Stabilized Reduced Basis Methods for the approximation of parametrized viscous flows: increasing Reynolds number

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Abstract

Starting from the state of the art [1, 2, 3] we study stabilization techniques for parametrized viscous flows in a reduced basis setting. We are interested in the approximation of the velocity and pressure. Offline-online computational splitting is implemented and offline-only, offline-online stabilization are compared (as well as without stabilization approach). Different test cases are illustrated and several stabilization classical approaches (SUPG, GaLS, Brezzi-Pitkaranta, Franca, Hughes, etc) are recast into a parametric reduced order setting. This approach is then compared with the other supremizer approach to guarantee the approximation stability by increasing the corresponding parametric inf-sup constant. The goal is two-fold: to guarantee stable parametrized viscous flows with increasing Reynolds numbers and to look for online computational savings by reducing the dimension of the online reduced basis system.

Stabilization Options for FE: Stokes Problem

$$\begin{cases} \text{Find } \boldsymbol{u_h}(\mu) \in V_h,\ p_h(\mu) \in Q_h : \\ a(\boldsymbol{u_h}(\mu), \boldsymbol{v_h}; \mu) + b(\boldsymbol{v_h}, p(\mu)) = (\boldsymbol{f}, \boldsymbol{v_h}) - \psi_h^\rho(\boldsymbol{v_h}) & \forall \boldsymbol{v_h} \in V_h \\ b(\boldsymbol{u_h}(\mu), q_h) = \phi_h(q_h) & \forall q_h \in Q_h \end{cases}$$

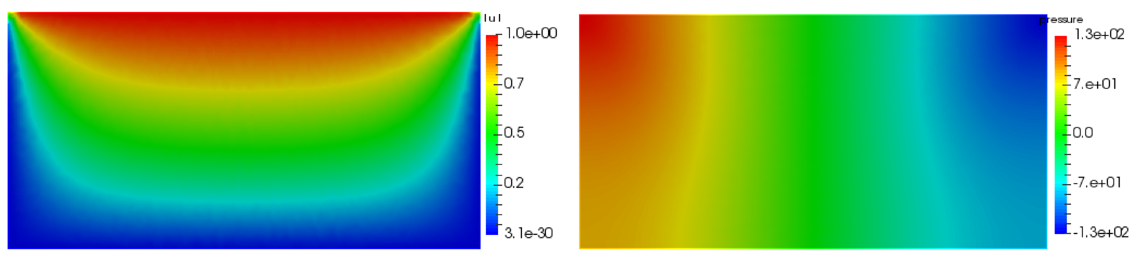
where V_h, Q_h are suitable FE spaces

$\psi_h^\rho = 0$ and $\phi_h(q_h) := \sum_K h_K^2 \int_K \nabla p_h \cdot \nabla q_h$ [Brezzi-Pitkaranta (BP),1984]
$\phi_h(q_h) := \delta \sum_K h_K^2 \int_K (-\nu\Delta \boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) \cdot \nabla q_h$ [Franca-Hughes (FH), 1986]
$\psi_h^\rho(\boldsymbol{v}_h) := \delta \sum_K h_K^2 \int_K (-\nu\Delta \boldsymbol{u}_h + \nabla p_h - \boldsymbol{f}) \cdot (-\rho\nu\Delta \boldsymbol{v}_h)$ and $\rho = 0, 1, -1$ corresponds to Franca, Hughes (1986); Franca, Hughes, Hulbert GALS (1989); Douglas, Wang DW (1989), respectively
For $\mathbb{P}_1/\mathbb{P}_1$ finite element pair, GALS and DW are same as FH.

