(P) \to u_k(P) + \oint_\gamma \omega_k, \quad k = 1, \ldots, g,$$

where γ is some cycle on S. Decomposing it with respect to the basis of cycles, $\gamma \simeq \sum m_j \alpha_j + \sum n_j \beta_j$ we get that:

$$u_k(P) \to u_k(P) + m_k + \sum_j B_{kj} n_j, \quad k = 1, \ldots, g.$$

The increment on the right-hand side is the kth coordinate of the period lattice vector $2\pi i M + NB$ where $M = (m_1, \ldots, m_g)$, $N = (n_1, \ldots, n_g)$. The lemma is proved. □

Finally we can start talking about Theta functions on our Riemann surface S.
Generally speaking, chosen $B = (B_{jk})$ a symmetric $g \times g$ matrix with positive-definite imaginary part and $z = (z_1, \ldots, z_g)$ and $N = (N_1, \ldots, N_g)$ some g-dimensional vectors, we can define the Riemann theta function (associated to B) as its multiple Fourier series,

$$\Theta(z) = \Theta(z; B) = \sum_{N \in \mathbf{Z}^g} \exp\left(\pi i \langle NB, N\rangle + 2\pi i \langle N, z\rangle\right), \tag{14.22}$$

where the angle brackets denote the Euclidean inner product:

$$\langle N, z\rangle = \sum_{k=1}^{g} N_k z_k, \quad \langle NB, N\rangle = \sum_{j,k=1}^{g} B_{kj} N_j N_k.$$

The summation in 14.22 is over the lattice of integer vectors $N = (N_1, \ldots, N_g)$. The obvious estimate $\mathrm{Re}\,(i\langle NB, N\rangle) \leq -b\langle N, N\rangle$, where $b > 0$ is the smallest eigenvalue of the matrix $\mathrm{Im}\,(B)$, implies that this series defines an entire function of the variables $z_1, \ldots, z_g$.

Proposition 14.13 *The theta-function has the following properties.*

1. $\Theta(-z\,;\,B) = \Theta(z\,;\,B)$.
2. *For any integer vectors* $M, K \in \mathbf{Z}^g$*,*
$$\Theta(z + K + MB; B) = \exp\left(-\pi i \langle MB, M\rangle - 2\pi i \langle M, z\rangle\right) \Theta(z; B). \tag{14.23}$$
3. *It satisfies the heat equation*
$$\begin{aligned} \frac{\partial}{\partial B_{ij}} \Theta(z\,;\,B) &= \frac{1}{2\pi i} \frac{\partial^2}{\partial z_i z_j} \Theta(z\,;\,B), \quad i \neq j \\ \frac{\partial}{\partial B_{ii}} \Theta(z\,;\,B) &= \frac{1}{4\pi i} \frac{\partial^2}{\partial z_i^2} \Theta(z\,;\,B). \end{aligned} \tag{14.24}$$

Note that, from the equation 14.23 in the second point of the above proposition, the theta-function is an analytic multivalued function on the g-dimensional torus $T^{2g} = \mathbf{C}^g/\{N + MB\}$.
For $e = (e_1, \ldots, e_g) \in \mathbf{C}^g$ we define the function $F : S \to \mathbf{C}$ as

$$F(P) = \Theta(\mathcal{A}(P) - e; B). \tag{14.25}$$

Then using the properties of the holomorfic differentials and the periodicity of the $\Theta-$function given by 14.23, we have that $F(P)$ transforms in the following way:

- $F(P + \alpha_j) = F(P)$ $\quad$ (14.26)
- $F(P + \beta_j) = F(P) \exp\left(-\pi i B_{jj} - 2\pi i \int_{P_0}^{P} \omega_j + 2\pi i e_j\right).$ $\quad$ (14.27)

Also $F(P)$ is non single-valued on S from its definition. But we can study the set of its zeros as the study of the intersection of the Abel mapping's image $\mathcal{A}(P) \subset J(S)$ with the set of zeros of the Riemann $\Theta-$function. For this we have the following result.

Theorem 14.14 (Riemann Vanishing Theorem) *If $F(P)$ is not identically zero, then it has g zeros $Q_1, \dots, Q_g$, counted with their multiplicity.*[1]
Furthermore the following equality holds:

$$\sum_{j=1}^{g} \int_{P_0}^{Q_j} \omega = e + K_{P_0}, \tag{14.28}$$

where K_{P_0} is the vector of Riemann constants, that depends only on the surface and its marking and on P_0.

15 Bloch eigenfunctions

In this section we return to the spectral problem for the operator $\mathcal{L}_{b,a}$, analyzed before. In particular, we consider the $k-$shifted spectral equation for $\mathcal{L}_{b,a}$, i.e. the equation 4.5 where the coefficients a_n, b_n are replaced by a_{k+n}, b_{k+n}, and we show the relations between its fundamental solutions and the fundamental solutions for the non-shifted spectral equation. Through these new functions we find an expression for every b_k, that will give at the end the explicit solution.
Then we introduce some particular eigenfunctions of the matrix L^+, called Bloch eigenfunctions, that will be the main mean for the explicit formulation of b_k. Indeed, considering these functions on the Riemann surface S, and studying their analytical properties we will find a way to relate them to the expression for b_k.

Definition 15.1 We define the two fundamental solutions $c(k,n), s(k,n)$ of the $k-$shifted spectral equation 4.5, the ones with initial conditions:

$$\begin{cases} c(k,0) = 1 \\ c(k,1) = 0 \end{cases}, \begin{cases} s(k,0) = 0 \\ s(k,1) = 1 \end{cases}.$$

As seen in the case of the non-shifted equation 4.5, also these fundamental solutions $c(k,n), s(k,n)$ are polynomials in λ with respectively degree $n-2$ and $n-1$. Indeed, following an analogue proof with respect to the one done for the proposition 5.4 we can

[1] Actually, here we are considering $F(P)$ as a single-valued analytic function on $\tilde{S}$, that is the Poincaré polygon of S. This is a $4g-$polygon obtained cutting and gluening all the cycles, and their inverse, of the canonical homology basis.

find the following expressions:

$$c(k,n) = -a_k \prod_{i=1}^{n-1} a_i^{-1} \lambda^{n-2} + \ldots, \quad s(k,n) = \prod_{i=1}^{n-1} a_i^{-1} \lambda^{n-1} + \ldots. \tag{15.1}$$

In particular, defining the $k-$ shifted matrix of L_2^N, of dimension $N-1$:

$$(L_{(k)}) = \begin{pmatrix} b_{k+2} & a_{k+2} & 0 & \ldots & 0 \\ a_{k+2} & b_{k+3} & a_{k+3} & \ldots & 0 \\ 0 & a_{k+3} & \ddots & & \vdots \\ \vdots & & & & \vdots \\ 0 & \ldots & & a_{k+N-1} & b_{k+N} \end{pmatrix},$$

we can find that:

$$c(k, N+1) = -a_k \prod_{i=1}^{n-1} a_i^{-1} \det(L_{(k)} - \lambda I). \tag{15.2}$$

This formula will be useful later on. With the next proposition we establish the relation between $c(k,n), s(k,n)$ and the old fundamental solutions.

