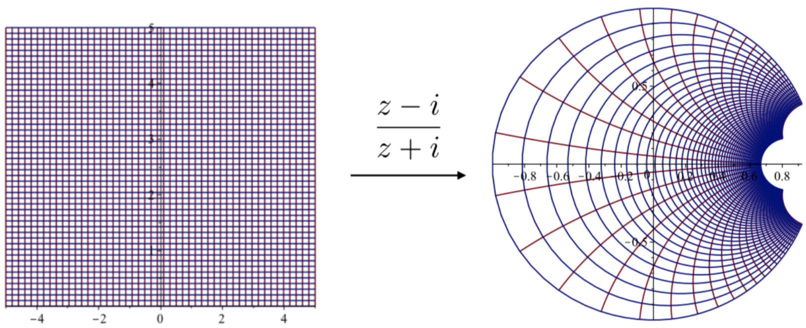
Complex Function Theory

Lecturer: Tamara Grava
10 Homework and they are due on Thursday at noon.
First homework due on Thursday the 26th of September
Homework 4 and 8 are assessed.
The weight of assessed homework is 10%
Time-Table
Monday 9.00-10.00 FRY BLDG LG.20
Tuesday 13.00-14.00 FRY BLDG G.13
Wednesday 11.00-12.00 FRY BLDG LG.20
Office hour Monday 10.00-11.00 Room 2A.02
Clinic with Besfort Shala tx22306@bristol.ac.uk

from second week Friday 4.00-5.00pm FRY BLDG LG.20



December 29, 2024

Contents

Notations

$B_r(z_0)$	*open ball* $B_r(z_0) = \{z \in \mathbb{C} : \|z - z_0\| < r\}$ of radius $r > 0$, centered at $z_0 \in \mathbb{C}$
$\bar{B}_r(z_0)$	*closed ball* $\bar{B}_r(z_0) = \{z \in \mathbb{C} : \|z - z_0\| \leq r\}$ of radius $r > 0$, centered at $z_0 \in \mathbb{C}$
$S_r(z_0)$	*circle* $S_r(z_0) = \{z \in \mathbb{C} : \|z - z_0\| = r\}$ of radius $r > 0$, centered at $z_0 \in \mathbb{C}$
$A_{r,R}(z_0)$	*open annulus* $A_{r,R}(z_0) = \{z \in \mathbb{C} : r < \|z - z_0\| < R\}$ with radii $0 \leq r < R \leq +\infty$, centered at $z_0 \in \mathbb{C}$
$\mathbb{C}_+$	open upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$
$S_r^+(z_0)$	*positively oriented circle* parameterized by $\gamma(t) = z_0 + re^{it}$ $(0 \leq t \leq 2\pi)$
$[a, b]$	*line segment* between points $a, b \in \mathbb{C}$, parameterized by $\gamma(t) = (1-t)a + tb$ $(0 \leq t \leq 1)$

Recommended Texts

1. I. Stewart, D. Tall, *Complex Analysis*, Cambridge University Press, Cambridge, 1983. Ebook from Bristol library

2. Foluso Ladeinde, *Applications of Complex Variables Asymptotics and Integral Transforms*, https://doi-org.bris.idm.oclc.org/10.1515/9783111351179

3. S. Raghavan Narasimhan Yves Nieverg, *Complex analysis in one variable*, Springer–Verlag, New York – Berlin, 2001.

More advanced text-book

4. Jane P. Gilman, Irwin Kra, Rubi E. Rodriguez, *Complex analysis*, Springer 2007. Ebook from Bristol library

5. J.B. Conway, *Functions of one complex variable*, Springer–Verlag, New York – Berlin, 1978.Ebook from Bristol library

[1]

The *Schaum's outline of theory and problems of complex variables* by M. R. Spiegel (McGraw–Hill, London, New–York, 1974) is a good additional source of problems.

WEEK 1

1 Holomorphic functions

1.1 Introduction:complex numbers, complex plane

We define the following structure

$$\mathbb{C} := \{z := (x, y) \mid x, y \in \mathbb{R}\}$$

and endowed $\mathbb{C}$ with the following operations

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$$

with neutral element $(0, 0)$ for addition and $(1, 0)$ for multiplication.

Lemma 1.1. *The set* $(\mathbb{C}, +, \cdot)$ *satisfies the axioms of a* **field** *with multiplication inverse*

$$z = (x, y) \neq (0, 0), \quad z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$$

so that

$$z \cdot z^{-1} = (1, 0).$$

We will use the following short-cut

$$1 := (1, 0), \quad i := (0, 1), \quad 0 := (0, 0)$$

so that we can write any complex number in the form

$$z = 1 \cdot x + iy, \quad x, y \in \mathbb{R}.$$

Note that $i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -(1, 0) = -1$.

We also define the real part and imaginary part of a complex number $z = x + iy$ as follows

$$\mathrm{Re}(z) := x, \quad \mathrm{Im}(z) = y,$$

and the operation of complex conjugation

$$\begin{aligned} - :& \mathbb{C} \to \mathbb{C} \\ & z \to \bar{z} := x - iy. \end{aligned}$$

Then we have the following elementary properties

$$\begin{aligned} &\mathrm{Re}(z) = \frac{z + \bar{z}}{2}, \quad \mathrm{Im}(z) := \frac{z - \bar{z}}{2i} \\ &\overline{z \cdot w} = \bar{z} \cdot \bar{w} \\ &\overline{z \pm w} = \bar{z} \pm \bar{w} \\ &\bar{\bar{z}} = z. \end{aligned}$$

Another important map is the *modulus*

$$|\;|\;:\;\mathbb{C}\to\mathbb{R}_{\geq 0},$$

defined as

$$|z|=\sqrt{z\bar{z}}=\sqrt{x^2+y^2},\quad z=x+iy$$

with the properties for any $z,w\in\mathbb{C}$

$$\begin{aligned}&|z|\geq 0,\text{and } |z|=0 \text{ if and only if } z=0\\&|z\,w|=|z|\,|w|\\&|z+w|\leq|z|+|w|\,.\end{aligned}$$

The *distance* between two complex number $z_1=x_1+iy_1, z_2=x_2+iy_2$ is then given by

$$d(z_1,z_2)=|z_1-z_2|=\sqrt{x_1-x_2)^2+(y_1-y_2)^2}.$$

Polar coordinates. A nonzero complex number $z\in\mathbb{C}$ can be represented by the standard rectangular coordinates $z=x+iy$ or polar coordinates $r>0$ and $\phi\in[0,2\pi)$ by

$$\begin{aligned}z=x+iy=&\sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}}+\frac{iy}{\sqrt{x^2+y^2}}\right)=r(\cos\phi+i\sin\phi),\\r=|z|\quad&\cos\phi=\frac{\mathrm{Re}(z)}{|z|},\;\;\sin\phi=\frac{\mathrm{Im}(z)}{|z|}.\end{aligned}$$

The angle ϕ is denoted as the argument of the complex number, namely $\arg(z)=\phi(\mathrm{mod}2\pi)$.

Exercise 1.2 (Euler's formula)**.** For $z=x+iy$ show that

$$e^{x+iy}=e^x(\cos y+i\sin y).$$

From Euler's formula, it follows that any complex number $z=x+iy$ can be written in the form

$$z=x+iy=|z|(\cos\phi+i\sin\phi)=|z|e^{i\phi},\quad \phi=\arg z.$$

Exercise 1.3. Show that a linear map $A:\mathbb{R}^2\to\mathbb{R}^2$ that is a rotation of an angle ϕ with respect to the origin and a dilatation by ρ can be represented in an orthonormal basis in $\mathbb{R}^2$ in the form

$$A=\rho\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}.$$

Show that such transformation can be represented in the form $z\to\lambda z$ for $z\in\mathbb{C}$ for a suitable λ. Calculate λ.

Solution. A rotation in $\mathbb{R}^2$ by an angle ϕ is given by the matrix $\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$, and a dilation by ρ is simply given by $\rho\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Combining the two operations one gets the matrix A.
Setting $\lambda=\rho(\cos\phi+i\sin\phi)$ then the map $z\to\lambda z$ is equivalent to $\begin{pmatrix}x\\ y\end{pmatrix}\to A\begin{pmatrix}x\\ y\end{pmatrix}$.

Exercise 1.4. Show that

$$z_1\, z_2 = |z_1||z_2|\, e^{i\phi_1 + i\phi_2}, \quad \phi_k = \arg z_k,\ k = 1, 2,$$
$$\arg(z_1\, z_2) = \arg(z_1) + \arg(z_2) \mod 2\pi.$$

Solution.

$$z_k = |z_k|(\cos\phi_k + i\sin\phi_k) = |z_k|e^{i\phi_k},$$

so that

$$\begin{aligned} z_1 z_2 &= |z_1||z_2|e^{i\phi_1}e^{i\phi_2} = |z_1||z_2|(cos\phi_1 + i\sin\phi_1)(cos\phi_2 + i\sin\phi_2) \\ &= |z_1||z_2|[\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 + i(\cos\phi_1\sin\phi_2 + \cos\phi_2\sin\phi_1)] \\ &= |z_1||z_2|(\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)) = |z_1||z_2|e^{i\phi_1+i\phi_2} \end{aligned}$$

so that $\arg(z_1 z_2) = \phi_1 + \phi_2 = \arg z_1 + \arg z_2 \mod 2\pi$.

As a consequence of the above exercise we have that for n integer

$$z^n = (|z|e^{i\phi})^n = |z|^n e^{in\phi} = |z|^n(\cos(n\phi) + i\sin(n\phi)).$$

1.2 Sequences, series, and convergence

The space $(\mathbb{C}, |\,.\,|)$ is a **metric space** which can be identified with $\mathbb{R}^2$ with the euclidean distance.

Definition 1.5. A sequence $\{z_n\}_{n\in\mathbb{N}}$ converges to a point $w \in \mathbb{C}$ if for $\epsilon > 0$ there exist a $n_0 \in \mathbb{N}$ such that $n > n_0$, implies $|z_n - w| < \epsilon$. The limit is denoted as

$$\lim_{n\to\infty} z_n = w\,.$$

A sequence of complex numbers $\{z_n\}_{n\in\mathbb{Z}_{>0}}$ is called Cauchy sequence if for given $\epsilon > 0$ there exist a $n_0 \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon$ for all $n, m > n_0$.

It is not hard to see that $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $\{\mathrm{Re} z_n\}_{n\in\mathbb{N}}$ and $\{\mathrm{Im} z_n\}_{n\in\mathbb{N}}$ are Cauchy sequences. It follows that since $\mathbb{R}^2$ is a **complete** metric space so is $\mathbb{C}$.

Lemma 1.6. *The space* $(\mathbb{C}, |\,|)$ *is a complete metric space, namely every Cauchy sequence of complex numbers converges to a complex number.*

As for real numbers the converge of series is defined according to the converge of partial sum, namely the series

$$\sum_{n=0}^{\infty} z_n$$

converges iff the sequence of partial sums $s_n = \sum\limits_{j=0}^{n} z_j$ converges.

We also recall the Cauchy rule from real analysis: the series of real numbers $\sum_{n=0}^{\infty} b_n$, $b_n \geq 0$ converges if $\limsup_{n\to\infty} |b_n|^{1/n} < 1$ and diverges to $+\infty$ if $\limsup_{n\to\infty} |b_n|^{1/n} > 1$ ($\limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \sup\{|a_n|^{1/n}, |a_{n+1}|^{1/n+1}, \dots\}$). Thefore the series $\sum_{n=0}^{\infty} |z_n|$ converges if

$$\limsup_{n\to\infty} |z_n|^{1/n} < 1.$$

Exercise 1.7. Show that if the series $\sum_{n=0}^{\infty} |z_n|$ converges, then the series $\sum_{n=0}^{\infty} z_n$ converges as well and

$$\left| \sum_{n=0}^{\infty} z_n \right| \leq \sum_{n=0}^{\infty} |z_n|.$$

If $\sum_{n=0}^{\infty} |z_n|$ converges we say that $\sum_{n=0}^{\infty} z_n$ is absolutely convergent.

1.3 Differentiation and holomorphic functions

Definition 1.8. A subset G of $\mathbb{C}$ is called a *domain* if it is open and path connected.

Next we will study the properties of complex functions

$$f : G \to \mathbb{C}.$$

We will use the notation

$$z = x + iy, \quad f(z) = u(x, y) + iv(x, y),$$

where $u(x, y)$ and $v(x, y)$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. We can also write

$$\mathrm{Re}(f) = u, \quad \mathrm{Im}(f) = v.$$

As for real functions we can introduce the concepts of limit and continuity.

Definition 1.9. The function f has a limit c at a point $z_0 \in G$,

$$\lim_{z \to z_0} f(z) = c, \quad z, z_0 \in G,$$

if

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ s.t. |z - z_0| < \delta, \text{ then } \ |f(z) - c| < \epsilon.$$

Let us make an example of a function f that does not have a limit.

Example 1.10. Let

$$f(z) = \begin{cases} \frac{2xy}{x^2+y^2} = \frac{2\frac{y}{x}}{1+\left(\frac{u}{x}\right)^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

The function does not have a limit at $z = 0$. Indeed the approach to zero along the line $\frac{y}{x^2} = k$ for any fixed k will give the limit $\frac{2k}{1+k^2}$.

Definition 1.11. The function f is continuous at a point $z_0 \in G$ if the limit of f at z_0 exists and

$$\lim_{z \to z_0} f(z) = f(z_0)$$

If f is continuous at every point $z \in G$ then f is continuous in G.

The concept of continuity of f is clearly the concept of continuity for any map in a metric space. For example the function $f = u + iv$ is continuous in G if u and v are continuous in G.

Definition 1.12. Let G be a domain in $\mathbb{C}$ and $f : G \to \mathbb{C}$ a complex function on G. The function f is (complex) differentiable at a point $z_0 \in G$ iff the limit

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \qquad (h \neq 0,\ z_0 + h \in G)$$

exists and is finite. The limit is also denoted as $\frac{d}{dz} f(z)|_{z=z_0}$.

The above definition implies that the limit is independent from the direction in the complex plane in which h goes to zero.

Example 1.13. The function $f(z) = z^2$ is differentiable. Indeed

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h)^2 - z^2}{h} = \frac{2zh + h^2}{h} = 2z + h$$

and

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = 2z$$

so that $f(z) = z^2$ is differentiable at any point of $z \in \mathbb{C}$ and $f'(z) = 2z$.

Example 1.14. The function $f(z) = z\bar{z}$ is not differentiable for $z \neq 0$. Indeed

$$\frac{f(z+h) - f(z)}{h} = \frac{(z+h)\overline{(z+h)} - z\bar{z}}{h} = \frac{z\bar{h} + \bar{z}h + h\bar{h}}{h} = z\frac{\bar{h}}{h} + \bar{z} + \bar{h}$$

and

$$\lim_{h\to 0} \frac{\bar{h}}{h}$$

does not exist. Indeed we can write $h = \rho e^{i\phi}$ so that $\frac{\bar{h}}{h} = e^{-2i\phi}$ which clearly does not have a limit for $h \to 0$ since we can choose ϕ in an arbitrary way. So the function $f(z) = z\bar{z}$ is not differentiable for any $z \neq 0$. For $z = 0$ we can see that

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \bar{h} = 0.$$

This means that the function $f(z) = z\bar{z}$ is complex differentiable only at $z = 0$.

1.4 Basic properties of complex differentiation

The concept of complex differentiability shares several properties with real differentiability: it is linear and obeys the product rule, quotient rule, and chain rule. The exercises below describe these basic properties.

Exercise 1.15. Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$, $g : G \to \mathbb{C}$ complex functions on G, $z_0 \in G$, $a \in \mathbb{C}^*$. Suppose that the functions f, g are differentiable at the point z_0. Then the following functions $f + g$, af, fg are differentiable at the point z_0 and

$$(f+g)'(z_0) = f'(z_0)+g'(z_0), \qquad (af)'(z_0) = af'(z_0), \qquad (fg)'(z_0) = f'(z_0)g(z_0)+g'(z_0)f(z_0).$$

Exercise 1.16. Let $G \subseteq \mathbb{C}$ be an open set and $f : G \to \mathbb{C}$, $g : G \to \mathbb{C}$ complex functions on G, $z_0 \in G$. Suppose that the functions f, g are differentiable at the point z_0 and $g'(z_0) \neq 0$. Then f/g is differentiable at the point z_0 and

$$(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - g'(z_0)f(z_0)}{g^2(z_0)}.$$

Exercise 1.17. Prove that if f is differentiable at a point $z_0 \in G$ then f is continuous at z_0.

Exercise 1.18. (Chain Rule) Let $G \subseteq \mathbb{C}$, $B \subseteq \mathbb{C}$ be open sets. Suppose that $f : G \to \mathbb{C}$ is differentiable at $z_0 \in G$, $f(G) \subseteq B$ and $g : B \to \mathbb{C}$ is differentiable at $f(z_0) \in B$. Prove that then the composition $g \circ f : G \to \mathbb{C}$ is differentiable at z and

$$(g(f))'(z_0) = g'(f(z_0))f'(z_0).$$

Next we introduce the definition of holomorphic function.

Definition 1.19. A function $f : G \to \mathbb{C}$ defined on a domain G is called *holomorphic* in G if it has a complex derivative at all points of the domain G. A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is called *entire*.

1.5 Cauchy-Riemann equations

We start observing that the function $f(z) = \bar{z}$ does not have a complex derivative in the sense of definition 1.12. However it is a differentiable function as seen as a function from $\mathbb{R}^2 \to \mathbb{R}^2$. Indeed in this case $(x, y) \to (u = x, v = -y)$ which is clearly a differentiable map from $\mathbb{R}^2 \to \mathbb{R}^2$. Clearly complex differentiation requires extra conditions on the functions $u(x, y)$ and $v(x, y)$.

Theorem 1.20. (Cauchy–Riemann Theorem 1) *If the function $f : G \to \mathbb{C}$, $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, is complex differentiable in $z_0 \in G$ then the functions $u(x, y)$ and $v(x, y)$ have partial derivatives at $(x_0, y_0) \in G$ and satisfy the following Cauchy–Riemann equations:*

$$\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) & = & \frac{\partial v}{\partial y}(x_0, y_0), \\ \frac{\partial u}{\partial y}(x_0, y_0) & = & -\frac{\partial v}{\partial x}(x_0, y_0). \end{cases} \tag{1.1}$$

In this case, the derivative of f at z can be represented by the formula

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0). \tag{1.2}$$

Proof. Assume that the function f is differentiable at $z_0 = x_0 + iy_0$. Then we can chose $h = t \in \mathbb{R}$

$$\begin{aligned} f'(z_0) &= \lim_{t \to 0} \frac{f(z_0 + t) - f(z_0)}{t} \\ &= \lim_{t \to 0} \frac{u(x_0 + t, y_0) - u(x_0, y_0) + i(v(x_0 + t, y_0) - v(x_0, y_0))}{t} = u_x(x_0, y_0) + iv_x(x_0, y_0)\,. \end{aligned}$$

We can also chose $h = it \in i\mathbb{R}$ so that

$$\begin{aligned} f'(z_0) &= \lim_{t\to 0} \frac{f(z_0+it)-f(z_0)}{it} \\ &= \lim_{t\to 0} \frac{u(x_0,y_0+t)-u(x_0,y_0)+i(v(x_0,y_0)-v(x_0,y_0+t))}{it} = -iu_y(x_0,y_0)+v_y(x_0,y_0). \end{aligned}$$

Comparing the above two expressions of $f'(z_0)$ we obtain (1.1). □

1.6 Another perspective on complex differentiation

We start by recalling the concept of Fréchet differentiation.

Definition 1.21. (Differentiability for normed vector spaces). Let V and W be two normed vector spaces. Then $g : V \to W$ is Fréchet differentiable at $x \in V$ iff there exists a bounded linear map $L : V \to W$ such that

$$\lim_{h\to 0} \frac{||g(x+h)-g(x)-Lh||_W}{||h||_V} = 0.$$

This notion of differentiability is called Frechet differentiability and the linear map L is the Jacobian $J(g)$ of g, namely $L_{jk} = J(g)_{jk} = \dfrac{\partial g_k}{\partial x_j}$.

Remark 1.22. Note that g having partial derivatives is actually not enough for it to be Fréchet differentiable. A sufficient condition is that the partial derivatives $\dfrac{\partial g_k}{\partial x_j}$ exist and are continuous in an open neighborhood.

If we consider the holomorphic function $f = u + iv$ as a real map $F : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$, then the Jacobian of F takes the form

$$JF = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

The Cauchy-Riemann (CR) equations imply that

$$JF = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}. \tag{1.3}$$

Recalling that the multiplication of complex numbers $x + iy \to (a + ib)(x + iy) = (ax - by) + i(bx + ay)$ can be lifted to a matrix acting on a 2-dimensional vector

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

see exercise 1.3, we see from the above expression of JF that the linear map $L = JF$ of definition 1.21 is a $\mathbb{C}$-linear map, namely the Jacobian JF is a rotation of an angle $\arg(f'(z))$ followed by a dilatation of $|f'(z)|$. We summarize this consideration in the following theorem

Theorem 1.23. *A function $f : C \to \mathbb{C}$ is complex differentiable if the associated map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is Fréchet differentiable with the further restriction that the linear map $L = JF$ is $\mathbb{C}$-linear, namely of the form* (1.3)*. Furthermore*

$$\det(JF)(z) = u_x^2 + v_x^2 = |f'(z)|^2.$$

Next we want to represent the Cauchy-Riemann equations in a different form. Let $f : G \to \mathbb{C}$ be a a complex valued function and suppose that f_x and f_y exist in G. We introduce the Wirtinger derivatives that are defined as the following linear partial differential operators of first order:

$$\frac{\partial}{\partial \bar{z}} f := \frac{1}{2}\left(\frac{\partial}{\partial x} f + i\frac{\partial}{\partial y} f\right), \quad \frac{\partial}{\partial z} f := \frac{1}{2}\left(\frac{\partial}{\partial x} f - i\frac{\partial}{\partial y} f\right). \tag{1.4}$$

Lemma 1.24. *If the function $f = u + iv$ is holomorphic in a domain $G \subset \mathbb{C}$, the Cauchy-Riemann equations can be written in the form*

$$\frac{\partial}{\partial \bar{z}} f = 0. \tag{1.5}$$

Proof. Let us calculate

$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}(u_x + iu_y) + \frac{i}{2}(v_x + iv_y) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(v_x + u_y).$$

Then the Cauchy-Riemann equations are equivalent to $\frac{\partial}{\partial \bar{z}} f = 0$. ☐

Therefore for a holomorphic function f in a domain G, the Cauchy-Riemann equations can be interpreted as the independence from $\bar{z}$ of the function f.

Example 1.25. Consider the function

$$f(z) = \begin{cases} \dfrac{|z|^4}{z^2} & \text{for } z \in \mathbb{C}\backslash\{0\}, \\ 0 & \text{for } z = 0. \end{cases}$$

Determine where it is differentiable. The function f is continuous in $\mathbb{C}$. The function $f = \frac{|z|^4}{z^2} = \frac{z^2\bar{z}^2}{z^2} = \bar{z}^2$ for $z \neq 0$. Hence applying Lemma 1.24 we have

$$\frac{\partial}{\partial \bar{z}} f = 2\bar{z} \neq 0, \quad \text{for } z \neq 0$$

that shows that f is not differentiable for $z \neq 0$. To calculate the derivative at $z = 0$ we apply the definition and calculate the limit

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\bar{h}^2}{h} = 0$$

which implies that the function f is differentiable at $z = 0$.

The validity of the Cauchy Riemann equations in a point does not guarantee differentiability in that point as the following example shows

Example 1.26. Consider $f:\mathbb{C}\to\mathbb{C}$ defined by

$$f(z)=\begin{cases} 0, & xy=0,\\ 1+i, & xy\neq 0.\end{cases}$$

Then $u=\mathrm{Re}(f)=H, v=\mathrm{Im}(f)=H$ where

$$H(x,y)=\begin{cases} 0, & xy=0,\\ 1, & xy\neq 0.\end{cases}$$

In this case the function $f(z)$ is not complex nor real differentiable in $(0,0)$ since it is not continuous in a neighbourhood of $(0,0)$. However the partial derivatives in $(0,0)$ do exist, indeed

$$\frac{\partial u}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{u(0+\Delta x,0)-u(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0,$$
$$\frac{\partial u}{\partial y}(0,0)=\lim_{\Delta y\to 0}\frac{u(0,0+\Delta y)-u(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0,$$

and the same relations hold for $v_x(0,0)=0$, $v_y(0,0)=0$. Therefore the Cauchy Riemann equations are satisfied at $z=0$, but the function is not continuous or differentiable at $z=0$.

In the following we are going to give conditions for a functions f to be complex differentiable.

Definition 1.27. The function $f:G\to\mathbb{C}$ is a $\mathcal{C}^1$-complex valued function with $f=u+iv$ if u and v have continuous partial derivatives in G or equivalently if u and v are Fréchet differentiable in G.

Definition 1.28. Let $w(z)$ be a complex function in a neighbourhood of the point $z=0$. The function $w(z)=o(z)$, if

$$\lim_{z\to 0}\frac{w(z)}{z}=0.$$

The function $w(z)=O(z)$ if

$$\lim_{z\to 0}\frac{w(z)}{z}=c,\quad c\neq 0.$$

The following theorem shows that first partial derivatives give a linear approximation of a $\mathcal{C}^1$ complex function in a neighbourhood of a point.

Theorem 1.29. *If the function $f:G\to\mathbb{C}$, $f(z)=u(x,y)+iv(x,y)$, $z=x+iy$, is a $\mathcal{C}^1$-complex valued function in G then for any $z_0, z\in G$ with z in a neighbourhood of z_0 we have*

$$f(z)-f(z_0)=(z-z_0)f_z(z_0)+\overline{(z-z_0)}f_{\bar z}(z_0)+o(z-z_0). \tag{1.6}$$

Proof. Let us define $u_0=u(x_0,y_0)$ and $v_0=v(x_0,y_0)$, $\Delta u=u(x,y)-u(x_0,y_0)$, $\Delta z=z-z_0$, $\Delta x=x-x_0$, $\Delta y=y-y_0$. By the hypothesis of existence of partial derivatives we define

$$\epsilon_1(z,z_0):=\frac{\Delta u-u_x(z_0)\Delta x-u_y(z_0)\Delta y}{\Delta z} \tag{1.7}$$

The goal is to show that $\lim_{z\to z_0}\epsilon_1(z,z_0)=0$. We observe that

$$\Delta u=u(x,y)-u(x,y_0)+u(x,y_0)-u(x_0,y_0)$$

and by the continuity of first partial derivatives and the real mean value theorem there are real numbers a and b (depending on z and z_0) such that

$$\Delta u = u_y(x,b)\Delta y + u_x(a,y_0)\Delta x$$

and therefore

$$\epsilon_1(z,z_0) = \frac{(u_x(a,y_0) - u_x(x_0,y_0))\Delta x + (u_y(x,b) - u_y(x_0,y_0))\Delta y}{\Delta z}$$

Evaluating the modulus we obtain

$$|\epsilon_1(z,z_0)| \le |u_x(a,y_0) - u_x(x_0,y_0)| + |u_y(x,b) - u_y(x_0,y_0)|$$

which show, by continuity of the first partial derivative that that $\lim_{z\to z_0} \epsilon_1(z,z_0) = 0$. We conclude that

$$u(x,y) - u(x_0,y_0) = (x-x_0)u_x(x_0,y_0) + (y-y_0)u_y(x_0,y_0) + (z-z_0)\epsilon_1(z,z_0).$$

In a similar way we can obtain

$$v(x,y) - v(x_0,y_0) = (x-x_0)v_x(x_0,y_0) + (y-y_0)v_y(x_0,y_0) + (z-z_0)\epsilon_2(z,z_0)$$

where $\lim_{z\to z_0} \epsilon_2(z,z_0) = 0$. Summing up the first equation with the second one multiplied by i we obtain

$$\begin{aligned} f(z) - f(z_0) = \Delta u + i\Delta v &= \Delta x f_x(z_0) + \Delta y f_y(z_0) + (z-z_0)(\epsilon_1(z,z_0) + i\epsilon_2(z,z_0)) \\ &= \frac{\Delta z + \Delta\bar{z}}{2} f_x(z_0) + \frac{\Delta z - \Delta\bar{z}}{2i} f_y(z_0) + (z-z_0)\epsilon(z,z_0) \\ &= \Delta z f_z(z_0) + \Delta\bar{z} f_{\bar{z}}(z_0) + (z-z_0)\epsilon(z,z_0) \end{aligned}$$

where $\epsilon = \epsilon_1 + i\epsilon_2$ and in the last identity we use (1.4). It then follows that $(z-z_0)\epsilon(z,z_0) = o(z-z_0)$ and we have the statement. □

The above lemma expresses the linear approximation of the complex $\mathcal{C}^1$ function f in a neighborhood of z_0. Note that however the limit of the quantity

$$\frac{f(z) - f(z_0)}{z - z_0} = f_z(z_0) + \frac{\overline{(z-z_0)}}{z-z_0} f_{\bar{z}}(z_0) + \frac{o(z-z_0)}{z-z_0}$$

as $z \to z_0$ does not exist because $\frac{\overline{(z-z_0)}}{z-z_0}$ does not have a limit.

Theorem 1.30. (Cauchy–Riemann Theorem 2) *If the function $f : G \to \mathbb{C}$, $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$, is a $\mathcal{C}^1$-complex valued function in a neighbourhood of a point $z_0 \in G$ and the functions $u(x,y)$ and $v(x,y)$ satisfy the Cauchy–Riemann equations* (1.1) *at $z_0 = x_0 + iy_0$, then f is complex differentiable at z_0.*

Proof. To prove the statement we use Theorem 1.29 and the assumption that u and v satisfies CR equations so that by lemma 1.24 $f_{\bar{z}}(z_0) = 0$. Then using (1.6) we have

$$f(z) - f(z_0) = (z-z_0)f_z(z_0) + \overline{(z-z_0)} f_{\bar{z}}(z_0) + o(z-z_0) = (z-z_0)f_z(z_0) + o(z-z_0). \tag{1.8}$$

Now we observe that

$$f_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv)$$
$$= \frac{1}{2}(u_x + iv_x) + \frac{1}{2}(-iu_y + v_y) \overset{(1.1)}{=} u_x + iv_x.$$

The quantity

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = u_x(x_0, y_0) + iv_x(x_0, y_0) + \frac{o(z - z_0)}{z - z_0}$$

has a limit as $z \to z_0$ and therefore

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = u_x(z_0) + iv_x(z_0) = f_z(z_0).$$

□

Geometric interpretation of Cauchy Riemann equations. Let $f = u + iv$ be an holomorphic function in $\mathbb{C}$ and let us consider the level curve

$$u(x, y) = c_0, \quad v(x, y) = c_1, \quad c_0, c_1 \in \mathbb{C}.$$

Then the vectors $\vec{\nabla} u = (u_x, u_y)$ and $\vec{\nabla} v = (v_x, v_y)$ are orthogonal to each level surface respectively. Now let us consider their scalar product and apply Cauchy Riemann equation

$$(\vec{\nabla} u, \vec{\nabla} v) = u_x v_x + u_y v_y = u_x v_x - v_x u_x = 0\,.$$

Namely $\vec{\nabla} u$ and $\vec{\nabla} v$ are orthogonal to each other. It follows that $\vec{\nabla} u$ is tangent to the level surface $v(x, y) = c_1$ and $\vec{\nabla} v$ is tangent to the level surface $u(x, y) = c_0$, (see figure 1).

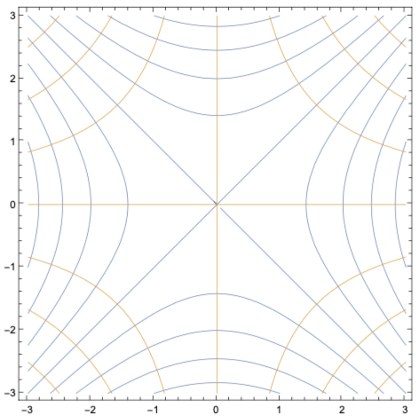


Figure 1: The function $f(z) = z^2$, and the level set of $u(x, y) = x^2 - y^2$ and $v(x, y) = xy$.

Mathematician of this section:

- *Baron Augustin-Louis Cauchy (1789 - 1857), French mathematician*
- *Georg Friedrich Bernhard Riemann (1826 - 1866), German mathematician,*
- *Wilhelm Wirtinger (15 July 1865 – 15 January 1945) Austrian mathematician.*

Week 2

2 Integration of complex functions

2.1 Paths and contours on the plane

Let $[\alpha, \beta] \subset \mathbb{R}$ be a closed interval of real numbers and $G \subset \mathbb{C}$ an open set.

Definition 2.1. A path is a continuous map

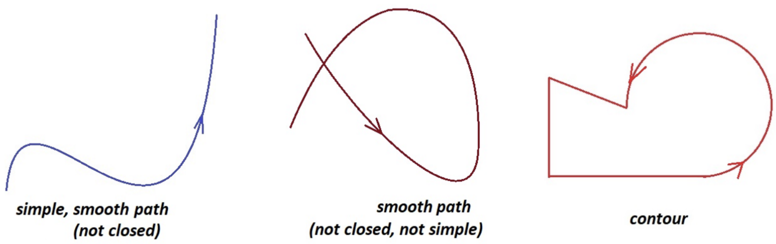
$$\gamma : [\alpha, \beta] \to G.$$

We call $\gamma(\alpha)$ the initial point, and $\gamma(\beta)$ the final point of the path γ. We say that γ is

- closed if $\gamma(\alpha) = \gamma(\beta)$;
- simple if γ does not intersect itself (except at the end points in case of a close path);
- smooth if the derivative $\gamma'(t)$ exists and it is continuous on $[a, b]$:

$$\gamma'(t) = \lim_{\epsilon \to 0} \frac{\gamma(t+\epsilon) + \gamma(t)}{\epsilon}, \quad \gamma'_+(a) = \lim_{t \to a_+} \gamma'(t), \;\; \gamma'_-(b) = \lim_{t \to b_-} \gamma'(t).$$

where $\gamma'_+(a)$ and $\gamma'_-(b)$ are respectively the right and left derivatives.



Example 2.2. (a) Let $a, b \in \mathbb{C}$ be points on the complex plane. Then

$$\gamma(t) = (1-t)a + tb, \qquad t \in [0, 1]$$

is called the line segment from a to b. This is frequently denoted by $[a, b]$. Observe that the path $\tilde{\gamma}(t) = ta + (1-t)b$ $(t \in [0, 1])$ is the same line segment having the "opposite direction" from b to a.
(b) Let $a \in \mathbb{C}$ and $r > 0$. The path

$$\gamma(t) = a + re^{it}, \qquad t \in [0, 2\pi]$$

is called the positively oriented circle of radius r centered at a. This is frequently denoted by $S_r^+(a)$. Observe that, e.g., the path $\gamma_1(t) = a + r(\cos(2\pi t) + i\sin(2\pi t))$ $(t \in [0, 1])$ represents the same "geometrical" circle with the same "positive orientation".

A path

$$\gamma : [\alpha, \beta] \to \mathbb{C}$$

is piecewise smooth, if there exists a partition of $[\alpha, \beta]$

$$t_0 = \alpha \leq t_1 \leq t_2 \leq \cdots \leq t_n = \beta$$

so that γ is smooth on each interval $[t_k, t_{k+1}]$. Denoting by γ_k the $k-th$ smooth path defined on the interval $[t_{k-1}, t_k]$ we can write

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

Definition 2.3. A *curve* is a piecewise smooth and simple path and a *contour* is a piecewise smooth and close path.

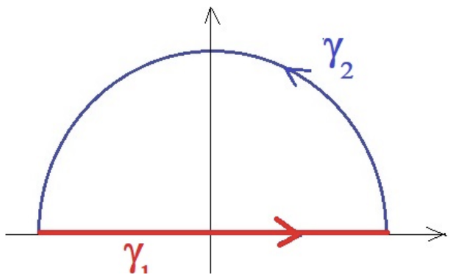
Example 2.4. Let γ_1 and γ_2 be given by

$$\gamma_1(t) = t, \quad t \in [-1, 1]$$

and

$$\gamma_2(t) = e^{\pi i(t-1)}, \quad t \in [1, 2].$$

Then the contour $\gamma = \gamma_1 + \gamma_2$ is the semicircle shown in the figure.



Definition 2.5. Let $\gamma_1 : [\alpha_1, \beta_1] \to G$ be a path. A path $\gamma_2 : [\alpha_2, \beta_2] \to G$ is called a reparametrization of γ_1 iff there is a continuously differentiable function $\phi : [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$ with $\phi'(t) > 0$, $\phi(\alpha_1) = \alpha_2$, $\phi(\beta_1) = \beta_2$, such that $\gamma_1 = \gamma_2 \circ \phi$.

Taking $\phi(t) = \frac{t-\alpha}{\beta-\alpha}$, every path can be reparametrized to the path with parameter interval in $[0, 1]$.

Remark 2.6. The condition $\phi'(t) > 0$ (hence ϕ is increasing) and $\phi(\alpha_1) = \alpha_2$, $\phi(\beta_1) = \beta_2$ means that γ_2 has the same orientation, initial and final points as γ_1.

2.2 Integration of complex functions

Integral over the real segment. Let $F : [\alpha, \beta] \to \mathbb{C}$ be a continuous function, $F(t) = u(t) + iv(t)$. Define the integral of F over the real segment $[\alpha, \beta]$ to be

$$\int_\alpha^\beta F(t)dt := \int_\alpha^\beta u(t)dt + i \int_\alpha^\beta v(t)dt,$$

where integrals of u and v are the usual (real) Riemann integrals. Such defined integral respects the usual properties of the Riemann integral. Note however that the integral of F over $[\alpha, \beta]$ is a complex number!

Now we are ready to define the integral of a complex function along a path.

Definition 2.7. Let $G \subset \mathbb{C}$ be an open set, $f : G \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a smooth path. Define the integral of f along γ to be

$$\int_\gamma f = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt,$$

This is also frequently written $\int_\gamma f(z)dz$.