Proposition 15.2 *[17]. For every fixed $k \in \mathbf{N}$, and for every $n \in \mathbf{Z}$ the following relations hold:*

$$\begin{pmatrix} c(k,n) \\ s(k,n) \end{pmatrix} = \begin{pmatrix} a_0^{-1} a_k s(k+1) & -a_0^{-1} a_k c(k+1) \\ -a_0^{-1} a_k s(k) & a_0^{-1} a_k c(k) \end{pmatrix} \begin{pmatrix} c(k+n) \\ s(k+n) \end{pmatrix}. \tag{15.3}$$

Proof. Once k is fixed, we proceed by induction over n.
For $n = 0$ the relations hold, indeed:

$$c(k,0) = 1 = a_0^{-1} \underbrace{a_k \left(s(k+1)c(k) - c(k+1)s(k)\right)}_{=W(k)},$$

since we know from the property of the Wronskian 5.2, shown in proposition 5.3 that $W(k) = a_0$ for every k. And

$$s(k,0) = 0 = a_0^{-1} a_k \left(-s(k)c(k) + c(k)s(k)\right).$$

For the same reason also for $n = 1$ the identities hold:

$$c(k,1) = 0 = a_0^{-1} a_k \left(s(k+1)c(k+1) - c(k+1)s(k+1)\right),$$
$$s(k,1) = 1 = a_0^{-1} a_k \left(-c(k+1)s(k) + c(k)s(k+1)\right).$$

Then supposing that they hold for every $j \leq n-1$, we obtain the thesis for n, indeed using the spectral equation:

$$\begin{aligned}
c(k,n) &= a_{k+n-1}^{-1}\left((\lambda - b_{k+n-1})c(k,n-1) - a_{k+n-2}c(k,n-2)\right) \\
&= a_{k+n-1}^{-1}\left((\lambda - b_{k+n-1})a_0^{-1}a_k\left(s(k+1)c(k+n-1) - c(k+1)s(k+n-1)\right)\right) + \\
&+ a_{k+n-1}^{-1}\left(-a_{k+n-2}a_0^{-1}a_k\left(s(k+1)c(k+n-2) - c(k+1)s(k+n-2)\right)\right) \\
&= a_0^{-1}a_k\left(s(k+1)a_{k+n-1}^{-1}\left((\lambda - b_{k+n-1})c(k+n-1) - a_{k+n-2}c(k+n-2)\right)\right) + \\
&+ a_0^{-1}a_k\left(-c(k+1)a_{k+n-1}^{-1}\left((\lambda - b_{k+n-1})s(k+n-1) - a_{k+n-2}s(k+n-2)\right)\right) \\
&= a_0^{-1}a_k\left(s(k+1)c(k+n) - c(k+1)s(k+n)\right),
\end{aligned}$$

That is exactly our thesis for $c(k,n)$ and with the same argument is obtained the formula for $s(k,n)$. □

Definition 15.3 For a fixed k, we denote by $\mu_j(k)$ for $j = 1, \ldots, N-1$ the roots of the polynomial $c(k, N+1)$.

Remark 15.4 For $k = 0$, we have that $\mu_j(k)$ are exactly the Dirichlet eigenvalues μ_j, so we refer to the $\mu_j(k)$ as *shifted* Dirichlet eigenvalues. As well as the μ_j, these $\mu_j(k)$ are no more constant quantities in time.
One can observe that also the $\mu_j(k)$ belong to the interval $[\lambda_{2j}, \lambda_{2j+1}]$, so they coincide to the corrispondent Dirichlet eigenvalue, whenever the interval degeneretes to one point.

We recall now the useful expression 8.4, that gives b_1 in terms of Dirichlet eigenvalues and we show how we can write something similar for every b_k, using the shifted Dirichlet eigenvalues defined above.
During the proof of the theorem 8.1, that gave the solution for the inverse spectral problem for the matrix Q, we used the fact that:

$$b_1 = \frac{1}{2}Tr(Q) - Tr(L_2^N) = \frac{1}{2}\sum_{i=1}^{2N}\lambda_i - \sum_{j=1}^{N-1}\mu_j.$$

This can also be obtained looking at the coefficient of the term λ^{N-2} of the polynomial $c(N+1)$ and at the coefficient of the term λ^{2N-1} of $\Delta^2 - 4$ in two different ways. Indeed, for $c(N+1)$, since it is, up to a multiplicative factor, the characteristic polynomial for the matrix L_2^N, we can write it as:

$$c(N+1) = \prod_{i=1}^{N-1} a_i^{-1}\left(\lambda^{N-1} - Tr(L_2^N)\lambda^{N-2} + \frac{1}{2}((Tr(L_2^N))^2 - Tr(L_2^N)^2)\lambda^{N-3} + \ldots\right).$$

On the other side, calling the roots of Δ_2^N as μ_j, we can write $c(N+1)$ using the *Viéte's formulas* and we have:

$$c(N+1) = \prod_{i=1}^{N-1} a_i^{-1} \left(\lambda^{N-1} - (\sum_{j=1}^{N-1} \mu_j)\lambda^{N-2} + (\sum_{1\leq i<j\leq N-1} \mu_i\mu_j)\lambda^{N-3} + \dots \right),$$

and so $\sum_{j=1}^{N-1} \mu_j = Tr(L_2^N)$.
The same argument is used for $\Delta^2 - 4$ to obtain $Tr(Q) = \sum_{i=1}^{2N} \lambda_i$.
Now we can reapeat all this for $c(k, N+1)$. Indeed, using the expression in 15.2 we have that the coefficient c_{N-2} of the term λ^{N-2} is (up to a multiplicative factor):

$$c_{N-2} = -Tr(L_{(k)}) = -\sum_{\substack{n=1 \\ n\neq k+1}}^{N} b_n,$$

from the periodicity of b_n. On the other side, using that it has roots $\mu_j(k)$, we obtain through the following formulas that:

$$c_{N-2} = -\sum_{j=1}^{N-1} \mu_j(k).$$

Then, for every $k \geq 1$, we can write that:

$$b_{k+1} = \frac{1}{2}Tr(Q) - Tr(L_{(k)}) = \frac{1}{2}\sum_{i=1}^{2N} \lambda_i - \sum_{j=1}^{N-1} \mu_j(k). \tag{15.4}$$

This formula plays a crucial role for our work. Indeed, recalling that the eigenvalues of $L^{\pm}$ are constants of motion for the periodic Toda lattice, if we denote with

$$\Lambda = \frac{1}{2}\sum_{i=1}^{2N} \lambda_i,$$

with the formulas 8.4, 15.4 we expressed b_k in such a way that its temporal evolution is totally contained in the term $\sum_{j=1}^{N-1} \mu_j(k)$.
So, these quantities are exactly the ones that we are going to study on the Riemann surface S, looking for an explicit time formulation through Theta functions of S. Finally the temporal evolution of b_k will come from:

$$b_k(t) = \Lambda - \sum_{j=1}^{N-1} \mu_j(k-1, t), \quad k = 1, \dots, N. \tag{15.5}$$

Now, we are going to define the Bloch eigenfunctions and to show some of their properties in connection to what we have done untill now.

Definition 15.5 We call Bloch eigenfunctions $\psi(k)$ the eigenfunctions for the matrix L^+, i.e. such that:

$$\left(L^+\psi\right)(k) = \lambda\psi(k),$$

with the normalization $\psi(1) = 1$. Furthermore the following condition must hold:

$$\begin{pmatrix} \psi(N) \\ \psi(N+1) \end{pmatrix} = \begin{pmatrix} c(N) & s(N) \\ c(N+1) & s(N+1) \end{pmatrix} \begin{pmatrix} \psi(0) \\ 1 \end{pmatrix} = \xi \begin{pmatrix} \psi(0) \\ 1 \end{pmatrix}, \tag{15.6}$$

for ξ some eigenvalue of the matrix M_N already defined in 7.3.

Recalling that $Tr(M_N) = \Delta$, the eigenvalues of M_N have the form:

$$\xi_\pm = \frac{\Delta \pm \sqrt{\Delta^2 - 4}}{2}.$$

Then using the second equation from condition 15.6:

$$\psi(0)c(N+1) + s(N+1) = \xi_\pm,$$

we can rewrite $\psi(0)$ in the following way:

$$\psi(0) = \frac{\Delta \pm \sqrt{\Delta^2 - 4} - 2s(N+1)}{2c(N+1)} = \frac{(c(N) - s(N+1)) \pm \sqrt{\Delta^2 - 4}}{2c(N+1)}.$$

Finally, using the fact that every eigenfunction of L^+ can be written as linear combination of the fundamental solutions c, s, we find a particular expression for the Bloch eigenfunctions:

$$\psi^\pm(k+1) = s(k+1) + \frac{(c(N) - s(N+1)) \pm \sqrt{\Delta^2 - 4}}{2c(N+1)} c(k+1), \tag{15.7}$$

for every $k \geq 0$.
Clearly, the plus or minus sign for the Bloch eigenfunction is chosen accordingly to the sign in front of the squareroot. In this way we have two Bloch eigenfunctions and, if we consider their product $\psi^+(k+1)\psi^-(k+1)$, we find a compact formula that joins them, $c(N+1)$ and $c(k, N+1)$.
In order to prove it, we need the following lemma.

Lemma 15.6 *The following relations hold:*

$$\begin{pmatrix} c(k+N) \\ s(k+N) \end{pmatrix} = \begin{pmatrix} c(N) & c(N+1) \\ s(N) & s(N+1) \end{pmatrix} \begin{pmatrix} c(k) \\ s(k) \end{pmatrix}, \tag{15.8}$$

for every $k \geq 0$.

Proof. The proof comes from the periodicity of a_k, b_k. We proceed by induction over k. For $k = 0$, we have that:

$$\begin{cases} c(N)c(0) + c(N+1)s(0) = c(N) \\ s(N)c(0) + s(N+1)s(0) = s(N) \end{cases},$$

just using the initial conditions of the two fundamental solutions. The same for $k = 1$, indeed:

$$\begin{cases} c(N)c(1) + c(N+1)s(1) = c(N+1) \\ s(N)c(1) + s(N+1)s(1) = s(N+1) \end{cases}.$$

Now supposing that the relations are true for every $j \leq k-1$, we obtain that also for k, using the spectral equation:

$$\begin{aligned} c(k+N) &= a_{k+N-1}^{-1}\left((\lambda - b_{k+N-1})c(k+N-1) - a_{k+N-2}c(k+N-2)\right) \\ &= a_{k+N-1}^{-1}\left((\lambda - b_{k+N-1})(c(N)c(k-1) + c(N+1)s(k-1))\right) + \\ &- a_{k+N-1}^{-1}\left(a_{k+N-2}(c(N)c(k-2) + c(N+1)s(k-2))\right) \\ &= c(N)\left(a_{k-1}^{-1}(\lambda - b_{k-1})c(k-1) - a_{k-2}c(k-2)\right) + \\ &+ c(N+1)\left(a_{k-1}^{-1}(\lambda - b_{k-1})s(k-1) - a_{k-2}s(k-2)\right) \\ &= c(N)c(k) + c(N+1)s(k), \end{aligned}$$

where we use the periodicity of $a_{k-1}, a_{k-2}, b_{k-1}, b_{k-2}$.
In the same way is obtained the formula for $s(k+N)$, and then the thesis hold for every k. □

Finally we can give the formula that matches the three types of eigenfunctions we introduced in this section. It will be used later on, in order to show the analytical properties of Bloch eigenfuntions considered as a function on the Riemann surface S.

Proposition 15.7 *The product of the two Bloch eigenfunctions is such that:*

$$\psi^+(k+1)\psi^-(k+1) = a_0 a_k^{-1} \frac{c(k, N+1)}{c(N+1)}, \tag{15.9}$$

for every $k \geq 0$.

Proof. We note first that for $k = 0$ the thesis is true since $c(0, N+1) = c(N+1)$ and $\psi^+(1) = \psi^-(1) = 1$.