Example 2.8. Let $\gamma = S_r^+(a)$. Then

$$\int_\gamma (z-a)^n dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{if } n \in \mathbb{Z},\ n \neq -1. \end{cases}$$

Proof. We may write $\gamma(t) = a + re^{it}$ $(t \in [0, 2\pi])$. Then $\gamma'(t) = ire^{it}$. We obtain

$$\begin{aligned} \int_\gamma (z-a)^n dz &= \int_0^{2\pi} (a + re^{it} - a)^n ire^{it} dt \\ &= \int_0^{2\pi} \left(re^{it}\right)^n ire^{it} dt = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt \\ &= \begin{cases} i \int_0^{2\pi} dt & = 2\pi i \quad \text{if } n = -1, \\ ir^{n+1} \left.\frac{e^{i(n+1)t}}{i(n+1)}\right|_0^{2\pi} & = 0 \quad \text{if } n \in \mathbb{Z},\ n \neq -1, \end{cases} \end{aligned}$$

by taking into account the periodicity $e^z = e^{z + 2\pi n i}$ $(z \in \mathbb{C},\ n \in \mathbb{Z})$. □

Exercise 2.9. Let us consider the function $f(z) = \bar{z}$ and compute the integral

$$\int_{\gamma = S_1^+(0)} f(z)dz\,.$$

We have using $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$ that

$$\begin{aligned} \int_{S_1^+(0)} f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt \\ &= \int_0^{2\pi} \overline{e^{it}} ie^{it} dt == \int_0^{2\pi} e^{-it} ie^{it} dt = i \int_0^{2\pi} dt = 2\pi i\,. \end{aligned}$$

Now let us define the integral along a curve. If $\gamma = \gamma_1 + \gamma_2$, then we just need to integrate along γ_1 and γ_2 and then add the result. If the curve γ has the form

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$$

then the integral of f along r γ is

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f + \cdots + \int_{\gamma_n} f\,.$$

Exercise 2.10. Let $f(z) = z^3$ and calculate the integral along the contour $\gamma = \gamma_1 + \gamma_2$ where $\gamma_1(t) = t$ with $t \in [-1, 1]$ and $\gamma_2 = e^{it}$, $t \in [0, \pi]$.

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f.$$

$$\int_{\gamma_1} f = \int_{-1}^{1} t^3 dt = \frac{t^4}{4}\bigg|_{-1}^{1} = 0$$

$$\int_{\gamma_2} f = \int_0^\pi (e^{it})^3 i e^{it} dt = i \int_0^\pi e^{4it} dt = i\, \frac{e^{4it}}{4i}\bigg|_0^\pi = 0$$

so that

$$\int_\gamma f = 0\,.$$

Now we define a very useful concept, the lenght of a path.

Definition 2.11. The length of a smooth path $\gamma : [\alpha, \beta] \to G$ is defined by

$$L(\gamma) = \int_\alpha^\beta |\gamma'(t)| dt,$$

If $\gamma(t) = x(t) + iy(t)$ we have that

$$|\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}.$$

We can interpret $\gamma(t)$ as the position of the particle at time t along the path γ and $|\gamma'(t)|$ as the speed of the particle. Let $z_0 = \gamma(t_0)$, $t_0 \in [\alpha, \beta]$. If $\gamma'(t_0) \neq 0$ then the tangent line at z_0 along γ is in the direction of $\gamma'(t_0)$.

For a curve $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ we have that

$$L(\gamma) = L(\gamma_1) + L(\gamma_2) + \cdots + L(\gamma_n).$$

We will use the length of the smooth piecewise path γ to estimate the contour integral of f along γ. First we need the following exercise

Exercise 2.12. Prove that if $F : [\alpha, \beta] \to \mathbb{C}$ is continuous then

$$\left|\int_\alpha^\beta F(t) dt\right| \leq \int_\alpha^\beta |F(t)| dt. \tag{2.1}$$

Lemma 2.13. (Estimation Lemma) *Let $G \subset \mathbb{C}$ be an open set, $f : G \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a curve. If $\sup_\gamma |f| \leq M$ then*

$$\left|\int_\gamma f\right| \leq ML(\gamma).$$

Proof. Assume for simplicity that γ is a smooth simple path. Then, by using (2.1),

$$\left|\int_\gamma f\right| = \left|\int_\alpha^\beta f(\gamma(t))\gamma'(t) dt\right| \leq \int_\alpha^\beta |f(\gamma(t))||\gamma'(t)| dt \leq M \int_\alpha^\beta |\gamma'(t)| dt = ML(\gamma).$$

□

2.3 General properties of complex integrals

A very simple consequences of the definition of integral of a complex function f along a curve γ:

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt$$

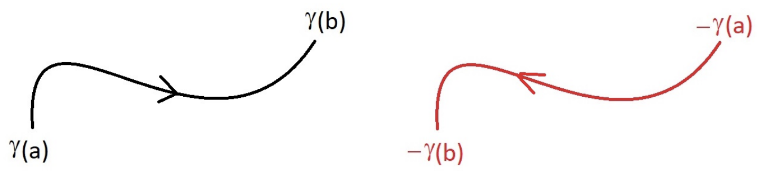
is that the integral is linear

$$\int_\gamma (\lambda f + \mu g) = \lambda \int_\gamma f + \mu \int_\gamma g \qquad (\lambda, \mu \in \mathbb{C});$$

For any path $\gamma : [a,b] \to G$ we may define the reverse path $-\gamma : [a,b] \to G$ by

$$-\gamma(t) = \gamma(a+b-t).$$

This is merely γ traversed in the opposite direction.



Exercise 2.14. Prove that

$$\int_{-\gamma} f = -\int_\gamma f.$$

The above exercise says that the value of the integral *does depend* on the orientation of the curve. We are going to show that it does not depend on the particular choice of parametrization.

Lemma 2.15. *Let $G \subset \mathbb{C}$ be an open set, $f : G \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a smooth path. Let $\tilde{\gamma} : [\tilde{\alpha}, \tilde{\beta}] \to G$ be a reparametrization of γ, namely $\gamma = \tilde{\gamma} \circ \phi$ with $\phi : [\alpha, \beta] \to [\tilde{\alpha}, \tilde{\beta}]$, $\phi(\alpha) = \tilde{\alpha}$, $\phi(\beta) = \tilde{\beta}$ and $\phi' > 0$. Then*

$$\int_\gamma f = \int_{\tilde{\gamma}} f.$$

Proof. By definition,

$$\int_\gamma f = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt.$$

By the real Chain Rule applied separately to real and imaginary parts of γ we obtain

$$\gamma'(t) = (\tilde{\gamma}(\phi(t)))' = \tilde{\gamma}'(\phi(t))\phi'(t).$$

Set $s = \phi(t)$ be a new variable. Then

$$\int_\alpha^\beta f(\gamma(t))\gamma'(t)dt = \int_\alpha^\beta f(\tilde{\gamma}(\phi(t))) \left\{\tilde{\gamma}'(\phi(t))\phi'(t)\right\} dt = \int_{\tilde{\alpha}}^{\tilde{\beta}} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)ds = \int_{\tilde{\gamma}} f\,.$$

The changing of variables in a complex integral over a real segment is justified by applying the usual rule to its real and imaginary parts. □

Lemma 2.16. (Curve Chain Rule) *Let $G \subset \mathbb{C}$ be an open set, $F : G \to \mathbb{C}$ a differentiable function and $\gamma : [\alpha, \beta] \to G$ a smooth path. Then*

$$(F(\gamma(t)))' = F'(\gamma(t))\gamma'(t).$$

Proof. Set $F = u + iv$ and $\gamma = \gamma_1 + i\gamma_2$. Applying the real Chain Rule to real and imaginary parts of $F(\gamma(t))$ and using the Cauchy–Riemann equations we obtain

$$\begin{aligned}(F(\gamma(t)))' &= (u(\gamma_1(t), \gamma_2(t)) + iv(\gamma_1(t), \gamma_2(t)))' \\ &= \frac{\partial u}{\partial x}(\gamma(t))\gamma_1'(t) + \frac{\partial u}{\partial y}(\gamma(t))\gamma_2'(t) + i\frac{\partial v}{\partial x}(\gamma(t))\gamma_1'(t) + i\frac{\partial v}{\partial y}(\gamma(t))\gamma_2'(t) \\ &= \left(\frac{\partial u}{\partial x}(\gamma(t)) - i\frac{\partial u}{\partial y}(\gamma(t))\right)(\gamma_1'(t) + i\gamma_2'(t)) = F'(\gamma(t))\gamma'(t).\end{aligned}$$

□

Theorem 2.17. (Fundamental Theorem) *Let $G \subset \mathbb{C}$ be an open set, $f : G \to \mathbb{C}$ a continuous function and $\gamma : [\alpha, \beta] \to G$ a smooth path with initial point $a = \gamma(\alpha)$ and final point $b = \gamma(\beta)$. Suppose there is a differentiable function $F : G \to \mathbb{C}$ such that*

$$F'(z) = f(z) \quad \text{in } G.$$

Then

$$\int_\gamma f = F(b) - F(a).$$

In particular, if γ is a contour (close curve) then $\int_\gamma f = 0$.

Proof. By the Curve Chain Rule we obtain

$$\int_\gamma f = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt = \int_\alpha^\beta F'(\gamma(t))\gamma'(t)dt = \int_\alpha^\beta (F(\gamma(t)))'dt = F(b) - F(a),$$

which is required. □

Application of the Fundamental Theorem can save a lot of effort in computations.

Example 2.18. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a curve. Then

$$\int_\gamma (z-a)^n dz = \frac{(\gamma(\beta) - a)^{n+1}}{n+1} - \frac{(\gamma(\alpha) - a)^{n+1}}{n+1} \qquad (n \in \mathbb{N} \cup 0),$$

since $(z-a)^n = \left(\frac{(z-a)^{n+1}}{n+1}\right)'$ in $\mathbb{C}$. In particular,

$$\int_{S_r^+(a)} (z-a)^n dz = 0 \qquad (n \in \mathbb{N} \cup 0).$$

Antiderivatives. The Fundamental Theorem suggests the following definition.

Definition 2.19. Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a continuous function. We say that f has an antiderivative in D iff there is a differentiable function $F : D \to \mathbb{C}$ such that

$$F'(z) = f(z) \quad \text{in } D.$$

Example 2.20. Let $n \in \mathbb{N}$, $n \geq 2$. The function $F(z) = z^{-n}$ is an antiderivative of the function $f(z) = -nz^{-n-1}$ in any annulus $A_{r,R}(0) := \{z \in \mathbb{C} : r < |z| < R\}$ $(0 < r < R < +\infty)$.

Exercise 2.21. The function $f(z) = z^{-1}$ has no antiderivatives in an annulus $A_{r,R}(0) := \{z \in \mathbb{C} : r < |z| < R\}$ $(0 < r < R < +\infty)$.

Proof. It follows by the Fundamental Theorem from the fact that $\int_{S_\rho^+(0)} z^{-1}dz = 2\pi i \neq 0$ for any $\rho \in (r, R)$ (see Example 2.8). □

2.4 Converse to the Fundamental Theorem

Let $G \subseteq \mathbb{C}$ be an open set. We recall that G is path connected, namely it consists of one piece, if, for every $a, b \in G$, there is a continuos path $\gamma : [\alpha, \beta] \to G$ with $\gamma(\alpha) = a$ and $\gamma(\beta) = b$. In this case G is called a domain.

Theorem 2.22. *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a differentiable function. If $f'(z) = 0$ on D then f is constant on D.*

Proof. Let $a, b \in D$ and $\gamma : [\alpha, \beta] \to D$ be a smooth path with $\gamma(\alpha) = a$ and $\gamma(\beta) = b$. By the Contour Chain Rule

$$(f(\gamma(t)))' = f'(\gamma(t))\gamma'(t) = 0,$$

since $f'(z) = 0$. Thus writing $f = u + iv$ we conclude that $(u(\gamma(t)))' = 0$ and $(v(\gamma(t)))' = 0$ for all $t \in [\alpha, \beta]$. From Analysis, we know this implies that $u(\gamma(t))$ and $v(\gamma(t))$ are constant functions of t. Comparing the values at $t = a$ and $t = b$ we conclude that $f(a) = f(b)$. □

Remark 2.23. Clearly connectedness is needed. If a set $G \subset \mathbb{C}$ consists of "two pieces" then we could let f to be equal 1 on one "piece" and 0 on another. Thus f' vanishes on G but f would not be constant on G.

Converse to the Fundamental Theorem. According to Fundamental Theorem, if a function has an antiderivative, then the integral does not depend on the path of integration. We are going to show the converse, namely if the integral of a function does not depend on the path of integration then the function has an antiderivative. Firstly we will prove the following statement.

Lemma 2.24. *Let $D \subseteq \mathbb{C}$ be a domain, $z \in D$, $f : D \to \mathbb{C}$ a continuous function. Then*

$$\lim_{h\to 0} \frac{1}{h} \int_{[z,z+h]} f = \lim_{h\to 0} \frac{1}{h} \int_{[z,z+h]} f(w)dw = f(z).$$

Here $[z, z + h]$ is the segment from the point z to the point $z + h$ which is assumed to be contained in D.

Proof. We observe that

$$\int_{[z,z+h]} dw = (z+h) - z = h$$

Since D is open there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(z) \subset D$. Hence for any $h \in B_{\varepsilon_0}(0)$ the line segment $[z, z+h]$ lies in D. Therefore

$$\begin{aligned}\frac{1}{h}\int_{[z,z+h]} f(w)dw - f(z) &= \frac{1}{h}\int_{[z,z+h]} f(w)dw - \frac{f(z)}{h}\int_{[z,z+h]} dw\\ &= \frac{1}{h}\int_{[z,z+h]} (f(w)-f(z))dw.\end{aligned}$$

Then by the Estimation Lemma and continuity of f we obtain

$$|\int_{[z,z+h]} \frac{f(w)-f(z)}{h} dw| \leq \frac{1}{|h|} L([z,z+h]) \sup_{w\in[z,z+h]} |f(w)-f(z)| =$$

$$= \sup_{w\in[z,z+h]} |f(w)-f(z)| \to 0 \quad \text{as} \ \ h \to 0.$$

□

The construction of an antiderivative is described in the following lemma.

Lemma 2.25. *Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ a continuous function. Assume that for any path $\gamma : [\alpha, \beta] \to G$ the integral $\int_\gamma f$ depends only on the initial and final points of γ. Fix a point $a \in D$. Define a function $F_a : D \to \mathbb{C}$ by*

$$F_a(z) := \int_{\gamma_{a,z}} f, \tag{2.2}$$

where $\gamma_{a,z} : [\alpha, \beta] \to D$ is a smooth path with initial point a and final point z. Then

$$F_a'(z) = f(z) \quad \text{in} \quad D,$$

that is F_a is an antiderivative of f in D.

Remark 2.26. The integral in (2.2) correctly defines a (single–valued) function $F_a(z)$ from D to $\mathbb{C}$ because, according to the assumption, $\int_{\gamma_{a,z}} f$ depends only on the initial and final points of the smooth piecewise path $\gamma_{a,z}$.

Proof. Let z be a point in D. Since D is open there exists $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(z) \subset D$. Hence for any $h \in B_{\varepsilon_0}(z)$ the line segment $[z, z+h]$ lies in D. Then

$$F_a(z+h) := \int_{\gamma_{a,z}} f + \int_{[z,z+h]} f.$$

Thus

$$\frac{F_a(z+h) - F_a(z)}{h} = \frac{1}{h}\int_{[z,z+h]} f.$$

Hence by Lemma 2.24 we conclude that

$$F_a'(z) = \lim_{h\to 0} \frac{F_a(z+h) - F_a(z)}{h} = \lim_{h\to 0} \frac{1}{h} \int_{[z,z+h]} f = f(z)$$

in D, as required. □

The theorem below summarizes relations between the existence of antiderivatives and independence of the integral on the path of integration.

Theorem 2.27. (Converse to the Fundamental Theorem) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a continuous function.*

The following statements are equivalent:

(a) f has an antiderivative in D;

(b) For any smooth piecewise path $\gamma : [\alpha, \beta] \to D$, $\int_\gamma f$ depends only on the initial and final points of γ;

(c) For any closed contour $\gamma : [\alpha, \beta] \to D$, then $\int_\gamma f = 0$.

Proof. The implication $(a) \Rightarrow (c)$ is already proved in the Fundamental Theorem.

In order to prove the implication $(c) \Rightarrow (b)$, let $a, b \in D$. Suppose $\gamma_1, \gamma_2 : [\alpha, \beta] \to D$ are two smooth piecewise paths from a to b. Then $\gamma := \gamma_1 - \gamma_2$ is a closed contour. Hence

$$0 = \int_\gamma f = \int_{\gamma_1} - \int_{\gamma_2} f.$$

So $\int_{\gamma_1} f = \int_{\gamma_2} f$, which is (b).

Finally, the implication $(b) \Rightarrow (a)$ has been proved in Lemma 2.25. □

2.5 Complex exponential, logarithm and power

In this section we study the complex exponential, logarithm and power functions. The logarithm and power functions are *multivalued functions*. The logarithmic function is defined as the inverse of the exponential function. We start the section with the properties of the exponential function.

The complex exponential. Let $z = x + iy$. The complex exponential is the function

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y).$$

This function is holomorphic in the whole complex plane. Indeed decomposing it in real and imaginary parts we see that

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y$$

are real differentiable functions in $\mathbb{R}^2$ and the Cauchy-Riemann equations (1.1) are satisfied:

$$u_x = e^x \cos y = v_y, \quad u_y = -e^x \sin y = -v_x, \quad (x, y) \in \mathbb{R}^2.$$

The differential of e^z is then , using (1.1) and (1.2)

$$(e^z)' = e^z.$$

Recall also that

$$e^{z_1+z_2} = e^{z_1}e^{z_2}, \tag{2.3}$$

that can be verified using the addition formulas of trigonometric functions. The modulus of the exponential function is

$$|e^z| = |e^{x+iy}| = |e^x e^{iy}| = |e^x||e^{iy}| = e^x.$$

Finally we observe that the exponential is a periodic function with period $2\pi i$. In fact

$$e^{z+2\pi i} = e^z. \tag{2.4}$$

Therefore the complex exponential is not a one to one function in sharp contrast with the real exponential. From the definition we have

$$e^{z_1} = e^{z_2}, \quad \text{if and only if } z_1 = z_2 + 2\pi i n, \;\; n \in \mathbb{Z}.$$

We can divide the complex plane in horizontal strips of height 2π

$$S_n = \{x + iy \in \mathbb{C} \; s.t. \; (2n-1)\pi \le y < (2n+1)\pi\}, \quad n = 0, \pm 1, \pm 2, \dots.$$

in such a way that in each strip the exponential is one to one. In this case if z_1 and z_2 belong to the same strip S_n, then $e^{z_1} = e^{z_2}$ implies $z_1 = z_2$.

The complex logarithm. We define

$$\log z = \log|z| + i \arg z.$$

Clearly $\arg z$ is a multivalued function because it is defined modulo $2\pi i n$ with $n \in \mathbb{Z}$. Observe that the exponential function is the inverse of the logarithmic function

$$e^{\log(z)} = e^{\log|z| + i\arg(z) + 2\pi i n} = |z|e^{i\arg(z)}e^{2\pi i n} = z,$$

while

$$\log(e^z) = \log|e^z| + i\arg(e^z) = \log|e^x| + i\arg(e^x e^{iy}) = x + i\arg(e^{iy}) = x + iy + 2\pi i\mathbb{Z} = z + 2\pi i\mathbb{Z}.$$

Example 2.28. Let us find the value of $\log(1)$.

$$\log(1) = \log|1| + i\arg(1) = \{z \in \mathbb{C} \, : \, z = 2\pi i n, \quad n \in \mathbb{Z}\}.$$

In a similar way

$$\log(-1) = \log|-1| + i\arg(-1) = 0 + i\pi + 2\pi i n \quad n \in \mathbb{Z}.$$

To make the logarithmic a single valued function we need to make the $\arg(z)$ a single valued function. We introduce the following definition.

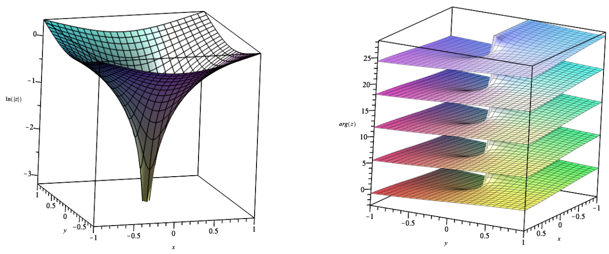


Figure 2: The real (left) and imaginary part (right) of the logarithmic function plotted as a multivalued function.

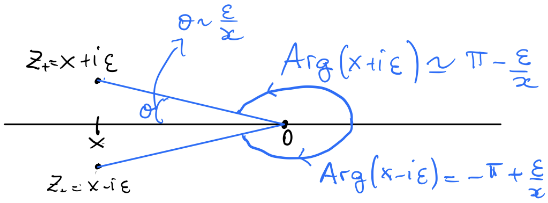
Definition 2.29. We define $\mathrm{Arg}(z)$ the principal value of the argument as

$$\mathrm{Arg}(z) = \theta \longleftrightarrow z = |z|(\cos\theta + i\sin\theta), \quad \theta \in (-\pi, \pi].$$

To make the logarithm a single valued function we restrict $\arg(z)$ to its principal branch $\mathrm{Arg}(z)$. We define the *principal branch* Log of the logarithmic function

$$\mathrm{Log}(z) := \log|z| + i\,\mathrm{Arg}\,z, \quad -\pi < \mathrm{arg} z \leq \pi. \tag{2.5}$$

Clearly $\mathrm{Log}(z)$ is a discontinuous function on the negative real axis. Indeed let $z_\pm = x \pm i\epsilon$, with $x < 0$. As $\epsilon \to 0$ the points $z_\pm$ tend to the same point x on the negative real axis, from the upper and lower complex plane, but the argument $\mathrm{Arg}(z_\pm)$ is as in the figure.



The logarithm gives

$$\lim_{\epsilon\to 0}\mathrm{Log}(z_+) = \lim_{\epsilon\to 0}[\log|x+i\epsilon| + i\mathrm{Arg}(x+i\epsilon)] = \log|x| + i\pi$$
$$\lim_{\epsilon\to 0}\mathrm{Log}(z_-) = \lim_{\epsilon\to 0}[\log|x-i\epsilon| + i\mathrm{Arg}(x-i\epsilon)] = \log|x| - i\pi.$$

Therefore this function is not continuous anywhere on the negative real axis or the origin. The negative real axis is called a *branch cut* for the logarithm and the origin is called a *branch point.*

Definition 2.30.

Instead of chosing $\mathrm{Arg}(z) \in (-\pi, \pi]$ we could actually chose

$$\mathrm{Arg}_\beta(z) \in (-\beta, -\beta + 2\pi], \quad \beta \in [0, 2\pi].$$

This is also a choice of the branch for the multivalued argument function and therefore also for the logarithmic. Notice that

$$\mathrm{Arg} = \mathrm{Arg}_{\beta=\pi}.$$

Then the line $e^{i\beta}[0,\infty)$ is a branch cut for the function Arg_β and the corresponding logarithmic. The branch point $z = 0$ is the same for all the choices of branch cuts.

Exercise 2.31. Show that $\mathrm{Log}(z)$ is a differentiable function in $\mathbb{C}\backslash(-\infty, 0]$.

Solution. The logarithmic is clearly a continuous function in its domain of definition. It can be represented in the form

$$\mathrm{Log}z = \log|z| + i\mathrm{Arg}(z) = u(x,y) + iv(x,y)$$

where now $u(x,y) = \frac{1}{2}\log(x^2+y^2)$ and $v = \mathrm{arctg}\frac{y}{x}$. Clearly u and v are $\mathcal{C}^1$ functions in $\mathbb{C}\backslash(-\infty, 0]$ and

$$u_x = \frac{x}{x^2+y^2}, \quad v_y = \frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}$$

so that $u_x = v_y$ and similarly one obtains $u_y = -v_x$ and the Cauchy-Riemann equations are satisfied. Therefore $\mathrm{Log}(z)$ is a complex differentiable function in its domain of definition The differential is

$$(\mathrm{Log}z)' = u_x + iv_x = \frac{x}{x^2+y^2} - i\frac{\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \frac{x-iy}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}\,.$$

Complex powers. With the logarithmic, we are able to define complex powers of complex numbers. Let α be a nonzero complex number. For $z \neq 0$ we define the α-th power z^α as

$$z^\alpha = e^{\alpha\log z} = e^{\alpha\log|z| + i\alpha\mathrm{arg}(z)}.$$

The multivaluedness of the argument $\arg(z)$ means that generically there will be an infinite number of values of z^α and the value of each branch of the logarithm gives a branch of the complex power. To make more manifest the multivaluedness we write

$$z^\alpha = e^{\alpha\mathrm{Log}(z) + 2\pi i\alpha n}, \quad n \in \mathbb{Z}$$

where $\mathrm{Log}(z)$ has been defined in (2.5). Depending on the value of α there is one, finitely many or infinitely many values of $e^{2\pi i\alpha n}$. In particular when $\alpha \in \mathbb{Z}$ such definition of complex power agrees with the usual integer powers of z because in this case $e^{2\pi i\alpha n} = 1$.

When $\alpha \notin \mathbb{Z}$ to make the complex power a single valued function we need to choose a branch and we define the *principal* branch to be

$$z^\alpha = e^{\alpha\mathrm{Log}(z)}, \quad -\pi < \arg(z) \leq \pi.$$

As for the logarithmic, the function z^α for α not an integer is not continuos on the negative real axis. Defining $z_\pm = x \pm i\epsilon$, with $x > 0$, we have

$$\lim_{\epsilon\to 0} z_+^\alpha = \lim_{\epsilon\to 0} e^{\alpha\mathrm{Log}(z_+)} = |x|^\alpha e^{i\pi\alpha}$$

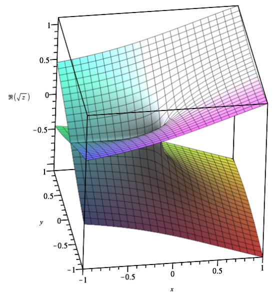


Figure 3: The real part of the square root function plotted as a multivalued function. The first sheet corresponds to the choice $-\pi < \arg(z) \leq \pi$.

and

$$\lim_{\epsilon \to 0} z_-^\alpha = \lim_{\epsilon \to 0} e^{\alpha \mathrm{Log}(z_-)} = |x|^\alpha e^{-\pi i \alpha}.$$

In particular for $\alpha = 1/2$ the values of the function $\sqrt{z}$ on the two side of the branch cut $(-\infty, 0]$ differ by a sign.

The power z^α is holomorphic in the same domain where Log is holomorphic and by the chain rule we have

$$\frac{d}{dz} z^\alpha = \frac{d}{dz} e^{\alpha \mathrm{Log}(z)} = e^{\alpha \mathrm{Log}(z)} \frac{\alpha}{z} = z^\alpha \frac{\alpha}{z}, \quad z \in \mathbb{C} \backslash (-\infty, 0].$$

Exercise 2.32. Calculate a branch cut of the function $f(z) = \sqrt{z(z-1)}$.

Complexity of functions. Let us recall the complexity of numbers: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. To pass from the rational numbers $\mathbb{Q}$ to the real, one has to consider irrational numbers: these are the transcendental numbers like π and the algebraic numbers that are roots of polynomial equations with integer coefficients, like $x^2 - 2 = 0$ that gives $x = \pm\sqrt{2}$. Note that not all algebraic numbers are irrational (for example the roots of $x^2 - 4 = 0$ are not irrational numbers). An analogous classification roughly holds for complex functions:

- polynomials $p_n(z) = a_n z^2 + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_j \in \mathbb{C}$;
- rational functions $f(z) = \dfrac{p_n(z)}{q_m(z)}$, where p_n and q_m are polynomials;
- algebraic functions $f(z)$ that are solutions of a polynomial equation in two variables $P(z, w) = 0$ with complex coefficients. For example $f(z) = \sqrt{z}$ is the solution of $w^2 - z = 0$ that gives $w = \pm\sqrt{z}$. Algebraic functions are usually multivalued functions.
- Transcendental functions are solutions to linear differential equations in the complex domain. For example $f(z) = e^z$ is the solution of the equation $f'(z) = f(z)$. Note that the inverse of a transcendental function is transcendental as the logarithmic, and the sum of two transcendental functions like $e^z + e^{-z} = 2\cosh(z)$ is also transcendental.
- New transcendental functions are solutions to *nonlinear* differential equations in the complex domain.

Week 3

3 Cauchy's theorems

In this section we continue to study the integral of a complex differentiable function and in particular the property of independence of the integral from the path of integration. We first need to introduce the concept of homotopy.

3.1 Homotopy

Definition 3.1. Let $D \subseteq \mathbb{C}$ be a domain. Two closed paths $\gamma_0, \gamma_1 : [\alpha, \beta] \to D$ are called homotopic in D if there exists a continuous map $H = H(t,s) : [\alpha,\beta] \times [0,1] \to D$ such that

(a) $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$ $(t \in [\alpha,\beta])$;

(b) $H(\alpha,s) = H(\beta,s)$ $(s \in [0,1])$.

We write $\gamma_0 \sim \gamma_1$ in D.

A similar definition holds when the curves are open.

Definition 3.2. Let $D \subseteq \mathbb{C}$ be a domain. Two paths $\gamma_0, \gamma_1 : [\alpha,\beta] \to D$ with

$$a = \gamma_0(\alpha) = \gamma_1(\alpha), \quad b = \gamma_0(\beta) = \gamma_1(\beta)$$

are called homotopic with fixed end-points if there exists a continuous map $H = H(t,s) : [\alpha,\beta] \times [0,1] \to D$ such that

(a) $H(t,0) = \gamma_0(t)$ and $H(t,1) = \gamma_1(t)$ $(t \in [\alpha,\beta])$;

(b) $H(\alpha,s) = a, \quad H(\beta,s) = b$ $(s \in [0,1])$.

The idea of these definitions is that the path γ_0 can be continuously deformed to γ_1 without passing outside of the domain D. Or, in other words, as s changes from 0 to 1, we have a family of paths that continuously ("without cutting") change its shape from γ_0 to γ_1 without leaving D. Note that the paths can be self–intersecting!

Exercise 3.3. Prove that the homotopy of two closed paths in a domain D is an equivalence relation.

Definition 3.4. A set $G \subset \mathbb{C}$ is called convex iff, for every $a, b \in G$, the line segment $[a,b]$ is contained in G. Clearly every open convex set is connected.

Definition 3.5. We say that a set $D \subseteq \mathbb{C}$ is a starshaped domain iff D is open and there exists a point $a \in D$, called a star–center of D, such that for any $z \in D$ one has that the segment $[a,z] \subset D$.

Exercise 3.6. Let $D \subset \mathbb{C}$ be a convex domain. Then any two closed paths $\gamma_0, \gamma_1 : [\alpha,\beta] \to D$ are homotopic in D.

Proof. Define the linear homotopy between γ_0 and γ_1 by

$$H(t,s) := (1-s)\gamma_0(t) + s\gamma_1(t).$$

Since D is convex, H maps $[\alpha, \beta] \times [0, 1]$ into D. Since $\gamma_0, \gamma_1 : [\alpha, \beta] \to D$ are continuous, so is H. Clearly $H(t, 0) = \gamma_0(t)$, $H(t, 1) = \gamma_1(t)$ $(t \in [\alpha, \beta])$ and $H(\alpha, s) = H(\beta, s)$ as $s \in [0, 1]$. Thus $\gamma_0 \sim \gamma_1$ in D. □

Exercise 3.7. Prove that:
(a) a starshaped domain is a domain (i.e. prove that any starshaped domain is connected);
(b) $\mathbb{C}$ is a starshaped domain;
(c) a convex domain is a starshaped domain (cf. Definition 3.4);
(d) $\mathbb{C} \setminus (-\infty, 0]$ is a starshaped domain with a star–center 1;
(e) $\mathbb{C} \setminus \{0\}$ is not a starshaped domain.

Exercise 3.8. Let $D \subset \mathbb{C}$ be a domain. Let $a, b \in \mathbb{C}$ and $r, R > 0$ be such that:

(a) $B_r(a) \subset B_R(b)$ (but not necessarily $B_r(a) \subset D$);

(b) $A := \{z \in \mathbb{C} : |z - b| \leq R, |z - a| \geq r\} \subset D$.

Prove that the positively oriented circles $S_R^+(b)$ and $S_r^+(a)$ are homotopic in D.

Hint. Consider the linear homotopy between $S_R^+(b)$ and $S_r^+(a)$.

Let $D \subseteq \mathbb{C}$ be a domain and $a \in D$ a fixed point. Denote by $\gamma_a : [\alpha, \beta] \to D$ the constant path

$$\gamma_a(t) := \{a\}.$$

Obviously γ_a is a closed smooth path in D with $|\gamma'(t)| = 0$.

Definition 3.9. We say that a closed path $\gamma : [\alpha, \beta] \to D$ is homotopic to a point in D if there is a point $a \in D$ such that $\gamma \sim \gamma_a$ in D.

Definition 3.10. A domain $D \subseteq \mathbb{C}$ is called simply connected if every closed path $\gamma : [\alpha, \beta] \to D$ is homotopic to a point in D.

Intuitively *simply connected* means that the domain "has no holes". An equivalent definition of simply connected domain, is to request that any two curves with the same end points are homotopic.

Exercise 3.11. Let $D \subseteq \mathbb{C}$ be a domain and $\gamma : [\alpha, \beta] \to D$ a closed path. Let $a \in D$ be a given point. Assume that $\gamma \sim \gamma_a$ in D. Prove that $\gamma \sim \gamma_b$ in D for any other point $b \in D$.

Exercise 3.12. Prove that:
(a) a convex domain is simply connected;
(b) a starshaped domain is simply connected.

According to Theorem 2.27 a *continuous* function f has an antiderivative iff $\int_\gamma f = 0$ for any closed contour γ. Cauchy's Theorems put forward conditions under which $\int_\gamma f = 0$ when there is no initial reason to have an antiderivative. Instead, *differentiability* of f will be assumed.

We first recall Stoke theorem from real analysis.

Theorem 3.13. Stoke theorem. *Let $P(x, y)$ and $Q(x, y)$ be continuously differentiable functions in a connected domain $G \subset \mathbb{R}^2$. Let $D \subset G$ be a simply connected domain with the closure $\overline{D} \subset G$ compact and bounded and such that the boundary $\partial D := \overline{D} \backslash D$ consists of* *piece-wise smooth paths. Let the orientation of ∂D be such that the domain D remains on the left while moving along the oriented boundary. Then*

$$\int_{\partial D}(P(x,y)dx + Q(x,y)dy) = \int_D (Q_x - P_y)dxdy. \tag{3.1}$$

Proof of Stokes theorem. We give a sketch of the proof by restricting to the case in which the domain D is a rectangle. The general proof consists in showing that the domain D can be approximated by a covering of rectangles and then by applying the cancellation of integrals over common boundaries oppositely oriented. See MARSDEN, pp.99–113 for the full proof.

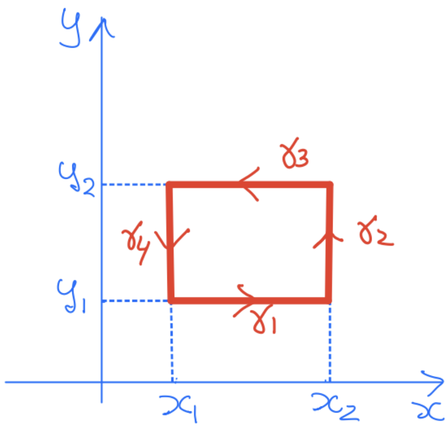


Figure 4: The rectangular domain D.

Let D be a rectangle like in the figure 4 with boundary $\partial D = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ and . Then

$$\begin{aligned}
\int_D (Q_x - P_y)dxdy &= \int_{y_1}^{y_2}\left(\int_{x_1}^{x_2} Q_x(x,y)dx\right)dy - \int_{x_1}^{x_2} dx \int_{y_1}^{y_2} P_y(x,y)dy \\
&= \int_{y_1}^{y_2}[Q(x_2,y) - Q(x_1,y)]dy - \int_{x_1}^{x_2} dx[P(x,y_2) - P(x,y_1)] \\
&= \int_{\gamma_2+\gamma_4} Qdy + \int_{\gamma_1+\gamma_3} Pdx \\
&= \int_{\sum_{j=1}^4 \gamma_j}(Pdx + Qdy) = \int_{\partial D}(Pdx + Qdy)
\end{aligned}$$

where in the identity in the last line we use the fact that Pdx is constant along $\gamma_2 + \gamma_4$ and Qdy is constant along $\gamma_1 + \gamma_3$. □

Now let us apply Stoke theorem to the complex differentiable function $f = u + iv$ in the same setting as Stoke theorem.

Corollary 3.14. *Let $f : G \to \mathbb{C}$ be a complex function, $f = u + iv$ and u, v continuous differentiable functions, $D \subset G$ a simply connected domain with piece-wise smooth oriented boundary. Then*

$$\int_{\partial D} f dz = 2i \int_D \frac{\partial f}{\partial \bar{z}} dxdy \tag{3.2}$$

where we recall that $\dfrac{\partial}{\partial \bar z} := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$.

Proof. The above formula is a direct application of Stokes theorem:

$$\begin{aligned}
\int_{\partial D} f dz &= \int_{\partial D} (u+iv)(dx+idy) = \int_{\partial D} (udx - vdy) + i\int_{\partial D} (vdx + udy) \\
&\overset{(3.1)}{=} -\int_D (v_x + u_y) dxdy + i\int_D (u_x - v_y) dxdy \\
&= i\int_D \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) udxdy - i\int_D \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) vdxdy \\
&= 2i\int_D \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) dxdy \\
&= 2i\int_D \frac{\partial f}{\partial \bar z} dxdy.
\end{aligned}$$

□

We are now ready to prove a weak form of Cauchy theorem.