Then for every $k > 0$, we use the expressions obtained in 15.7 and we rewrite the product of the Bloch eigenfunctions:

$$\begin{aligned}
\psi^+(k+1)\psi^-(k+1) &= \left(s(k+1) + \frac{(c(N) - s(N+1)) + \sqrt{\Delta^2 - 4}}{2c(N+1)} c(k+1) \right) \times \\
&\times \left(s(k+1) + \frac{(c(N) - s(N+1)) - \sqrt{\Delta^2 - 4}}{2c(N+1)} c(k+1) \right) . \\
&= s^2(k+1) + s(k+1)c(k+1)\frac{c(N) - s(N+1)}{c(N+1)} + \\
&c^2(k+1)\frac{(c(N) - s(N+1))^2 - (\Delta^2 - 4)}{4c^2(N+1)} = \dots
\end{aligned}$$

Working on the numerator of last term, using the definition of Δ, we can rewrite it as:

$$(c(N) - s(N+1))^2 - (\Delta^2 - 4) = -4c(N)s(N+1) + 4 = 4(-c(N+1)s(N)),$$

where in the last equality we used the well known property of the Wronskian 5.2, namely for $k = N$ $W(N) = a_0$, that implies $c(N)s(N+1) - c(N+1)s(N) = 1$.
Then we can proceed with the chain of equalities started before substituting the quanitity above and making common denominator of the three terms, we have that:

$$\begin{aligned}
\dots &= \frac{s^2(k+1)c(N+1) + s(k+1)c(k+1)(c(N) - s(N+1)) - s(N)c^2(k+1)}{c(N+1)} \\
&= \frac{1}{c(N+1)} \left(s(k+1) \underbrace{(s(k+1)c(N+1) + c(N)c(k+1))}_{=c(k+N+1)} \right) \\
&+ \frac{1}{c(N+1)} \left(-c(k+1) \underbrace{(s(k+1)s(N+1) + s(N)c(k+1))}_{=s(k+N+1)} \right) = \dots
\end{aligned}$$

where we used the relations from the lemma 15.6.
At the end, recalling the expression for the shifted solution $c(k, N+1)$ in function of the fundamental solutions, given in 15.3, we conclude that:

$$\begin{aligned}
\dots &= \frac{1}{c(N+1)} (s(k+1)c(k+N+1) - c(k+1)s(k+N+1)) \\
&= a_0 a_k^{-1} \frac{c(k, N+1)}{c(N+1)},
\end{aligned}$$

and this is exactly our thesis. □

In particular, we will have that one between $\psi^+(k+1)$ and $\psi^-(k+1)$ has some zeros in $\mu_j(k)$ and some poles in μ_j.

16 Explicit integration

In this section we will combined all the results obtained in the previous section in order to arrive at the formulation of b_k through Riemann $\Theta-$functions. Similar formulas have been obtained for the theory of the Korteweg de Vries equation with periodic potential in [**?**].

Theorem 16.1 (Explicit integration for the periodic Toda lattice) *The Flaschka coordinates b_k for the periodic Toda lattice have the following temporal evolution:*

$$b_k(t) = \Lambda^* - \sum_{l=1}^{g} \int_{\alpha_l} \lambda \omega_l(\lambda) + \frac{1}{2}\frac{d}{dt}\left(\log \frac{\Theta\left(tV + (k-1)U + \phi_0, B\right)}{\Theta\left(tV + kU + \phi_0, B\right)} \right), \tag{16.1}$$

for $k = 2, \ldots, N$.
Here $\Lambda^* = \frac{1}{2}\sum_{j=1}^{2g+2} \lambda_j$ with $g \leq N-1$ and U and V are the $g-$dimensional vectors defined in (14.14) *and* (14.17) *respectively, namely :*

$$U = \left(\int_{\infty^+}^{\infty^-} \omega_1, \quad \ldots, \quad \int_{\infty^+}^{\infty^-} \omega_g \right) \tag{16.2}$$

$$V = \left(V_1, \quad \ldots, \quad V_g\right),$$

$$V_l = \frac{1}{2\pi i} \int_{\beta_l} \Omega^{(1)}_{\infty^\pm} \qquad l = 1, \ldots, g, \tag{16.3}$$

where $\omega_1, \ldots, \omega_g$ are the g holomorphic differentials and $\Omega^{(1)}_{\infty^\pm}$ is the normalised second kind differential defined in (14.15) *and ϕ_0 is a constant real vector. The quantity B is the $g \times g$ period matrix* (14.10) *associated to the curve S and $\Theta(z, B)$ is the Riemann Theta-function defined in* (14.22)*.*

Remark 16.2 Note that the vectors U and V are real, and the period matrix B is pure imaginary. It follows that the value of the Theta-function is real. Furthermore, we observe the linear dependence on k and t, in the argument of the Theta functions. We also observe that the integrals $\int_{\alpha_l} \lambda \omega_l(\lambda)$ are real, since the holomorphic differentials ω_l are real on the α-cycles.

Then, as a conseguence of this, we will obtain similar formulations for all the others coordinates of the periodic Toda lattice.
To start with, let reconsider the Riemann surface defined in 14.1 as the zero locus in $\mathbf{C}^2$ of the polynomial $w^2 - R(\lambda)$, where we recall that this polynomial is given by:

$$R(\lambda) = \prod_{j=1}^{2g+2} (\lambda - \lambda_j),$$

with $\lambda_1 < \cdots < \lambda_{2g+2}$, the different eigenvalues of Q.
We also recall that we renamed $\mu_1 < \cdots < \mu_g$ the Dirichlet eigenvalues for which the interval of instability is non-degenerate, i.e. $\lambda_{2j} < \mu_j < \lambda_{2j+1}$ for each $j = 1, \ldots, g$. The same is true for $\mu_j(k)$.
The crucial point, now, is to see these $\mu_j(k)$ as simple poles for a certain meromorphic differential on S, that we are going to define. Starting from this, we arrive at the most useful result of this section, that describes the sum of $\mu_j(k)$ in terms of Riemann Theta functions, for every $k \geq 1$.
The proof of theorem 16.1, namely the derivation of the explicit expression (16.1) is based on the equation (15.5),

$$b_k(t) = \Lambda^* - \sum_{j=1}^{g} \mu_j(k-1,t), \quad k = 1, \ldots, g,$$

and the fact that when $\lambda_{2j} = \mu_j = \lambda_{2j+1}$ these quantities disappear from the above sum, and the following theorem.

Theorem 16.3 *[Formula for the sum of k−shifted eigenvalues] The following relation holds:*

$$\sum_{j=1}^{g} \mu_j(k) = \sum_{l=1}^{g} \int_{\alpha_l} \lambda\omega_l(\lambda) - \frac{1}{2}\frac{d}{dt}\left(\log \frac{\Theta\left(Vt + kU + \phi_0, B\right)}{\Theta\left(Vt + (k+1)U + \phi_0, B\right)}\right) \tag{16.4}$$

for $k \geq 1$, where U and V are the vectors defined in (16.2) and (16.3) *respectively and ϕ_0 is a constant phase.*

In order to prove theorem 16.3 we first need to introduce several quantities.