Theorem 3.15. (Cauchy Theorem) *Let* $G \subseteq \mathbb{C}$ *be a simply connected domain and* $f : G \to \mathbb{C}$ *a complex differentiable function. Let* $\gamma : [\alpha, \beta] \to G$ *be a (closed) contour. Then*

$$\int_\gamma f dz = 0.$$

Proof. We prove a weak version of the theorem by assuming that f is continuously complex differentiable. Then the proof is a simple application of Stokes theorem, in particular we use the equation (3.2). For any closed path γ in G we define by $\mathrm{Int}(\gamma) \subset G$ the domain whose boundary is γ. Then by applying (3.2) we obtain

$$\int_\gamma f(z)dz = -\int_{\mathrm{Int}(\gamma)} f_{\bar z} dxdy = 0$$

where in the last identity we use the Cauchy Riemann equations (1.1) written in the compact form $f_{\bar z} = 0$. □

Remark 3.16. The Cauchy theorem under the assumption that f is continuously differentiable was proved by Cauchy himself, while it was proved by Goursat assuming only differentiability of f.

Exercise 3.17. Prove that $\int_{S_1(0)} e^{z^2} dz = 0$.

An important consequence of Stokes theorem is the invariance of the integrals under homotopic change of the contour.

Theorem 3.18. (Deformation Theorem) *Let* $G \subseteq \mathbb{C}$ *be a domain and* $f : G \to \mathbb{C}$ *a* $\mathcal{C}^1$ *complex differentiable function. Let* $\gamma_0, \gamma_1 : [\alpha, \beta] \to G$ *be two homotopic closed contours in* G *or two homotopic curves in* G *such that* $\gamma_0(\alpha) = \gamma_1(\alpha) = a$ *and* $\gamma_0(\beta) = \gamma_1(\beta) = b$. *Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz. \tag{3.3}$$

Proof. We consider the case in which γ_0 and γ_1 are curves (open piece-wise smooth paths). The case in which γ_0 and γ_1 are contours is left as an exercise (not easy). Let us assume that γ_0 and γ_1 do not intersect except at the end points. Since the curves γ_0 and γ_1 are homotopic there is a continuous map $H : [\alpha, \beta] \times [0, 1] \to G$ that maps γ_0 into γ_1. Let $\overline{D}$ be the image of the rectangle $[\alpha, \beta] \times [0, 1]$ through H. The domain $\overline{D} \subset G$ is compact since H is continuous and $[\alpha, \beta] \times [0, 1]$ is compact and the boundary of $\overline{D}$ is $\gamma_0 - \gamma_1$, (see figure 5). We conclude that D does not have holes and it is simply connected.

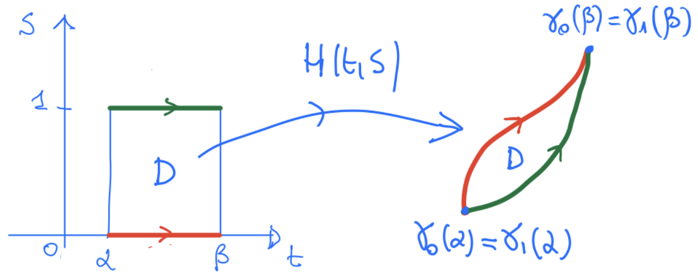


Figure 5: The contours γ_0 and γ_1 and the domain D.

Then by Stokes theorem and Cauchy theorem

$$\int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = \int_{\gamma_0-\gamma_1} f(z)dz = \int_{\partial D} f(z)dz \overset{\text{Cauchy Th}}{=} \int_D f_{\bar{z}} dxdy = 0,$$

because when f is holomorphic in G, then $f_{\bar{z}}$ by Cauchy Riemann equations (1.1).

□

Exercise 3.19. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a close smooth contour encircling the point $z = 0$. Show that

$$\int_\gamma \frac{dz}{z} = 2\pi i.$$

Exercise 3.20. Prove that $\mathbb{C} \setminus \{0\}$ is not simply connected.

Remark 3.21. When we consider a complex holomorphic function $f : D \to \mathbb{C}$ and D simply connected, by Cauchy Theorem 3.15

$$\int_\gamma f(z)dz = 0$$

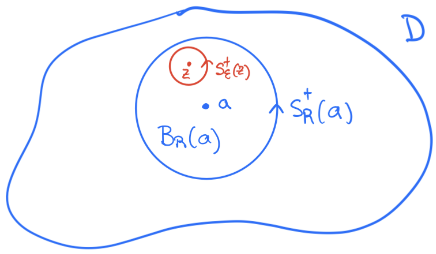
for any closed contour γ in D. By the converse of the Fundamental theorem 2.27, it also follows that f has an antiderivative F in D. To arrive to this conclusion it is essential that D is simply connected.

3.2 Cauchy Integral Formula

The Cauchy Integral Formula shows that the values of a differentiable function on a disc is determined completely by its values on the boundary circle of the disc

Theorem 3.22. (Cauchy Integral Formula) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ complex differentiable function. Let $\bar{B}_R(a) \subset D$. Then for any $z \in B_R(a)$ one has*

$$f(z) = \frac{1}{2\pi i} \int_{S_R^+(a)} \frac{f(w)}{w-z} dw. \tag{3.4}$$



Proof. Observe that the function

$$g(w) := \frac{f(w)}{w-z}$$

is differentiable in $D \setminus \{z\}$. Choose $0 < \varepsilon < R - |z-a|$. By the Deformation Theorem and Exercise 3.8

$$\int_{S_\varepsilon^+(z)} g = \int_{S_R^+(a)} g \Leftrightarrow \int_{S_\varepsilon^+(z)} \frac{f(w)}{w-z} dw = \int_{S_R^+(a)} \frac{f(w)}{w-z} dw.$$

By Example 2.8 we know that

$$\int_{S_\varepsilon^+(z)} \frac{f(w)-f(z)}{w-z} dw = \int_{S_R^+(a)} \frac{f(w)}{w-z} dw - 2\pi i f(z).$$

By the Estimation Lemma we obtain

$$\begin{aligned} |\int_{S_\varepsilon^+(z)} \frac{f(w)-f(z)}{w-z} dw| &\leq L(S_\varepsilon^+(z)) \sup_{w \in S_\varepsilon(z)} \frac{|f(w)-f(z)|}{|w-z|} \\ &= 2\pi\epsilon \sup_{w \in S_\varepsilon(z)} \frac{|f(w)-f(z)|}{\epsilon} \to 0 \quad \text{as } \epsilon \to 0. \end{aligned}$$

We conclude that

$$0 = \lim_{\epsilon \to 0} \int_{S_\varepsilon^+(z)} \frac{f(w)-f(z)}{w-z} dw = \int_{S_R^+(a)} \frac{f(w)}{w-z} dw - 2\pi i f(z).$$

which gives the statement of the theorem. □

Remark 3.23. Observe that if $z \in D \setminus \bar{B}_R(a)$ then

$$\frac{1}{2\pi i} \int_{S_R^+(a)} \frac{f(w)}{w-z} dw = 0.$$

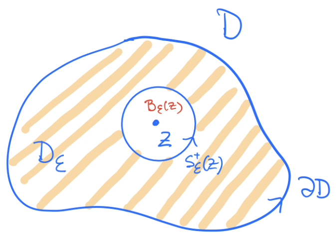
This follows from the Cauchy Theorem because $\frac{f(w)}{w-z}$ is differentiable in $B_{R+\varepsilon}(a)$, for sufficiently small $\varepsilon > 0$.

What can be said If the function f is not holomorphic? The answer is the following generalized Cauchy integral formula

Theorem 3.24. (Cauchy-Pompeiu integral formula). *Let $D \subset G \subset \mathbb{C}$ be domains and D simply connected and bounded by a simple close curve, $f : G \to \mathbb{C}$ and $f = u + iv$ with u, v continuous differentiable functions in G. Then we have the generalized Cauchy formula for $z \in D$*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\iint_{D}\frac{\frac{\partial f}{\partial \overline{\zeta}}}{\zeta - z}dA(\zeta) \tag{3.5}$$

where $dA(\zeta)$ is the area measure in the coordinate ζ.



Proof. For $\epsilon > 0$ we consider a small disk around $B_\epsilon(z) \subset D$ and the region $D_\epsilon = D \backslash B_\epsilon(z)$ with boundary $\partial D \cup S^-_\epsilon(z)$. By Stoke theorem formula (3.2) we consider the integral in the region D_ϵ. This is possible because even if D_ϵ is not simply connected we can split it in two simply connected regions. Then we have

$$\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta - \int_{S^+_\epsilon(z)}\frac{f(\zeta)}{\zeta - z} = 2i\iint_{D_\epsilon}\frac{\frac{\partial f}{\partial \overline{\zeta}}}{\zeta - z}dA(\zeta)$$

For the integral along $S^+_\epsilon(z)$ we make the change $\zeta - z = \epsilon e^{i\theta}$

$$\int_{S^+_\epsilon(z)}\frac{f(\zeta)}{\zeta - z} = \int_0^{2\pi}\frac{f(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}i\epsilon e^{i\theta}d\theta = i\int_0^{2\pi} f(z+\epsilon e^{i\theta})e^{i\theta}d\theta \to 2\pi i f(z)$$

and $\epsilon \to 0$ by the continuity of f. Regarding the area integral, we verify that the integral over the disk $B_\epsilon(z)$ goes to zero:

$$\left|\iint_{B_\epsilon(z)}\frac{\frac{\partial f}{\partial \overline{\zeta}}}{\zeta - z}dA(\zeta)\right| = \left|\int_0^{\epsilon}\int_0^{2\pi}\frac{\frac{\partial f}{\partial \overline{\zeta}}}{re^{i\theta}}ire^{i\theta}drd\theta\right| \leq 2\pi M\epsilon$$

where we use the continuity of $\dfrac{\partial f}{\partial \overline{\zeta}}$ in a bounded region, that implies there exist $M > 0$ such that

$$\left|\frac{\partial f}{\partial \overline{\zeta}}\right| < M, \quad \forall \zeta \in \overline{D}.$$

So we have obtained

$$\int_{\partial D} \frac{f(w)}{w-z} dw - 2\pi i f(z) = 2i \iint_D \frac{\frac{\partial f}{\partial \bar{\zeta}}}{\zeta - z} dA(\zeta).$$

which is equivalent to the statement of the theorem. □

Theorem 3.25. (Liouville Theorem) *Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded differentiable function. Then f is constant.*

Proof. Since f is bounded there exists a constant $M \in (0, +\infty)$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Fix $a, b \in \mathbb{C}$. Let $R \in (0, +\infty)$ such that $R > |a|$, $|R > |b|$. By the Cauchy Integral Formula

$$f(a) - f(b) = \frac{1}{2\pi i} \int_{S_R^+(0)} \frac{f(w)}{w-a} dw - \frac{1}{2\pi i} \int_{S_R^+(0)} \frac{f(w)}{w-b} dw = \frac{1}{2\pi i} \int_{S_R^+(0)} \frac{f(w)(a-b)}{(w-a)(w-b)} dw.$$

Hence by the Estimation Lemma

$$|f(a) - f(b)| \leq \frac{2\pi R}{2\pi} \sup_{w \in S_R(0)} \frac{|f(w)||a-b|}{|w-a||w-b|} \leq \frac{RM|a-b|}{(R-|a|)(R-|b|)}.$$

Therefore

$$|f(a) - f(b)| \leq \lim_{R \to +\infty} \frac{RM|a-b|}{(R-|a|)(R-|b|)} = 0 \Rightarrow f(a) = f(b).$$

□

The Cauchy Integral Formula might be extremely useful in computations.

Example 3.26. In order to evaluate the integral

$$\int_{S_1^+(0)} \frac{e^w}{w} dw$$

we may use (3.4) with $f(w) := e^w$ and $z := 0$ Clearly f is differentiable in $\mathbb{C}$. Then

$$\int_{S_1^+(0)} \frac{e^w}{w} dw = 2\pi i \, e^0 = 2\pi i.$$

Example 3.27. Evaluate

$$\int_\gamma \frac{e^w + w}{w+2} dw,$$

where (a) $\gamma = S_1^+(0)$; (b) $\gamma = S_3^+(0)$.

Solution. Note that the integrand $\frac{e^w+w}{w+2}$ is differentiable in $\mathbb{C} \setminus \{-2\}$.
(a) It is clear that $S_1^+(0)$ is homotopic into a point in $\mathbb{C} \setminus \{-2\}$. Thus by the Deformation Theorem

$$\int_{S_1^+(0)} \frac{e^w + w}{w+2} dw = 0.$$

(b) We may use (3.4) with $f(w) := e^w + w$ and $z := -2$. Clearly f is differentiable in $\mathbb{C}$ and we obtain

$$\int_{S_3^+(0)} \frac{e^w + w}{w+2} dw = 2\pi i(e^{-2} - 2).$$

Example 3.28. Evaluate
$$\int_{S_1^+(i)} \frac{dw}{w^2+1}.$$

Solution. Observe that
$$\frac{1}{w^2+1} = \frac{1}{(w-i)(w+i)}.$$
Set $f(w) := \frac{1}{w+i}$ and let $z := i$. Thus f is differentiable in $\mathbb{C} \setminus \{-i\}$. Hence by the Cauchy Integral Formula
$$\int_{S_1^+(i)} \frac{dw}{w^2+1} = \int_{S_1^+(i)} \frac{f(w)}{w-i} = 2\pi i f(i) = \pi.$$
In the same way we compute that
$$\int_{S_1^+(-i)} \frac{dw}{w^2+1} = -\pi.$$

Remark 3.29. Note that in Cauchy's Integral Formula the function f, not the integrand $f(w)/(w-z)$, is holomorphic. If $z \in B_R(a)$ then the integrand is holomorphic only on $B_R(a) \setminus \{z\}$, so we can not use Cauchy's Theorem to conclude that the integral is zero.

Mathematicians of this section:

- *Baron Augustin-Louis Cauchy (21 August 1789 – 23 May 1857), French mathematician*
- *Joseph Liouville (24 March 1809 – 8 September 1882), French mathematician*
- *Sir George Gabriel Stokes (13 August 1819 – 1 February 1903) was an Irish physicist and mathematician.*
- *Dimitrie D. Pompeiu (4 October 1873 – 8 October 1954) was a Romanian mathematician.*

Week 4

4 Taylor and Laurent series

4.1 Analytic functions

In this section we introduce analytic functions, namely complex functions defined by power series. We will then show that such functions are infinitely many times differentiable.

4.1.1 Revision on convergence of series of complex and real numbers

Let us recall from Section 1 that the space $(\mathbb{C}, |\,.\,|)$ is a **complete metric space**. As for real numbers the converge of series is defined according to the converge of partial sum, namely the series

$$\sum_{n=0}^{\infty} z_n$$

converges iff the sequence of partial sums $s_n = \sum_{j=0}^{n} z_j$ converges, namely the $\lim_{n\to\infty} s_n = S$ is finite. Alternatively being $\mathbb{C}$ a complete metric space the notion of convergence is equivalent to $\{s_n\}_{n\in\mathbb{N}}$ being a Cauchy sequence, namely for all $\epsilon > 0$ there is a $n_0 > 0$ such that for all $n > m > n_0$

$$|s_n - s_m| = |\sum_{j=m}^{n} z_j| < \epsilon$$

The series $\sum_{n=0}^{\infty} z_n$ is called absolutely convergent if $\sum_{n=0}^{\infty} |z_n|$ converges. We also recall the Cauchy-Hadamard formula from real analysis: the series of real numbers $\sum_{n=0}^{\infty} b_n$, $b_n \geq 0$ converges if

$$\limsup_{n\to\infty} b_n^{1/n} < 1, \tag{4.1}$$

and diverges to $+\infty$ if $\limsup_{n\to\infty} b_n^{1/n} > 1$ and it is undefined if $\limsup_{n\to\infty} b_n^{1/n} = 1$. *(Remember that* $\limsup_{n\to\infty} b_n^{1/n} = \lim_{n\to\infty} \sup\{b_n^{1/n}, b_{n+1}^{1/n+1}, \dots\}$*).*
Alternatively using the ration test if

$$\limsup_{n\to\infty} \frac{b_{n+1}}{b_n} < 1,$$

then the series converges and it diverges if $\limsup_{n\to\infty} \dfrac{b_{n+1}}{b_n} > 1$.

Therefore the series $\sum_{n=0}^{\infty} |z_n|$ converges if

$$\limsup_{n\to\infty} |z_n|^{1/n} < 1 \quad \text{or} \quad \limsup_{n\to\infty} \frac{|z_{n+1}|}{|z_n|} < 1.$$

4.1.2 Series of complex holomorphic functions

We consider a series of holomorphic functions in a domain D

$$f_1(z), f_2(z), \dots, f_n(z), \dots \quad z \in D$$

and their partial sum

$$s_n(z) = \sum_{j=0}^{n} f_j(z).$$

We say that the series $\sum_{j=0}^{\infty} f_n(z)$ converges *point-wise* to a function $f(z)$, if the sequence of partial sums $\{s_n(z)\}_{n\in\mathbb{N}}$ converges to a function $f(z)$ with $z \in D$. This means that for each $z \in D$ and $\forall\epsilon > 0$ there is a natural number $n_0 = n_0(\epsilon, z)$ such that for all $n > n_0(\epsilon, z)$

$$|s_n(z) - f(z)| < \epsilon.$$

The series converges *uniformly* in D if $\forall\epsilon > 0$ there is a natural number $n_0 = n_0(\epsilon)$ independent from z such that for all $n > n_0(\epsilon)$

$$|s_n(z) - f(z)| < \epsilon. \quad \forall z \in D.$$

The series $\sum_{j=0}^{\infty} f_n(z)$ is said to converge *absolutely* in D if

$$\sum_{j=0}^{\infty} |f_n(z)|$$

converges for all $z \in D$.

Definition 4.1. Let $D \subseteq \mathbb{C}$ be a domain. A function $f : D \to \mathbb{C}$ is said to be analytic at a point $z_0 \in D$ iff for some ball $B_r(z_0) \subseteq D$ there is a power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

that converges absolutely for all $|z - z_0| < r$ and

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n, \qquad z \in B_r(z_0). \tag{4.2}$$

A function $f : D \to \mathbb{C}$ is said to be analytic in a domain D iff f is analytic at every point $z \in D$.

Theorem 4.2. *Consider the power series*

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n\,, \tag{4.3}$$

where $(a_n)_{n=0}^{n=+\infty}$ is a sequence of complex numbers, and let $0 \le R \le +\infty$ be its radius of convergence given by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{4.4}$$

Then the following properties are satisfied:

(a) if $|z - z_0| < R$ then the series (4.3) converges absolutely;

(b) if $|z - z_0| > R$ then the series (4.3) is absolutely divergent;

(c) if $r \in (0, R)$ then the series (4.3) converges absolutely and uniformly on $\overline{B_r(z_0)}$.

Proof. We first prove (c). Fix $r > 0$ and let $|z - z_0| \le r < R$. Then

$$|\sum_{n=0}^{\infty} a_n (z - z_0)^n| \le \sum_{j=0}^{\infty} |a_j| r^j.$$

By the Cauchy-Hadamard rule (4.1) the series $\sum_{j=0}^{\infty} |a_j| r^j$ converges if

$$\limsup_{j \to \infty} (|a_j| r^j)^{1/j} = \frac{r}{R} < 1,$$

that is true because $r < R$. Let us now consider the partial sums $s_n(z) = \sum_{j=0}^{n} a_j (z - z_0)^j$. For all $\epsilon > 0$ there is a $n_0 > 0$ such that for all $n, m > n_0$, $n > m$, we have that

$$|s_n(z) - s_m(z)| \le \sum_{j=m}^{n} |a_j| r^j < \epsilon, \quad \forall z \in \overline{B_r(z_0)},$$

because $\sum_{j=0}^{\infty} |a_j| r^j$ is a convergent series and in particular a Cauchy sequence. Let fix $m > n_0$. Given $z \in B_r(z_0)$ chose $n > n_0$ sufficiently large (and possibly depending on z) so that

$$|f(z) - s_n(z)| < \sum_{j=n}^{\infty} |a_j| r^j < \epsilon\,.$$

Then for $m > n_0$ (and now m independent from z)

$$|f(z) - s_m(z)| \le |f(z) - s_n(z)| + |s_m(z) - s_n(z)| < 2\epsilon,$$

which shows the uniform convergence $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ for every $|z - z_0| \le r < R$.
Next we prove (b). The series clearly diverges absolutely for $|z - z_0| > R$, let say $|z - z_0| = R + \delta$, $\delta > 0$, because the series

$$\sum_{j=0}^{\infty} |a_j||z - z_0|^j = \sum_{j=0}^{\infty} |a_j||R + \delta|^j$$

is divergent. We have thus proved (b) and (c). To prove (a) we observe that given z such that $|z - z_0| < R$, there is $r > 0$, (depending on z) such that $|z - z_0| < r < R$ and repeating the argument in (a) we conclude that the series (4.3) is absolutely convergent. □

Remark. **Ratio Test**. Alternatively, if the limit $\lim_{n \to +\infty} (|a_{n+1}|/|a_n|)$ exists, then the radius of convergence satisfies the relation $\frac{1}{R} = \lim_{n \to +\infty} \frac{|a_{n+1}|}{|a_n|}$.

Example 4.3. Let $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{r^n}$, $\quad r > 0$. This is the geometric series that converges for $|z| < r$.

Exercise 4.4. Find the radius of convergence of each of the following power series:

a) $\sum_{n=0}^{\infty} \frac{z^n n^2}{2^n}$, *b)* $\sum_{n=1}^{\infty} \frac{27^{2n} (z-2)^{3n}}{n^2}$, *c)* $\sum_{n=1}^{\infty} \frac{(z-5)^n}{n^n}$, *d)* $\sum_{n=1}^{\infty} \frac{z^{n^2}}{n^n}$.

Finally we state without proof the following result that will be used.

Theorem 4.5. *Let $\sum_{n\geq 0} a_n(z-z_0)^n$ and $\sum_{j\geq 0} b_j(z-z_0)^j$ be convergent power series in $B_r(z_0)$, $r>0$. Then their sum and their product are a convergent power series in $B_r(z_0)$. In particular*

$$\sum_{n\geq 0} a_n(z-z_0)^n \sum_{j\geq 0} b_j(z-z_0)^j \overset{j+n=m}{=} \sum_{m\geq 0}(z-z_0)^m \sum_{n=0}^{m} a_n b_{m-n}.$$

From the above theorem it follows that if $f(z)$ and $g(z)$ are analytic functions in $B_r(z_0)$ their sum and product are also analytic functions $B_r(z_0)$.

We introduced complex analytic functions defined by a power series. Now we show that such functions have an infinite number of complex derivatives. Then we will show that any complex differentiable function has also an infinite number of complex derivatives and it is analytic.

Lemma 4.6. *Suppose the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R>0$, then the power series*

$$\sum_{n=0}^{\infty} n a_n z^{n-1}$$

has the same radius of convergence. Let us define

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad g(z)=\sum_{n=0}^{\infty} n a_n z^{n-1},$$

Then

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=g(z), \quad \forall z\in B_R(0)\,.$$

Proof. Let R' be the radius of convergence of $\sum_{n=0}^{\infty} n a_n z^{n-1}$, then

$$\frac{1}{R'}=\limsup_{n\to\infty}|na_n|^{1/(n-1)}=\limsup_{n\to\infty} n^{\frac{1}{n-1}}\left(|a_n|^{\frac{1}{n}}\right)^{\frac{n}{n-1}}=\frac{1}{R}$$

because $\limsup_{n\to\infty} n^{\frac{1}{n-1}}=1$. Next let us calculate the derivative of $f(z)$. We have

$$\begin{aligned} f(z+h)-f(z) &= \sum_{n=0}^{\infty} a_n\left[(z+h)^n-z^n\right]=\sum_{n=0}^{\infty} a_n\left[\sum_{k=0}^{n}\binom{n}{k} z^{n-k}h^k - z^n\right] \\ &= h\sum_{n=0}^{\infty} a_n\left[nz^{n-1}+\sum_{k=2}^{n}\binom{n}{k} z^{n-k}h^{k-1}\right] \end{aligned}$$

so that

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\sum_{n=0}^{\infty} a_n\left[nz^{n-1}+\sum_{k=2}^{n}\binom{n}{k} z^{n-k}h^{k-1}\right]=\sum_{n=0}^{\infty} n a_n z^{n-1}=g(z).$$

□

Next we generalize the above result to any number of derivative. Namely any analytic function is infinitely many times differentiable.

Theorem 4.7. *Suppose the power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *has radius of convergence* $R > 0$. *Then the function* $f : B_R(z_0) \to \mathbb{C}$ *defined by*

$$f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is infinitely many times differentiable in $B_R(z_0)$ *and for any* $k \in \mathbb{N}$ *the formula*

$$f^{(k)}(z) := \sum_{n=k}^{\infty} \{n(n-1)\dots(n-k+1)\}\, a_n(z-z_0)^{n-k} \qquad z \in B_R(z_0) \tag{4.5}$$

holds. For instance $f^{(n)}(z_0) = n!a_n$.

Recall, that a (complex) polynomial of degree n is a function of the form

$$p_n(z) = a_n z^n + a_{n-1}z^{n-1} + \dots + a_1 z_1 + a_0,$$

where $a_1, \dots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Clearly every polynomial is analytic in $\mathbb{C}$. Examples of analytic functions are

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1), \qquad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (|z| < \infty),$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \quad (|z| < \infty), \qquad \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad (|z| < \infty),$$

Now let us prove that the coefficients a_n of a power series expansion of an analytic function are uniquely defined.

Theorem 4.8. *Let* $r \in (0, +\infty)$, z_0 *be a point in* $\mathbb{C}$, *and* $(a_n)_{n=0}^{n=+\infty}$, $(b_n)_{n=0}^{n=+\infty}$ *be sequences of complex numbers. Assume that*

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n \qquad z \in B_r(z_0)$$

(we suppouse that the series are convergent for any $z \in B_r(z_0)$.

Then $a_n = b_n$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Proof. Define $f : B_r(z_0) \to \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n \qquad (|z-z_0| < r)$$

Then by theorem 4.7 $f^{(n)}(z_0) = n!a_n$, $f^{(n)}(z_0) = n!b_n$ which implies $a_n = b_n$. □

4.2 Taylor Series Theorem

Using the Cauchy Integral Formula we can now prove that *every differentiable complex function is infinitely many times differentiable and analytic.*

Theorem 4.9. (Taylor Series Theorem) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ differentiable function. Then f is analytic in D and for any ball $B_R(z_0) \subseteq D$ the power series expansion*

$$f(z_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} h^n \qquad (|h| < R) \tag{4.6}$$

is valid. Further, if $r \in (0,R)$ then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{S_r^+(z_0)} \frac{f(w)}{(w-z_0)^{n+1}} dw. \tag{4.7}$$

Remark 4.10. *i*) Power series expansion (4.6) is called the *Taylor series expansion* of f at z_0.
ii) By the Deformation Theorem the integral in the right hand side of (4.7) does not depend on $r \in (0,R)$.

Proof. Fix $z_0 \in D$ and a ball $B_R(z_0) \subseteq D$. Choose h such that $z_0 + h \in B_R(z_0)$ and r such that $|h| < r < R$. By the Cauchy Integral Formula applied at the point $z_0 + h \in B_r(z_0)$ we know that

$$f(z_0+h) = \frac{1}{2\pi i} \int_{S_r^+(z_0)} \frac{f(w)}{w-(z_0+h)} dw.$$

By the (complex) geometric sum formula

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^m + \frac{q^{m+1}}{1-q} \tag{4.8}$$

applied with $q := \frac{h}{w-z_0}$ we obtain

$$\begin{aligned} \frac{1}{w-(z_0+h)} &= \frac{1}{w-z_0} \frac{1}{1-\frac{h}{w-z_0}} \\ &= \frac{1}{w-z_0} \Big\{ 1 + \tfrac{h}{w-z_0} + \tfrac{h^2}{(w-z_0)^2} + \cdots + \tfrac{h^m}{(w-z_0)^m} + \tfrac{h^{m+1}}{(w-z_0)^m (w-(z_0+h))} \Big\}. \end{aligned}$$

Therefore

$$\begin{aligned} f(z_0+h) &= \frac{1}{2\pi i} \int_{S_r^+(z_0)} \frac{f(w)}{w-(z_0+h)} dw \\ &= \frac{1}{2\pi i} \int_{S_r^+(z_0)} f(w) \Big\{ \tfrac{1}{w-z_0} + \tfrac{h}{(w-z_0)^2} + \cdots + \tfrac{h^m}{(w-z_0)^{m+1}} + \tfrac{h^{m+1}}{(w-z_0)^{m+1}(w-(z_0+h))} \Big\} dw \\ &= \sum_{n=0}^{m} a_n h^n + A_m, \end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw \quad \text{and} \quad A_m = \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(w)h^{m+1}}{(w-z_0)^{m+1}(w-(z_0+h))}dw.$$

We are going to show that $\lim_{m\to\infty} A_m = 0$.

Indeed, f is differentiable in $B_R(z_0)$ and hence continuous on $S_r(z_0)$. So f is bounded on $S_r(z_0)$ (because $S_r(z_0)$ is compact). This means that there exists $M > 0$ such that

$$|f(w)| \le M \qquad (w \in S_r(z_0)).$$

Now $|h| < r$, $|w-z_0| = r$ and $|w-(z_0+h)| \ge ||w-z_0|-|h|| = r-|h|$. Then by the Estimation Lemma

$$|A_m| \le \frac{1}{2\pi}\frac{M|h|^{m+1}}{r^{m+1}(r-|h|)}\underbrace{L(S_r(z_0))}_{2\pi r} = \frac{Mh}{r-|h|}\left(\frac{|h|}{r}\right)^m.$$

Since $|h| < r$ we conclude that $\lim_{m\to\infty} A_m = 0$.

Therefore

$$\lim_{m\to\infty}\left(f(z_0+h) - \sum_{n=0}^{m} a_n h^n\right) = 0.$$

This means that f is analytic at z_0 with the power series expansion

$$f(z_0+h) = \sum_{n=0}^{\infty} a_n h^n,$$

and this expansion is valid for all $|h| < R$. The coefficients a_n of the expansion are

$$a_n = \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}dw$$

where r is such that $|h| < r$. However the restriction $|h| < r$ is unnecessary by the Remark 4.10. To complete the proof we simply observe that $a_n = \frac{f^{(n)}(z_0)}{n!}$ by Theorem 4.7. □

Remark 4.11. The Taylor Series Theorem states, in particular, that *every differentiable complex function is infinitely many times differentiable and analytic*. A corresponding notion of analyticity can be similarly introduced for real functions. However, the following example demonstrates that for real functions existence of all higher order derivatives does not imply analyticity. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Then f is infinitely many times differentiable and in particular at $x = 0$

$$f^{(n)}(0) = 0 \qquad (n \in \mathbb{N} \cup \{0\}).$$

However $f \not\equiv 0$ in a neighbourhood of $x = 0$ and therefore f is not analytic at $x = 0$.

Example 4.12. The Taylor series of $\exp(z)$ about $z_0 = 0$ is given by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!}z^n \qquad (z \in \mathbb{C}).$$

Indeed the required expansion can be obtained by using (4.6) and $\exp^{(n)}(0) = 1$. Since $\exp(z)$ is differentiable for every $z \in \mathbb{C}$, the radius of convergence of the series is $R = +\infty$.

In practice, we determine the radius of convergence of the Taylor series of a differentiable function as *the radius of the largest ball contained in the domain of differentiability of the function.*

Example 4.13. Find the Taylor expansion of the function $f(z) = \frac{1}{1+z^2}$ about $z_0 = 3i$.

Solution. One can obtain the required expansion by computing the derivatives $f'(n)(3i)$ explicitly. However it might be more efficient to represent f as partial fractions

$$\frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Then we compute

$$f^{(n)}(z) = \frac{1}{2i}\left(\frac{(-1)^n n!}{(z-i)^{n+1}} - \frac{(-1)^n n!}{(z+i)^{n+1}}\right).$$

hence

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2i}\left(\frac{1}{(2i)^{n+1}} - \frac{1}{(4i)^{n+1}}\right)(z-3i)^n.$$

Observe that $1 + z^2 = 0$ iff $z = \pm i$. Thus the domain of differentiability of f is $\mathbb{C} \setminus \{i, -i\}$. Since $B_2(3i) \subset \mathbb{C} \setminus \{i, -i\}$ is a ball centered at $z_0 = 3i$ contained in $\mathbb{C} \setminus \{i, -i\}$, we conclude that the series converges for $|z - 3i| < 2$.

4.3 Morera Theorem

The following theorem gives a "converse" to Cauchy's type theorems.

Theorem 4.14. (Morera Theorem) *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ continuous function. Assume that for any closed contour $\gamma : [\alpha, \beta] \to D$ one has*

$$\int_\gamma f(z)dz = 0.$$

Then f is differentiable in D.

Proof. We show that f has an anti-derivative and then by the Taylor series theorem the antiderivative is analytic. Fix any point $a \in D$. Since D is connected then for any z in D there is a contour $\gamma_{a,z}$ joining a and z and the integral

$$F_a(z) = \int_{\gamma_{a,z}} f(z)dz$$

does not depend on the choice of the contour of integration. Therefore $F_a(z)$ is well defined. By the Converse to the Fundamental Theorem (Theorem 2.27), the function $F_a(z)$ is an antiderivative of f. Namely $F_a(z)$ is complex differentiable in D. By the Taylor series theorem it follows that $F_a(z)$ is analytic and it has an infinite number of derivaties, therefore, $f(z)$ is also differentiable. □

Remark 4.15. Not that the converse is not always true. For example let $D = \mathbb{C}\backslash\{0\}$ and $f(z) = \frac{1}{z}$. Then f is differentiable in D but

$$\int_{S_r^+(0)} \frac{dz}{z} = 2\pi i \quad \forall r > 0.$$

4.4 Cauchy Estimates and the corollaries

The next lemma is an immediate consequence of the formula (4.7) but is stated separately in view of its importance.

Lemma 4.16. (Cauchy Estimate) *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic function. Suppose that $\bar{B}_R(z_0) \subset D$ and*

$$|f(z)| \leq M \qquad (z \in S_R(z_0)).$$

Then for any $n \in \mathbb{N} \cup \{0\}$

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}. \tag{4.9}$$

Proof. From the Taylor Series Theorem we know that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{S_R^+(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw.$$

Hence, by the Estimation Lemma,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} \underbrace{L(S_R^+(z_0))}_{2\pi R} = \frac{n!M}{R^n},$$

as required. □

Definition 4.17. An entire function is a function which is defined and analytic on the whole complex plane $\mathbb{C}$.

Example 4.18. Clearly every polynomial is an entire function. Functions $\exp(z)$, $\sin(z)$, $\cos(z)$ are entire. By the Taylor Series Theorem every entire function f has a power series expansion $\sum_{n=0}^{\infty} a_n z^n$ with infinite radius of convergence, so entire functions can be viewed as polynomials of "infinite" degree.

As a simple corollary of the Liouville Theorem (see Theorem 3.25) we prove that every polynomial has a root in the complex plane.

Theorem 4.19. (Fundamental Theorem of Algebra) *Let p_n be a polynomial of degree $n \geq 1$. Then there is a point $z_0 \in \mathbb{C}$ such that $p_n(z_0) = 0$.*

Proof. Suppose that $p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \neq 0$ for all $z \in \mathbb{C}$. Here $a_j \in \mathbb{C}$ and $a_n \neq 0$. Then

$$g(z) := \frac{1}{p_n(z)}$$

is an entire function.

On the other hand, writing

$$p_n(z) = a_n z^n \left(1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + \cdots + \frac{a_0}{a_n z^n}\right)$$

with appropriate constants $b_1, \dots, b_n$, we see that[2]

$$\lim_{|z|\to\infty} |p_n(z)| = \infty.$$

Hence

$$\lim_{|z|\to\infty} |g(z)| = 0.$$

In particular, by definition of limit we have that $\forall \epsilon > 0$ there is a radius $R > 0$ such that $|g(z)| < \epsilon$ for all $|z| > R$. But $|g(z)|$ is continuous on $\overline{B_R(0)}$ so there is a constant $M > 0$ such that $|g(z)| < M$ if $|z| \le R$. Therefore g is a bounded entire function. By the Liouville Theorem g must be constant. This is a contradiction. □

Remark 4.20. The fundamental theorem of algebra was first proved by Gauss.

Solving differential equations. The concept of a function being analytic is very useful to solve linear differential equations. In particular let us consider a second order linear differential equations of the form

$$f''(z) + A(z)f'(z) + B(z)f(z) = 0, \quad f(z_0) = c_0, \ f'(z_0) = c_1 \tag{4.10}$$

where the function $A(z)$ and $B(z)$ are analytic in a domain $D \subset \mathbb{C}$ and $z_0 \in D$. The question is: is there a solution of the above differential equation in a neighbourhood of z_0? The answer is yes and it is provided by the Cauchy-Kowaleskaya theorem.

Theorem 4.21. *The initial value problem* (4.10) *with the functions $A(z)$ and $B(z)$ analytic in D and $z_0 \in D$ has a unique analytic solution in a neighbourhood of z_0.*

We do not prove the theorem but infer how to obtain a solution of the equation in a power series expansion with an example

Example 4.22. Find the solution of the differential equation

$$f''(z) + [a_0 + a_1(z - z_0)]f'(z) + [b_0 + b_1(z - z_0)]f(z) = 0, \quad f(z_0) = c_0, \ f'(z_0) = c_1$$

We look for a solution in the form

$$f(z) = \sum_{n=0}^{\infty} f_n(z - z_0)^n$$

where $f(z_0) = f_0 = c_0$ and $f'(z_0) = f_1 = c_1$. Plugging the series into the differential equation we obtain

$$\sum_{n=0}^{\infty} n(n-1)f_n(z-z_0)^{n-2} + [a_0+a_1(z-z_0)]\sum_{n=0}^{\infty} nf_n(z-z_0)^{n-1} + [b_0+b_1(z-z_0)]\sum_{n=0}^{\infty} f_n(z-z_0)^n = 0\,.$$

Re-ordering the summation we obtain

$$\sum_{m=0}^{\infty}(m+2)(m+1)f_{m+2}(z-z_0)^m + a_0\sum_{m=0}^{\infty}(m+1)f_{m+1}(z-z_0)^m + a_1\sum_{m=0}^{\infty} mf_m(z-z_0)^m$$
$$+ b_0\sum_{m=0}^{\infty} f_m(z-z_0)^m + b_1\sum_{m=1}^{\infty} f_{m-1}(z-z_0)^m = 0\,.$$

[2] Prove this as an exercise.