On S, we can consider the meromorphic function $\psi(k+1)$ which is the extension on S of the Bloch eigenfunctions (15.7) namely $\psi^+(k+1)$ defined in the upper sheet and $\psi^-(k+1)$ in the lower sheet, for every $k \geq 0$.
Then, being $\psi(k+1,\lambda)$ a meromorphic function on S, it is completely defined by its zeros and poles: these informations come from what we already proved in the previous section. Indeed, the formula 15.9 given in the last proposition allow us to say that $\psi(k+1,\lambda)$ has simple zeros at $\mu_j(k)$ and simple poles at μ_j.
Furthermore, if we use the expression 15.7 for the two branches of $\psi(k+1,\lambda)$, we can make some considerations about the infinity points. On the upper sheet, for λ large enough, we have:

$$\psi^+(k+1) = \underbrace{s(k+1)}_{deg=k} + \underbrace{\frac{(c(N) - s(N+1)) + \sqrt{\Delta^2 - 4}}{2c(N+1)}}_{=\simeq 0 - \frac{\lambda}{2} + \frac{\lambda}{2}} \underbrace{c(k+1)}_{deg=k-1},$$

then we conclude that $\psi(k+1,\lambda)$ has a pole of order k at ∞^+. Instead, on the lower sheet:

$$\psi^-(k+1) = \underbrace{s(k+1)}_{deg=k} + \underbrace{\frac{(c(N)-s(N+1)) - \sqrt{\Delta^2-4}}{2c(N+1)}}_{\simeq 0-\frac{\lambda}{2}-\frac{\lambda}{2}} \underbrace{c(k+1)}_{deg=k-1},$$

then $\psi(k+1,\lambda)$ has a zero of order k at ∞^-.
Now we define a meromorphic differential on S using this funcion.

Definition 16.4 We denote by $\omega(k+1,\lambda)$ the following $1-$form:

$$\omega(k+1,\lambda) = \frac{\partial}{\partial\lambda}\log\left(\psi(k+1,\lambda)\right)d\lambda = \psi^{-1}(k+1,\lambda)\frac{\partial\psi(k+1,\lambda)}{\partial\lambda}d\lambda, \tag{16.5}$$

for every $k \geq 0$.

This differential has simple poles at $\mu_j(k)$ and μ_j for every $j = 1,\dots,g$ with residue $+1$ and -1 respectively. It has simple poles at ∞^+ and ∞^- with residue respectively $-k$ and k.
Then, from the study of differentials on Riemann surface done in the paragraph 14, we know that on S every meromorphic differential can be represented as a combination of abelian differentials of second and third kind plus some linear combination of holomorphic differentials. So, we define the following normalized abelian differential of the third kind:

- $$\Omega_{\mu_j(k),\mu_j}, j = 1,\dots,g, \tag{16.6}$$

  that has just simple poles at $\mu_j(k)$ with residue $+1$ and at μ_j with residue -1.

- $$\Omega_{\infty^-,\infty^+} \tag{16.7}$$

  that has just simple poles at ∞^- with residue $+1$ and at ∞^+ with residue -1.

The canonical homology basis is the one described in Figure 4 and the corresponding canonical basis of holomorphic differentials on S, is the one constructed in 14.9 and denoted by ω_j, for $j = 1,\dots,g$,. We conclude that we can represent the differential $\omega(k+1,\lambda)$ as:

$$\omega(k+1,\lambda) = \sum_{j=1}^{g}\Omega_{\mu_j(k),\mu_j} + k\Omega_{\infty^-,\infty^+} + \sum_{j=1}^{g}d_j\omega_j. \tag{16.8}$$

where d_j are some constants to be determined.
Now fixing some $P_0 = (\lambda_0, w_0) \in S$, using (16.5) and (16.8) we can represent the function

$\psi(k+1)$ in the form

$$\psi(k+1,P) = \exp\left(\sum_{j=1}^{g}\int_{P_0}^{P}\Omega_{\mu_j(k),\mu_j} + k\int_{P_0}^{P}\Omega_{\infty^-,\infty^+} + \sum_{j=1}^{g} d_j\int_{P_0}^{P}\omega_j\right).$$

The function $\psi(k+1,P)$ is a single-valued function on S. This means that for every $\alpha-$cycle and every $\beta-$cycle, and for every $P\in S$ the condition:

$$\begin{cases}\psi(k+1,P+\alpha_l) = \exp(2\pi i n_l)\psi(k+1,,P)\\ \psi(k+1,P+\beta_l) = \exp(2\pi i m_l)\psi(k+1,P)\end{cases}, l=1,\dots,g, \tag{16.9}$$

must hold for some constants $n_l, m_l \in \mathbf{Z}$.
The first equality can be rewritten as (we are dropping the dependence on $k+1$):

$$\begin{aligned}\exp(2\pi i n_l)\psi(P) &= \exp\left(\sum_{j=1}^{g}\int_{P_0}^{P+\alpha_l}\Omega_{\mu_j(k),\mu_j} + k\int_{P_0}^{P+\alpha_l}\Omega_{\infty^-,\infty^+} + \sum_{j=1}^{g} d_j\int_{P_0}^{P+\alpha_l}\omega_j\right)\\ &= \psi(P)\exp\left(\sum_{j=1}^{g}\int_{\alpha_l}\Omega_{\mu_j(k),\mu_j} + k\int_{\alpha_l}\Omega_{\infty^-,\infty^+} + \sum_{j=1}^{g} d_j\int_{\alpha_l}\omega_j\right)\\ &= \psi(P)\exp(d_l),\end{aligned}$$

since all the abelian differentials $\Omega_{\mu_j(k),\mu_j}, \Omega_{\infty^-,\infty^+}$ are all normalized.
So we have that the constants d_l are fixed:

$$d_l = 2\pi i n_l, \quad l=1,\dots,g. \tag{16.10}$$

Doing the same steps, the second equality in (16.9) becomes:

$$\begin{aligned}\exp(2\pi i m_l)\psi(P) &= \exp\left(\sum_{j=1}^{g}\int_{P_0}^{P+\beta_l}\Omega_{\mu_j(k),\mu_j} + k\int_{P_0}^{P+\beta_l}\Omega_{\infty^-,\infty^+} + \sum_{j=1}^{g} d_j\int_{P_0}^{P+\beta_l}\omega_j\right)\\ &= \psi(P)\exp\left(\sum_{j=1}^{g}\int_{\beta_l}\Omega_{\mu_j(k),\mu_j} + k\int_{\beta_l}\Omega_{\infty^-,\infty^+} + \sum_{j=1}^{g} d_j\int_{\beta_l}\omega_j\right),\end{aligned}$$

and so we must have that:

$$2\pi i m_l = \sum_{j=1}^{g}\int_{\beta_l}\Omega_{\mu_j(k),\mu_j} + k\int_{\beta_l}\Omega_{\infty^-,\infty^+} + 2\pi i\sum_{j=1}^{g} n_j B_{lj},$$

where we used the condition 16.10, and we recall that B_{lj} is the period matrix of S for the fixed canonical homology basis and the fixed holomorphic differentials basis.
Next we can apply the lemma 14.10 that gives us a way to calculate the $\beta-$periods of normalized Abelian differentials of the third kind through holomorphic differentials.
So the equation above becomes:

$$2\pi i m_l = 2\pi i \sum_{j=1}^{g} \int_{\mu_j}^{\mu_j(k)} \omega_l + 2\pi i k \int_{\infty^+}^{\infty^-} \omega_l + 2\pi i \sum_{j=1}^{g} n_j B_{lj}.$$

Finally, choosing a fixed point $P_0 \in S$, we can split all the integrals as:

$$\int_{\mu_j}^{\mu_j(k)} \omega_l = \int_{P_0}^{\mu_j(k)} \omega_l - \int_{P_0}^{\mu_j} \omega_l,$$

and we conclude that:

$$\sum_{j=1}^{g} \int_{P_0}^{\mu_j(k)} \omega_l = \sum_{j=1}^{g} \int_{P_0}^{\mu_j} \omega_l - k \int_{\infty^+}^{\infty^-} \omega_l - \sum_{j=1}^{g} n_j B_{l,j} + m_l. \tag{16.11}$$

We denote by v the vector formed by the right hand side of the above equation:

$$v_l := \sum_{j=1}^{g} \int_{P_0}^{\mu_j} \omega_l - k \int_{\infty^+}^{\infty^-} \omega_l - \sum_{j=1}^{g} n_j B_{l,j} + m_l, \quad l = 1 \ldots, g. \tag{16.12}$$

Next we we consider the function

$$F(P) = \Theta(\mathcal{A}_{P_0}(P) - e, B) = \Theta\left(\int_{P_0}^{P} \omega - e, B\right),$$

where the complex vector $e = v - K_{P_0}$, with v the vector defined above and K_{P_0} is the vector of Riemann constants. We apply the Riemann Vanishing theorem 14.14 to the function $F(P)$ and conclude that its g zeros $Q_1, \ldots, Q_g$ satisfy the following relation

$$\sum_{j=1}^{g} \int_{P_0}^{Q_j} \omega = e + K_{P_0} = v = \sum_{j=1}^{g} \int_{P_0}^{\mu_j(k)} \omega, \tag{16.13}$$

namely such zeros Q_j coincide with $\mu_j(k)$. This statement is true because the $\mu_j(k)$ are all distinct and therefore the associated divisor of degree g is not special.
Now we are ready to prove theorem 16.3: namely we are solving the Jacobi inversion problem, inverting the formula 16.13. This means find symmetric functions in $\mu_j(k)$ given the vector v.

Proof. (Formula for the sum of the $k-$shifted eigenvalues). We introduce the differential:

$$G(\lambda) = \lambda \frac{\partial}{\partial \lambda} \log F(\lambda) d\lambda.$$

The differential $G(\lambda)$ has the following analytical properties

- it is multivalued on the surface S and single-valued on the Poincaré polygon $\tilde{S}$ obtained from S;
- it has g simple poles at the points $(\mu_1(k), \dots \mu_g(k))$;
- it has simple poles at $\infty^{\pm}$.

Therefore we can evaluate the integral $\int_{\partial \tilde{S}} G$, where $\partial \tilde{S}$ is the boundary of $\tilde{S}$, using the residue theorem. We obtain

$$\frac{1}{2\pi i} \int_{\partial \tilde{S}} G(\lambda) d\lambda = \sum_{j=1}^{g} \text{Res}_{\lambda=\mu_j(k)} G(\lambda) + \text{Res}_{\lambda=\infty^{\pm}} (G(\lambda)) = \sum_{j=1}^{g} \mu_j(k) + \text{Res}_{\lambda=\infty^{\pm}} (G(\lambda)).$$

We conclude that

$$\sum_{j=1}^{g} \mu_j(k) = \frac{1}{2\pi i} \int_{\partial \tilde{S}} G(\lambda) d\lambda - \text{Res}_{\lambda=\infty^{\pm}} (G(\lambda)). \tag{16.14}$$

From the explicit calculation of the r.h.s. of the above expression we will find a representation for the sum of $k-$shifted Dirichlet eigenvalues through the Riemann $\Theta-$functions.

Lemma 16.5 *The following relation is satisfied*

$$\text{Res}_{\lambda=\infty^+} (G(\lambda)) + \text{Res}_{\lambda=\infty^-} (G(\lambda)) = \frac{1}{2} \frac{d}{dt} \left(\log \frac{\Theta\left(Vt + k \int_{\infty^+}^{\infty^-} \omega + \phi_0, B\right)}{\Theta\left(Vt + (k+1) \int_{\infty^+}^{\infty^-} \omega + \phi_0; B\right)} \right), \tag{16.15}$$

where the vector V has been defined in (14.17) *and ϕ_0 is a constant vector.*

Proof. We first start to compute the residue of G at ∞^+, i.e. :

$$\text{Res}_{\lambda=\infty^+} (G(\lambda)) = \text{Res}_{\lambda=\infty^+} \frac{\lambda \sum_{j=1}^{g} \left(\Theta(\int_{P_0}^{\lambda} \omega - (v - K_{P_0}), B)_j \omega_j(\lambda) \right)}{F(P)},$$

where the $\lambda-$logarithmic-derivative is computed recalling that $\Theta(\cdot, B)$ is a vectorial function and $\Theta(\int_{P_0}^{\lambda} \omega - (v - K_{P_0}), B)_j$ is the derivative with respect to the j-argument. First we observe that ∞^+ is a simple pole for G. To do this, we use the explicit form of the canonical basis of holomorhpic differentials ω_j on S, already defined in 14.9. Using the local parameter $\lambda = \frac{1}{s}$ we rewrite the holomorphic differentials as:

$$\begin{aligned}\omega_j(s) &= \frac{\sum_{k=1}^{g} \tilde{A}_{jk}\lambda^{k-1}d\lambda}{\sqrt{R(\lambda)}} \\ &= \frac{\sum_{k=1}^{g} \tilde{A}_{jk}\frac{1}{s^{k-1}}\left(-\frac{ds}{s^2}\right)}{\frac{1}{s^{g+1}}\sqrt{\prod_{i=1}^{2g+2}(1-\lambda_i s)}} \\ &= -\frac{\sum_{k=1}^{g} \tilde{A}_{jk}s^{g-k}ds}{\sqrt{\prod_{i=1}^{2g+2}(1-\lambda_i s)}},\end{aligned}$$