By the uniqueness of the Taylor series expansion, we conclude that we can settle equal to zero the terms of the same power.

In particular for $m = 0$ we obtain

$$2f_2 + a_0 f_1 + b_0 = 0$$

and for $m \geq 1$

$$f_{m+2} = -\frac{1}{(m+1)(m+2)}\left(a_0(m+1)f_{m+1} + (ma_1 + b_0)f_m + b_1 f_{m-1}\right). \tag{4.11}$$

By the Cauchy-Kowaleskaya theorem the solution $f(z) = \sum_{n=0}^{\infty} f_n(z-z_0)^n$ is analytic in a neighobourhood of z_0. Let us assume $\frac{1}{R} = \lim_{m\to\infty} \frac{|f_{m+1}|}{|f_m|} < \infty$. Then using the recursive equation (4.11) for f_m we obtain that

$$\begin{aligned}\frac{1}{R^3} &= \lim_{m\to\infty} \frac{|f_{m+2}||f_{m+1}||f_m|}{|f_{m+1}||f_m||f_{m-1}|} = \lim_{m\to\infty} \frac{|f_{m+2}|}{|f_{m-1}|}\\ &\leq \lim_{m\to\infty} \frac{1}{(m+1)(m+2)}\left(|a_0|(m+1)\frac{|f_{m+1}|}{|f_{m-1}|} + |ma_1 + b_0|\frac{|f_m|}{|f_{m-1}|} + |b_1|\right) = 0\,,\end{aligned}$$

that show that the series has radius of convergence $R = \infty$.

Remark 4.23. By the Cauchy-Kowaleskay theorem the solution of the differential equation (4.10) is analytic in a neighbourhood of z_0. However in the above example we obtain that the solution exists for all $z \in \mathbb{C}$. This is the case because the coefficients $A(z)$ and $B(z)$ that we chose are polynomials, and therefore analytic in the whole $\mathbb{C}$.

4.5 Laurent Series

By a Laurent series we mean a series

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n. \tag{4.12}$$

This is to be thought of as a compact notation for the sum of two series

$$\underbrace{\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n}_{\text{Laurent series}} := \underbrace{\sum_{n=0}^{\infty} a_n(z-z_0)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}}_{\text{singular part}}.$$

Laurent series (absolutely) *converges* in a set $A \subset \mathbb{C}$ iff both regular and singular parts (absolutely) converge at every $z \in A$. Note that the singular part of a Laurent series is not defined at $z = z_0$.

We know that the natural domain of convergence of a power series is a ball.

Consequently, the singular part of a Laurent series converges outside a ball.

Theorem 4.24. *Let $(a_n)_{n=-\infty}^{n=-1}$ be a sequence of complex numbers, z_0 be a point in $\mathbb{C}$. Then there exist the number $r \in [0,+\infty) \cup \{+\infty\}$ such that the following properties are satisfied :*
(a) if $|z-z_0| > r$ then the series $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges absolutely;
(b) if $|z-z_0| < r$ then the series $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ diverges;
(c) if $\rho \in (r,+\infty)$ then the $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges uniformly on $\overline{A_{\rho,+\infty}(z_0)}$.
Moreover, r can be computed by the formula

$$r = \limsup_{n\to+\infty} |a_{-n}|^{1/n}.$$

The prove of the theorem follows from Theorem 4.2 by setting $w := \frac{1}{z-z_0}$ and consider the power series $\sum_{n=1}^{\infty} a_n w^n$.
We define the open anulus

$$A_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z-z_0| < R\}, \quad 0 \le r < R \le +\infty, \quad z_0 \in \mathbb{C}.$$

Theorem 4.25. *Let $(a_n)_{n=-\infty}^{n=+\infty}$ be a sequence of complex numbers, z_0 be a point in $\mathbb{C}$. Define $r, R \in [0,+\infty]$ by*

$$r = \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}}, \qquad R^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}, \tag{4.13}$$

where we agree that $+\infty^{-1} = 0$, $0^{-1} = +\infty$. Consider the Laurent series

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n. \tag{4.14}$$

Then the following conditions are satisfied:
a) the series (4.14) converges for all z such that $r < |z-z_0| < R$,
b) the series (4.14) diverges for all $z \in \mathbb{C} \setminus \overline{A_{r,R}(z_0)}$
c) if $r < \rho_1 < \rho_2 < R$ then (4.14) converges uniformly on $\overline{A_{\rho_1,\rho_2}(z_0)}$.

Hint. Apply the formula for the radius of convergence of power series separately to the regular and singular parts of the Laurent series.

Definition 4.26. The function $f(z)$ defined in $A_{r,R}(z_0)$ admits an expansion in a Laurent series if there is a series of the form (4.14) that converges for every $z \in A_{r,R}(z_0)$ to the function $f(z)$

As for Taylor series, Laurent series can be differentiated term by term, and therefore, Laurent series define holomorphic functions.

Theorem 4.27. *If the Laurent series of a function $f(z)$ defined in $A_{r,R}(z_0)$ exists, then it is uniquely defined.*

Proof. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ for $z \in A_{r,R}(z_0)$. The Laurent series converges uniformly for $r < r' < \rho < R' < R$. We consider the quantity $\frac{f(z)}{(z-z_0)^{k+1}}$ and integrate term by term over a circle $S_\rho^+(z_0)$

$$\int_{S_\rho^+(z_0)} \frac{f(z)}{(z-z_0)^{k+1}} dz = \sum_{n=-\infty}^{\infty} \int_{S_\rho^+(z_0)} a_n(z-z_0)^{n-k-1} dz = 2\pi i a_k$$

by Cauchy theorem. We obtain

$$a_n = \frac{1}{2\pi i} \int_{S_\rho^+(z_0)} \frac{f(w)}{(w - z_0)^{n+1}} dw. \tag{4.15}$$

By the deformation theorem the result does not depend on the choice of ρ and therefore the coefficients a_n are uniquely defined. □

Formula (4.15) for the coefficients a_n is not very practical for computing the Laurent series of a given function. Instead, tricks can be used to obtain an expansion of the required form. Then the uniqueness of the expansion indicates that this is the desired one.

Example 4.28. Let $f(z) = \exp(z) + \exp(\frac{1}{z})$. We have

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \quad (|z| < \infty); \qquad \exp(\frac{1}{z}) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \quad (|z| > 0).$$

Therefore $f(z)$ has the Laurent series expansion in $A_{0,\infty}(0)$ given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where $a_n = \frac{1}{n!}$ for $n \geq 0$, $a_0 = 2$ and $a_n = \frac{1}{(-n)!}$ for $n \leq -1$.

Example 4.29. Let $f(z) = \frac{1}{z} + \frac{1}{1-z}$. Then, by using the geometric series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ $(0 < |z| < 1)$ we see that

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} z^n \quad (0 < |z| < 1) \tag{4.16}$$

is the Laurent expansion of f in $A_{0,1}(0)$. On the other hand, by using transformation $w := \frac{1}{z}$ we obtain

$$f(w) = w + w\left(-\frac{1}{1-w}\right) = w - w\sum_{n=0}^{\infty} w^n = -\sum_{n=2}^{\infty} w^n \quad (|w| < 1).$$

Therefore

$$f(z) = -\sum_{n=-\infty}^{-2} z^n \quad (|z| > 1) \tag{4.17}$$

is the Laurent expansion of f in $A_{1,\infty}(0)$.

Remark 4.30. Observe that both (4.16) and (4.17) are Laurent expansions of $f(z)$. The first case is a Laurent expansion near $z_0 = 0$. The second case is a Lauren expansion near $w_0 = 0$ or $z_0 = \infty$.

Example 4.31. Consider

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{1-z} + \frac{1}{z-2}.$$

We can expand each fraction near $z_0 = 0$ or infinity as a geometric series to obtain different Laurent expansions:

$$f(z) = \sum_{n=0}^{\infty} \left(1 - 2^{-(n+1)}\right) z^n \quad (z \in B_1(0)), \tag{4.18}$$

$$f(z) = -\sum_{n=-\infty}^{-1} z^n - \sum_{n=0}^{\infty} 2^{-(n+1)} z^n \quad (z \in A_{1,2}(0)), \tag{4.19}$$

$$f(z) = \sum_{n=-\infty}^{-1} \left(2^{-(n+1)} - 1\right) z^n \quad (z \in A_{2,\infty}(0)). \tag{4.20}$$

Note that (4.18) is simply the Taylor expansion at $z_0 = 0$ while the others are genuinely Laurent expansions involving the singular part.

Week 5

Theorem 4.32. (Laurent Series Theorem) *Let $f : A_{r,R}(z_0) \to \mathbb{C}$ be a holomorphic function. Then f has a Laurent series expansion*

$$f(z) = \sum_{m=-\infty}^{\infty} a_m (z - z_0)^m, \qquad (z \in A_{r,R}(z_0)).$$

Further, if $\rho \in (r, R)$ then

$$a_m = \frac{1}{2\pi i} \int_{S^+_\rho(z_0)} \frac{f(w)}{(w - z_0)^{m+1}} dw. \tag{4.21}$$

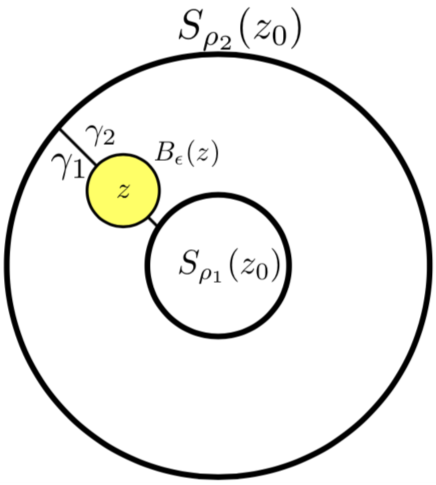
Remark 4.33. *i*) By the Deformation Theorem the integral in the right hand side of (4.21) does not depend on $\rho \in (r, R)$.

ii) Note that in contrast to the Taylor Series we can not longer assert that $a_m = \frac{f^{(m)}(z_0)}{m!}$ even for $m \geq 0$, since f need not be differentiable at z_0 under the hypothesis of the Laurent Series Theorem.

In order to prove the Laurent Series Theorem we need a modification of the Cauchy Integral Formula for annular domains.

Lemma 4.34. Anular Cauchy integral formula. *Let $f : A_{r,R}(z_0) \to \mathbb{C}$ be a holomorphic function. Let $z \in A_{r,R}(z_0)$. Let ρ_1, ρ_2 and ε be such that $r < \rho_1 < \rho_2 < R$ and $\bar{B}_\varepsilon(z) \subset A_{\rho_1,\rho_2}(z_0)$. Then*

$$f(z) = \frac{1}{2\pi i} \int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_{S^+_{\rho_1}(z_0)} \frac{f(w)}{w - z} dw, \quad z \in A_{\rho_1,\rho_2}(z_0).$$



Proof. Let $S^+_\varepsilon(z)$ be a circle of radius ϵ inside the anulus $A_{\rho_1,\rho_2}(z_0)$. Join $S^+_{\rho_1}(z_0)$ with $S^+_\varepsilon(z)$ and $S^+_\varepsilon(z)$ with $S^+_{\rho_2}(z_0)$ via straight line segments γ_2 and γ_2, traversed in opposite directions each.

The obtained closed contour is contractible into a point in $A_{r,R}(z_0)$. Hence

$$\int_{S^+_\varepsilon(z)} \frac{f(w)}{w-z}dw + \int_{S^+_{\rho_1}(z_0)} \frac{f(w)}{w-z}dw - \int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{w-z}dw = 0$$

by the Deformation Theorem. Therefore

$$\int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{w-z}dw - \int_{S^+_{\rho_1}(z_0)} \frac{f(w)}{w-z}dw = \int_{S^+_\varepsilon(z)} \frac{f(w)}{w-z}dw = 2\pi i f(z)$$

by the Cauchy Integral formula, applied at the point $z \in \bar{B}_\varepsilon(z)$. □

<u>Proof of the Laurent Series Theorem.</u> Fix $z \in A_{r,R}(z_0)$. Choose ρ_1, ρ_2 and ε such that $r < \rho_1 < \rho_2 < R$ and $\bar{B}_\varepsilon(z) \subset A_{\rho_1,\rho_2}(z_0)$. Then by Lemma 4.34,

$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{w-z}dw}_{\text{regular part}} - \underbrace{\frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)} \frac{f(w)}{w-z}dw}_{\text{singular part}}.$$

All we need to do now is to work out the two integrals as power series and calculate the coefficients.

As in the proof of the Taylor Series Theorem we obtain

$$\frac{1}{2\pi i}\int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{w-z}dw = \sum_{m=0}^{\infty} a_m (z-z_0)^m,$$

where

$$a_m = \frac{1}{2\pi i}\int_{S^+_{\rho_2}(z_0)} \frac{f(w)}{(w-z_0)^{m+1}}dw.$$

The treatment of the second integral is similar. Using the geometric sum (4.8) with $q := \frac{w-z_0}{z-z_0}$ we obtain

$$\begin{aligned} -\frac{1}{w-z} &= \frac{1}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}} \\ &= \frac{1}{z-z_0}\Big\{1+\tfrac{w-z_0}{z-z_0}+\cdots+\tfrac{(w-z_0)^{n-1}}{(z-z_0)^{n-1}}+\tfrac{(w-z_0)^n}{(z-z_0)^{n-1}(z-w)}\Big\}. \end{aligned}$$

Therefore

$$\begin{aligned} &-\frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)}\frac{f(w)}{w-z}dw \\ &\qquad = \frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)} f(w)\Big\{\tfrac{1}{z-z_0}+\tfrac{w-z_0}{(z-z_0)^2}+\cdots+\tfrac{(w-z_0)^{n-1}}{(z-z_0)^n}+\tfrac{(w-z_0)^n}{(z-z_0)^n(z-w)}\Big\}dw \\ &\qquad = \sum_{m=1}^{n} a_{-m}(z-z_0)^{-m}+B_n, \end{aligned}$$

where

$$a_{-m} = \frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)} f(w)(w-z_0)^{m-1}dw \quad \text{and} \quad B_n = \frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)} f(w)\frac{(w-z_0)^n}{(z-z_0)^n(z-w)}dw.$$

We are going to show that $\lim_{n\to\infty} B_n = 0$.

Indeed, f is holomorphic in $A_{r,R}(z_0)$ and hence bounded on $S_{\rho_1}(z_0)$ (because $S_{\rho_1}(z_0)$ is compact). This means that there exists $M > 0$ such that

$$|f(w)| \le M \qquad (w \in S_{\rho_1}(z_0)).$$

Set $\delta := |z-z_0|-\rho_1 > 0$. Now $|w-z_0| = \rho_1$, $|z-z_0| = \rho_1+\delta$ and $|z-w| \ge ||z-z_0|-|w-z_0|| = \delta$. Then by the Estimation Lemma

$$|B_n| \le \frac{1}{2\pi}\frac{M\rho_1^n}{(\rho_1+\delta)^n\delta}\underbrace{L(S_{\rho_1}(z_0))}_{2\pi\rho_1} = \frac{M\rho_1}{\delta}\left(\frac{\rho_1}{\rho_1+\delta}\right)^n.$$

We conclude that $\lim_{n\to\infty} B_n = 0$.

Therefore

$$-\frac{1}{2\pi i}\int_{S^+_{\rho_1}(z_0)}\frac{f(w)}{w-z}dw = \sum_{m=1}^{\infty} a_{-m}(z-z_0)^{-m}$$

and

$$f(z) = \underbrace{\sum_{m=0}^{\infty} a_m(z-z_0)^n}_{\text{regular part}} + \underbrace{\sum_{m=1}^{\infty} a_{-m}(z-z_0)^{-m}}_{\text{singular part}} \qquad (z \in A_{r,R}(z_0)),$$

which the required Laurent expansion. To finish the proof we observe that if $\rho \in (r,R)$ then $S^+_\rho(z_0) \sim S^+_{\rho_1}(z_0) \sim S^+_{\rho_2}(z_0)$ in $A_{r,R}(z_0)$. Thus the integrals in the expressions for a_n and b_n do not depend on the particular choice of $\rho \in (r,R)$. □

As a consequence of the above theorem we have:

Corollary 4.35. *Let $f : A_{r,R}(z_0) \to \mathbb{C}$ be a holomorphic function. Then there are two functions $f_0(z)$ holomorphic for $|z-z_0| < R$ and $f_\infty(z)$ holomorphic for $|z-z_0| > r$ such that*

$$f(z) = f_0(z) + f_\infty(z), \quad z \in A_{r,R}(z_0).$$

If

$$\lim_{|z|\to\infty} f_\infty(z) = 0,$$

then this decomposition is unique.

Proof. From the Laurent series theorem we have

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n, \quad z \in A_{r,R}(z_0).$$

We set

$$f_0(z) := \sum_{n=0}^{+\infty} a_n (z-z_0)^n, \quad f_\infty(z) := \sum_{n<0} a_n (z-z_0)^n.$$

This functions satisfy the requested properties, namely they are holomorphic in $|z-z_0| < R$ and $|z-z_0| > r$ respectively. Now suppose to have another decomposition

$$f(z) = \tilde{f}_0(z) + \tilde{f}_\infty(z), \quad \lim_{|z|\to\infty} \tilde{f}_\infty(z) = 0.$$

Then $\tilde{f}_0(z) - f_0(z) = f_\infty(z) - \tilde{f}_\infty(z)$ for $z \in A_{r,R}(z_0)$, so that the function

$$h(z) = \begin{cases} \tilde{f}_0(z) - f_0(z) & \text{for } |z-z_0| < R \\ f_\infty(z) - \tilde{f}_\infty(z) & \text{for } |z-z_0| > r \end{cases}$$

is holomorphic in $\mathbb{C}$ and $\lim_{|z|\to\infty} h(z) = 0$. It follows from Liouville theorem that $h(z) = 0$. □

Mathematician of this section:

- *Brook Taylor FRS (18 August 1685 – 29 December 1731), British mathematician,*
- *Pierre Alphonse Laurent (18 July 1813 – 2 September 1854), French mathematician,*
- *Johann Carl Friedrich Gauss (30 April 1777 – 23 February 1855), German mathematician and physicist,*
- *Giacinto Morera (18 July 1856 – 8 February 1909), Italian engineer and mathematician.*
- *Sofya Vasilyevna Kovalevskaya (15 January 1850 – 10 February 1891), Russian mathematician.*

5 Isolated singularities

Definition 5.1. Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be a complex function and $R > 0$. We say that z_0 is an isolated singularity of f if the function f is holomorphic in $A_{0,R}(z_0)$ and not holomorphic (or is not defined) at z_0.

For example, $f(z) = \frac{1}{z}$ has an isolated singularity at $z_0 = 0$ since it is analytics in $A_{0,R}(0)$ for any $R > 0$. The function $f(z) = \frac{1}{\sqrt{z}}$ is singular at $z = 0$ but it is not analytic in any neighbourhood of $z = 0$ since the function $\frac{1}{\sqrt{z}}$ is a *multivalued* function. Therefore $\frac{1}{\sqrt{z}}$ does not have an isolated singularity at $z = 0$.

Definition 5.2. Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be a complex function. We say that z_0 is a branch point singularity of f if the function $f(z)$ is multivalued in $A_{0,R}(z_0)$ and holomorphic in

$$\{z \in \mathbb{C} \ \ s.t. \ \ 0 < |z - z_0| < R, \ \ \epsilon < \arg(z - z_0) < 2\pi - \epsilon\}, \quad \epsilon > 0, \tag{5.1}$$

and is singular at $z = z_0$.

As examples of branch point singularities we have $\log(z)$ or z^a with $-1 < a < 0$. Indeed both functions are multivalued in a neighbourhood of the origin, they are singular at $z = 0$ and they are holomorphic in the domain (5.1). Notice that the choice of the domain (5.1) is not unique.

5.1 Classification of isolated singularities

Observe that if z_0 is an isolated singularity of a function f then f has a Laurent series expansion

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}}_{\text{singular part}} + \underbrace{\sum_{m=0}^{\infty} a_m (z - z_0)^m}_{\text{regular part}} \qquad (z \in A_{0,R}(z_0)).$$

We classify isolated singularities according to the number of nonzero coefficients in the singular part of the Laurent expansion of f at z_0. Three different situations are possible.

Removable singularity. We say that z_0 is a removable singularity of f if for all $n > 0$ one has $a_{-n} = 0$. In this case the Laurent expansion of f at z_0 consists only of the regular part, e.g.

$$f(z) = \underbrace{\sum_{m=0}^{\infty} a_m (z - z_0)^m}_{\text{Taylor series}} \qquad (z \in A_{0,R}(z_0)).$$

Then the singularity at z_0 can be *removed*, by defining $f(z_0) := a_0$, and we obtain a function which is analytic on $B_R(z_0)$. For example, consider

$$f(z) = \frac{\sin(z)}{z} \qquad (z \neq 0).$$

Clearly f is analytic on $A_{0,\infty}(0)$ and z_0 is an isolated singularity of f, because f is not defined at $z_0 = 0$. The Laurent expansion of f at $z_0 = 0$ has the form

$$f(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots \qquad (z \in A_{0,\infty}(0)).$$

So z_0 is a removable singularity. By defining $f(0) := 1$ we get a function which is analytic on $\mathbb{C}$.

Pole. We say that z_0 is a pole of order k of f if there exists $k \in \mathbb{N}$ such that $a_{-k} \neq 0$ and for all $n > k$ one has $a_{-n} = 0$. Sometimes a pole of order 1 is called a *simple* pole.

For a pole of order k the Laurent expansion of f at z_0 has the form

$$f(z) = \underbrace{\sum_{n=1}^{k} \frac{a_{-n}}{(z-z_0)^n}}_{\text{"finite" singular part}} + \underbrace{\sum_{m=0}^{\infty} a_m (z-z_0)^m}_{\text{regular part}} \qquad (z \in A_{0,R}(z_0)).$$

For example, consider

$$f(z) = \frac{\sin(z)}{z^4} \qquad (z \neq 0).$$

Clearly f is analytic on $A_{0,\infty}(0)$ and z_0 is an isolated singularity of f. The Laurent expansion of f at $z_0 = 0$ has the form

$$f(z) = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} - \frac{z^3}{7!} + \dots \qquad (z \in A_{0,\infty}(0)).$$

So z_0 is a pole of order 3.

Essential singularity. We say that z_0 is an essential singularity of f if for any $k \in \mathbb{N}$ there exists $n > k$ such that $a_{-n} \neq 0$. In this case the Laurent expansion of f at z_0 has the form

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}}_{\text{"infinite" singular part}} + \underbrace{\sum_{m=0}^{\infty} a_m (z-z_0)^m}_{\text{regular part}} \qquad (z \in A_{0,R}(z_0)).$$

For example, consider

$$f(z) = \sin(\frac{1}{z}) \qquad (z \neq 0).$$

Clearly f is analytic on $A_{0,\infty}(0)$ and z_0 is an isolated singularity of f. The Laurent expansion of f at $z_0 = 0$ has the form

$$f(z) = \frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - \frac{1}{7!}\frac{1}{z^7} + \dots \qquad (z \in A_{0,\infty}(0)).$$

So z_0 is an essential singularity.

5.2 Local behavior of holomorphic functions

Removable singularities are simple because they always can be removed. The next proposition allows to recognize removable singularities.

Proposition 5.3. *Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be a holomorphic function. Assume that z_0 is an isolated singularity of f. Then the following are equivalent:*

(a) z_0 is a removable singularity;

(b) $\lim_{z \to z_0} f(z)$ exists and finite;

(c) $|f(z)|$ is bounded on $A_{0,r}(z_0)$ for every $r \in (0, R)$.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are obvious.
$(c) \Rightarrow (a)$. Let $r \in (0, R)$. Suppose that $|f(z)|$ is bounded on $A_{0,r}(z_0)$, e.g. there exists $M > 0$ such that

$$|f(z)| \leq M \qquad (z \in A_{0,r}(z_0)).$$

Consider the Laurent expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n} + \sum_{m=0}^{\infty} a_m (z - z_0)^m \qquad (z \in A_{0,r}(z_0)).$$

We are going to prove that $a_{-n} = 0$ for all $n > 0$. Indeed, for $n > 0$ and $\rho \in (0, r)$ we have

$$a_{-n} = \frac{1}{2\pi i} \int_{S_\rho^+(z_0)} f(w)(w - z_0)^{n-1} dw.$$

By the Estimation Lemma

$$|a_{-n}| \leq \frac{1}{2\pi} M \rho^{n-1} L(S_\rho^+(z_0)) = M \rho^n.$$

If we let $\rho \to 0$ it follows that $|a_{-n}| = 0$. □

When z_0 is a removable singularity the function $f(z) : A_{0,R}(z_0) \to C$ has an analytic extension $\hat{f}(z)$ in $z = z_0$. Indeed let us define

$$\hat{f}(z) = \begin{cases} f(z) & \text{for } z \in A_{0,R}(z_0), \\ \lim_{z \to z_0} f(z) & \text{for } z = z_0. \end{cases}$$

The function $\hat{f}(z)$ is not only continuos in $z = z_0$ but also analytic. Indeed we leave as an exercise to show that

$$\lim_{h \to 0} \frac{\hat{f}(z) - \hat{f}(z_0)}{h}$$

exists and is finite.

Exercise 5.4. Let z_0 be a non removable isolated singularity of a function f. Prove that

$$\limsup_{z \to z_0} |f(z)| = +\infty.$$

There is a similar criterion for recognizing poles.

Proposition 5.5. *Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be a holomorphic function. Assume that z_0 is an isolated singularity of f. Then the following are equivalent:*
(a) *z_0 is a pole of order k;*
(b) $\lim_{z \to z_0} (z - z_0)^k f(z) = w \neq 0$.

Proof. The implications $(a) \Rightarrow (b)$ is obvious.
$(b) \Rightarrow (a)$. Suppose $\lim_{z \to z_0} (z - z_0)^k f(z) = w \neq 0$. Then

$$g(z) := (z - z_0)^k f(z)$$

has a removable singularity at z_0 by Proposition 5.3. Set $g(z_0) := w$. Hence g is holomorphic on $B_R(z_0)$ and has a Taylor expansion

$$g(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \qquad (z \in B_R(z_0)),$$

where $a_0 = w$. Therefore

$$f(z) = \sum_{n=1}^{k} \frac{a_{k-n}}{(z - z_0)^n} + \sum_{m=0}^{\infty} a_{k+m} (z - z_0)^m \qquad (z \in A_{0,R}(z_0))$$

where $a_0 \neq 0$, so f has a pole of order k at z_0. □

Exercise 5.6. Let z_0 be a pole of a function f. Then

$$\lim_{z \to z_0} |f(z)| = +\infty.$$

Exercise 5.7. A rational function is a function of the form

$$f(z) = \frac{q_m(z)}{p_n(z)},$$

where $q_m(z)$ and $p_n(z)$ are polynomials of orders m and n. Assume that $z_0 \in \mathbb{C}$ is a root of p_n of multiplicity $k > 0$, i.e. there exists a polynomial $p_{n-k}(z)$ of order $n - k$ such that

$$p_n(z_0) = (z - z_0)^k p_{n-k}(z), \quad p_{n-k}(z_0) \neq 0.$$

Prove that if $q_m(z_0) \neq 0$ then z_0 is a pole of order k of the rational function $f(z)$.

Definition 5.8. A set of points $S \subset \mathbb{C}$ is discrete if for any point $z_0 \in S$ there exists a neighbourhood $B_r(z_0) = \{z \in \mathbb{C} \ \ s.t. \ \ |z - z_0| < r\}$ with $r > 0$ such that $B_r(z_0) \cap S = \{z_0\}$.

For example the set $\{z_0 + 2\pi k\}$ for $k \in \mathbb{Z}$ is discrete in $\mathbb{C}$.

Definition 5.9. A function $f : \mathbb{C} \to \mathbb{C}$ is called meromorphic in the domain $D \subset \mathbb{C}$, if f is holomorphic in $D \backslash S$ where S is a *discrete* subset of D and f has poles in S.

For example rational functions are meromorphic in $\mathbb{C}$. Let $f(z)$ and $g(z)$ be two holomorphic functions in D and suppose that $g(z)$ is not identically zero and has a discrete number of zeros. Then the ratio $f(z)/g(z)$ is a meromorphic function in D. An example of a non rational meromorphic function is $f(z) = \frac{\cos(z)}{\sin(z)}$. Indeed the set of zeros is given by πk, $k \in Z$.

Exercise 5.10. Show that the set of meromorphic function in a domain D is a field. Further show that if f is meromorphic in D also f' is meromorphic in D.

Week 6

5.2.1 Extended complex plane.

Sometimes it is convenient to extend the complex plane by adding to $\mathbb{C}$ a single "point" ∞.

Denote by $\overline{\mathbb{C}}$ the set $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. We call $\overline{\mathbb{C}}$ the extended complex plane together with the following algebraic properties: for any $z \in \mathbb{C}$

- $\infty + z = z + \infty = \infty$
- if $z \neq 0$ then $z \cdot \infty = \infty \cdot z = \infty$
- $\infty \cdot \infty = \infty$
- $\frac{z}{\infty} = 0$

We say that a sequence of complex numbers $(z_n)_{n\in\mathbb{N}}$ converges to ∞ in $\overline{\mathbb{C}}$ if

$$\forall R > 0 \;\; \exists N \in \mathbb{N} \text{ such that } \forall n \geq N \Rightarrow |z_n| > R.$$

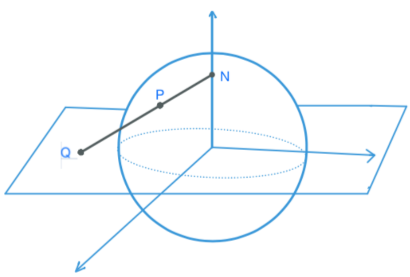
The points of $\overline{\mathbb{C}}$ can be identified with the points of the unit sphere $\mathbf{S}^2$ in $\mathbb{R}^3$ via the stereographic projection. Let

$$\mathbf{S}^2 = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$$

be the unit sphere in $\mathbb{R}^3$. The stereographic projection from the north pole $N = (0, 0, 1)$ associates to the point $P = (X, Y, Z) \in \mathbf{S}^2$, the point $(x, y, \zeta = 0) \in \mathbb{R}^3$ obtained from the intersection of the line passing through N and P and the plane $\zeta = 0$. Let $r(t)$ be the line that passes through N and P, then

$$r(t) = N + t(P - N) = (tX, tY, 1 + t(Z - 1)), \quad t \in \mathbb{R}.$$

The line intersects the plane $\zeta = 0$ when $1 + t(Z - 1) = 0$ namely at the value of $t = \frac{1}{1-Z}$



which corresponds to the point $Q = (\frac{X}{1-Z}, \frac{Y}{1-Z}, 0)$. Introducing the complex coordinate

$$z = x + iy = \frac{X + iY}{1 - Z}$$

we have a map $\sigma_1 : \mathbf{S}^2 \to \overline{\mathbb{C}}$ defined as

$$P = (X, Y, Z) \to z = x + iy := \sigma_1(P) = \frac{X + iY}{1 - Z}, \text{ when } Z \neq 1 \text{ and } \quad \sigma_1((0,0,1)) = \infty.$$

The map σ_1 establishes a one to one correspondence between $\mathbf{S}^2$ and $\overline{\mathbb{C}}$. Indeed the map has an inverse that can be constructed in a similar way.

Exercise 5.11. Construct the inverse map to σ_1.

Solution. Let $Q = (x, y, 0)$ be a point on the plane and consider the line $\ell(t)$ that passes through Q and N,

$$\ell(t) = N + t(Q - N) = (tx, ty, 1 - t), \quad t \in \mathbb{R}.$$

Then a point $(tx, ty, 1 - t)$ on the line $\ell(t)$ intersects the sphere $\mathbf{S}^2$ when

$$(tx)^2 + (ty)^2 + (1 - t)^2 = 1 \longrightarrow t = \frac{2}{x^2 + y^2 + 1} = \frac{2}{|z|^2 + 1}, \quad z = x + iy,$$

which corresponds to the point on the sphere

$$\left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, 1 - \frac{2}{|z|^2 + 1} \right).$$

Recalling that $x = \mathrm{Re}z$ and $y = \mathrm{Im}z$ we have obtained the inverse map $\sigma_1^{-1} : \overline{\mathbb{C}} \to \mathbf{S}^2$

$$\sigma_1^{-1}(z) = \left(\frac{2\mathrm{Re}z}{|z|^2 + 1}, \frac{2\mathrm{Im}z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right), \quad \sigma_1^{-1}(\infty) = N.$$

□

Note that a similar construction can be done using the stereographic projection to the south pole $S = (0, 0, -1)$. In this case the intersection of the line passing from S and P with the plane $(x, y, \zeta = 0)$ is the point Q' with coordinate

$$\sigma_2 : \mathbf{S}^2 \to \mathbb{C}, \quad \sigma_2(P) = Q' = z' = \frac{X - iY}{1 + Z}.$$

Note the relation between z and z':

$$zz' = \frac{X^2 + Y^2}{1 - Z^2} = 1.$$

The two open set $U_1 := \mathbf{S}^2 \backslash \{N\}$, and $U_2 := \mathbf{S}^2 \backslash \{S\}$ together with the maps σ_1 and σ_2 give to $\mathbf{S}^2$ the structure of a one dimensional complex compact manifold[3]

[3]A one dimensional (two real dimensions) complex manifold M is a topological space endowed with an atlas, namely a collection of charts $\{U_\alpha, \sigma_\alpha\}_{\alpha \in I}$ with $I = \{1, 2, \dots\}$ such that U_α is an open set, $\cup_{\alpha \in I} U_\alpha = M$ and each $\sigma_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism, namely a bijective continuous map with continuous inverse. When $U_\alpha \cap U_\beta \neq \emptyset$, $\alpha, \beta \in I$, the composition map (called transition function)

$$\sigma_\beta \circ \sigma_\alpha^{-1} : \sigma_\alpha(U_\alpha \cap U_\beta) \to \mathbb{C}$$

is holomorphic. (If the transition function is simply differentiable, the manifold is called a differentiable manifold). The most elementary one dimensional complex manifold is the complex plane $\mathbb{C}$ with just one chart $I = \{1\}$, $U_1 = \mathbb{C}$ and map σ_1 coincides with the identity. The Riemann sphere $\mathbb{S}$ is the next example, where now two charts are needed (U_1, σ_1) and (U_2, σ_2) (see above).

Indeed on the intersection $U_1 \cap U_2$ the maps $\sigma_j(U_1 \cap U_2) = \mathbb{C}\backslash\{0\}$, $j = 1, 2$ and the composition maps

$$\sigma_2 \circ \sigma_1^{-1}(z) = \frac{1}{z'}, \quad \sigma_1 \circ \sigma_2^{-1}(z') = \frac{1}{z}$$

are holomorphic maps from $\mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$.

Because of this correspondence, which was introduced by Riemann, the extended complex plane sometimes is called also the *Riemann sphere*.

A disk of radius R centred at infinity is the set of points $z \in \overline{\mathbb{C}}$ such that

$$\frac{1}{R} < |z| \leq \infty.$$

A holomorphic function on the circle of radius R centred at $z = \infty$ is the function holomorphic in $z' = \frac{1}{z}$ on the set $|z'| < R$. Such function admits the Taylor expansion in z'

$$f(z') = \sum_{n=0}^{\infty} c_n (z')^n$$

In the variable z such function takes the form

$$f(z'(z)) = \sum_{n=0}^{\infty} c_n z^{-n}.$$

Singularities at infinity. Let $f : A_{1/R,\infty}(0) \to \mathbb{C}$ be a holomorphic function. In order to classify the singularity of the function f at $z = \infty$ we consider the change of variable $z' = \frac{1}{z}$ and we classify the singularity of the holomorphic function

$$f(z(z')) : A_{0,R}(0) \to \mathbb{C}$$

at $z' = 0$. Such function has an *isolated* singularity at $z' = 0$.

Removable singularity at infinity. We say that $f(z)$ has a removable singularity at $z = \infty$ if $f(z(z'))$ has a removable singularity at $z' = 0$ For example, the function $f(z) = z^{-m}$ has a removable singularity at infinity for any positive integer m or $m = 0$.

Exercise 5.12. Let $f : A_{1/R,\infty}(0) \to \mathbb{C}$ be a holomorphic function. Prove that the following statements are equivalent:

(a) f has a removable singularity at infinity;

(b) $\lim_{z\to\infty} f(z)$ exists and is finite;

(c) $|f(z)|$ is bounded on $\bar{A}_{r,\infty}(0)$ for every $r \in (1/R, \infty)$.

Pole. We say that $f(z)$ has a pole of order k at infinity if $f(z(z'))$ has a pole of order k at zero.

Exercise 5.13. Let $f : A_{1/R,\infty}(0) \to \mathbb{C}$ be a holomorphic function. Prove that the following statements are equivalent:

(a) f has a pole of order k at infinity;

(b) $\lim_{z\to\infty} \frac{f(z)}{z^k} = w \neq 0$.

Essential singularity. We say that $f(z)$ has an essential singularity at $z = \infty$ if $f(z(z'))$ has an essential singularity at zero. For example, $\exp(z)$, $\sin(z)$, $\cos(z)$ have essential singularities at infinity.

Lemma 5.14. *The only holomorphic function on $\overline{\mathbb{C}}$ is the constant function.*

Proof. If $f(z)$ is holomorphic on $\mathbb{C}$ then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}$$

where the series converges for any $z \in \mathbb{C}$. Now let us check the behaviour of this function at infinity, by considering the function $f(z(z'))$ where $z = \frac{1}{z'}$. We have that

$$f(z(z')) = \sum_{n=0}^{\infty} a_n (z')^{-n}$$

which has a singularity in $z' = 0$ unless $a_n = 0$ for all $n > 0$. It follows that $f(z) = a_0$, namely a constant. □

Meromorphic functions in the extended complex plane. A function meromorphic in $\overline{\mathbb{C}}$ is a function that is holomorphic in $\overline{\mathbb{C}} \backslash \mathcal{S}$ where $\mathcal{S}$ is a discrete set in $\overline{\mathbb{C}}$ and f has poles at the points of $\mathcal{S}$. We claim that $\mathcal{S}$ contains a finite number of isolated points. Indeed if $\mathcal{S}$ had infinite cardinality, the limit point z_0 of the discrete set $\mathcal{S}$ would lie in $\overline{\mathbb{C}}$ because $\overline{\mathbb{C}}$ is compact, and therefore z_0 would be an accumulation point for $\mathcal{S}$, contradicting the fact the $\mathcal{S}$ is discrete. Since meromorphic functions in $\overline{\mathbb{C}}$ have only discrete set of poles, it follows that $\mathcal{S}$ must be finite. Rational functions are meromorphic functions in $\overline{\mathbb{C}}$, while the function $f(z) = \dfrac{1}{\sin(z)}$ is not a meromorphic function in $\overline{\mathbb{C}}$ because it has poles at $z_n = \pi n$ and the set $\mathcal{S} = \{z \in \mathbb{C} |\, z = \pi n,\ n \in \mathbb{Z}\}$ has an accumulation point at $\infty \in \overline{\mathbb{C}}$ and therefore it is not discrete.

One can show that the only meromorphic functions on $\overline{\mathbb{C}}$ are rational functions, namely ratio of polynomials.

5.3 Essential singularities

The behavior of a function f near a pole z_0 is relatively simple. In particular, we know that $\lim_{z \to z_0} |f(z)| = \infty$. However in the case of an essential singularity $|f(z)|$ will not have a limit as $z \to z_0$. The following result is classical.

Theorem 5.15. (Casorati–Weierstrass Theorem) *Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be an analytic function. Assume that z_0 is an essential singularity of f. Then the image of any punctured neighbourhood of z_0 is dense in $\mathbb{C}$, namely the set*

$$f(A_{0,r}(z_0)),$$

is dense in $\mathbb{C}$ for all $0 < r < R$.

Proof. Suppose the conclusion of the theorem is false. Then there exist $w_0 \in \mathbb{C}$, $r \in (0, R)$ and $c > 0$ such that

$$|f(z) - w_0| \geq c \qquad (z \in A_{0,r}(z_0)).$$

Let

$$g(z) := \frac{1}{f(z) - w_0}.$$

Then g is analytic on $A_{0,r}(z_0)$. Moreover,

$$|g(z)| \leq \frac{1}{c} \qquad (z \in A_{0,r}(z_0)).$$

Hence z_0 is a removable singularity of function g by Proposition 5.3 and

$$g(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m, \qquad z \in B_r(z_0).$$

Let a_k be the first nonzero coefficient of the above expansion, namely

$$g(z) = \sum_{m=k}^{\infty} a_m (z - z_0)^m, \qquad z \in B_r(z_0).$$

Then

$$\lim_{z \to z_0} (z - z_0)^k (f(z) - w_0) = \lim_{z \to z_0} \frac{(z - z_0)^k}{g(z)} = \frac{1}{a_k} \neq 0.$$

By Propositions 7.3,7.2 we conclude that z_0 is a pole of order k (when $k > 1$) or a removable singularity (when $k = 0$) of the function $f(z) - w_0$. Observe that the Laurent series of $f(z) - w_0$ differs from the Laurent series of $f(z)$ only in the zero–order coefficient. So z_0 is a pole or removable singularity of $f(z)$, which is a contradiction. □

In fact a much stronger result, known as Picard Theorem, is true. The proof requires techniques considerably beyond the reach of this course.

Theorem 5.16. (PICARD THEOREM) *Let $f : A_{0,R}(z_0) \to \mathbb{C}$ be a holomorphic function. Assume that z_0 is an essential singularity of f. Then the image of the function f of each $A_{0,r}(z_0)$, $0 < r < R$, assumes all complex values except possibly one.*

Proof. See CONWAY, pp.302–303. □

Remark 5.17. The exception mentioned in the theorem can really occur. For example, the function $f(z) = \exp(\frac{1}{z})$ misses the value 0 but attains all others. In other words, for any $\varepsilon > 0$ one has $f(A_{0,\varepsilon}(0)) = \mathbb{C} \setminus \{0\}$. On the other hand the functions $f(z) = \sin(\frac{1}{z})$ attains every value, that is for any $\varepsilon > 0$ one has $f(A_{0,\varepsilon}(0)) = \mathbb{C}$, as can easily be verified.

We recall that we defined entire functions as functions that are holomorphic in the whole $\mathbb{C}$. Clearly the polynomials are entire functions. All entire functions that are not polynomials are called *transcendental* entire functions. Examples are e^z, $\sin(z)$, $\cos(z)$. The following result is just an obvious consequence of the material of the section.

Lemma 5.18. *The entire function $f : \mathbb{C} \to \mathbb{C}$ is transcendental if and only if f has an essential singularity at infinity.*

Proof. Let us suppose that f is entire and transcendental, then it admits a Taylor series expansion convergent in $\mathbb{C}$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where an infinite number of coefficients $a_n \neq 0$, otherwise f would be a polynomial. Then we make the change of variable $z = \frac{1}{z'}$ and we have that

$$f(z(z')) = \sum_{n=0}^{\infty} a_n (z')^{-n}$$

which has a singularity in $z' = 0$ and the singularity is essential because an infinite number of coefficients $a_n \neq 0$. It follows that f as an essential singularity at $z = \infty$. The converse statement is quite straightforward. □

Mathematicians in this section

- *Karl Theodor Wilhelm Weierstrass (31st October 1815 – Berlin, 19th February 1897), German mathematician,*
- *Felice Casorati (17 December 1835 – 11 September 1890), Italian mathematician,*
- *Charles Emile Picard, FRS and FRSE (24 July 1856 – 11 December 1941) French mathematician.*

6 Residue Theorem

6.1 Winding number of a curve

In Example 2.8 we found that

$$\frac{1}{2\pi i}\int_{S_\rho^+(a)}\frac{1}{z-a}dz = 1.$$

If $S_\rho^-(a) := -S_\rho^+(a)$ is the circle traversed in the negative (clockwise) direction then obviously

$$\frac{1}{2\pi i}\int_{S_\rho^-(a)}\frac{1}{z-a}dz = -1.$$

If $\gamma_n(t) = a + \rho e^{it}$ $(0 \le t \le 2\pi n)$ then $\gamma_n = S_\rho^+(a) + \cdots + S_\rho^+(a)$ winds n times around the point a and we find that

$$\frac{1}{2\pi i}\int_{\gamma_n}\frac{1}{z-a}dz = n.$$

Now suppose that $\tilde{\gamma} : [0, 2\pi n] \to \mathbb{C}\setminus\{a\}$ is a closed contour that is homotopic to γ_n in $\mathbb{C}\setminus\{a\}$. Then by the Deformation Theorem again

$$\frac{1}{2\pi i}\int_{\tilde{\gamma}}\frac{1}{z-a}dz = n.$$

Exercise 6.1. Prove that

$$\frac{1}{2\pi i}\int_{S_\rho^+(a)}\frac{1}{z-z_0}dz = \begin{cases} 1, & \text{if } z_0 \text{ lies } \textit{inside } S_\rho^+(a)\text{, i.e. } |z_0 - a| < \rho, \\ 0, & \text{if } z_0 \text{ lies } \textit{outside } S_\rho^+(a)\text{, i.e. } |z_0 - a| > \rho. \end{cases}$$

Hint. Use Deformation Theorem and Exercise 3.8 when z_0 is inside $S_\rho^+(a)$. Use Cauchy Theorem when z_0 is outside $S_\rho^+(a)$.

These observations lead to the following definition.

Definition 6.2. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed contour and $z_0 \notin \gamma$. The winding number of γ with respect to z_0 indicated as $W(\gamma, z_0)$ (also called the *index* of γ with respect to z_0) is defined by

$$W(\gamma, z_0) := \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_0}dz.$$

Geometrically, this means that the contour γ winds $W(\gamma, z_0)$ times around the point z_0.

The definition of the winding number would be improper if the following result were not true.

Theorem 6.3. *Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a closed contour and $z_0 \notin \gamma$. Then $W(\gamma, z_0) \in \mathbb{Z}$.*

Proof. Assume for simplicity that γ is a smooth contour. Since $z_0 \notin \gamma$ the function θ : $[\alpha, \beta] \to \mathbb{C}$,

$$\theta(t) = \int_\alpha^t \frac{\gamma'(s)}{\gamma(s) - z_0}ds$$

is well defined, continuous and differentiable. Moreover,

$$\theta(\alpha) = 0, \quad \theta(\beta) = 2\pi i W(\gamma, z_0)$$

and performing the integral we obtain

$$\theta(t) = \log(\gamma(s) - z_0)|_{s=\alpha}^{s=t}.$$

In particular

$$\theta(\beta) = \log \frac{\gamma(\beta) - z_0}{\gamma(\alpha) - z_0} = \log(1),$$

because $\gamma(\beta) = \gamma(\alpha)$. This implies, because of the properties of the complex logarithm,

$$\theta(\beta) = 2\pi i n, \quad n \in \mathbb{Z}.$$

□

Exercise 6.4. For each given $n \in \mathbb{Z}$, construct a closed contour γ_n such that $W(\gamma, 0) = n$.

The following summarizes the most important properties of the winding number.

Exercise 6.5. Prove the following properties of the winding number:
(a) $W(-\gamma, z_0) = -W(\gamma, z_0)$;
(b) $W(\gamma_1 + \gamma_2, z_0) = W(\gamma_1, z_0) + W(\gamma_2, z_0)$;
(c) if $\gamma_1 \sim \gamma_2$ in $\mathbb{C} \setminus \{z_0\}$ then $W(\gamma_1, z_0) = W(\gamma_2, z_0)$.

Hint. (a) and (b) follows from Exercise 2.14, (c) follows from the Deformation Theorem.

Remark 6.6. In fact, the converse to the property (c) above is valid. Namely, if $W(\gamma_1, z_0) = W(\gamma_2, z_0)$ then $\gamma_1 \sim \gamma_2$ in $\mathbb{C} \setminus \{z_0\}$ (for the proof of this deep topological result, see Conway, p.90 and p.252). So the winding number completely describes the classes of equivalence of contours which are homotopic in $\mathbb{C} \setminus \{z_0\}$.

6.2 The residue Theorem

Let z_0 be an isolated singularity of a function f. By the Laurent Series Theorem f has a Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad (z \in A_{0,R}(z_0)). \tag{6.1}$$

Definition 6.7. The residue of a function f at an isolated singularity z_0 is the coefficient a_{-1} of the Laurent expansion (6.1) of f about z_0. This is denoted by

$$\operatorname{Res}(f, z_0) := a_{-1}.$$

Remark 6.8. By the formula (4.21) for the Laurent coefficients we know that

$$a_{-1} = \frac{1}{2\pi i} \int_{S_\rho^+(z_0)} f(w)dw \qquad (\rho \in (0, R)).$$

Therefore we can define the residue in the equivalent form

$$\operatorname{Res}(f, z_0) := \frac{1}{2\pi i} \int_{S_\rho^+(z_0)} f(w)dw \qquad (\rho \in (0, R)\,.$$

Thus if we know the Laurent coefficients we can evaluate certain integrals and vice versa. We will see later that in some cases, however, one can compute the residue of a function without having to find the full Laurent expansion or computing integrals.

6.3 Residue Theorem

The Residue Theorem, which is proved in this section, is one of the main results of Complex Analysis. It includes Cauchy's Theorem and Cauchy's Integral Formula as special cases and leads quickly to important applications. In particular, it becomes one of the most powerful tools of Analysis for evaluation of definite integrals.

Theorem 6.9. *Let $f : \mathbb{C}\backslash\{z_0\} \to \mathbb{C}$ be a holomorphic function with an isolated singularity in z_0 and $\gamma : [\alpha, \beta] \to \mathbb{C}\backslash\{z_0\}$ a closed contour. Then*

$$\int_\gamma f(z)dz = 2\pi i W(\gamma, z_0)\operatorname{Res}(f, z_0).$$

Proof. Since f is holomorphic in $\mathbb{C}\backslash\{z_0\}$ it has a Laurent expansion

$$f(z) = \sum_{n \neq -1} a_n (z - z_0)^n + \frac{a_{-1}}{z - z_0}, \quad z \in \mathbb{C}\backslash\{z_0\}.$$

Let us define the function $g(z)$ as

$$g(z) := \sum_{n \neq 1} a_n (z - z_0)^n.$$

The function $g(z)$ admits and anti-derivative in $\mathbb{C}\backslash\{z_0\}$ and the anti-derivative can be obtained by integrating the Laurent series term by term:

$$g(z) = G'(z), \quad G(z) := \sum_{n \neq -1} a_n \frac{(z - z_0)^{n+1}}{n + 1}.$$

Then by applying the fundamental theorem to the function $g(z)$, we obtain

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_\gamma g(z)dz + a_{-1} \int_\gamma \frac{dz}{z - z_0} \\
&= a_{-1} \int_\gamma \frac{dz}{z - z_0} \\
&= 2\pi i a_{-1} W(\gamma, z_0) \\
&= 2\pi i W(\gamma, z_0)\operatorname{Res}(f, z_0).
\end{aligned}$$

□

This result has a clear generalization to the case in which the function f has many isolated singularities. Now, the idea of the residue theorem is quite simple. Suppose that we integrate

$f(z)$ around a contour that encloses many isolated singularities of $f(z)$. We attempt to deform this contour to a point, but it gets snagged on the singularities. The snagged bits of the contour reduce to integrals around circles lying close to the singularity so that it can be evaluated as in theorem 6.9.

Theorem 6.10. *Let $D \subset \mathbb{C}$ be a domain and $\mathcal{S} = \{z_1, \ldots, z_m\}$ be a discrete set of points in D and let $f : D\backslash\mathcal{S} \to \mathbb{C}$ be a holomorphic function with isolated singularities in $\mathcal{S}$. Suppose that $\gamma : [\alpha, \beta] \to D\backslash\mathcal{S}$ is a close simple curve. Then the following relation holds*

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{z_k\in\mathcal{S}} W(\gamma, z_k)\mathrm{Res}(f, z_k).$$

Proof. The proof of this theorem follows the lines of the previous theorem. Let us first define for convenience the set $\mathcal{S}'$ of points $z_k \in \mathcal{S}$ such that $W(\gamma, z_k) \neq 0$. These points lie in the connected domain D' with boundary γ (because γ is a close simple curve). Then we consider $\epsilon > 0$ sufficiently small so that each open set $B_\epsilon(z_k) \subset D'$ and $B_\epsilon(z_k) \cap B_\epsilon(z_j) = \emptyset$ for $k \neq j$. We define the domain V as

$$V = D' \backslash \cup_{z_k\in\mathcal{S}'} \overline{B_\epsilon(z_k)}.$$

and its positively oriented boundary ∂V consists of

$$\partial V = \gamma - \sum_{z_k\in\mathcal{S}'} S_\epsilon^+(z_k)$$

where now we are also assuming that γ is positively oriented (anticlockwise). If γ is negatively oriented the oriented boundary of V is

$$\partial V = -\gamma - \sum_{z_k\in\mathcal{S}'} S_\epsilon^+(z_k).$$

Then the function f is holomorphic on the compact set $V \cup \partial V$ and applying Cauchy Theorem it follows that the integral on the boundary is equal to zero. Hence

$$\int_{\partial V} f(z)dz = 0 \to \int_{\pm\gamma} f(z)dz = \sum_{z_k\in\mathcal{S}'} \int_{S_\epsilon^+(z_k)} f(z)dz,$$

where the sign $\pm\gamma$ refers to a anticlockwise/clockwise oriented contour. In order to calculate the integral of each term $\int_{S_\epsilon^+(z_k)} f(z)dz$ we need to repeat the steps of the proof of the previous theorem. Indeed $f(z)$ near each singular point z_k has a Laurent expansion that can be written in the form

$$f(z) = \frac{a_{-1}^{(k)}}{z - z_k} + g_k(z), \quad g_k(z) = \sum_{n\neq -1} a_n^{(k)}(z - z_k)^n.$$

Then the function $g_k(z)$ has an antiderivative in $\overline{B_\epsilon(z_k)}$ so that we can conclude that

$$\int_{S_\epsilon^+(z_k)} f(z)dz = \int_{S_\epsilon^+(z_k)} g_k(z)dz + a_{-1}^{(k)} \int_{S_\epsilon^+(z_k)} \frac{dz}{z - z_k} = 2\pi i \mathrm{Res}(f, z_k).$$

Therefore

$$\int_{\pm\gamma} f(z)dz = \sum_{z_k\in\mathcal{S}'} \int_{S_\epsilon^+(z_k)} f(z)dz = 2\pi i \sum_{z_k\in\mathcal{S}'} \mathrm{Res}(f, z_k).$$

We recall that the winding number $W(\gamma, z_k)$ keeps track of the orientation of γ and whether z_k is in D' so that we can rewrite the above formula as

$$\int_\gamma f(z)dz = 2\pi i \sum_{z_k \in \mathcal{S}} W(\gamma, z_k)\text{Res}(f, z_k).$$

□

The above theorem can be generalized to any contour γ non necesseraly simple. By using the definition of winding number we arrive to the residue theorem in its general form.

Theorem 6.11. (Residue theorem) *Let $D \subseteq \mathbb{C}$ be a simply connected domain, and $\mathcal{S} = \{z_1, z_2, \cdots, z_m\} \subset D$ be a finite subset of D. Let $f : D \setminus \mathcal{S} \to \mathbb{C}$ be an analytic function with isolated singularities in $\mathcal{S}$ and $\gamma : [\alpha, \beta] \to D \setminus \mathcal{S}$ be a closed contour. Then*

$$\int_\gamma f = 2\pi i \sum_{z_k \in \mathcal{S}} W(\gamma, z_k)\text{Res}(f, z_k). \tag{6.2}$$

The proof of this theorem is omitted, but it is a simple generalization of the proof of theorem 6.10.

The Residue Theorem includes Cauchy's Theorem as a special case if we assume that $\mathcal{S} = \emptyset$ (note however that the proof of the Residue Theorem relies on Cauchy's theorem).

6.3.1 Singular integrals and principal value

Consider the integral

$$\int_a^b \frac{dx}{x-c}, \quad a < c < b\,.$$

We can compute it as an improper integral in the form

$$\begin{aligned}\int_a^b \frac{dx}{x-c} &= \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \left[-\int_a^{c-\epsilon_1} \frac{dx}{c-x} + \int_{c+\epsilon_2}^b \frac{dx}{c-x} \right] \\ &= \log\frac{b-c}{c-a} + \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_2 \to 0}} \log\frac{\epsilon_1}{\epsilon_2}.\end{aligned}$$

Clearly the limit of the last expression depends on how ϵ_1 and ϵ_2 goes to zero. Consequently the integral regarded as improper integral does not exist. However it can be given a meaning by chosing $\epsilon_1 = \epsilon_2 = \epsilon$.

Definition 6.12. The Cauchy principal value of the improper integral

$$\int_a^b \frac{dx}{x-c}, \quad a < c < b$$

is the expression

$$p.v. \int_a^b \frac{dx}{x-c} := \lim_{\epsilon \to 0} \left[\int_a^{c-\epsilon} \frac{dx}{x-c} + \int_{c+\epsilon}^b \frac{dx}{c-x} \right] = \log\frac{b-c}{c-a}.$$

Now consider a contour $\gamma : [\alpha, \beta] \to \mathbb{C}$, $t_0 \in [\alpha, \beta]$ and $\gamma(t_0) = z_0$, $\gamma(\alpha) = a$, $\gamma(\beta) = b$. Consider a circle of radius ϵ centred at z_0 with ϵ sufficiently small so that $\zeta_1(\epsilon)$ and $\zeta_2(\epsilon)$ are the points of intersection of this circle with the oriented contour γ and let L_ϵ be the path contained in γ connecting ζ_1 to ζ_2. Then we can defined the principal value of the contour integral

$$p.v. \int_\gamma \frac{dz}{z - z_0} = \lim_{\epsilon \to 0} \int_{\gamma - L_\epsilon} \frac{dz}{z - z_0} = \lim_{\epsilon \to 0} \left[\int_a^{\zeta_1(\epsilon)} \frac{dz}{z - z_0} + \int_{\zeta_2(\epsilon)}^b \frac{dz}{z - z_0} \right]$$
$$= \log \frac{b - z_0}{a - z_0} + \lim_{\epsilon \to 0} \left[\log(\zeta_1(\epsilon) - z_0) - \log(\zeta_2(\epsilon) - z_0) \right].$$

By construction $|\zeta_1(\epsilon) - z_0| = |\zeta_2(\epsilon) - z_0|$ so that

$$\lim_{\epsilon \to 0} \left[\log(\zeta_1(\epsilon) - z_0) - \log(\zeta_2(\epsilon) - z_0) \right] = i \lim_{\epsilon \to 0} \left[\arg(\zeta_1(\epsilon) - z_0) - \arg(\zeta_2(\epsilon) - z_0) \right] = i\pi$$

We conclude the following

Lemma 6.13. *Let γ be a contour in $\mathbb{C}$ and z_0 a point on the contour γ. Then*

$$p.v. \int_\gamma \frac{dz}{z - z_0} = \log \frac{b - z_0}{a - z_0} + i\pi.$$

If the contour is close

$$p.v. \int_\gamma \frac{dz}{z - z_0} = i\pi,$$

We are now ready to prove a generalisation of Cauchy integral theorem.

Theorem 6.14. (Generalized Cauchy Integral Formula) *Let $f : D \to \mathbb{C}$ be a meromorphic function with poles in $z_0, z_1, \ldots, z_N \in D$ and let us suppose that z_0 is a simple pole for f. Let us consider a close positively oriented contour $\gamma \in D$ and let $z_0 \in \gamma$ while the other poles do not lie on γ. Then*

$$p.v. \int_\gamma f(z)dz = \pi i \text{Res}(f, z_0) + 2\pi i \sum_{j=1}^N W(\gamma, z_j) \text{Res}(f, z_j) \tag{6.3}$$

Proof. The p.v. of the integral is not a close contour but $\gamma - L_\epsilon$ where L_ϵ is the path from $\zeta_1(\epsilon)$ to $\zeta_2(\epsilon)$ with ϵ sufficiently small so that these points are the only intersection of $S_\epsilon(z_0) \cap \gamma$. To make a close contour we add an arc C_ϵ of a circle of radius ϵ, oriented clockwise, centred at z_0 and connecting $\zeta_1(\epsilon)$ to $\zeta_2(\epsilon)$ so that $\Gamma_\epsilon = \gamma - L_\epsilon + C_\epsilon$ is a close contour that does not contain z_0. Then we can apply the Cauchy residue theorem

$$\int_{\Gamma_\epsilon} f(z)dz = 2\pi i \sum_{j=1}^N W(\Gamma_\epsilon, z_j) \text{Res}(f, z_j)$$

On the other hand, since the poles z_j do not lie on the contour Γ_ϵ by deformation theorem

$$\int_{\Gamma_\epsilon} f(z)dz = 2\pi i \sum_{j=1}^N W(\gamma, z_j) \text{Res}(f, z_j)$$

that is equivalent

$$\int_{\gamma - L_\epsilon} f(z)dz = -\int_{C_\epsilon} f(z)dz + 2\pi i \sum_{j=1}^{N} W(\gamma, z_j)\text{Res}(f, z_j).$$

To calculate $\int_{C_\epsilon} f(z)dz$ we consider the Laurent series expansion of f at z_0 convergent in $A_{0,\rho}(z_0)$, $\rho > \epsilon$:

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

so that

$$\begin{aligned}
\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)dz &= \lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{a_{-1}dz}{z - z_0} + \lim_{\epsilon \to 0} \int_{C_\epsilon} \sum_{j=0}^{\infty} a_j (z - z_0)^j \\
&= \lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{a_{-1}dz}{z - z_0} \\
&= a_{-1} \lim_{\epsilon \to 0} [\log(\zeta_2(\epsilon) - z_0) - \log(\zeta_1(\epsilon) - z_0)] = -\pi i a_{-1} = -\pi i \text{Res}(f, z_0).
\end{aligned}$$

We conclude that

$$\begin{aligned}
p.v. \int_{\gamma} f(z)dz = \lim_{\epsilon \to 0} \int_{\gamma - L_\epsilon} f(z)dz &= -\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)dz + 2\pi i \sum_{j=1}^{N} W(\gamma, z_j)\text{Res}(f, z_j) \\
&= \pi i \text{Res}(f, z_0) + 2\pi i \sum_{j=1}^{N} W(\gamma, z_j)\text{Res}(f, z_j).
\end{aligned}$$

□

6.4 Calculating residues

It is important that in many cases residues can be calculated without finding the full Laurent expansion of a function.

Example 6.15. Find the residue of $f(z) = \frac{\sin(z)}{z^2}$ at $z_0 = 0$.

Solution. We have

$$\frac{\sin(z)}{z^2} = \frac{1}{z^2} \underbrace{\left(z - \frac{z^3}{3!} + \dots\right)}_{\sin(z)} = \frac{1}{z} + \dots.$$

Hence $\text{Res}(f, 0) = 1$.

In general, if we are given a function f with an isolated singularity at z_0 then we proceed in the following way. First we decide whether we can find easily the first few terms of the Laurent expansion in $A_{0,R}(z_0)$. If so, the residue of f at z_0 will be the coefficient a_{-1} in the expansion. If not some other methods or rules can be used, according to the type of singularity. Consider several examples.

Removable singularity. If z_0 is a removable singularity of a function f, then the Laurent expansion of f about z_0 has no singular part. Hence $\text{Res}(f, z_0) = 0$, so removable singularities do not contribute to the value of integral in (6.2).

Pole. In the case of a simple pole one can develop easy criteria for calculating residues.

Lemma 6.16. *Let z_0 be a pole of order one of a function f. Then*

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Proof. If z_0 is a pole of order one of f then

$$f(z) = \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (z \in A_{0,R}(z_0)),$$

for some $R > 0$. Hence

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} \left\{ a_{-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1} \right\} = a_{-1},$$

as required. □

Lemma 6.16 can be easily extended to the case of higher order poles.

Lemma 6.17. *Let z_0 be a pole of order k of a function f. Set $\varphi(z) := (z - z_0)^k f(z)$. Then*

$$\text{Res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z \to z_0} \varphi^{(k-1)}(z).$$

Proof. If z_0 is a pole of order k of f then

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \frac{a_{-k+1}}{(z - z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (z \in A_{0,R}(z_0)),$$

for some $R > 0$. Hence

$$\varphi(z) = a_{-k} + a_{-k+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{k-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{k+n} \qquad (z \in A_{0,R}(z_0)).$$

Therefore

$$\varphi^{(k-1)}(z) = (k-1)! a_{-1} + \sum_{n=0}^{\infty} a_n \frac{(k+n)!}{(1+n)!} (z - z_0)^{1+n} \qquad (z \in A_{0,R}(z_0))$$

and

$$\text{Res}(f, z_0) = a_{-1} = \frac{1}{(k-1)!} \lim_{z \to z_0} \varphi^{(k-1)}(z)\,.$$

□

Example 6.18. Find the residues of $f(z) = \frac{z^2}{(z-1)^3(z+1)}$ at $z_0 = \pm 1$.

Solution. Clearly $z_0 = -1$ is a pole of order one. By Lemma 6.16 we compute

$$\mathrm{Res}(f,-1) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}\frac{z^2}{(z-1)^3} = -\frac{1}{8}.$$

We see also that $z_0 = 1$ is a pole of order $k = 3$. Thus

$$\varphi(z) := (z-z_0)^k f(z) = \frac{z^2}{z+1}, \qquad \varphi''(z) = \frac{2}{z+1} - \frac{4z}{(z+1)^2} + \frac{2z^2}{(z+1)^3}.$$

By Lemma 6.17 we compute

$$\mathrm{Res}(f,1) = \frac{1}{2!}\lim_{z\to 1}\varphi''(z) = \frac{1}{8}.$$

Exercise 6.19. Let $D \subseteq \mathbb{C}$ be a simply connected domain. Let $f : D \to \mathbb{C}$ be an analytic function, $z_0 \in D$ and $\gamma : [\alpha,\beta] \to D \setminus \{z_0\}$ a closed contour. Prove that

$$W(\gamma,z_0)f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}dw. \tag{6.4}$$

Compare this formula with (4.7).

Hint. Apply Lemma 6.17 to $g(z) = \frac{f(z)}{(z-z_0)^{n+1}}$.

Example 6.20. Evaluate

$$\int_{S_2(0)} \frac{1}{(z+1)(z+3)}dz.$$

Solution. The only pole of the integrand $f(z) = \frac{1}{(z+1)(z+3)}$ inside $S_1^+(0)$ is a simple pole at $z_1 = -1$. By Lemma 6.16, we obtain

$$\mathrm{Res}(f,z_1) = \lim_{z\to -1}(z+1)f(z) = \frac{1}{2}.$$

Further, it is geometrically clear that $W(S_2(0),-1) = 1$. Thus by the Residue Theorem,

$$\int_{S_2(0)} f(z)dz = 2\pi i\, W(S_2(0),-1)\mathrm{Res}(f,-1) = \pi i,$$

Example 6.21. Evaluate

$$\int_\gamma \frac{z^2}{(z-1)^3(z+1)}dz,$$

where γ is a square $[2,2i] + [2i,-2] + [-2,-2i] + [-2i,2]$.

Solution. The singularities of the integrand $f(z) = \frac{z^2}{(z-1)^3(z+1)}$ occur at ± 1. Thus, by the Residue Theorem

$$\int_\gamma f(z)dz = 2\pi i\left\{W(\gamma,-1)\mathrm{Res}(f,-1) + W(\gamma,1)\mathrm{Res}(f,1)\right\}.$$

It is geometrically evident that $W(\gamma,\pm 1) = 1$. Using calculations from Example 6.18 we evaluate

$$\int_\gamma f(z)dz = 2\pi i\left\{-\frac{1}{8} + \frac{1}{8}\right\} = 0.$$

Essential singularity. In the case of an essential singularity there are no simple rules like for poles, so we must rely on our ability to find the Laurent expansion.

Example 6.22. Find the residue of $f(z) = \exp(z + z^{-1})$ at $z_0 = 0$.

Solution. We have

$$f(z) = \exp(z + z^{-1}) = \exp(z)\exp(z^{-1}) = \left(1 + z + \frac{z^2}{2!} + \dots\right)\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots\right).$$

Gathering terms involving $\frac{1}{z}$ we get

$$\frac{1}{z}\left\{1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \dots\right\}.$$

Thus the residue is

$$\mathrm{Res}(f, 0) = 1 + \frac{1}{2!} + \frac{1}{2!3!} + \frac{1}{3!4!} + \dots.$$

One can not sum the series explicitly.

Residue at infinity. We have seen that the residue of a complex function f holomorphic in $A_{0,R}(z_0)$, $z_0 \in \mathbb{C}$, and $R > 0$, with Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ can equivalently be defined as the coefficient a_{-1},

$$\mathrm{Res}(f, z_0) := a_{-1} \tag{6.5}$$

or as a contour integral

$$\mathrm{Res}(f, z_0) := \frac{1}{2\pi i}\int_{S_\rho^+(z_0)} f(z)dz \qquad (\rho \in (0, R)). \tag{6.6}$$

Namely if we take (6.5) as a definition then (6.6) holds. Viceversa, if we take (6.6) as a definition, then (6.5) holds.

The point at infinity is a point on the sphere S^2 and we need to define a counter-clock wise contour around the point at infinity We consider a function $f : A_{1/R,\infty}(0) \to \mathbb{C}$ analytic for $R > 0$. Any circle $S_{1/\rho}^-(0)$ with $\rho < R$ is positively oriented with respect to the domain containing the point at infinity.

Definition 6.23. Let $f : A_{1/R,\infty}(0) \to \mathbb{C}$, $R > 0$ be analytic. The residue at infinity of the function f is defined as

$$\mathrm{Res}(f, z = \infty) := \frac{1}{2\pi i}\int_{S_{1/\rho}^-(0)} f(z)dz \qquad (0 < \rho < R). \tag{6.7}$$

To calculate such integral we cannot use directly the residue theorem because the function in not analytic inside the circle $S_{1/\rho}^-(0)$ and the following Lemma is needed.

Lemma 6.24. $f : A_{1/R,\infty}(0) \to \mathbb{C}$, $R > 0$ *be analytic with Laurent expansion*

$$f = \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}.$$

Then

$$\operatorname{Res}(f, z=\infty) = -\operatorname{Res}\left(\frac{f(z(z'))}{(z')^2}, z'=0\right) = -a_{-1}, \tag{6.8}$$

where $z = \frac{1}{z'}$.

Proof. To calculate the contour integral in (6.7) we need to make the change of coordinates $z' = 1/z$ and consider the function $f(z(z'))$ that is analytic in the punctured disc $0 < |z'| < R$ and it has an isolated singularity at $z' = 0$. The circle $S^-_{1/\rho}(0)$ defined in the variable $z = \frac{1}{\rho}e^{-it}$, $t \in [0, 2\pi]$ becomes the circle $S^+_\rho(0)$ in the variable $z' = \frac{1}{z}$. With the change of variable $z' = 1/z$, the contour integral becomes

$$\begin{aligned}
\operatorname{Res}(f, z=\infty) &:= \frac{1}{2\pi i}\int_{S^-_{1/\rho}(0)} f(z)dz \\
&= \frac{1}{2\pi i}\int_{S^+_\rho(0)} f(z(z'))\left(-\frac{dz'}{(z')^2}\right) = -\sum_{n=1}^{\infty}\frac{1}{2\pi i}\int_{S^+_\rho(0)}\frac{a_{-n}}{(z')^{2-n}}dz \\
&= -\sum_{n=1}^{\infty}\operatorname{Res}\left(\frac{a_{-n}}{(z')^{2-n}}, z'=0\right) = -a_{-1}.
\end{aligned}$$

where we exchange the integral with the sum because of uniform convergence of Laurent series, and in the last relation we use the definition of residue (6.7). □

The next lemma, applies to a function that has a finite number of isolated singularities. It states that the sum of all its residues, including the point at infinity is equal to zero.

Lemma 6.25. *Let* $f : \mathbb{C} \to \mathbb{C}$ *be a complex function analytic in* $\mathbb{C}\backslash S$ *where* $S = \{z_1, \dots z_m\}$ *is the finite set of isolated singularities of* f. *Then*

$$\sum_{j=1}^{m}\operatorname{Res}(f(z), z=z_j) = -\operatorname{Res}(f, z=\infty).$$

Proof. We consider a circle of radius R, with R sufficiently small so that $z_j \in B_{1/R}(0)$ for all $j = 1, \dots, m$. In the complement region $A_{1/R,\infty}(0)$ the function f is analytic. Then

$$\begin{aligned}
\sum_{j=1}^{m}\operatorname{Res}(f(z), z=z_j) &= \frac{1}{2\pi i}\int_{S^+_{1/R}(0)} f(z)dz \\
&= -\frac{1}{2\pi i}\int_{S^-_{1/R}(0)} f(z)dz \\
&= -\operatorname{Res}(f, z=\infty).
\end{aligned}$$

where in the last relation we use (6.8). □

Example 6.26. Let us consider the function $f(z) = \frac{1}{z-1}$ it has a pole at $z = 1$ with residue equal to one. Its Laurent expansion at $z = \infty$ is

$$f(z) = \frac{1}{z}\sum_{k=0}^{\infty}\frac{1}{z^n}$$

so that $\text{Res}(f(z), z = \infty) = -1$ and this shows that

$$\text{Res}(f(z), z = 1) + \text{Res}(f(z), z = \infty) = 0$$

Example 6.27. Let us consider the function $f(z) = \frac{1}{\prod_{j=1}^{m}(z-j)}$. It has clearly simple poles at the integer $j = 1, \ldots, m$. Using the lemma 6.25 we can claim that

$$\sum_{j=1}^{m} \text{Res}(f(z), z = j) = -\text{Res}(f(z), z = \infty)$$

On the other hand the residue at infinity is

$$\text{Res}(f(z), z = \infty) = -\text{Res}\left(\frac{f(z(z'))}{(z')^2}, z' = 0\right) = -\text{Res}\left(\frac{(z')^{m-2}}{\prod_{j=1}^{m}(1 - jz)}, z' = 0\right) = 0$$

and therefore

$$\sum_{j=1}^{m} \text{Res}(f(z), z = j) = -\text{Res}(f(z), z = \infty) = 0.$$

Week 7

6.5 Evaluating Integrals

We now illustrate the practical use of the Residue Theorem in evaluating some integrals.

It is remarkable that in some cases systematic methods for evaluating *real* integrals can be developed using the Residues Theorem of *complex* analysis. We consider only simplest results of this type. For other applications of the Residue Theorem in evaluation of real integrals see Stewart and Tall, Marsden, or any other textbook on Complex Analysis.

Integrals of type $\int_{\mathbb{R}} f(x)dx$. Consider integrals of the form

$$\int_{-\infty}^{+\infty} f(x)dx.$$

In what follows, $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ denotes the upper half–plane in $\mathbb{C}$, while $\bar{\mathbb{C}}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$ stands for the closed half–plane.