where we recall that the matrix $\tilde{A}$ is the inverse matrix of the $\alpha-$periods matrix for the holomorphic differentials η_k, not still normalized.
In such coordinates, the product $\lambda\omega_j(\lambda)$ becomes:

$$\frac{1}{s}\omega_j(s) = -\frac{\sum_{k=1}^{g} \tilde{A}_{jk}s^{g-k-1}}{\sqrt{\prod_{i=1}^{2g+2}(1-\lambda_i s)}}ds = -\sum_{k=1}^{g} \tilde{A}_{jk}s^{g-k-1}ds(1+o(s)),$$

that for $k = g$ has a simple pole in $s = 0$. The residue at $\lambda = \infty^+$ is then calculated as:

$$\mathrm{Res}_{\lambda=\infty^+}\lambda\omega_j(\lambda) = \mathrm{Res}_{s=0}\frac{1}{s}\omega_j(s) = -\tilde{A}_{j,g},$$

the coefficient of the term s^{-1}.
Applying this result to the residue of G at ∞^+ we obtain:

$$\begin{aligned}\mathrm{Res}_{\lambda=\infty^+}G(\lambda) =& \mathrm{Res}_{\lambda=\infty^+}\frac{\lambda\sum_{j=1}^{g}\left(\Theta(\int_{P_0}^{\lambda}\omega-(v-K_{P_0}),B)_j\omega_j(\lambda)\right)}{F(P)} \\ &= -\frac{\sum_{j=1}^{g}\left(\Theta(\int_{P_0}^{\infty^+}\omega-(v-K_{P_0}),B)_j\tilde{A}_{j,g}\right)}{\Theta(\int_{P_0}^{\infty^+}\omega-(v-K_{P_0}),B)}.\end{aligned} \tag{16.16}$$

Now, with some manipulations, we can rewrite this quantity as a logarithmic derivative with respect to the time of the Theta function. Indeed, we recall that, with our choice of v in (16.12) we have :

$$v - K_{P_0} = \left(\sum_{j=1}^{g}\int_{P_0}^{\mu_j}\omega - k\int_{\infty^+}^{\infty^-}\omega - nB + m - K_{P_0}\right),$$

where the vectors $n = (n_l)_{l=1}^g$, $m = (m_l)_{l=1}^g$ and B is the period matrix of S and $\mu_j = \mu_j(0,t)$.
The crucial point here is that, in this sum only one of the term that appears depends on time, and it is:

$$\sum_{j=1}^{g} \int_{P_0}^{\mu_j(0,t)} \omega, \tag{16.17}$$

that appear in the calculation of the angle variables (12.3). Combining the expression of the angle variables (12.3) with the definition of the period vector V in (14.17) we obtain that

$$\sum_{j=1}^{g} \int_{P_0}^{\mu_j(0,t)} \omega_l = 2t\tilde{A}_{lg} + t_0 = -tV_l + t_0 \tag{16.18}$$

where t_0 is a constant vector and $\tilde{A}$ is the inverse of the period matrix of the non normalized differentials (14.6).

Combining (16.16) and (16.18) we conclude that

$$\frac{d}{dt}\left(\log\Theta\left(\int_{P_0}^{\infty^+} \omega - (v - K_{P_0}), B\right)\right) = 2\mathrm{Res}_{\lambda=\infty^+}\left(G(\lambda)\right), \tag{16.19}$$

In the same way we can calculate the residue at ∞^- obtaining

$$\frac{d}{dt}\left(\log\Theta\left(\int_{P_0}^{\infty^-} \omega - (v - K_{P_0}), B\right)\right) = -2\mathrm{Res}_{\lambda=\infty^-}\left(G(\lambda)\right), \tag{16.20}$$

Combining the above expressions and setting $P_0 = \infty^+$ we obtain (16.15). $\square$

Finally, the last term we have to compute is the integral:

$$\frac{1}{2\pi i}\int_{\partial S} G(\lambda) d\lambda.$$

We have the following lemma.

Lemma 16.6 *The following relation is satisfied:*

$$\frac{1}{2\pi i}\int_{\partial\tilde{S}} G(\lambda) d\lambda = \sum_{l=1}^{g} \int_{\alpha_l} \lambda\omega_l(\lambda). \tag{16.21}$$

Proof. We perform the integral on the boundary of the Poincaré polygon $\tilde{S}$ associated to S, so that we can rewrite the integral as a summation of integrals on all the $\alpha, \beta-$cycles

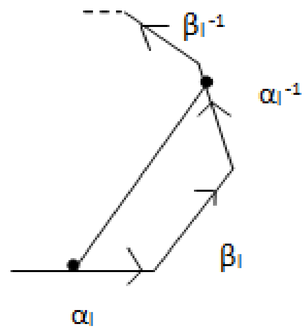

Figure 5: Each point of α_l is connected to the correspondent one of α_l^{-1} through the cycle β_l.

and their correspondent inverse, and then we can use the properties of F (due to Θ) to calculate each one of them. So we consider:

$$\frac{1}{2\pi i}\int_{\partial\tilde{S}} G(\lambda)d\lambda = \frac{1}{2\pi i}\left(\sum_{l=1}^{g}\left(\int_{\beta_l} G(\lambda) + \int_{\alpha_l} G(\lambda) + \int_{\alpha_l^{-1}} G(\lambda) + \int_{\beta_l^{-1}} G(\lambda)\right)\right),$$

and first, for every $l = 1, \ldots, g$, we take the sum of the integrals on α_l and on α_l^{-1}. This can be rewritten as:

$$\int_{\alpha_l} \lambda\frac{\partial \log F(\lambda)}{\partial\lambda}d\lambda - \int_{\alpha_l} \lambda\frac{\partial \log F(\lambda+\beta_l)}{\partial\lambda}d\lambda,$$

since the points on α_l^{-1} are the same of the ones on α_l less then a $\beta-$cycles, as shown in the Figure 5. Then we work on the function F, and we see that:

$$F(\lambda+\beta_l) = \Theta\left(\int_{P_0}^{\lambda+\beta_l}\omega - (v - K_{P_0}), B\right) = \Theta\left(\int_{P_0}^{\lambda}\omega + \underbrace{\int_{\beta_l}\omega}_{=e^l B} - (v - K_{P_0}), B\right).$$

Now we recall the property of the function F, given in 14.27, and we apply it in our case (taking $K = 0$ and $M = e^l$), so we have:

$$F(P+\beta_l) = F(P)\exp\left(-\pi i B_{ll} - 2\pi i\int_{P_0}^{P}\omega_l + 2\pi i e_l\right).$$