Theorem 6.28. *Let f be a complex analytic function in the domain $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ except for a finite number of isolated singularities $\{z_1, \dots, z_m\} \subset \mathbb{C}_+$ and continuous in $\mathbb{R}$. Suppose also that there exist $R > 0$, $M > 0$ and $\epsilon > 0$ such that*

$$|f(z)| \leq \frac{M}{|z|^{1+\epsilon}} \qquad (|z| > R,\ z \in \mathbb{C}_+ \cup \mathbb{R}). \tag{6.9}$$

Then

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f, z_k). \tag{6.10}$$

Proof. Let $r > R$. Consider the contour $\gamma_r = \tilde{\gamma}_r + [-r, r]$, where $\tilde{\gamma}_r = re^{it}$ ($t \in [0, \pi]$) is the "upper" half–circle. Assumption (6.9) implies that all the singularities of f lie inside γ_r. It is clear also that $W(\gamma_r, z_k) = 1$, for each k. Hence

$$\int_{\gamma_r} f = 2\pi i \sum_{k=1}^{m} \mathrm{Res}(f, z_k)$$

by the Residue Theorem. But

$$\int_{\gamma_r} f(z)dz = \int_{[-r,r]} f(z)dz + \int_{\tilde{\gamma}_r} f(z)dz = \int_{-r}^{r} f(x)dx + \int_0^{\pi} f(re^{is})ire^{is}ds.$$

By (2.1) and (6.9) we obtain

$$\left|\int_0^{\pi} f(re^{is})ire^{is}ds\right| \leq r\int_0^{\pi} \underbrace{|f(re^{is})|}_{\leq M/r^{1+\epsilon}} \underbrace{|e^{is}|}_{=1} ds \leq \frac{\pi M}{r^{\epsilon}} \to 0 \quad \text{as } r \to \infty,$$

since $\epsilon > 0$. Therefore

$$\lim_{r\to\infty}\int_{-r}^{r} f(x)dx = \underbrace{\lim_{r\to\infty}\int_{\gamma_r} f(z)dz}_{=2\pi i\sum_{k=1}^m \mathrm{Res}(f,z_k)} - \underbrace{\lim_{r\to\infty}\int_0^{\pi} f(re^{is})ire^{is}ds}_{\to 0} = 2\pi i\sum_{k=1}^{m}\mathrm{Res}(f,z_k).$$

On the other hand, from Calculus we know that $f(x)$ is integrable on $\mathbb{R}$ and

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{r} f(x)dx,$$

because f is continuous on $\mathbb{R}$ and because of the condition (6.9). Thus (6.15) follows. □

Remark 6.29. If the function f is analytic in the lower half space $C_- \cup \mathbb{R}$ and it has the decaying properties in (6.9) for $|z| > R$ and $z \in \mathbb{C}_- \cup \mathbb{R}$, then to evaluate the real integral (6.15) one has to calculate the sum of residues in the lower half space. Note that if $f(\mathbb{R}) \subset \mathbb{R}$ then the integral is real.

Example 6.30. Evaluate

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} \qquad (a, b > 0,\ a \neq b).$$

Solution. The singularities of the integrand

$$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

occur at $\pm ia$ and $\pm ib$. In particular, f is analytic in $\mathbb{C}_+$, except for poles of order one $\{ia, ib\} \subset \mathbb{C}_+$. Obviously, f satisfies the assumption (6.9). So all the conditions of Theorem 6.28 are verified and hence

$$\int_{-\infty}^{+\infty} f(x)dx = 2\pi i\left\{\mathrm{Res}(f, ia) + \mathrm{Res}(f, ib)\right\}.$$

By Lemma 6.16 we compute

$$\mathrm{Res}(f, ia) = \lim_{z\to ia}\frac{z-ia}{(z^2+a^2)(z^2+b^2)} = \frac{1}{2ia(b^2-a^2)}.$$

In the similar way,

$$\mathrm{Res}(f, ib) = \lim_{z\to ib}\frac{z-ib}{(z^2+a^2)(z^2+b^2)} = \frac{1}{2ib(a^2-b^2)}.$$

Therefore

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}.$$

Exercise 6.31. (Cauchy integral formula for the half–space) Let f be a complex function such that $f(\mathbb{R}) \subseteq \mathbb{R}$. Suppose that f is analytic in the closed upper half–plane $\bar{\mathbb{C}}_+$ and that there exist $M > 0$ and $\epsilon > 0$ such that

$$|f(z)| \leq \frac{M}{|z|^\epsilon} \qquad (z \in \mathbb{C}_+).$$

Prove the integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(x)}{x - z_0} dx \qquad (z_0 \in \mathbb{C}_+).$$

Hint. Use Theorem 6.28.

Trigonometric Integrals. Consider integrals of the form

$$\int_0^{2\pi} Q(\cos(\varphi), \sin(\varphi))\, d\varphi,$$

where $Q = Q(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is a *rational function*, that is

$$Q(x, y) = \frac{q_m(x, y)}{p_n(x, y)},$$

where $q_m(x, y)$ and $p_n(x, y)$ are real polynomials of orders m and n.

Assume that $p_n(x, y)$ has no roots on the unit circle $S_1(0)$. Then $Q(x, y)$ is continuous on the unit circle. The substitution $z = e^{i\varphi}$ may be used to convert such trigonometric integrals to those involving rational complex functions. Indeed, after the substitution $z = e^{i\varphi}$ we obtain

$$\cos(\varphi) = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin(\varphi) = \frac{1}{2i}\left(z - \frac{1}{z}\right) \qquad (z = e^{i\varphi},\ \varphi \in [0, 2\pi]).$$

Introduce the function

$$f(z) = \frac{Q\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)}{iz}. \tag{6.11}$$

Thus $f : \mathbb{C} \to \mathbb{C}$ is a rational function (cf. Exercise 5.7) that has no poles on $S_1(0)$ and

$$f(x + iy) = Q(x, y) \qquad (z = x + iy \in S_1(0)).$$

Then by the Residue Theorem we obtain the formula

$$\int_0^{2\pi} Q(\cos(\varphi), \sin(\varphi)) d\varphi = \int_0^{2\pi} f(e^{i\varphi}) i e^{i\varphi} d\varphi = \int_{S_1^+(0)} f(z) dz = 2\pi i \sum_{z_k \in \mathcal{S}} \operatorname{Res}(f, z_k), \tag{6.12}$$

where $\mathcal{S} = \{z_1, \ldots, z_m\}$ is the set of poles of f inside of the unit circle $S_1(0)$.

Example 6.32. For $a > 1$, evaluate

$$\int_0^{2\pi} \frac{d\varphi}{a + \sin(\varphi)}.$$

Solution. From (6.11) we obtain

$$f(z) = \frac{2}{z^2 + 2iaz - 1}.$$

The only pole inside $S_1^+(0)$ is a pole of order one at $z_1 = -ia + i\sqrt{a^2 - 1}$. By Lemma 6.16, we obtain

$$\operatorname{Res}(f, z_1) = \lim_{z \to z_1} (z - z_1) f(z) = \frac{1}{i\sqrt{a^2 - 1}}.$$

Thus by (6.12) we conclude that

$$\int_0^{2\pi} \frac{d\varphi}{a + \sin(\varphi)} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

Evaluation of integrals of the form $\int_0^\infty x^{a-1}f(x)dx$ with a real non integer constant.

Theorem 6.33. *Suppose that f is analytic in $\mathbb{C}$ except for a finite number of poles $\{z_1, \dots, z_m\}$ with $z_j \in \mathbb{C}\backslash[0,\infty)$ for $j = 1, \dots, m$ and suppose that $f(\mathbb{R}) \subseteq \mathbb{R}$. Let $a \in \mathbb{R}\backslash\mathbb{Z}$ and suppose also that there exist constants $R > 0$, $M_1 > 0$ and $\delta_1 > 0$ such that*

$$|f(z)z^{a-1}| \leq \frac{M_1}{|z|^{1+\delta_1}} \qquad |z| > R, \tag{6.13}$$

and there exists constants $r > 0$, $M_2 > 0$ and $\delta_2 > 0$ such that

$$|f(z)z^{a-1}| \leq M_2|z|^{\delta_2 - 1} \qquad |z| < r. \tag{6.14}$$

Then

$$\int_0^{+\infty} x^{a-1}f(x)dx = \frac{2\pi i}{1 - e^{2\pi i(a-1)}} \sum_{k=1}^m \mathrm{Res}(f(z)z^{a-1}, z_k). \tag{6.15}$$

Proof. By conditions (6.13) and (6.14) the function $x^a f(x) \to 0$ as $x \to 0$ and $x \to \infty$ so that the integral $\int_0^\infty x^{a-1}f(x)dx$ converges at the upper and lower limit of integration. The new feature is now that $z^{a-1}f(z)$ is not a single-valued function. This is however the circumstance that makes it possible to find the integral form 0 to ∞. We chose the principal branch of z^{a-1} as $0 \leq \arg z < 2\pi$ so that the line $(0,\infty)$ is a branch cut for the function z^{a-1} and for

$$z \to ze^{2\pi i} \implies z^{a-1} \to z^{a-1}e^{2\pi i(a-1)}.$$

The function $z^{a-1}f(z)$ is analytic in $\mathbb{C}\backslash[0,\infty)$ except for a finite number of poles for the rational function $f(z)$. In order to compute the integral we consider the close contour $\Gamma = S_\epsilon^-(0) \cup \mathcal{S}_{1/\epsilon}^+(0) \cup \gamma_+ \cup \gamma_-$ where $\gamma_+ = [\epsilon, 1/\epsilon]$ on the upper side of the branch cut $(0,\infty)$ and $\gamma_- = [1/\epsilon, \epsilon]$ on the lower side of the branch cut. We chose ϵ sufficiently small so that the contour Γ contains all the poles $z_1, \dots, z_m$ of the function $f(z)$. Then, by applying the residue theorem we obtain

$$\int_\Gamma z^{a-1}f(z)dz = 2\pi i \sum_{j=1}^m \mathrm{Res}(f, z_j).$$

On the other hand

$$\begin{aligned}\int_\Gamma z^{a-1}f(z)dz &= \int_{S_\epsilon^-(0)} z^{a-1}f(z)dz + \int_{S_{1/\epsilon}^+(0)} z^{a-1}f(z)dz + \int_\epsilon^{1/\epsilon} x^{a-1}f(x)dx + e^{2\pi i(a-1)}\int_{1/\epsilon}^\epsilon x^{a-1}f(x)dx \\ &= \int_{S_\epsilon^-(0)} z^{a-1}f(z)dz + \int_{S_{1/\epsilon}^+(0)} z^{a-1}f(z)dz + (1 - e^{2\pi i(a-1)})\int_\epsilon^{1/\epsilon} x^{a-1}f(x)dx.\end{aligned}$$

Combining the last two equations we have that

$$\begin{aligned}\int_\epsilon^{1/\epsilon} x^{a-1}f(x)dx &= \frac{2\pi i}{1 - e^{2\pi i(a-1)}} \sum_{j=1}^m \mathrm{Res}(f(z)z^{a-1}, z_j) \\ &\quad - \frac{\left(\int_{S_\epsilon^-(0)} z^{a-1}f(z)dz + \int_{S_{1/\epsilon}^+(0)} z^{a-1}f(z)dz\right)}{1 - e^{2\pi i(a-1)}}.\end{aligned} \tag{6.16}$$

We need to show that the last term in the above identity tends to zero as $\epsilon \to 0$. One has by the estimation lemma and (6.14)

$$\left|\int_{S_\epsilon^-(0)} z^{a-1}f(z)dz\right| \leq 2\pi\epsilon \sup_{z\in S_\epsilon(0)} |z^{a-1}f(z)| < 2\pi\epsilon M_2\epsilon^{\delta_2-1} = 2\pi M_2\epsilon^{\delta_2},$$

where we assume ϵ sufficiently small so that $\epsilon < r$, with r defined in (6.14). In the same way by the estimation lemma and (6.13) we have

$$\left|\int_{S_{1/\epsilon}^+(0)} z^{a-1}f(z)dz\right| \leq 2\pi/\epsilon \sup_{z\in S_{1/\epsilon}(0)} |z^{a-1}f(z)| < \frac{2\pi}{\epsilon}M_1\epsilon^{\delta_1+1} = 2\pi M_1\epsilon^{\delta_1},$$

where again we assume ϵ sufficiently small so that $1/\epsilon > R$ with R defined in (6.13). Therefore we conclude that in the limit $\epsilon \to 0$ the relation (6.16) becomes (6.15).

□

Example 6.34. Let us calculate the integral

$$\int_0^\infty \frac{x^{a-1}}{x+1}dx, \quad 0 < a < 1.$$

We clearly have that the function $\frac{z^{a-1}}{z+1}$ satisfies the conditions (6.13) and (6.14). The function $1/(z+1)$ has the only simple pole at $z = -1$ so that

$$\int_\epsilon^{1/\epsilon} \frac{x^{a-1}}{x+1}dx = \frac{2\pi i}{1-e^{2\pi i(a-1)}}\text{Res}(\frac{z^{a-1}}{z+1}, z=-1) - \frac{\left(\int_{S_\epsilon^-(0)} \frac{z^{a-1}}{z+1}dz + \int_{S_{1/\epsilon}^+(0)} \frac{z^{a-1}}{z+1}dz\right)}{1-e^{2\pi i(a-1)}}.$$

where $z = -1 = e^{\pi i}$. In the limit $\epsilon \to 0$ we obtain

$$\int_0^\infty \frac{x^{a-1}}{x+1}dx = \frac{2\pi i}{1-e^{2\pi i(a-1)}}(e^{\pi i})^{a-1} = \frac{\pi}{\sin(\pi(1-a))}.$$

Example 6.35. Singular integrals. In some cases one needs to calculate a real integral that, using complex analysis, becomes a singular integral. Let us consider the classical example

$$\int_0^\infty \text{sinc}(x)dx$$

where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$. Note that $\lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} = 1$ and $x = 0$ is a removable singularity. Since $\text{sinc}(x)$ is an even function, we can consider this integral on the whole real axis and as the imaginary part of $\frac{e^{i\pi z}}{\pi z}$, namely

$$\int_0^\infty \text{sinc}(x)dx = \frac{1}{2}\text{Im}\int_{-\infty}^\infty \frac{e^{i\pi z}}{\pi z}dz,$$

where the r.h.s. is a singular integral. Indeed the function $\frac{e^{i\pi z}}{\pi z}$ has a pole at $z = 0$ on the contour of integration and no other poles. Since $e^{iz\pi}$ is exponentially small for $\text{Im}(z) > 0$ we can use contour deformation and close the integral on the upper half space with a semicircle of radius R and let $R \to \infty$. Then we apply the generalized Cauchy integral formula of theorem 6.14 to obtain

$$p.v. \int_{-\infty}^{\infty} \frac{e^{i\pi z}}{\pi z} dz = \pi i \text{Res}(f, 0) = i$$

We conclude that

$$\int_0^{\infty} \text{sinc}(x) dx = \frac{1}{2}.$$

6.6 Discrete application of the Residue theorem

Infinite sum. Consider a series of the form

$$S = \sum_{n=-\infty}^{+\infty} f(n)$$

where $f(z)$ is a holomorphic function with possibly some finite number of poles off the real axis.

Theorem 6.36. *Let f be a holomorphic function with possibly some finite number of poles off the real axis. Furthermore we assume that there is $\epsilon > 0$ such that*

$$\lim_{|z|\to\infty} |f(z)||z|^{1+\epsilon} = c \tag{6.17}$$

where c is a constant. Then

$$\sum_{n\in\mathbb{Z}} f(n) = -\sum_{z=poles\,of\,f} \text{Res}(f(z)\pi\cot(\pi z), z). \tag{6.18}$$

Proof. In order to perform the sum, we find an auxiliary function $g(z)$ with simple poles at the integers and with residue equal to 1. Typically

$$g(z) = \pi\cot(\pi z) = \frac{\pi\cos(\pi z)}{\sin(\pi z)}$$

which has simple poles at the integers $n \in \mathbb{Z}$ with residue equal to one. Let γ_N be the positively oriented square with vertices $N + \frac{1}{2} \pm i(N + \frac{1}{2})$ and $-(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$ with N positive integer. Namely each side of the square has length $2N + 1$. We consider the integral

$$\frac{1}{2\pi i} \int_{\gamma_N} f(z) \frac{\pi\cos(\pi z)}{\sin(\pi z)} dz.$$

We consider N sufficiently large so that all poles of f are inside γ_N and we can apply the residue theorem:

$$\frac{1}{2\pi i} \int_{\gamma_N} f(z) \frac{\pi\cos(\pi z)}{\sin(\pi z)} dz = \sum_{n\in\mathbb{Z}} f(n) + \sum_{z=poles\,of\,f} \text{Res}(f(z)\pi\cot(\pi z), z).$$

Next we want to show that the integral $\int_{\gamma_N} f(z)\dfrac{\pi\cos(\pi z)}{\sin(\pi z)}dz$ is equal to zero. Given the decay condition (6.17) of $f(z)$ it can be checked, using the Estimation Lemma, that for N sufficiently large

$$\left|\int_{\gamma_N} f(z)\frac{\pi\cos(\pi z)}{\sin(\pi z)}dz\right| \leq 4\pi(2N+1)\sup_{z\in\gamma_N}|\frac{\cos(\pi z)}{\sin(\pi z)}|\frac{c_1}{N^{1+\epsilon}} \to 0 \text{ as } N\to\infty$$

where $c_1 > c > 0$ and we use the fact that $\sup_{z\in\gamma_N}|\dfrac{\cos(\pi z)}{\sin(\pi z)}|$ is bounded as $N\to\infty$. Indeed we have

$$\cot(z) = \frac{\cos(x+iy)}{\sin(x+iy)} = i\frac{\cos x\cosh y - i\sin x\sinh y}{-\cos x\sinh y + i\sin x\cosh y}$$

The right vertical side of the square γ_N is parametrized by $(N+\frac{1}{2})(1+it)$, with $t\in[-1,1]$ so that the cotangent can be bounded by a constant, namely

$$|\cot\left(\pi(N+\frac{1}{2})(1+it)\right)| = \left|\frac{\cos(\pi(N+\frac{1}{2}))\cosh(t\pi(N+\frac{1}{2})) - i\sin(\pi(N+\frac{1}{2}))\sinh(t\pi(N+\frac{1}{2}))}{-\cos(\pi(N+\frac{1}{2}))\sinh(t\pi(N+\frac{1}{2})) + i\sin(\pi(N+\frac{1}{2}))\cosh(\pi t(N+\frac{1}{2}))}\right|$$

$$= \left|\frac{\sinh(\pi t(N+\frac{1}{2}))}{\cosh(\pi t(N+\frac{1}{2}))}\right| < M, \quad t\in[-1,1]$$

because $\tanh(\pi t(N+\frac{1}{2}))$ is uniformly bounded by a constant M when $t\in[-1,1]$. In a similar way one can show that $\cot(\pi z)$ is bounded on all other sides of the square γ_N. Therefore applying the residue theorem we obtain

$$0 = \lim_{N\to\infty}\frac{1}{2\pi i}\int_{\gamma_N} f(z)\frac{\pi\cos(\pi z)}{\sin(\pi z)}dz = \sum_{n\in\mathbb{Z}} f(n) + \sum_{z=poles\,of\,f}\text{Res}(f(z)\pi\cot(\pi z), z)$$

so that

$$\sum_{n\in\mathbb{Z}} f(n) = -\sum_{z=poles\,of\,f}\text{Res}(f(z)\pi\cot(\pi z), z). \tag{6.19}$$

□

Example 6.37. The goal is to show that

$$S = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

For the purpose we consider the sum

$$I(a) = \sum_{n\in\mathbb{Z}}\frac{1}{n^2+a^2}$$

where $a\in\mathbb{R}$ and we take $f(z) = \dfrac{1}{z^2+a^2}$ which has simple poles at $z = \pm ia$. Then by (6.19)

$$I(a) = \sum_{n\in\mathbb{Z}}\frac{1}{n^2+a^2} = -\text{Res}\left(\frac{\pi\cot(\pi z)}{z^2+a^2}, z = ia\right) - \text{Res}\left(\frac{\pi\cot(\pi z)}{z^2+a^2}, z = -ia\right) = \pi\frac{\coth(\pi a)}{a}$$

Next we observe that

$$\frac{1}{2}\lim_{a\to 0}(I(a)-\frac{1}{a^2})=\frac{1}{2}\lim_{a\to 0}\sum_{n\in\mathbb{Z},n\neq 0}\frac{1}{n^2+a^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}=S$$

So we need just to calculate

$$\frac{1}{2}\lim_{a\to 0}(I(a)-\frac{1}{a^2})=\frac{1}{2}\lim_{a\to 0}\left(\pi\frac{\coth(\pi a)}{a}-\frac{1}{a^2}\right)=\frac{\pi^2}{6}.$$

We conclude that $S=\sum_{n=1}^{\infty}\frac{1}{n^2}=\pi^2/6$. Note that in this way we have calculated the Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

at $s=2$.

Exercise 6.38. Applying the same ideas as above you can show that

$$\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^3}{90},\quad \zeta(6)=\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^5}{945},\quad \zeta(8)=\sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^7}{4725\times 2},\quad \dots$$

Remark 6.39. For alternating serie $\sum_{n=-\infty}^{+\infty}(-1)^n f(n)$ one uses instead of $\cot(\pi z)$ the function $g(z)=\frac{\pi}{\sin(\pi z)}$ that has residue $(-1)^n$ at the integer n.

Week 8

7 Zeros of analytic functions

7.1 Zeros.

Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic function. We say that a point $z_0 \in D$ is a zero of f iff $f(z_0) = 0$.

Classification of zeros. Let z_0 be a zero of an analytic function f. Expanding f in Taylor series about z_0 we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (z \in B_R(z_0))$$

where $B_R(z_0) \subset D$. Then $a_0 = f(z_0) = 0$ and two different possibility can occur:

i) all the coefficients a_n are zero, and then $f(z) \equiv 0$ in $B_R(z_0)$,

ii) there exists $m \in \mathbb{N}$ such that $a_0 = a_1 = \cdots = a_{m-1} = 0$, but $a_m \neq 0$.

In the case (ii) we say that z_0 is a zero of f of order m. Sometime zeros of order 1 are called *simple* zeros. For example, the function $f(z) = z^m$ has a zero of order m at $z_0 = 0$.

Example 7.1. Let $p_n(z)$ be a polynomial and z_0 be a root of p_n of multiplicity m (see Exercise 5.7). Then z_0 is a zero of p_n of order m.

Exercise 7.2. Prove that the following conditions are equivalent:

a) z_0 is a zero of a function f of order m;

b) $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$;

c) f can be represented as

$$f(z) = (z - z_0)^m h(z) \qquad (z \in B_R(z_0)),$$

where h is analytic in $B_R(z_0)$ and $h(z_0) \neq 0$.

d) z_0 is a pole of order m of the function $\frac{1}{f(z)}$.

Remark 7.3. Proposition 5.5 implies that if z_0 is a pole of order m of the function $f(z)$ then the function $\frac{1}{f(z)}$ is analytic at z_0, and z_0 is its zero of order m. However, if z_0 is an essential singularity of $f(z)$ then $\frac{1}{f(z)}$ can not be analytic at z_0. This follows from the Casorati–Weierstrass Theorem (explain how). Moreover if z_0 is an essential singularity of $f(z)$, then there is more then one option for $1/f(z)$:

- z_0 is an essential singularity of $\frac{1}{f(z)}$
- z_0 is a *cluster point* of singularities, i.e. a limit point of isolated singularity.

For example $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$ and also the function $1/f(z) = e^{-\frac{1}{z}}$. The function $f(z) = \sin(1/z)$ has an essential singularity at $z = 0$ while the function $1/f(z) = 1/\sin(\frac{1}{z})$ has poles at $z = \frac{1}{2\pi n}$ that are clustering at $z = 0$ as $n \to \infty$. So in this case $z = 0$ is a cluster point of singularities and not an essential singularity.

Example 7.4. Find zeros and their orders of the function

$$f(z) = \frac{(z^2+1)^2}{z}.$$

Solution. It is clear that zeros of f occur at the roots of $(z^2+1)^2 = 0$, that is at $z = \pm i$. To find the order of zeros we represent f as follows:

$$\frac{(z^2+1)^2}{z} = \begin{cases} (z-i)^2 g(z), & \text{where } g(z) = \frac{(z+i)^2}{z} \text{ is analytic at } z = i \text{ and } g(i) = 4i \neq 0, \\ (z+i)^2 g(z), & \text{where } g(z) = \frac{(z-i)^2}{z} \text{ is analytic at } z = -i \text{ and } g(-i) = 4i \neq 0. \end{cases}$$

So by Exercise 7.2 (c) we conclude that $z = i$ and $z = -i$ are both zeros of f of order 2.

Example 7.5. Find zeros and their orders of the function

$$f(z) = \cos(z).$$

Solution. It is clear that zeros of f occur at $z_n = (n + \frac{1}{2})\pi$, $(n \in \mathbb{Z})$. Observe that

$$\cos'(z_n) = -\sin(z_n) = (-1)^{n+1} \neq 0.$$

So by Exercise 7.2 (b) we conclude that $z_n = (n + \frac{1}{2})\pi$ are zeros of f of order 1.

Definition 7.6. We say that a zero z_0 of an analytic function f is <u>isolated</u> iff there exists $r > 0$ such that $f(z) \neq 0$ on $A_{0,r}(z_0)$.

Lemma 7.7. *A zero of finite order of an analytic function is isolated.*

Proof. Write

$$f(z) = (z - z_0)^m g(z) \qquad (z \in B_R(z_0)),$$

where g is analytic in $B_R(z_0)$ and $g(z_0) \neq 0$. In particular, g is continuous at z_0. Hence there exists $r > 0$ such that for any $z \in B_r(z_0)$ one has $g(z) \neq 0$. But then $f(z) = (z - z_0)^m g(z) \neq 0$ on $A_{0,r}(z_0)$, since $g(z) \neq 0$ on $A_{0,r}(z_0)$. □

In the previous section we have considered infinite series representing analytic functions in a domain D. Now we consider infinite products.

7.2 Infinite products

The natural question is whether it is possible to represent analytic functions via their zeros. Let $\{a_j\}_{j=1}^{\infty}$, with $a_j \neq -1$, be a sequence of complex numbers and consider the product

$$P_n = \prod_{j=1}^{n} (1 + a_j).$$

If the limit $\lim_{n\to\infty} P_n$ exists and it is equal to P we write

$$P := \prod_{j=1}^{\infty}(1 + a_j)$$

Clearly, by taking the logarithmic, the infinite product becomes an infinite sum

$$S = \sum_{j=1}^{\infty} \log(1 + a_j)$$

where log is the principal branch of the logarithmic and the series means that the partial sums $S_n = \sum_{j=1}^{n} \log(1 + a_j)$ has a limit as $n \to \infty$. Clearly the sum converges if $|a_j| \to 0$ as $j \to \infty$.

Definition 7.8. Let $\mathrm{Re}(a_j) > -1$ for all $j \geq 1$, the product $\prod_{j=1}^{\infty}(1 + a_j)$ is said to converge absolutely if the series $\sum_{j=1}^{\infty} \log(1 + a_j)$ converges absolutely.

To have an estimate of the convergence we have the following lemma.

Lemma 7.9. *For* $|z| \leq \frac{1}{2}$

$$\frac{1}{2}|z| \leq |\log(1 + z)| \leq \frac{3}{2}|z| \tag{7.1}$$

Proof. We consider the Taylor series expansion of log near $z = 0$

$$\log(1 + z) = \sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots$$

that converges for $|z| < 1$. So for $|z| < 1$ we have

$$\begin{aligned} \left|1 - \frac{1}{z}\log(1 + z)\right| &= \left|\sum_{n=2}^{\infty}(-1)^n \frac{z^{n-1}}{n}\right| \leq \frac{1}{2}\sum_{n=1}^{\infty}|z^n| \\ &= \frac{1}{2}\frac{|z|}{1 - |z|} \overset{|z|\leq\frac{1}{2}}{\leq} \frac{1}{2} \end{aligned}$$

which implies

$$||z| - |\log(1 + z)|| \leq |z - \log(1 + z)| < \frac{1}{2}|z|$$

where in the first inequality we use the reverse triangular inequality. □

The above lemma enables to conclude that the series $\sum_{j=1}^{\infty} |z_j|$ converges, if and only if the series $\sum_{j=1}^{\infty} |\log(1 + z_j)|$ converges, because the terms of the logarithmic series are dominated from above by $\frac{3}{2}|z_j|$ and from below by $\frac{1}{2}|z_j|$. We have to keep in mind that $|z_j| < \frac{1}{2}$ for j sufficiently large (why?). Then the convergence of $P_n = e^{S_n}$ is established by the absolute convergence of S_n. We are now ready to prove the following

Theorem 7.10. *Let D be a domain in $\mathbb{C}$ and $\{f_n(z)\}_{n=1}^{\infty}$ a sequence of holomorphic functions in D. Suppose that there is $n_0 > 0$ such that for all $z \in D$ and $n > n_0$, $|f_n(z)| < M_n$ for some constant M_n and $\sum_{n=n_0+1}^{\infty} M_n < \infty$. Then the product*

$$\prod_{n=1}^{\infty}(1 + f_n(z))$$

converges uniformly to an analytic function $f(z)$ in D.

Proof. Since the series $\sum_{n=n_0+1}^{\infty} M_n$ is convergent, there is $N > n_0$ such that $M_n < \frac{1}{2}$ for all $n \geq N$. Let $k > m > N$, and $S_m = \sum_{n=N}^{m} \log(1 + f_n(z))$, then we have by Lemma 7.9

$$|S_m(z)| \leq \sum_{n=N}^{m} |\log(1 + f_n(z))| \leq \frac{3}{2}\sum_{n=N}^{m} |f_n(z)| \leq \frac{3}{2}\sum_{n=m}^{\infty} M_n := M < \infty \tag{7.2}$$

$$\begin{aligned} |S_m(z) - S_k(z)| &= |\sum_{n=m+1}^{k} \log(1 + f_n(z))| \leq \frac{3}{2}\sum_{n=m+1}^{k} |f_n(z)| \\ &\leq \frac{3}{2}\sum_{n=m+1}^{k} M_n \leq \frac{3}{2}\sum_{n=m+1}^{\infty} M_n = \epsilon_m \end{aligned} \tag{7.3}$$

where $\epsilon_m \to 0$ as $m \to \infty$. This shows that S_m is a uniform convergence Cauchy sequence. Now let $P_m = \prod_{n=N}^{m}(1 + f_n(z))$ so that

$$\begin{aligned} |P_k(z) - P_m(z)| &= \left|e^{S_m(z)}(e^{S_k(z)-S_m(z)} - 1)\right| \leq e^{|S_m(z)|}(e^{|S_k(z)-S_m(z)|} - 1) \\ &\leq e^{M}(e^{\epsilon_m} - 1), \end{aligned}$$

where we have used (7.2) and (7.3) and the fact that

$$|e^z - 1| = |\sum_{k=1}^{\infty}\frac{z^k}{k!}| \leq \sum_{k=1}^{\infty}\frac{|z|^k}{k!} = e^{|z|} - 1.$$

This shows that P_m is a uniform Cauchy sequence and therefore uniformly convergent to a continuous function $\tilde{P}(z)$ in D. By Morera theorem $\tilde{P}(z)$ is analytic in D. Indeed for any close contour $\gamma \in D$

$$\int_{\gamma} \tilde{P}(z)dz = \int_{\gamma} \lim_{m\to\infty} P_m(z)dz = \lim_{m\to\infty}\int_{\gamma} P_m(z)dz = 0$$

because each P_m is analytic. We conclude that

$$P(z) = (1 + f_1(z))(1 + f_2(z)) \ldots (1 + f_{N-1}(z))\tilde{P}(z) = \prod_{n=1}^{\infty}(1 + f_n(z))$$

is an absolutely and uniformly convergent sequence of analytic functions. □

Example 7.11. Let

$$P(z) = \pi z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2}).$$

In this casse $f_n(z) = \frac{z^2}{n^2}$ and inside the circle $|z| < R$ we have

$$\sum_{n=1}^{\infty} \frac{R^2}{n^2} < \infty,$$

that shows that $P(z)$ is analytic in $|z| < R$. Since we can take R arbitrarily large, $P(z)$ is analytic in $\mathbb{C}$. We observe that $\sin(\pi z)$ has the same zeros as $P(z)$. It can be shown indeed that the two functions coincides namely

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2}).$$

In a similar way

$$\cos(\pi z) = \prod_{n=0}^{\infty} (1 - \frac{z^2}{(n + \frac{1}{2})^2}).$$

7.3 Analytic continuation

Recall, that a point $z_0 \in \mathbb{C}$ is called a limit point of a set $\mathcal{S} \subset D \subseteq \mathbb{C}$ iff there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$, such that $z_n \neq z_0$ and $\lim_{n\to\infty} z_n = z_0$. In other words, for any $r > 0$ one has $S \cap A_{0,r}(z_0) \neq \emptyset$.

Lemma 7.12. *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic function. Let $\mathcal{N} \subseteq D$ be the set of zeros of f. If $\mathcal{N}$ has a limit point $z_0 \in D$ then $f(z) \equiv 0$ in any ball $B_R(z_0) \subset D$.*

Proof. Let $(z_n) \subset \mathcal{N}$ be a sequence such that $\lim_{n\to\infty} z_n = z_0 \in D$. Since f is analytic and, in particular, continuous in D, we conclude that

$$f(z_0) = \lim_{n\to\infty} f(z_n) = 0.$$

Therefore z_0 is a zero of f, which is not isolated. Hence z_0 is a zero of f of infinite order. On the other hand, since $f(z)$ is analytic in D for any ball $B_R(z_0) \subset D$ the function f has a convergent Taylor expansion

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \qquad (z \in B_R(z_0)),$$

The coefficients $a_n = 0$ for all $n \geq 0$ because the function f has a zero of infinite order in z_0. □

Theorem 7.13. (Identity Theorem) *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ analytic function. Let $\mathcal{N} \subseteq D$ be the set of zeros of f. If $\mathcal{N}$ has a limit point in D then $f(z) \equiv 0$ in D.*

Proof. By Lemma 7.12 we know that if z_0 is a limit point of $\mathcal{N}$ then $f(z) \equiv 0$ in any ball $B_R(z_0) \subset D$. For any other point $z \in D \backslash B_R(z_0)$ choose a contour $\gamma : [\alpha, \beta] \to D$ from z_0 to z. We are going to show that $f(\gamma(t)) = 0$ for all $t \in [\alpha, \beta]$.

By continuity of the map γ one can find $\delta > 0$ such that $\gamma(t) \in B_R(z_0)$ for each $t \in [\alpha, \alpha + \delta)$. Hence $f(\gamma(t)) = 0$ for all $t \in [\alpha, \alpha + \delta)$ and by continuity $f(\gamma(\alpha + \delta)) = 0$. Set

$$T := \sup\{\tau \in [\alpha, \beta] : f(\gamma(t)) = 0 \text{ for all } t \in [\alpha, \tau]\}.$$

Clearly $\alpha + \delta \leq T \leq \beta$. We need to prove that $T = \beta$.

Note that $f(\gamma(T)) = 0$ by continuity of γ. Assume that $T < \beta$. Then $z_1 = \gamma(T)$ is a non isolated zero of f. By Lemma 7.12 we know that $f(z) \equiv 0$ in a ball $B_{R_1}(z_1) \subset D$. Hence, as before, we can find $\delta_1 > 0$ such that $f(\gamma(t)) = 0$ for all $t \in [\alpha, T + \delta_1]$. But this contradicts the definition of T. Thus we conclude that $T = \beta$. □

Corollary 7.14. (Unique Continuation Theorem) *Let $D \subset \mathbb{C}$ be a domain and $f, g : D \to \mathbb{C}$ analytic functions. Let $\mathcal{S} \subset D$ be a subset of D and suppose that*

$$f(z) = g(z) \quad \textit{for all} \quad z \in \mathcal{S}.$$

If $\mathcal{S}$ has an accumulation point in D then $f(z) \equiv g(z)$ in D.

Proof. Apply the Identity Theorem to the function $f - g$. □

Analytic continuation. Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. We say that f is an extension (or a continuation) of a function $g : \mathcal{S} \to \mathbb{C}$ if $\mathcal{S} \subset D$ and $f(z) = g(z)$ for all $z \in \mathcal{S}$. According to the Unique Continuation Theorem, if $\mathcal{S}$ has an accumulation point in D then there exists *at most one* analytic extension of g from $\mathcal{S}$ to D. An important application is that real analytic functions $f : \mathbb{R} \to \mathbb{R}$ have at most one analytic extension. For example, the complex exponential function $(\exp, \mathbb{C})$ defined on $\mathbb{C}$ is the unique analytic extension of the real exponential function $(\exp, \mathbb{R})$, defined on $\mathbb{R}$. The same statement holds for the trigonometric functions $\sin(x)$, $\cos(x)$.

Example 7.15. Consider $g : B_1(0) \to \mathbb{C}$ given by $g(z) = \sum_{n=0}^{\infty} z^n$. The function $f(z) = \frac{1}{1-z}$ on $D = \mathbb{C} \setminus \{1\}$ is the unique analytic extension of g. Observe that f is the *maximal extension* of g in the sense that f can not be extended analytically beyond $\mathbb{C} \setminus \{1\}$.

The Unique Continuation Theorem provides a method for making the domain of an analytic function as large as possible, e.g., starting from a given real differentiable function, or starting from a convergent power series.

Example 7.16. Consider the principal branch of the logarithmic function

$$\mathrm{Log}(z) = \log|z| + i \arg z, \quad -\pi < \arg z \leq \pi.$$

Observe that such defined function $\mathrm{Log}(z)$ is the unique analytic extension to $\mathbb{C} \setminus (-\infty, 0]$ of the real logarithm $\log(x)$. The function $\mathrm{Log}(z)$ is also the unique analytic extension of the convergent power series

$$g(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n} \qquad (z \in B_1(1)),$$

simply because $\mathrm{Log}(z) = g(z)$ for $z \in B_1(1)$.

We can also analytically continue the function $\mathrm{Log}(z)$ along a path. For example if we consider a unit circle around the origin in the counterclockwise direction and start at the point 1 after a full circle one obtains the function $h(z) = 2\pi i + \mathrm{Log}(z)$.

Example 7.17. The function $f(z) = \sqrt{z} = e^{\frac{1}{2}\mathrm{Log}(z)}$ with $-\pi \leq \arg(z) < \pi$ is analytic in $\mathbb{C}\backslash(-\infty, 0]$. If we take the analytic continuation of $f(z)$ along a unit circle around the origin in the counterclockwise direction starting from the point $+1$ we end up with the function $g(z) = e^{\frac{1}{2}(\mathrm{Log}(z)+2\pi i)} = -\sqrt{z} = -f(z)$. This is the other solution of the equation

$$f(z)^2 = z.$$

If we continue analytically $g(z) = -\sqrt{z}$ along a unit circle centred at the origin, we recover $f(z)$ the original function. So the function $\sqrt{z}$ has an analytic extension to two copies of the complex plane glue along the negative real axis.

Further investigation of analytic continuation lead naturally to the concept of Riemann surfaces as domains where functions that are multivalued on the complex plane become single-valued.

Example 7.18. The Γ function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad \mathrm{Re}(z) > 0.$$

The function is a differentiable (and therefore analytic function) for $\mathrm{Re}(z) > 0$, where the integral is defined.

For n positive integer the function coincides with the factorial $\Gamma(n) = (n-1)!$ that can be verified by integration by parts of the integral $\int_0^\infty t^{n-1}e^{-t}dt$.

Further by integration by parts one obtains the functional relation

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt = \frac{t^z}{z}e^{-t}\Big|_0^\infty + \frac{1}{z}\int_0^\infty t^{z-1}e^{-t}dt = \frac{\Gamma(z+1)}{z}.$$

We notice that $\Gamma(z+1)$ is a well defined analytic function for $\mathrm{Re}(z) > -1$ and therefore the ratio $\frac{\Gamma(z+1)}{z}$ has a simple pole at $z = 0$. We conclude that the function

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

has an analytic extension in $\mathrm{Re}(z) > -1$ and $z \neq 0$. Repeating the integration by parts multiple times we arrive to

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \cdots = \frac{\Gamma(z+n)}{z(z+1)\ldots(z+n-1)}, \quad \forall n \in \mathbb{N}.$$

In the right hand-side of the above relation the function $\Gamma(z+n)$ is analytic for $\mathrm{Re}(z) > -n$. It follows that the function $\Gamma(z)$ has an analytic extension for $\mathrm{Re}(z) > -n$ and $z \neq 0, -1, -2, \ldots, n-1$ where it has simple poles. We can conclude that the function $\Gamma(z)$ has an analytic extension to the whole $\mathbb{C}\backslash\{0, -1, -2, -3\ldots\}$ and it has simple poles at the negative integers.

7.3.1 Fourier series

Let $f : [-1, 1] \to \mathbb{C}$ be an integrable function, namely

$$\int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta < \infty.$$

The Fourier coefficients of f are

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta, \quad z \in \mathbb{Z}.$$

The Fourier series of f is formally

$$f(e^{i\theta}) \simeq \sum_{n=-\infty}^{\infty} a_n e^{i\theta n}.$$

The use of Fourier series is crucial in harmonic analysis, quantum mechanics, wave propagation, signal processing.

We want to address the following question: what properties of the Fourier coefficients guarantees that the function f defined on the unit circle has an analytic extension to an annulus that contains the unit circle? Let us consider a function f that is analytic in $A_{r,R}(0)$, $0 < r < 1 < R$. Then by the Laurent series theorem the function f can be represented by the Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{7.4}$$

where

$$a_n = \frac{1}{2\pi i}\int_{S_\rho^+(0)} \frac{f(w)}{w^{n+1}}dw \quad r < \rho < R.$$

Clearly we can take $\rho = 1$ so that

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta, \quad n \in \mathbb{Z}$$

that coincide with the Fourier coefficients of f. The above considerations lead to the following proposition.

Proposition 7.19. *Let $\{a_n\}_{n\in\mathbb{Z}}$ be the Fourier coefficients of an integrable function g defined on the unit circle. If the Fourier coefficients a_n satisfy*

$$r = \limsup_{n\to\infty} |a_{-n}|^{\frac{1}{n}} < 1 < R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1},$$

then the function g has an analytic extension to the annulus $A_{r,R}(0)$. The analytic extension is given by the function f defined by Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad z \in A_{r,R}(0),$$

and $f(z)|_{S_1(0)} = g(z)$. If $a_n = 0$ for $n < 0$, then the function g has an analytic extension to the disc $B_R(0)$.

Next we consider the case of a function $f : \mathbb{R} \to \mathbb{C}$ with $\int_{-\infty}^{\infty} |f(x)|dx < \infty$. The Fourier transform is defined as

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx$$

and the inverse Fourier transform is formally defined (in the sense that we ignore convergence issues)

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

A similar question about decay of Fourier transform and analyticity can be posed. Let $\epsilon > 0$ and

$$\mathcal{S}_\epsilon = \{z \in \mathbb{C} \,|\, |\mathrm{Im}(z)| < \epsilon\}.$$

Let

$$\mathcal{A}_\epsilon = \{f : S_\epsilon \to \mathbb{C} \,|\, \text{f analytic and } \sup_{-\epsilon < y < \epsilon} \int_{-\infty}^{\infty} |f(x+iy)| dx < \infty\}.$$

We have the following lemma that we do not prove.

Lemma 7.20. *Paley-Wiener. Let $f \in \mathcal{A}_\epsilon$. Then for any $\delta \in [0, \epsilon)$*

$$|\hat{f}(\xi)| < B e^{-2\pi \delta \xi}, \quad \xi \in \mathbb{R}, \tag{7.5}$$

for some positive constant B. Viceversa, if the Fourier transform $\hat{f}(\xi)$ has the decay rate as in (7.5)*, then the function f*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi.$$

has an analytic extension off the real axis to a strip S_δ.

Example 7.21. Let us calculate the Fourier coefficients of $f(x) = \frac{1}{x^2+1}$, namely the integral

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i \xi x}}{x^2+1} dx.$$

We consider the case $\xi \geq 0$ and the case $\xi < 0$ separately. In the case $\xi \geq 0$ there are constants $M > 0$ and $R \gg 1$ so that

$$|\frac{e^{2\pi i z \xi}}{(z^2+1)}| < \frac{M}{R^2} \quad |z| > R, \mathrm{Im} z \geq 0.$$

Therefore we can close the contour on the upper half space and use residue theorem to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{2\pi i x \xi}}{(x^2+1)} dx &= 2\pi i \mathrm{Res}(\frac{e^{2\pi i z \xi}}{(z^2+1)}, z = i) \\
&= 2\pi i \lim_{z \to i} \left[(z-i) \frac{e^{2\pi i z \xi}}{(z^2+1)} \right] \\
&= \pi e^{-2\pi \xi}
\end{aligned}$$

In the case $\xi < 0$ the function $|\frac{e^{2\pi i z \xi}}{(z^2+1)}|$ is bounded for large $|z|$ and $\mathrm{Im} z < 0$, so in this case the contour should be closed in the lower half space and the total contour is oriented clockwise. By applying the residue theorem to the clockwise oriented contour we obtain

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i x \xi}}{(x^2+1)} dx = -2\pi i \mathrm{Res}(\frac{e^{2\pi i z \xi}}{(z^2+1)}, z = -i) = \pi e^{2\pi \xi}$$

Therefore $\hat{f}(\xi) = \pi e^{-2\pi |\xi|}$. From the Lemma 7.20 we conclude that the function $f(x) = \pi \int_{-\infty}^{\infty} e^{-2\pi|\xi|} e^{2\pi i \xi x} d\xi$ has an analytic extension to the strip $\mathcal{S}_1$. Since $f(x) = \frac{1}{1+x^2}$, the analytic extension is given by the function $1/(z^2+1)$ that has poles at $z = \pm i$.

7.4 Principle of the Argument

We now turn to an application of the Residue Theorem which gives information concerning the number of zeros and poles of a complex function. It is convenient to introduce the notation

$$\text{ind}(f, z_0) = \begin{cases} m, & \text{if } f \text{ has a zero of order } m \ (m \in \mathbb{N}) \text{ at } z_0, \\ -m, & \text{if } f \text{ has a pole of order } m \ (m \in \mathbb{N}) \text{ at } z_0, \\ 0, & \text{if } f \text{ is analytic (or has a removable singularity) at } z_0, \text{ and } f(z_0) \neq 0. \end{cases}$$

We say that $\text{ind}(f, z_0)$ is the index of f at z_0.

Theorem 7.22. (Principle of the Argument) *Let $D \subseteq \mathbb{C}$ be a simply connected domain and $\mathcal{S} = \{z_1, z_2, \cdots, z_p\} \subset D$, $p \geq 1$. Let $f : D \setminus \mathcal{S} \to \mathbb{C}$ be an analytic function which has no zeros or poles in $D \setminus \mathcal{S}$ and has a zero or a pole at each point of $\mathcal{S}$. Let $\gamma : [\alpha, \beta] \to D \setminus \mathcal{S}$ be a closed contour. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z_k \in \mathcal{S}} W(\gamma, z_k) \text{ind}(f, z_k). \tag{7.6}$$

Proof. By the Residue Theorem,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{z_k \in \mathcal{S}} W(\gamma, z_k) \text{Res}(f'/f, z_k).$$

We are going to show that:
(a) If $\text{ind}(f, z_k) = p > 0$ then f'/f has a pole of order 1 at z_k and $\text{Res}(f'/f, z_k) = p$;
(b) If $\text{ind}(f, z_k) = -p < 0$ then f'/f has a pole of order 1 at z_k and $\text{Res}(f'/f, z_k) = -p$.

Case (a). To prove (a) note that by Exercise 7.2 (c)

$$f(z) = (z - z_k)^p g(z) \qquad (z \in B_R(z_k)),$$

where $g : B_R(z_k) \to \mathbb{C}$ is analytic and $g(z_k) \neq 0$. Therefore

$$f'(z) = p(z - z_k)^{p-1} g(z) + (z - z_k)^p g'(z),$$

and hence

$$\frac{f'(z)}{f(z)} = \frac{p}{z - z_k} + \frac{g'(z)}{g(z)}$$

has a simple pole with residue p at z_k, because g'/g is analytic at z_k.

Case (b). In this case, similarly as in Proposition 5.5,

$$f(z) = (z - z_k)^{-p} g(z) \qquad (z \in A_{0,R}(z_k)),$$

where $g : B_R(z_k) \to \mathbb{C}$ is analytic and $g(z_k) \neq 0$. Therefore

$$f'(z) = -p(z - z_k)^{-p-1} g(z) + (z - z_k)^{-p} g'(z),$$

and hence

$$\frac{f'(z)}{f(z)} = \frac{-p}{z - z_k} + \frac{g'(z)}{g(z)},$$

which has a simple pole with residue $-p$ at z_k, because g'/g is analytic at z_k. □

If f is holomorphic in $\mathbb{C}$ except that for a finite set of poles and γ is a simple close and positively oriented curve in $\mathbb{C}$ and such that all the zeros and poles of f are contained inside γ we have that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \boxed{\begin{array}{l}\text{number of zeros of } f \text{ in}\\ \mathbb{C} \text{ counting multiplicity}\end{array}} - \boxed{\begin{array}{l}\text{number of poles of } f \text{ in}\\ \mathbb{C} \text{ counting multiplicity}\end{array}}. \tag{7.7}$$

If the function f does not have poles inside the simple contour γ, then the principle of the argument Theorem counts the number of zeros of f inside the contour γ.

In what follows, for an analytic function f defined on a domain $D \subseteq \mathbb{C}$ we denote $\mathcal{N}_f := \{z \in D : f(z) = 0\}$. Observe that by Lemma 7.12 the set N_f has no limit points in D, except for the case $f \equiv 0$ in D.

The theorem below is frequently used to locate zeros of a function by comparing it with another function whose zeros could be located more easily.

Theorem 7.23. (Rouché Theorem) *Let $D \subseteq \mathbb{C}$ be a simply connected bounded domain and $f, g : D \to \mathbb{C}$ analytic functions. Let $\gamma : [\alpha, \beta] \to D$ be a closed contour. Assume that*

$$|f(z) - g(z)| < |f(z)| \qquad (z \in \gamma). \tag{7.8}$$

Then

$$\sum_{z_k \in \mathcal{N}_f} W(\gamma, z_k)\text{ind}(f, z_k) = \sum_{z_k \in \mathcal{N}_g} W(\gamma, z_k)\text{ind}(g, z_k), \tag{7.9}$$

where $\mathcal{N}_f$ and $\mathcal{N}_g$ are the sets of zeros of f and g respectively.

Remark 7.24. Condition (7.9) implies that neither f nor g can be zero on γ.

Proof. Let $F(z) := \frac{g(z)}{f(z)}$. Then from (7.8) we have

$$|1 - F(z)| < 1 \qquad (z \in \gamma). \tag{7.10}$$

Let $\Gamma(t) = F(\gamma(t))$. Clearly $\Gamma : [\alpha, \beta] \to \mathbb{C}$ is a closed contour. Moreover, $\Gamma \subset B_1(1)$ by (7.10). Hence the winding number of the point $z = 0$ with respect to Γ is zero, namely $W(\Gamma, 0) = 0$ and therefore

$$\begin{aligned}\int_\gamma \frac{F'(z)}{F(z)}dz &= \int_\alpha^\beta \frac{F'(\gamma(t))}{F(\gamma(t))}\gamma'(t)dt = \int_\alpha^\beta \frac{\Gamma'(t)}{\Gamma(t)}dt\\ &= \int_\Gamma \frac{1}{z}dz = 2\pi i\, W(\Gamma, 0) = 0.\end{aligned}$$

But by direct computation

$$\frac{F'(z)}{F(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)},$$

which implies (7.9). □

Example 7.25. Let $p(z) = z^8 - 5z^3 + z - 2$. Prove that $p(z)$ has exactly 3 roots (counted with multiplicities) inside the unit circle.

Solution. Let $q(z) = -5z^3$. For $|z| = 1$ it is immediate that

$$|q(z) - p(z)| = | - z^8 - z + 2| < |q(z)| = 5.$$

By Rouché's Theorem p and q have the same number of zeros (counted with multiplicities) inside the unit circle, and the equation $5z^3 = 0$ obviously has one zero with multiplicity 3 inside the unit circle. □

Exercise 7.26. (Fundamental Theorem of Algebra) Let $p_n = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. Prove that p_n has exactly n zeros (counted with multiplicities).

Hint. Let $q(z) = a_n z^n$ and estimate $|p_n(z) - q(z)|$ on $\mathcal{S}_R^+(0)$ for large enough $R > 0$.

The following theorem provides an important property of non constant holomorphic functions. Recall that continuity requires that the preimage of an open set is open, namely $f^{-1}(U)$ is open in $\mathbb{C}$ when U is open in $\mathbb{C}$.

Definition 7.27. A function $f : D \to \mathbb{C}$ is called open if and only if it maps every open set $U \subset D$ to an open set $f(U) \subset \mathbb{C}$.

Theorem 7.28 (Open mapping theorem)**.** *Let $f(z) : D \to \mathbb{C}$ be a holomorphic non constant function. Then f is an open mapping. If f is a bijection to its image $f(D)$, then $f'(z) \neq 0$ for all $z \in D$.*

Proof. We use Rouché theorem to prove this result. Let $a \in D$ and let $b = f(a)$. We want to show that there is $\delta > 0$ so that $B_\delta(b) \subset f(D)$. Since the zeros of holomorphic functions are isolated, there is a sufficiently small and close neighbourhood $\overline{B_\epsilon(a)} \subset D$ so that $f(z) - b \neq 0$ for all $z \in \overline{B_\epsilon(a)}$ and $z \neq a$. This means that $f(z) - b$ is not zero in $S_\epsilon(a)$ and since $S_\epsilon(a)$ is compact we have

$$\delta = \inf_{z \in S_\epsilon(a)} |f(z) - b| > 0.$$

Next we apply the Rouché theorem to the functions $h(z) = f(z) - b$ and $g(z) = f(z) - w$ where $w \in B_\delta(b) \backslash \{b\}$. Then

$$|h(z) - g(z)| = |w - b| < \delta \leq |f(z) - b| = |h(z)|, \qquad z \in S_\varepsilon^+(a).$$

By the Rouché Theorem we conclude that $f(z) - b$ and $f(z) - w$ have the same number of zeros in $B_\varepsilon(a)$. Since $f(z) - b$ has a zero of order at least one at $z = a$, it follows that for every $w \in B_\delta(b) \backslash \{b\}$, there is at least one value of $z \in B_\epsilon(a) \backslash \{a\}$ so that

$$f(z) = w.$$

If f is a bijection to its image $f(D)$ it follows that for every $w \in f(D)$, there is only one value of $z \in D$ so that $f(z) - w = 0$ and this implies that the equation $f(z) - w = 0$ has a first order zero, or equivalently, $f'(z) \neq 0$ for all $z \in D$. □

7.5 Maximum Modulus Principle

The Maximum Modulus Principle states that if f is analytic in a domain D then $|f(z)|$ can have its maximum only the boundary of D. This property of analytic functions becomes crucial in the study of harmonic functions and boundary value problems. We start with a preliminary lemma.

Lemma 7.29. (Mean Value Property) *Let $f : B_R(z_0) \to \mathbb{C}$ be an analytic function. Then for any $r \in (0, R)$ one has*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\varphi})d\varphi. \tag{7.11}$$

Proof. By the Cauchy Integral Formula

$$f(z_0) = \frac{1}{2\pi i}\int_{S_r^+(z_0)} \frac{f(z)}{z - z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\varphi})}{re^{i\varphi}} rie^{i\varphi}d\varphi = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\varphi})d\varphi,$$

which is required. □

Remark 7.30. The Mean Value Property states that the value of f at the center of a ball $B_r(z_0)$ is the average its values on the spheres $S_r(z_0)$.

Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. We say that $z_0 \in D$ is a local maximum point of $|f|$ iff there exists $r > 0$ such that $|f(z)| \le |f(z_0)|$ for all $z \in B_r(z_0) \subset D$. We say that $z_0 \in D$ is a local minimum point of $|f|$ iff z_0 is a local maximum of $-|f|$.

Theorem 7.31. (Maximum Modulus Theorem) *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. If $|f|$ has a local maximum point in D then f is constant in D.*

Proof. Assume that $z_0 \in D$ is a local maximum point of $|f|$. Choose $R > 0$ such that $B_R(z_0) \subset D$. By the Mean Value Property applied at $z_0 \in D$, for any $r \in (0, R)$ one has

$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\varphi})|d\varphi.$$

On the other hand

$$\frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0 + re^{i\varphi})|}_{\le |f(z_0)|} d\varphi \le |f(z_0)|,$$

since z_0 is a local maximum point of $|f|$. We conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\varphi})|d\varphi = |f(z_0)|.$$

Rewrite the last equality as

$$\frac{1}{2\pi}\int_0^{2\pi} \underbrace{\left\{|f(z_0)| - |f(z_0 + re^{i\varphi})|\right\}}_{\ge 0,\text{ since } z_0 \text{ is a local maximum}} d\varphi = 0.$$

Therefore $|f(z_0)| - |f(z_0 + re^{i\varphi})| = 0$ on $S_r(z_0)$. Hence

$$\forall r \in (0, R) : \quad |f(z_0)| \equiv |f(z_0 + re^{i\varphi})| \quad \text{on } S_r(z_0).$$

We conclude that $|f(z_0)| \equiv |f(z_0 + re^{i\varphi})|$ in $B_R(z_0)$. This implies that $|f(z)| \equiv const$ in $B_R(z_0)$ and by Cauchy Riemann equations $f(z) \equiv const$ in $B_R(z_0)$. Thus $f(z)$ is constant in D by the Identity Theorem. □

Remark 7.32. In other words, if f is non constant holomorphic function in D then any local maximum point of the modulus $|f|$ may occur only on the boundary of D. Observe that f may have no local maxima on the boundary of D if D is not bounded. For example, if $f(z) = z$ and $D = \mathbb{C}_+$ is the upper half plane, then $|f(z)| = |z|$ has no local maximum points neither in D nor on the boundary of D. Another example is the function $f(z) = z^2$ on $D = \{z \in \mathbb{C} : |z| > 1\}$. Here every point of the boundary of D is a minimum point of $|f|$! Note also that in the trivial case when f is constant in D, the maximum of $|f|$ is attained at every point of D.

Remark 7.33. The maximum modulus principle is also a consequence of the open mapping theorem. We first remark that if $U \subset \mathbb{C}$ is an open set non containing the origin then $|.| : U \to (0, \infty)$ is an open mapping. Next assuming that $f(z)$ is non constant in D and $f(z) \neq 0$ for all $z \in D$ we consider the map $|f| : D \to (0, \infty)$ that is an open map. If there is $z_0 \in D$ such that $f(z_0) = b$ attains a maximum in D then $|f(B_\delta(z_0))| \leq |b|$ for every disk $B_\delta(z_0) \subset D$ and the upper bound is reached at $z = z_0$. Therefore $|f|$ cannot be an open map.

Example 7.34. If $f(z) = \exp(z)$ and $D = \bar{B}_1(0)$ then $|f(z)| = e^{\mathrm{Re}(z)}$. Hence $|f|$ has its maximum on $\bar{B}_1(0)$ at $z_0 = 1$.

Exercise 7.35. (Minimum Modulus Theorem) Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ an analytic function. If $|f|$ has a local minimum point $z_0 \in D$ then either $f(z_0) = 0$ or f is constant in D.

Hint. Apply the Maximum Modulus Theorem to the function $\frac{1}{f(z)}$.

Remark 7.36. In other words, if f is non constant in D then any local minimum point of the modulus $|f|$ may occur only at zeros of f or on the boundary of D.

Example 7.37. If $f(z) = z^2$ and $D = \bar{B}_1(0)$ then $|f(z)| = |z|^2$. Hence every $z \in S_1(0)$ is a maximum point of f and $z_0 = 0$ is the minimum point of $|f|$.

Mathematicians of this section
Raymond Edward Alan Christopher Paley (7 January 1907 – 7 April 1933), English Mathematician. Eugène Rouché (18 August 1832 – 19 August 1910) was a French mathematician. Norbert Wiener (November 26, 1894 – March 18, 1964) was an American computer scientist and mathematician.

8 Conformal mappings

In this chapter we consider a more global aspect of analytic functions and we describe geometrically what their effect is on various regions of the complex plane. The problems and ideas presented in this chapter are more geometric with respect to the previous chapters. We wish to define an equivalence relation between domains in $\mathbb{C}$. After doing this it will be shown that all simply connected domains $D \subset \mathbb{C}$ are equivalent to the open ball $B_1(0)$, and hence are equivalent to one another.

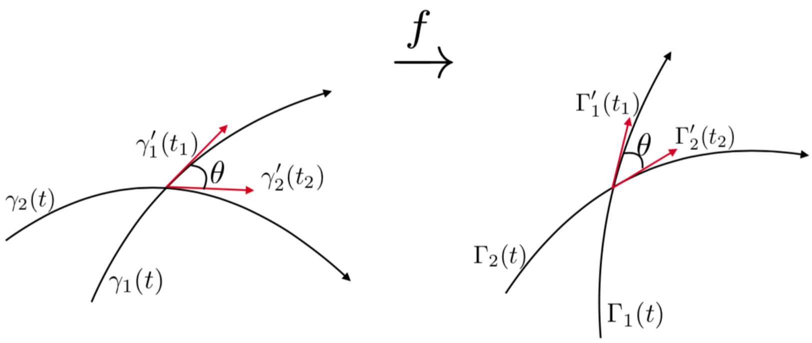
Let start with the following definition.

Definition 8.1. Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a holomorphic function. If f preserves angles between oriented paths passing at a point z_0 then f is said to be conformal at z_0. If f is conformal at every $z_0 \in D$ we say that f is conformal in D. In the latter case we also say that f is a conformal mapping of D onto $f(D)$.

To further explain the definition, let $D \subset \mathbb{C}$ be a domain and let $\gamma_1, \gamma_2 : [\alpha, \beta] \to D$ be two smooth intersecting paths at a point z_0, namely $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1) \neq 0$, $\gamma_2'(t_2) \neq 0$. The angle θ between the paths γ_1 and γ_2 at z_0 is by definition the angle bewteen its tangent vectors $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$, namely:

$$\theta := \text{Arg}(\gamma_2'(t_2)) - \text{Arg}(\gamma_1'(t_1)),$$

see the figure below.



Now let $f : D \to \mathbb{C}$ be a holomorphic function. Let $\Gamma_1(t) = f(\gamma_1(t))$ and $\Gamma_2(t) = f(\gamma_2(t))$. Then using the chain rule

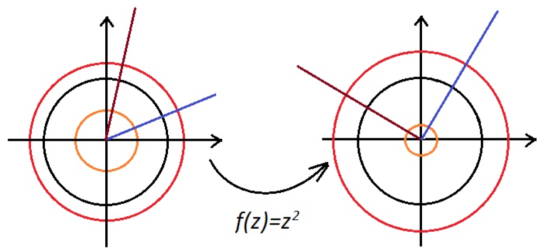
$$\Gamma_1'(t_1) = f'(\underbrace{\gamma_1(t_1)}_{z_0})\gamma_1'(t_1), \qquad \Gamma_2'(t_2) = f'(\underbrace{\gamma_2(t_2)}_{z_0})\gamma_2'(t_2).$$

Assuming that $f'(z_0) \neq 0$ we have

$$\begin{aligned} \text{Arg}(\Gamma_2'(t_2)) - \text{Arg}(\Gamma_1'(t_1)) &= \text{Arg}(f'(z_0)) + \text{Arg}(\gamma_2'(t_2)) - (\text{Arg}(f'(z_0)) + \text{Arg}(\gamma_1'(t_1))) \\ &= \text{Arg}(\gamma_2'(t_2)) - \text{Arg}(\gamma_1'(t_1)). \end{aligned}$$

This says that given any two paths through z_0, f maps these paths onto two paths through $w_0 = f(z_0)$ and, when $f'(z_0) \neq 0$, the angles between the paths are preserved by f both in magnitude and direction. Therefore, we have proved the following.

Theorem 8.2. *Let $D \subseteq \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a holomorphic function. Then f is conformal at each point $z_0 \in D$ such that $f'(z_0) \neq 0$.*



Example 8.3. Let us consider the function $f(z) = z^2$. We have for $z = r(\cos\theta + i\sin\theta)$

$$f(r(\cos\theta + i\sin\theta)) = r^2(\cos 2\theta + i\sin 2\theta);$$

so it is clear that

- the image of a circle $S_r(0)$ is the circle $S_{r^2}(0)$
- the image of a ray $R_\theta = r(\cos\theta + i\sin\theta)$ with θ fixed and $r \in (0, \infty)$ is the ray $R_{2\theta}$.

The angle of any circle centred at zero and any ray R_θ is $\pi/2$ and this angle is preserved since the map $f(z) = z^2$ maps circles to circles and rays to rays.

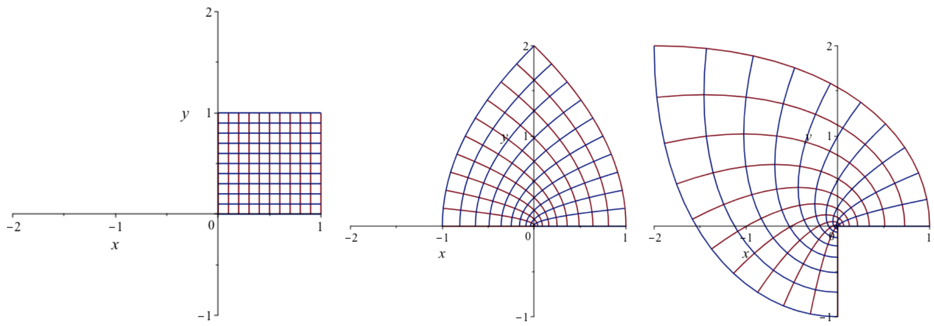


Figure 6: The square of size 1 under the map $f(z) = z^2$ and $f(z) = z^3$.

Example 8.4. The map $S(z) = az + b$, $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$ is a translation and a rotation, so it clearly preserves angle. The map $S(z) = 1/z$ is an inversion. It maps circles to circles and lines to lines.

8.1 Mapping properties of analytic functions

In this section we study mapping properties of analytic functions. In particular we are interested in studying the properties of a conformal map.

We recall the following important property of analytic functions that is given by the Open mapping theorem 7.28. *"Let $f(z) : D \to \mathbb{C}$ be a holomorphic non constant function. Then the image $f(D)$ is a open set of $\mathbb{C}$. If f is a bijection to its image $f(D)$, then $f'(z) \neq 0$ for all $z \in D$".*

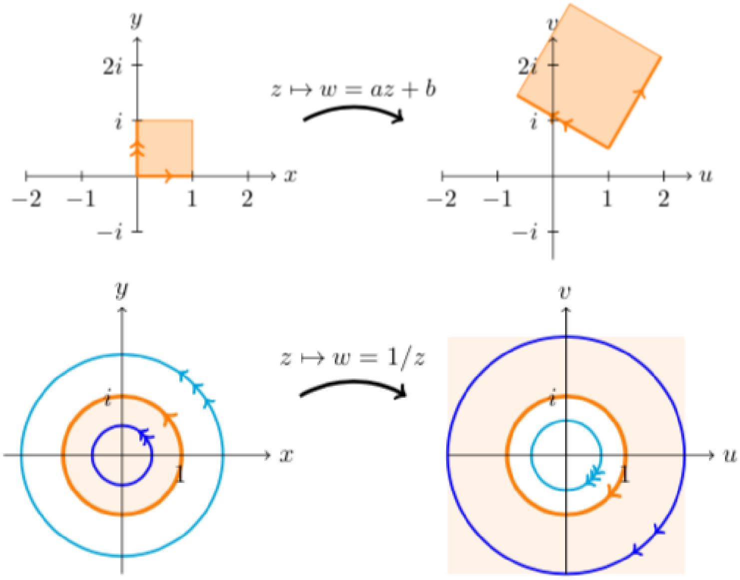


Figure 7: In the first figure the linear map $S(z) = az + b$, $a, b \in C\backslash\{0\}$ scales, rotates and translates the square. In the second picture the map $S(z) = 1/z$ leaves the unit circle invariant, while the circle of radius less then one is mapped to the circle of radius bigger then one and viceversa.

Definition 8.5. Let $U, V \subseteq \mathbb{C}$ be two domains of $\mathbb{C}$. A map

$$f : U \to V,$$

is called bi-holomorphic if f is analytic and it is a one to one (bijective) map from U to V. Namely there is an analytic function $g : V \to U$ such that $f \circ g = id_V$ and $g \circ f = id_U$. The function f is called an *automorphism* if $U = V$.

~~If f is an analytic function on an open set U and f is injective, then $f : U \to f(U)$ is a bi-holomorphic map from U to its image $f(U)$.~~

We remark that $f'(z) \neq 0$ is not sufficient to have an injective map.

Example 8.6. Consider $f(z) = \exp(z)$ with $z \in \mathbb{C}$. Thus $f'(z) \neq 0$ for each $z \in \mathbb{C}$. However f is not one–to–one, indeed $f(z) = f(z + 2\pi i)$ for any $z \in \mathbb{C}$.

Remark 8.7. In fact, according to Theorem 8.2, every one–to–one analytic function is conformal. The converse is not true, for example $f(z) = z^2$ is a conformal mapping of $\mathbb{C} \setminus \{0\}$ onto itself, but is not one–to–one. The use of the term *one–to–one conformal* instead of the equivalent term *one–to–one analytic* is a traditional terminology.

Example 8.8. Consider $f(z) = \exp(z)$, then f is conformal on $\mathbb{C}$ because f is analytic and $f'(z) \neq 0$ for all $z \in \mathbb{C}$. Let us look for a domain where f is bi-holomorphic to its image. Since $e^z = e^{z+2\pi i}$ we restrict the domain of definition to the strip $D = \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) < \pi\}$. Fix $c \in \mathbb{R}$ and consider $z = c + iy$, then $f(z) = re^{iy}$ for $r = e^c$. That is, for $\pi \leq y \leq \pi$, f maps the line $x = c$ onto the circle with center at the origin and of radius e^c. The image of the boundary of the strip D namely the lines $y = \pm\pi$ with $x \in \mathbb{R}$ are $f(z) = e^x e^{i\pi} < 0$ and $f(z) = e^x e^{-i\pi} < 0$, namely the negative real axis. So $f(D) = \mathbb{C} \setminus (-\infty, 0)$.

Example 8.9. The map $f(z) = \text{Log}(z)$ takes the upper half plane $\mathbb{H}$ to the strip $\{w = u + iv \in \mathbb{C} \,|\, u \in \mathbb{R},\ 0 < v < \pi\}$. Indeed $\text{Log}(z) = \log|z| + i\arg(z)$ where the range $u = \log|z|$ is the real axis while $0 < v = \arg(z) < \pi$. If we consider instead the half disk $D_+ = \{z \in \mathbb{C} \,|\, |z| < 1,\ \text{Im}(z) > 0\}$ then the image of D_+ via $\text{Log}(z)$ is given by $\{w = u + iv \in \mathbb{C} \,|\, u < 0,\ 0 < v < \pi\}$. Indeed now $u = \log|z| < 0$ and $0 < v = \arg(z) < \pi$.

Example 8.10. The function $f(z) = z^\alpha$ maps the sector $S = \{z \in \mathbb{C} \,|\, 0 < \arg(z) < \frac{\pi}{\alpha}\}$ to the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \text{Im}(z) > 0\}$. The inverse map is defined by $z = f^{-1}(w) = w^{\frac{1}{\alpha}}$, where the branch of the logarithm is chosen so that $0 < \arg(z) < \pi$. Thus f is a one to one conformal map between the sector S and $\mathbb{H}$. When $\alpha = 2$, f is a one–to–one conformal mappings between the first quadrant $D = \{z \in \mathbb{C} : \text{Re}(z) > 0, \text{Im}(z) > 0\}$ and the upper half plane.

Example 8.11. The function $f(z) = \frac{1}{2}(z + \frac{1}{z})$ is called Joukowski transformation. Setting $w = f(z)$ we see that we have to solve a quadratic equation to obtain the inverse $z_\pm = w \pm \sqrt{w^2 - 1}$, where $z_+ z_- = 1$. It is clear from the expression of $z_\pm$ that when $w \in [-1, 1]$ then $z_\pm \in S_1(0)$. So for any $w \in \mathbb{C} \backslash [-1, 1]$ there are two pre-images z_+ and z_-. Restricting the map to the half disk $D_+ = \{z \in \mathbb{C} \,|\, |z| < 1,\ \text{Im}(z) > 0\}$, we see that the image $f(D_+) = \{z \in \mathbb{C} \,|\, \text{Im}(z) < 0\}$.

Example 8.12. By combining the exponential map, a rotation and the Joukowski map, the function $f(z) = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ maps the strip $S = \{z \in \mathbb{C} \,|\, -\frac{\pi}{2}\text{Re}(z) < \frac{\pi}{2}, \text{Im}(z) > 0\}$ to the upper half plane.

8.2 Fractional linear transformation

An important example of conformal maps, are fractional linear transformations.

Definition 8.13. A fractional linear transformation is a function $S : \mathbb{C} \to \mathbb{C}$ of the form

$$S(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. These are also called *Moebius transformations* or bilinear transformations.

Note that

$$S'(z) = \frac{ad - bc}{(cz + d)^2},$$

so that the condition $ad - bc \neq 0$ implies that the map $S(z)$ is a conformal map.

Note that when $ad - bc = 0$ the map $S(z)$ is equal to a constant. Indeed $d = bc/a$ so that

$$S(z) = \frac{az + b}{cz + d} = \frac{\frac{a}{c}(cz + \frac{bc}{a})}{cz + d} = \frac{a}{c}.$$

When $ad - bc \neq 0$ the inverse map is

$$z = S^{-1}(w) = \frac{dw - b}{-cw + a}\,.$$

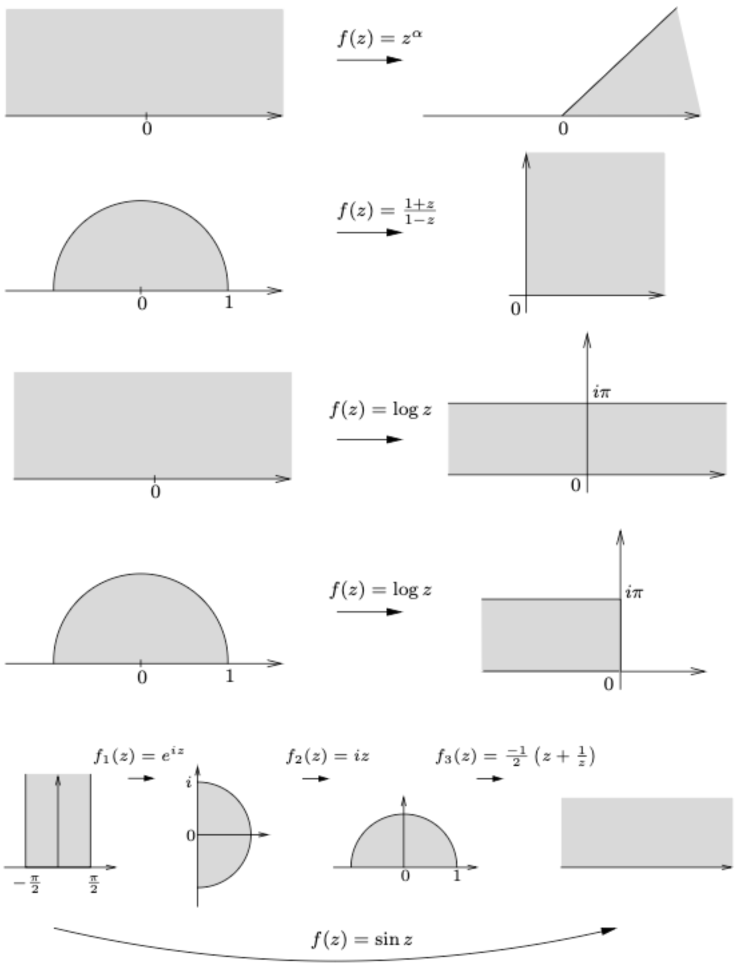


Figure 8: Several explicit conformal maps. In the first picture $0 < \alpha < 1$.

For $c \neq 0$ we can consider S as a map from $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ by observing that $S(\infty) = \dfrac{a}{c}$ and $S(-d/c) = \infty$.
The simplest linear fractional transformations are:

- translations: $w = z + b$, $b \in \mathbb{C}$;
- rotations: $w = az$ and $|a| = 1$, or rotation plus dilatation (homothetic transformation) if $w = az$, $a \in \mathbb{C}$ and $a \neq 0$;
- inversion: $w = 1/z$.