Then the integrals sum becomes:

$$\int_{\alpha_l} \lambda \frac{\partial \log F(\lambda)}{\partial \lambda} d\lambda - \int_{\alpha_l} \lambda \frac{\partial \log F(\lambda + \beta_l)}{\partial \lambda} d\lambda = \int_{\alpha_l} \lambda \frac{\partial \log F(\lambda)}{\partial \lambda} d\lambda$$
$$- \int_{\alpha_l} \lambda \frac{\partial \log(F(\lambda))}{\partial \lambda} - \int_{\alpha_l} \lambda \frac{\partial}{\partial \lambda} \left(-\pi i B_{ll} - 2\pi i \int_{P_0}^{\lambda} \omega_l + 2\pi i e_l \right) d\lambda = 2\pi i \int_{\alpha_l} \lambda \omega_l.$$

So this is exactly a constant contribute. It is also the only one, since if we take a sum of integrals on the cycles β_l and on β_l^{-1}, we obtain that this sum is null for every $l = 1, \ldots, g$. Indeed, with the same remarks done before, we can consider every point on the cycle β_l^{-1} as a point of β_l less then the α_l^{-1} cycle. In this way, we have that:

$$\int_{\beta_l} G(\lambda) + \int_{\beta_l^{-1}} G(\lambda) = \int_{\beta_l} \lambda \frac{\partial \log F(\lambda)}{\partial \lambda} - \int_{\beta_l} \lambda \frac{\partial \log F(\lambda - \alpha_l)}{\partial \lambda},$$

but this time, using the property of the funtion F given in 14.26, we have that $F(P - \alpha_l) = F(P)$, and so the quantity above is null. Then we conclude that:

$$\frac{1}{2\pi i} \int_{\partial S} G(\lambda) d\lambda = \sum_{l=1}^{g} \int_{\alpha_l} \lambda \omega_l(\lambda).$$

□

Finally, combining lemma 16.5 and lemma 16.6 we can conclude the proof of theorem 16.3 by re-writing (16.14) in the form (16.4). □

Remark 16.7 The integration of b_1 is obtained by observing that the total momentum is conserved and therefore

$$\sum_{k=1}^{N} b_k = \Lambda = \frac{1}{2} \sum_{j=1}^{2N} \lambda_j,$$

In this way, applying the theorem 16.1 we have that:

$$b_1(t) = \Lambda - \sum_{k=2}^{N} b_k$$
$$= \Lambda - (N-1) \left(\Lambda^* - \sum_{l=1}^{g} \int_{\alpha_l} \lambda \omega_l(\lambda) \right) + \frac{1}{2} \frac{d}{dt} \left(\log \frac{\Theta\left(tV + NU + \phi_0), B\right)}{\Theta\left(tV + U + \phi_0, B\right)} \right).$$

We obtain the integration of the Toda equations in the canonical variables by observing that

$$\dot{q}_k = p_k = -b_k.$$

Therefore, direct integration gives

$$q_k = -\left(\Lambda^* - \sum_{l=1}^{g} \int_{\alpha_l} \lambda\omega_l(\lambda)\right) t + \frac{1}{2}\left(\log \frac{\Theta\left(tV + (k-1)c + \phi_), B\right)}{\Theta\left(tV + kc + \phi_0, B\right)}\right) + K_k,$$

for some integration constant K_k.

References

[1] A. Cannas da Silva, A. Weinstein, Geometric models for noncommutative algebras, Amer. Math. Soc. 1999 (Berkeley Mathematics Lecture Notes).

[2] B. Dubrovin Y. Zhang, Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants. Preprint arXiv:math/0108160.

[3] B. Dubrovin Istituzioni di Fisica Matematica http://people.sissa.it/$\sim$ dubrovin

[4] J.P. Dufour, N.T. Zung, Normal forms of Poisson structures. Lectures on Poisson geometry, 109-169, Geom. Topol. Monogr., 17, Geom. Topol. Publ., Coventry, 2011.

[5] H. Flaschka, The Toda lattice. I. Existence of integrals, Phys. Rev. B 9, (1974) 1924 - 1925.

[6] Flaschka, H.: On the Toda lattice II, Prog. Theor. Phys.**51** (1974), 703-716.

[7] V. L. Ginzburg, Momentum mappings and Poisson cohomology, Internat. J. Math. 7 (1996), no. 3, 329 - 358.

[8] Y. Kodama, K. McLaughlin Explicit integration of the full Symmetric Today Hierarchy and the sorting problem, Lett. Math Physics **37**, (1996) 37-47.

[9] Y. Kosmann-Schwarzbach and F. Magri Lax - Nijenhuis operators for integrable systems, J. Math. Phys. 37, 6173 (1996).

[10] A. Lichnerowicz, Les variètès de Poisson et leurs algébres de Lie associées, J. Differential Geometry 12 (1977), 253-300.

[11] Manakov, S. V.: Complete integrability and stochastization of discrete dynamical systems, Soviet Phys. JETP 40 (1975), 703-716 (Zh. Exp. Teor. Fiz. 67 (1974), 543-555).

[12] Moser, J., Finitely many mass points on the line under the Influence of an exponential potential - An integrable system, in: J. Moser (ed), Dynamical Systems, Theory and Applications, Lecture Notes in Physics 38, Springer-Verlag, Berlin, New York, 1975, pp. 467-497.

[13] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19, (1978) 1156-1162.

[14] H. McKean, Compatible brackets in Hamiltonian mechanics, In: "Important development in soliton theory", 344-354, A.S. Fokas, V.E. Zakharov eds., Springer Series in Nonlinear dynamics, 1993.

[15] A. Nijenhuis, Jacobi-type identities for bilinear differential concomitants of certain ten- sor fields, Indag. Math. 17 (1955), 390- 403.

[16] Symes, W.,The QR algorithm and scattering for the finite non-periodic Toda lattice, Physica D **4** (1982), 275 - 278.

[17] E. Date and S. Tanaka. Analogue of inverse scattering theory for the discrete hill's equation and exact solutions for the periodic toda lattice. Progress of Theoretical Physics, 55(2):457–465, 1976.

[18] M. Kac and P. Van Moerbeke. A complete solution of the periodic toda problem. Proceedings of the National Academy of Sciences of the United States of America, 72(8):2879, 1975.

[19] M. Kac and P. Van Moerbeke. On some periodic toda lattices. Proceedings of the National Academy of Sciences, 72(4):1627–1629, 1975.

[20] P. van Moerbeke. The spectrum of Jacobi matrices. Invent. Math., 37(1):45–81, 1976.