The general linear fractional transformation is a composition of translations, inversions dilatation and rotations. Indeed for $c \neq 0$ we can write

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2(z+d/c)} + \frac{a}{c},$$

which shows that the general linear fractional transformation is composed by a translation, inversion, rotation and homothetic transformation followed by another translation.

Remark 8.14. It is easy to check that the set of linear fractional transformations $z \to \dfrac{az+b}{cz+d}$ such that $ad - bc \neq 0$ form a group with respect to composition. Indeed let

$$w_1 = \frac{a_1 z + b_1}{c_1 z + d_1}, \quad w_2 = \frac{a_2 w_1 + b_2}{c_2 w_1 + d_2}$$

then

$$w_2 = \frac{a_2 \frac{a_1 z + b_1}{c_1 z + d_1} + b_2}{c_2 \frac{a_1 z + b_1}{c_1 z + d_1} + d_2} = \frac{a_2(a_1 z + b_1) + (c_1 z + d_1) b_2}{c_2(a_1 z + b_1) + (c_1 z + d_1) d_2} = \frac{z(a_2 a_1 + b_2 c_1) + a_2 b_1 + d_1 b_2}{z(a_1 c_2 + c_1 d_2) + b_1 c_2 + d_2 c_1}. \tag{8.1}$$

Remark 8.15. Given three distinct points in the complex plane z_2, z_3 and z_4 there is always a linear fractional transformation that maps them to $1, 0$ and ∞. The transformation is given by the ratio

$$S(z) = \frac{z - z_3}{z - z_4} \frac{z_2 - z_4}{z_2 - z_3}.$$

Indeed we have $S(z_3) = 0$, $S(z_2) = 1$ and $S(z_4) = \infty$.

Next we consider important elements of the group of the fractional linear transformations.

Theorem 8.16. *Let*

$$\mathbb{H} = \{z \in \mathbb{C}, \mid \mathrm{Im} z > 0\},$$

and consider the map

$$S(z) = \frac{z - i}{z + i}.$$

The map $z \to w = S(z)$ is a bi-holomorphic map from the upper half plane $\mathbb{H}$ to the unit disk $B_1(0)$ with inverse $S^{-1}(w) = -i\dfrac{w+1}{w-1}$.

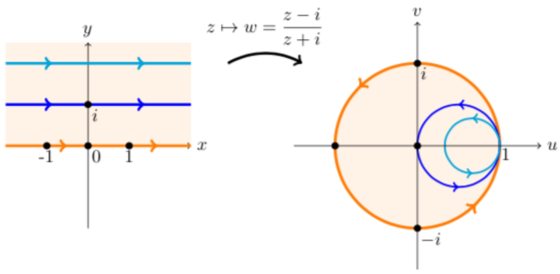


Figure 9: The map $S(z) = \frac{z-i}{z+i}$.

Proof. We show that the boundary of $\mathbb{H}$, namely the real axis is mapped to the unit circle and the upper half plane to the interior of the unit circle. Indeed let $z = x \in \mathbb{R}$ then

$$|w| = |S(x)| = \left|\frac{x-i}{x+i}\right| = \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} = 1$$

Now let $z = x + iy$ with $y > 0$, namely $z \in \mathbb{H}$. Then

$$|w| = |S(x+iy)| = \left|\frac{x+i(y-1)}{x+i(y+1)}\right| = \frac{\sqrt{x^2+(y-1)^2}}{\sqrt{x^2+(y+1)^2}} < 1$$

so that $S(\mathbb{H}) \subset B_1(0)$. To show that $S(\mathbb{H}) = B_1(0)$ we need to show that the map is surjective. Solving for z the equations $w = \dfrac{z-i}{z+i}$ we obtain

$$z = S^{-1}(w) = -i\frac{w+1}{w-1} = -i\frac{|w|^2 - 1 - 2i\text{Im}w}{|w-1|^2}.$$

We need to show that when $w \in B_1(0)$, then $\text{Im}(z) > 0$, that is

$$\text{Im}(S^{-1}(w)) = \frac{1-|w|^2}{|w-1|^2} > 0, \quad \text{when } w \in B_1(0),$$

namely $S^{-1}(w) \in \mathbb{H}$. This shows that $S : \mathbb{H} \to B_1(0)$ is surjective so that $S(\mathbb{H}) = B_1(0)$. □

Finally we consider the sub-set of transformations that maps the disk $|z| < 1$ to the disk $|w| < 1$. We have the following theorem.

Theorem 8.17. *Let $z_0 \in \mathbb{C}$ with $|z_0| < 1$ and $t \in \mathbb{R}$. The transformation*

$$S_{t,z_0}(z) = e^{it}\frac{z - z_0}{1 - \bar{z}_0 z} \tag{8.2}$$

is an automorphism of $B_1(0)$. All the automorphism of $B_1(0)$ have the above form.

Proof. Let $|z| < 1$, then

$$|z - z_0|^2 = (z - z_0)(\bar{z} - \bar{z}_0) = |z|^2 + |z_0|^2 - 2\text{Re}(z\bar{z}_0)$$

and

$$|1-\bar{z}_0 z|^2 = 1 + |z|^2|z_0|^2 - 2\text{Re}(z\bar{z}_0)$$

so that

$$|z-z_0|^2 - |1-\bar{z}_0 z|^2 = |z|^2 + |z_0|^2 - 2\text{Re}(z\bar{z}_0) - (1+|z|^2|z_0|^2 - 2\text{Re}(z\bar{z}_0)) = (1-|z_0|^2)(|z|^2-1) < 0$$

whenever $|z| < 1$ and $|z_0| < 1$. Hence for $z \in B_1(0)$

$$|w| = |(S_{t,z_0}(z)| = \left|\frac{z-z_0}{1-\bar{z}_0 z}\right| < 1,$$

namely it lies within the unit disk. It is left as an exercise to show that for any $w \in \overline{B_1(0)}$ there is $z \in \overline{B_1(0)}$ so that $z = S^{-1}_{t,z_0}(w)$. We conclude that the map $S_{t,z_0}(z)$ is a bijection of $\overline{B_1(0)}$ to itself. We skip the proof of the second part of the theorem regarding the statement that all automorphisms of $B_1(0)$ have this form. (See M. Stein and Rami Shakarchi, Complex Analysis page 218 "Schwartz Lemma"). □

8.3 Conformal equivalence

We start with a definition.

Definition 8.18. Let $D_1, D_2 \subseteq \mathbb{C}$ be domains. We say that D_1 is conformally equivalent to D_2 iff there is an analytic one-to-one function (bi-holomorphic function) $f : D_1 \to D_2$ such that $f(D_1) = D_2$.

Remark 8.19. Clearly conformal equivalence is an equivalence relation.

Exercise 8.20. Prove that $\mathbb{C}$ is not conformally equivalent to any bounded domain.
Solution. Let $f(z) : \mathbb{C} \to D$ be such conformal map, namely an analytic function from $\mathbb{C}$ to D with D a bounded domain. It follows that $|f(z)| < c_0$ for some constant c_0. Namely $f(z)$ is analytic in $\mathbb{C}$ and bounded, therefore by Liouville theorem f is constant. But $f'(z) = 0$ for all $z \in \mathbb{C}$ so a constant cannot be a conformal map from $\mathbb{C}$ to D.

The natural question is how to classify domains of the complex plane up to conformal equivalence. The theorem below is one of the fundamental results in the theory of conformal mappings.

Theorem 8.21. (Riemann Mapping Theorem) *Let $D \subset \mathbb{C}$ be a simply connected domain such that $D \neq \mathbb{C}$ and let $z_0 \in D$. Then there is a unique one–to–one conformal mapping f of D onto the unit ball $B_1(0)$, having the properties $f(z_0) = 0$ and $\text{Re}(f'(z_0)) > 0$, $\text{Im}(f'(z_0)) = 0$.*

In other words, among the simply connected domains there are only two equivalence classes; one consisting of $\mathbb{C}$ alone and another containing all proper (different from $\mathbb{C}$) simply connected domains.

Observe, that classification of conformally equivalent non simply connected domains becomes far more complicated. For example, an annulus $A_{1,2}(0)$ is not conformally equivalent to the punctured unit ball $A_{0,1}(0)$. We do not consider this topic in the course.

Corollary 8.22. *Let $D_1, D_2 \subset \mathbb{C}$ be simply connected domains such that $D_1, D_2 \neq \mathbb{C}$. Then D_1 and D_2 are conformally equivalent.*

Proof. We merely observe that D_1 is conformally equivalent to the unit ball $B_1(0)$, and $B_1(0)$ is conformally equivalent to D_2, via the inverse of the conformal mapping $f : D_2 \to B_1(0)$. □

We do not proof the existence of the Riemann mapping theorem but only the uniqueness. For the existence see Conway, pp.156–159. For the uniqueness we need to introduce the concept of conformal automorphism.

Conformal automorphism. Let $D \subseteq \mathbb{C}$ be a domain. Obviously, D is conformally equivalent to itself, the corresponding one–to–one conformal mapping is the identity mapping $f(z) = z$. Also, if $f, g : D \to D$ are one–to–one conformal mappings then the composition $g \circ f : D \to D$ is one–to–one and conformal. Finally, if $f : D \to D$ is a one–to–one and conformal map then $f^{-1} : D \to D$ is one–to–one and conformal. Because of these properties, the set of all one–to–one conformal mappings from D onto D forms a group (with respect to compositions), which is called the group of conformal transformations of D. Such group is also called *group of automorphism of* D defined as $\mathrm{Aut}(D)$.

Lemma 8.23. *The group of automorphism of* $\mathbb{C}$ *is given by the transformations*

$$z \to az + b, \quad a \neq 0, \quad a, b \in \mathbb{C}.$$

Proof. Let f be an automorphism of $\mathbb{C}$ to itself, it means that f is analytic and $f'(z) \neq 0$ in $\mathbb{C}$. Let us split the complex plane in two domains $B_1(0)$ and $D := \mathbb{C}\backslash\overline{B_1(0)}$. Since f is a one to one map, we have that $f(B_1(0)) \cap f(D) = \emptyset$. Applying Picard theorem 5.16 we conclude that f cannot have an essential singularity at infinity. Indeed the set of values of $f(D)$ is not dense in $\mathbb{C}$ since it does not contain $f(B_1(0))$. It follows that $f(z)$ has at most a pole at $z = \infty$, namely $f(z)$ is a polynomial in z. In order to have a one to one map, $f(z) = az + b$ with $a \neq 0$. □

Exercise 8.24. Prove that the group of conformal transformations f of the disk $B_1(0)$ to itself such that $f(0) = 0$ is given by $\{cz : c \in \mathbb{C}, |c| = 1\}$.
Solution. The element of the group of transformation of the unit disk to itself is $f(z) = S_{t,z_0}$ defined in Theorem 8.17. Then $f(0) = 0$ implies that $S_{t,z_0}(0) = e^{it}\dfrac{0 - z_0}{1 - \bar{z}_0 0} = -z_0 e^{it} = 0$ namely $z_0 = 0$ and t arbitrary, namely $S_{t,z_0}(z) = e^{it}z$. Setting $c = e^{it}$ we have the claim.

Proof of the uniqueness of Riemann mapping theorem. If there are two such functions f_1 and f_2, then the function $g(z) := f_1(f_2^{-1}(z))$ is a one to one mapping from the unit disk to itself with the property that $g(0) = 0$ and $g'(0) = f_1'(f_2^{-1}(0))/f_2'(z_0) > 0$. We know from exercise 8.24 that such maps are of the form $g(z) = cz$ with $|c| = 1$, and the condition $g'(0) = c > 0$, implies $c = 1$.

Mathematician of this section:
August Ferdinand Möbius (November 1790 – 26 September 1868) was a German mathematician and theoretical astronomer.
The proof of Riemann mapping theorem by Riemann, contained a fault that was fixed by Constantin Carathéodory (13 September 1873 – 2 February 1950, Greek mathematician) using the theory of Riemann surfaces, and separately by Paul Koebe (15 February 1882 – 6 August 1945, German mathematician).

9 Harmonic functions

Let $G \subseteq \mathbb{R}^2$ be an open set and $u = u(x, y) : G \to \mathbb{R}$ a real valued function. A natural question to ask is: *when u is a real (or imaginary) part of an analytic function $f : G \to \mathbb{C}$?*

Recall, that for a given real valued function $u = u(x, y)$, the differential expression

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is called the Laplacian of a u. Another way to write the Laplace equation is to use the Wirtinger derivatives (see Chapter 1)

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

so that we can write the Laplace equation in the form

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}u = \Delta u = 0. \tag{9.1}$$

Definition 9.1. Let $G \subseteq \mathbb{R}^2$ be an open set. A function $u : G \to \mathbb{R}$ is called harmonic in G iff u has continuous second partial derivatives in G and satisfies the Laplace equation

$$\Delta u = 0 \qquad (u \in G).$$

Example 9.2. The constant functions and the functions $x + y$, $x^2 - y^2$, xy, $\exp(x)\cos(y)$, are harmonic in $\mathbb{R}^2$ because they are the real or imaginary parts of analytic functions. The functions $\log(x^2 + y^2)$, $\arctan\frac{y}{x}$ are harmonic in $\mathbb{R}^2\backslash\{0\}$.

Example 9.3. If u and v are harmonic in G and $\lambda, \mu \in \mathbb{R}$ then $\lambda u + \mu v$ is harmonic in G.

In what follows we shall frequently use the polar coordinates (r, φ) on $\mathbb{R}^2 \setminus \{0\}$, where $x = r\cos(\varphi)$, $y = r\sin(\varphi)$ and $\varphi \in [0, 2\pi)$.

Exercise 9.4. Let (r, φ) be the polar coordinates on the plane. Prove that in polar coordinates the Laplacian of a function $u = u(x, y)$ is represented by

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}. \tag{9.2}$$

Hint. Use the (real) Chain Rule and direct computations of partial derivatives.

Exercise 9.5. Verify that $r^{\pm k}\cos(k\varphi)$ $(k \in \mathbb{N})$ is harmonic in $\mathbb{R}^2$ and $\log(r)$ is harmonic in $\mathbb{R}^2 \setminus \{0\}$.

The following lemma is an easy consequence of the Cauchy–Riemann equations.

Lemma 9.6. *Let $G \subseteq \mathbb{C}$ be an open set and $f = u + iv : G \to \mathbb{C}$ an analytic function. Then* $\mathrm{Re}(f) = u$ *and* $\mathrm{Im}(f) = v$ *are harmonic in G.*

The theorem below combined with the previous lemma states that a function is harmonic if and only if it is a real part of an analytic function.

Theorem 9.7. (HARMONIC CONJUGATE THEOREM) *Let $D \subset \mathbb{R}^2$ be a simply connected domain and $u : D \to \mathbb{R}$ a harmonic function. Then*

(a) u is infinitely many times differentiable in D

(b) there exists a function $v : D \to \mathbb{R}$, called a harmonic conjugate to u in D, such that the function $f = u + iv : D \to \mathbb{C}$ is complex analytic.

Remark 9.8. If v is a harmonic conjugate to u in D and $c \in \mathbb{R}$, then $v + c$ is also a harmonic conjugate to u. It follows from the Cauchy–Riemann equations that the converse is also true. Namely, if v and w are harmonic conjugates to u in D, then $v - w = const$ in D.

Proof. We prove (b) first. Set $z = x + iy$. Consider the function

$$g(z) = \frac{\partial}{\partial z}u = \frac{1}{2}\left(\frac{\partial}{\partial x}u - i\frac{\partial}{\partial y}u\right) \tag{9.3}$$

where $\frac{\partial}{\partial z}$ is the Wirganten derivative defined in Chapter 1. We show that g is analytic in D. Indeed from (9.1) we have that

$$\frac{\partial}{\partial \bar{z}}g = \frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}u = 0$$

and this is equivalent to the Cauchy Riemann equation for g. Furthermore, since u is $\mathcal{C}^2$ in D, the function g is $\mathcal{C}^1$ in D and by Theorem 1.30 is analytic. Next we define

$$G(z) = 2\int_{\gamma_{a,z}} g(w)dw$$

where $\gamma_{a,z} : [\alpha, \beta] \to D$ is a path from a to z in D with a fixed. Since D is simply connected, the function $G(z)$ is well-defined and analytic. Set $G = \tilde{u} + i\tilde{v}$. Then

$$\underbrace{G'}_{=g} = \frac{\partial \tilde{u}}{\partial x} + i\frac{\partial \tilde{v}}{\partial x} = \frac{1}{i}\left(\frac{\partial \tilde{u}}{\partial y} + i\frac{\partial \tilde{v}}{\partial y}\right) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Thus $\frac{\partial \tilde{u}}{\partial x} = \frac{\partial u}{\partial x}$ and $\frac{\partial \tilde{u}}{\partial y} = \frac{\partial u}{\partial y}$, namely $\tilde{u} - u = const$ in D. Set $f = G - const$. Then $\mathrm{Re}(f) = u$ and $\mathrm{Im}(f) = v$ as required.

To prove (a) simply observe that the real part of a complex analytic function is infinitely many times differentiable. □

Example 9.9. Find harmonic conjugates to the harmonic function $u(x, y) = x^2 - y^2$.

Solution. By direct computation we see that u is harmonic on $\mathbb{R}^2$. Then a harmonic conjugate to u in $\mathbb{R}^2$ can be constructed as the imaginary part of the integral of the function $g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2(x + iy) = 2z$.

$$G(z) = \int_{\gamma_{0,z}} 2wdw = z^2,$$

that is $v(x, y) = \mathrm{Im}(z^2) = 2xy$. □

Remark 9.10. We saw in Harmonic Conjugate Theorem that harmonic functions are infinitely many times differentiable. In fact, harmonic functions are also *real analytic* (as the real parts of complex analytic functions). Roughly speaking, a function is real analytic if it is expressed locally as a power series in the variables x, y. More precisely, a real valued function $u(x,y)$ is real analytic at a point (x_0, y_0) if

$$u(x,y) = \sum_{n=0}^{\infty} \sum_{\substack{m+k=n \\ m,k\in\mathbb{N}\cup\{0\}}} c_{m,k}(x-x_0)^m (y-y_0)^k,$$

where the series converges absolutely in a neighborhood of (x_0, y_0).

Exercise 9.11. Prove that if u is harmonic in a simply connected domain $D \subset \mathbb{R}^2$ then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are harmonic in D.

Hint. By the Harmonic Conjugate Theorem, u is infinitely many times differentiable in D.

Exercise 9.12. If f is analytic in a simply connected domain $D \subseteq \mathbb{C}$ and $f(z) \notin (-\infty, 0]$ for any $z \in D$, show that $u = \log|f|$ is harmonic in D.

9.1 Properties of Harmonic Functions.

One reason why Harmonic Conjugate Theorem is important is that it enables us to deduce properties of harmonic functions from corresponding properties of analytic functions. In what follows, if convenient, we identify a complex number $z = x+iy \in \mathbb{C}$ with the point $(x,y) \in \mathbb{R}^2$ on the real plane, without further notices.

Lemma 9.13. (Mean Value Property) *Let $u : B_R(z_0) \to \mathbb{R}$ be a harmonic function. Then for any $r \in (0,R)$ one has*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\varphi})d\varphi. \tag{9.4}$$

Proof. Let $v : B_R(z_0) \to \mathbb{R}$ be a harmonic conjugate to u and $f = u + iv : B_R(z_0) \to \mathbb{C}$ the corresponding analytic function. By the Mean Value Property for complex analytic functions, for any $r \in (0,R)$ one has

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\varphi})d\varphi.$$

Taking the real part of both sides of the equation gives the required result. □

Remark 9.14. In fact, one can prove the converse. Namely, if a continuous real function $u : B_R(z_0) \to \mathbb{R}$ satisfies for every $r \in (0,R)$ the Mean Value Property (9.4) then u is harmonic in $B_R(z_0)$.

Next we deduce a Maximum Principle for harmonic functions.

Theorem 9.15. (Maximum Principle) *Let $D \subset \mathbb{R}^2$ be a simply connected domain and $u : D \to \mathbb{R}$ a harmonic function. If u has a local maximum or a local minimum point in D then u is constant in D.*

Proof. Let $v : D \to \mathbb{R}$ be a harmonic conjugate to u and $f = u+iv : D \to \mathbb{C}$ the corresponding analytic function. Consider the function $g := \exp(f)$. Obviously $g : D \to \mathbb{C}$ is analytic and $|g| = e^{u(z)}$. Since the exponential is strictly increasing, the local maxima of u are the same as those of $|g|$. By the Maximum Modulus Theorem, if $|g|$ has a local maximum in D then $|g|$ is constant in D. Hence, u is constant in D.

Finally, if u has a local minimum in D then $-u$ has a local maximum in D. Thus we can repeat the previous argument for the function $-u$. □

The following refinement of the Maximum Principle for bounded domains is crucial in the study of boundary value problems. In what follows, $\bar{D}$ denotes the *closure* of a domain $D \subset \mathbb{R}^2$ and ∂D denotes the *boundary* of D. Then the set D is called the *interior* of $\bar{D}$.

Theorem 9.16. (Maximum Principle for bounded domains) *Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain and $u : \bar{D} \to \mathbb{R}$ a continuous functions that is harmonic in D. Then u attains its maximum and minimum values on ∂D.*

Proof. It is known from Analysis that every continuous function on a bounded closed set attains its maximum and minimum values. However a harmonic function u can not have local maxima or minima in the interior of $\bar{D}$. Thus the result follows. □

Remark 9.17. A harmonic function on a bounded domain D which is not continuous up to the boundary may not attain its maximum or minimum values on the boundary ∂D. Consider, for example $u(r, \varphi) = \log(r)$ on the punctured ball $A_{0,1}$. The function $\log(r)$ is divergent at $r = 0$ and so it is not defined on the closure of $A_{0,1}$.

Remark 9.18. Harmonic functions on an unbounded domain D may not attain its maximum or minimum values on the boundary ∂D (and hence on the entire domain $\bar{D}$). Consider, for example, $u(x, y) = \exp(x)\cos(y)$ on the strip $\Pi = \{x \in \mathbb{R}, y \in (-\pi, \pi)\}$ or $u(r, \varphi) = r\cos(\varphi)$ on the upper half–plane $\mathbb{R}^2_+ = \{x \in \mathbb{R}, y > 0\}$.

9.2 Dirichlet problem for harmonic functions

Let $D \subset \mathbb{C}$ be a *bounded* domain, ∂D the boundary and $\bar{D}$ the closure of D. The Dirichlet Problem for harmonic functions is: *given a piece-wise continuous*[4] *function $g : \partial D \to \mathbb{R}$, find a continuous function $u : \bar{D} \to \mathbb{R}$, that is harmonic in D and that equals g on ∂D*; or in other words, that satisfies

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = g & \text{on } \partial D. \end{cases} \tag{9.5}$$

Observe that if $g \equiv$ *const* then $u \equiv$ *const* is a solution the Dirichlet problem (9.5). One can easily see that the solution of the Dirichlet problem is always unique (if exists).

Theorem 9.19. *The Dirichlet Problem* (9.5) *has at most one solution.*

Proof. Let u_1 and u_2 be two solutions of (9.5). Let $w = u_1 - u_2$. Then w is harmonic in D and $w = 0$ on ∂D. By the Maximum Principle w attains its maximum and minimum values on the boundary ∂D. But since $w = 0$ on ∂D, we conclude that $w \equiv 0$ in $\bar{D}$. Thus $u_1 = u_2$. □

[4]The function f is piece-wise continuous on ∂D if it is continuous except for a finite number of discontinuities on ∂D

Remark 9.20. If the domain D is unbounded then the uniqueness statement of Theorem 9.19 fails. For example, let $D = A_{1,\infty}(0)$ and $g \equiv 0$ on $S_1(0)$. Then $u_1 \equiv 0$ and $u_2 = \log(r)$ are two different solutions of (9.5).

Exercise 9.21. Prove that if $g \gneqq 0$ on ∂D and u is a solution to (9.5) then $u > 0$ in D.

The Dirichlet problem on general bounded domains is rather complicated. We want to find a solution for the case where the domain D is the disk. To do this we derive an integral formula that explicitly expresses the values of a harmonic function on a ball in terms of its values on the boundary of the ball.

Theorem 9.22. (Poisson Formula) *Let $u : \overline{B_R(0)} \to \mathbb{R}$ be a continuous function that is harmonic in $B_R(0)$. Then for any $\rho \in (0, R)$ and $\varphi \in [0, 2\pi)$ one has*

$$u(\rho e^{i\varphi}) = \frac{R^2 - \rho^2}{2\pi} \int_0^{2\pi} \frac{u(Re^{i\theta})}{|Re^{i\theta} - \rho e^{i\varphi}|^2} d\theta. \tag{9.6}$$

Remark 9.23. If $\rho = 0$ then (9.6) is simply the Mean Value Property for u.

Proof. Let $v : B_R(0) \to \mathbb{R}$ be a harmonic conjugate to u and $f = u+iv : B_R(0) \to \mathbb{C}$ the corresponding analytic function. Let $\tilde{z}$ be defined as

$$\tilde{z} := \frac{s^2}{\bar{z}},$$

which is called the *reflection* of z with respect to $S_s^+(0)$. Thus if $z \in A_{0,s}(0)$ then $\tilde{z} \in A_{s,\infty}(0)$, and therefore

$$\frac{1}{2\pi i} \int_{S_s^+(0)} \frac{f(w)}{w - \tilde{z}} dw = 0 \qquad \tilde{z} \in A_{s,\infty}(0)$$

Thus we may subtract this integral to obtain

$$f(z) = \frac{1}{2\pi i} \int_{S_s^+(0)} f(w) \left\{ \frac{1}{w - z} - \frac{1}{w - \tilde{z}} \right\} dw \qquad z \in A_{0,s}(0).$$

Observing that $|w| = s$ we can simplify

$$\frac{1}{w - z} - \frac{1}{w - \tilde{z}} = \frac{1}{w - z} - \frac{1}{w - \frac{|w|^2}{\bar{z}}} = \frac{1}{w - z} - \frac{\bar{z}}{w(\bar{z} - \bar{w})} = \frac{|w|^2 - |z|^2}{w|w - z|^2}.$$

Hence we have

$$f(z) = \frac{1}{2\pi i} \int_{S_s^+(0)} f(w) \frac{|w|^2 - |z|^2}{w|w - z|^2} dw \qquad (z \in A_{0,s}(0)),$$

or, in polar coordinates $w = se^{i\theta}$, $z = \rho e^{i\varphi}$ we get

$$f(\rho e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} f(se^{i\theta}) \frac{s^2 - \rho^2}{|se^{i\theta} - \rho e^{i\varphi}|^2} d\theta \qquad (0 < \rho < s,\ \varphi \in [0, 2\pi)).$$

Taking the real parts on both sides of the equation, we obtain

$$u(\rho e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} u(se^{i\theta}) \frac{s^2 - \rho^2}{|se^{i\theta} - \rho e^{i\varphi}|^2} d\theta \qquad 0 < \rho < s.$$

Observe that this formula is true for any $r < R$.

To complete the proof we only need to show that the formula is still true for $s = R$ and $w = 0$, hence for any $w \in B_R(0)$.

Since u is continuous on $\bar{B}_R(0)$ and since $|se^{i\theta} - \rho e^{i\varphi}|$ is never zero when $\rho < s$, we conclude, that for a fixed $|z| \in (0, s)$ the function

$$U(s,\theta) := u(se^{i\theta})\frac{s^2 - |z|^2}{|se^{i\theta} - z|^2}$$

is continuous for (s, θ) on the compact set $\overline{A_{\frac{R+\rho}{2},R}(0)}$, $\rho = |z|$. Thus, by a basic result from Analysis, (continuity on a compact set implies uniform continuity) $U(s, \theta)$ is uniformly continuous on $\overline{A_{\frac{R+\rho}{2},R}(0)}$. Consequently,

$$\lim_{s \to R_-} U(s,\theta) = U(R,\theta),$$

uniformly in θ. Then we can exchange the limit with the integral so that

$$\lim_{s\to R} \frac{1}{2\pi}\int_0^{2\pi} u(se^{i\theta})\frac{s^2 - |z|^2}{|se^{i\theta} - z|^2}d\theta = \frac{(R^2 - |z|^2)}{2\pi}\int_0^{2\pi}\frac{u(Re^{i\theta})}{|Re^{i\theta} - z|^2}d\theta.$$

Assuming $z = \rho e^{i\varphi}$ one gets the statement of the theorem. Finally, if $z = 0$ then (9.6) simply expresses the Mean Value Property (9.4) for u. □

Definition 9.24. The quantity

$$P_\rho(\varphi) = \frac{(1 - |\rho|^2)}{2\pi}\frac{1}{|1 - \rho e^{i\varphi}|^2}$$

is called the Poisson kernel of the unit disk.

Remark 9.25. If $u : \overline{B_1(0)} \to \mathbb{R}$ is a continuous function that is harmonic in $B_1(0)$ then by the Poisson formula (9.6) we can write it in the form

$$u(\rho e^{i\varphi}) = u * P_\rho(\varphi) := \int_0^{2\pi} u(e^{i\theta})P_\rho(\varphi - \theta)d\theta \tag{9.7}$$

The Poisson Formula enables to settle the Dirichlet Problem for the case when the domain D is an open ball. Suppose that we are given a continuous function $g : S_R(0) \to \mathbb{R}$. Then we can plug g instead of u in the right hand side of (9.6).

Theorem 9.26. (Solution of the Dirichlet Problem for a ball) *Let $g : S_R(0) \to \mathbb{R}$ be a piecewise continuous function. Then*

i) there exists a function u that is harmonic in $B_R(0)$ and it is defined by the formula

$$u(\rho e^{i\varphi}) = \frac{R^2 - \rho^2}{2\pi}\int_0^{2\pi}\frac{g(Re^{i\theta})}{|Re^{i\theta} - \rho e^{i\varphi}|^2}d\theta \qquad (\rho \in (0, R),\ \varphi \in [0, 2\pi)) \tag{9.8}$$

ii) For those values of φ where g is continuous

$$\lim_{\rho \to R_-} u(\rho e^{i\varphi}) = g(Re^{i\varphi});$$

iii) if g is continuous on $S_R(0)$ then the limit

$$\lim_{\rho \to R_-} u(\rho e^{i\varphi}) = g(Re^{i\varphi}).$$

is uniform in φ

Sketch of the proof. To show that such defined function u is harmonic in $B_R(0)$ follows immediately from the Poisson formula (9.6). It is more difficult to prove ii) and iii). The integral expression for u must be examined in the critical case in which $\rho \to R$ and the integrand takes the values "cost/0" near $\theta = \varphi$. A complete proof of the theorem can be found in LANG, pp.244–251 or CONWAY, pp.258–262.

The basic method for solving the Dirichlet problem on general domains is as follows. Take the given domain D and transfer it by a *conformal map* to a ball, where the problem can be explicitly solved. This procedure is justified by the fact that under a conformal mapping, harmonic functions are transformed again into harmonic functions. When we have solved the problem on the ball, we can conformally transform the answer back to D.

9.3 Dirichlet Problem on Jordan domains

Theorem 9.27. *Let $D_1, D_2 \subseteq \mathbb{C}$ be conformally equivalent domains and $f : D_1 \to D_2$ a one–to–one conformal mapping of D_1 onto D_2. Let $u : D_2 \to \mathbb{R}$ be a harmonic function. Then the composition $u \circ f : D_1 \to \mathbb{R}$ is a harmonic function.*

Proof. Let $z_0 \in D_1$ and $w_0 = f(z_0) \in D_2$. Let $B_r(w_0) \subseteq D_2$ and $V = f^{-1}(B_r(w_0))$. Clearly $V \subseteq D_1$. Since $B_r(w_0)$ is simply connected, by the Harmonic Conjugate Theorem, there is an analytic function $g : B_r(w_0) \to \mathbb{C}$ such that $u = \mathrm{Re}(g)$. Hence $g \circ f : V \to \mathbb{C}$ is analytic as a composition of analytic functions. Therefore $u \circ f = \mathrm{Re}(g \circ f)$, as one easily sees. Thus $u \circ f : V \to \mathbb{C}$ is harmonic as it is the real part of an analytic function. Since $z_0 \in D_1$ was arbitrary, we conclude that $u \circ f$ is harmonic in D_1. □

Remark 9.28. One can think of a one–to–one conformal mapping of D_1 onto D_2 as a change of coordinates. Note, that definition of the Laplace operator Δ involves partial derivatives of u. So, *a priory*, definition of a harmonic function depends on the choice of coordinates. Theorem 9.27 says however, that a harmonic function remains harmonic after a conformal change of coordinates.

Recall, that a path $\gamma : [\alpha, \beta] \to \mathbb{C}$ is *simple* if γ is an injection, that is γ has no self intersections.

Definition 9.29. Let $D \subset \mathbb{C}$ be a bounded simply connected domain whose boundary ∂D is a simple closed path. Then D is called a Jordan domain.

Theorem 9.27, together with the Riemann Mapping Theorem, suggests a method of solving the Dirichlet Problem on Jordan domains.

Let $D \subset \mathbb{C}$ be a Jordan domain and $g : \partial D \to \mathbb{R}$ a piece-wise continuous function. The *Dirichlet Problem* is to find a harmonic function $u : D \to \mathbb{R}$ that satisfies

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = g & \text{on } \partial D \text{ a.e.}, \end{cases} \tag{9.9}$$

where a.e. means almost everywhere, namely except for a set of measure zero. Since D is simply connected and $D \neq \mathbb{C}$, by the Riemann Mapping Theorem, there is a one–to–one conformal mapping f of D onto the unit ball $B_1(0)$. Furthermore, it is known that *if D is a Jordan domain, then f is continuous up to the boundary ∂D (that is $f : \bar{D} \to \overline{B_1(0)}$ is continuous) and f is a one–to–one mapping of ∂D onto $S_1(0)$.* [5]

[5] The proof of this fact is rather technical. See, e.g., LANG, pp.351–358.

Thus the function g is transformed via the conformal mapping f^{-1} into piece-wise continuous function $G = g \circ f^{-1} : S_1(0) \to \mathbb{R}$. Let $U : \bar{B}_1(0) \to \mathbb{R}$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta U = 0 & \text{in } B_1(0), \\ U = G & \text{on } S_1(0) a.e. \end{cases}$$

Such solution is given by the convolution of the Poisson kernel with the function G, (see formula (9.7))

$$U(\rho e^{i\varphi}) = G * P_\rho(\varphi) = \frac{1-\rho^2}{2\pi} \int_0^{2\pi} \frac{G(e^{i\theta})}{|1-\rho e^{i(\varphi-\theta)}|^2} d\theta \qquad (\rho \in (0,1),\ \varphi \in [0, 2\pi)).$$

Then the function

$$u := U \circ f : D \to \mathbb{R}$$

is the solution of the original Dirichlet Problem (9.9). Therefore, we have just outlined the proof of the following result. [6]

Theorem 9.30. (Solution of the Dirichlet Problem for a Jordan domain) *Let $D \subset \mathbb{C}$ be a Jordan domain and $g : \partial D \to \mathbb{R}$ a piece-wise continuous function. Then there exists a unique function $u : \overline{D} \to \mathbb{R}$, such that u is harmonic in D and $u|_{\partial D} = g$ almost everywhere. If $f : D \to B_1(0)$ is a bi-holomorphic map and $f(z) = \rho e^{i\varphi}$, $0 < \rho < 1$, $0 \le \varphi \le 2\pi$, then*

$$u(z) = (g \circ f^{-1}) * P_\rho(\varphi) = \int_0^{2\pi} (g \circ f^{-1})(e^{i\theta}) P_\rho(\varphi - \theta) d\theta.$$

Example 9.31. Let us consider the Dirichlet problem on the strip $S = \{z \in \mathbb{C}\ | 0 \le \mathrm{Im}(z) \le i\pi\}$ with boundary condition $g(x) = g_0(x)$, $g(x + i\pi) = g_1(x)$ for $x \in \mathbb{R}$ with g_0 and g_1 continuous vanishing at infinity. The conformal map between the strip S and $B_1(0)$ is given by the composition of two standard maps:

- $f_1(z) = e^z$ maps $S \to \mathbb{H}$ (see example 8.8). For $x \in \mathbb{R}$, the line $x + i\pi$ is mapped to the negative real axis while the line x is mapped to the positive real axis
- $f_2(z) = \dfrac{z-i}{z+i}$ map $\mathbb{H} \to B_1(0)$ (see Theorem 8.16). The positive real line is mapped to the the unit half semicircle in the lower half plane while the negative real line is mapped to the half semicircle on the upper half plane.

Then $f = f_2 \circ f_1 = \dfrac{e^z - i}{e^z + i} : S \to B_1(0)$ is a bi-holomorphic map. The inverse map is $f^{-1}(w) = f_1^{-1} \circ f_2^{-1}(w) = \log\left(-i\dfrac{w+1}{w-1}\right)$ (see figure 8.1) The solution of the Dirichlet problem is given by

$$\begin{aligned} u(z) = (g \circ f^{-1}) * P_\rho(\varphi) &= \int_0^{2\pi} g\left(\log\left(-i\frac{e^{i\theta}+1}{e^{i\theta}-1}\right)\right) P_\rho(\varphi - \theta) d\theta \\ &= \int_0^{\pi} g_1\left(\log\left(-i\frac{e^{i\theta}+1}{e^{i\theta}-1}\right)\right) P_\rho(\varphi - \theta) d\theta + \int_\pi^{2\pi} g_0\left(\log\left(-i\frac{e^{i\theta}+1}{e^{i\theta}-1}\right)\right) P_\rho(\varphi - \theta) d\theta \end{aligned}$$

[6] For a complete proof see, e.g., R. Nevanlinna and V. Paatero, *Introduction to complex analysis*, Addison–Wesley, 1969; pp.305–324.

where ρ and φ are defined through z by $\frac{e^z - i}{e^z + i} = \rho e^{i\varphi}$.

Mathematicians of this section

Johann Peter Gustav Lejeune Dirichlet (13 February 1805 – 5 May 1859) was a German mathematician.
Marie Ennemond Camille Jordan (5 January 1838 – 22 January 1922) was a French mathematician.

[7]