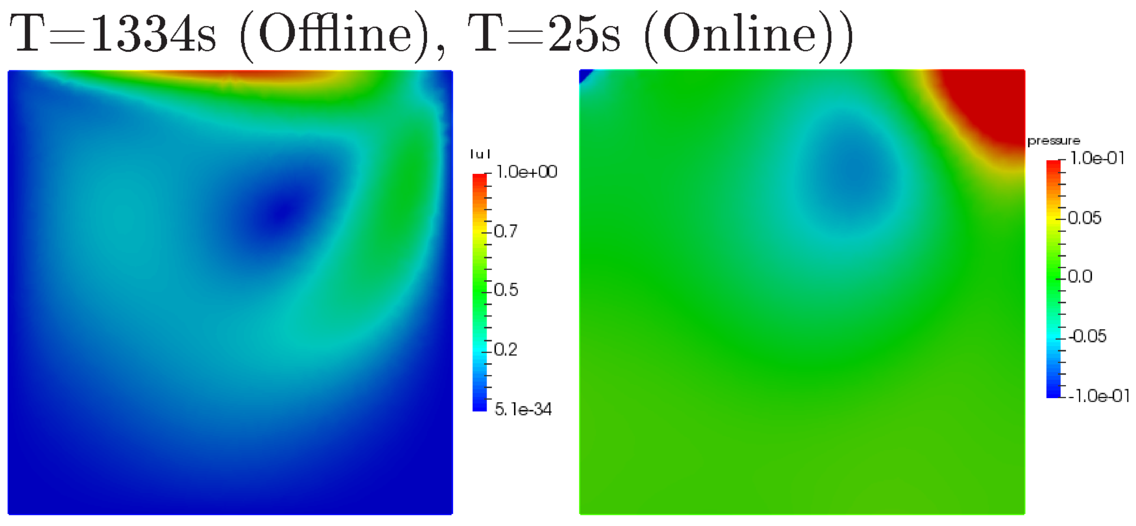
RB Stabilization Options

Offline-Only Stabilization:
$a(\boldsymbol{u}_h^N(\mu), \boldsymbol{v}_h^N; \mu) + b(\boldsymbol{v}_h^N, p_h^N(\mu)) = 0,\ \forall \boldsymbol{v}_h^N \in V_h^N$
$b(\boldsymbol{u}_h^N(\mu), q_h^N) = 0,\ \forall q_h^N \in Q_h^N$
Offline-Online Stabilization:
$a_s(\boldsymbol{u}_h^N(\mu), \boldsymbol{v}_h^N; \mu) + b_s(\boldsymbol{v}_h^N, p_h^N(\mu)) = 0, \forall \boldsymbol{v}_h^N \in V_h^N$
$b_s(\boldsymbol{u}_h^N(\mu), q_h^N) = 0, \forall q_h^N \in Q_h^N$
Supremizer:
$(\nabla \boldsymbol{s}, \nabla \boldsymbol{v}) + (p, \nabla \cdot \boldsymbol{v}) = 0,$ on $\Omega,$
$s = 0,$ on $\partial\Omega$
Offline-Only Stabilization with/without supremizer $(i) - (ii)$
Offline-Online Stabilization with/without supremizer $(iii) - (iv)$

Numerical Results

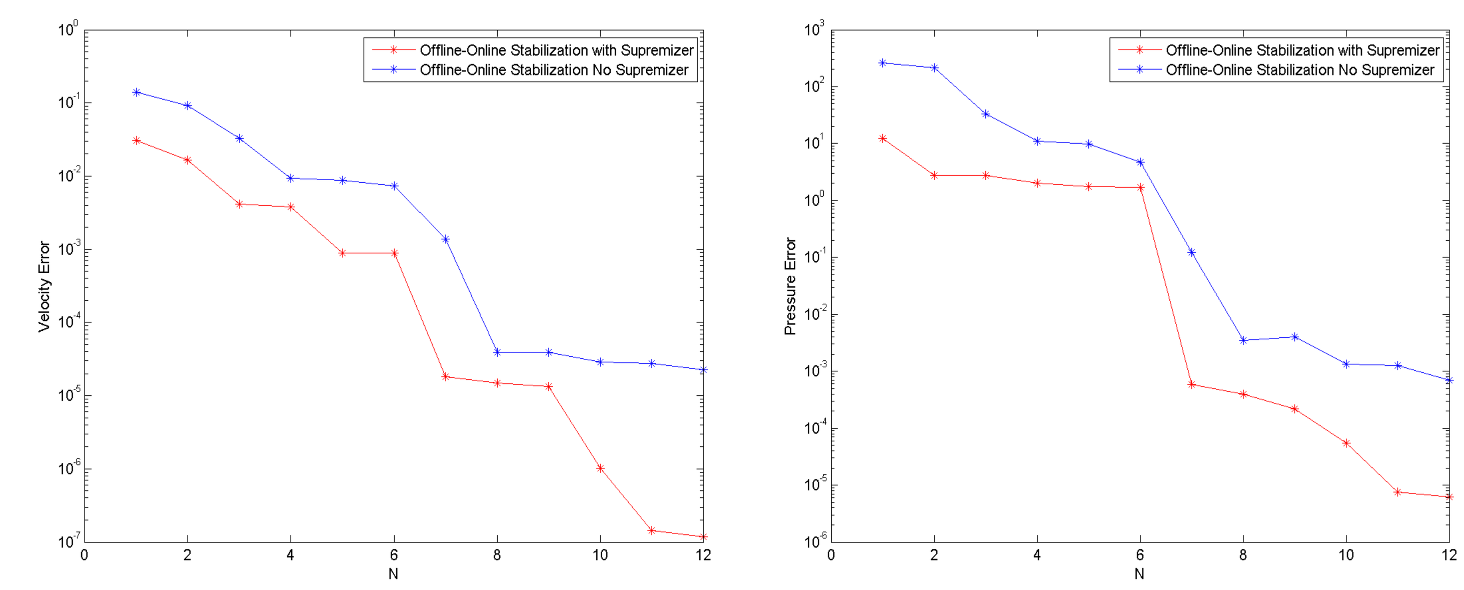


Stokes problem: Velocity (left) and Pressure (right) for Offline-Online Stabilization (FH) of Stokes problem: ($\nu = 100$, L=2, N=12, T=1334s (Offline), T=25s (Online))



Navier-Stokes problem: Velocity (left) and Pressure (right) for Offline-Online Stabilization (GALS) of Navier-Stokes problem at Re=200, T=2167s (Offline), T=81s (Online), N=7

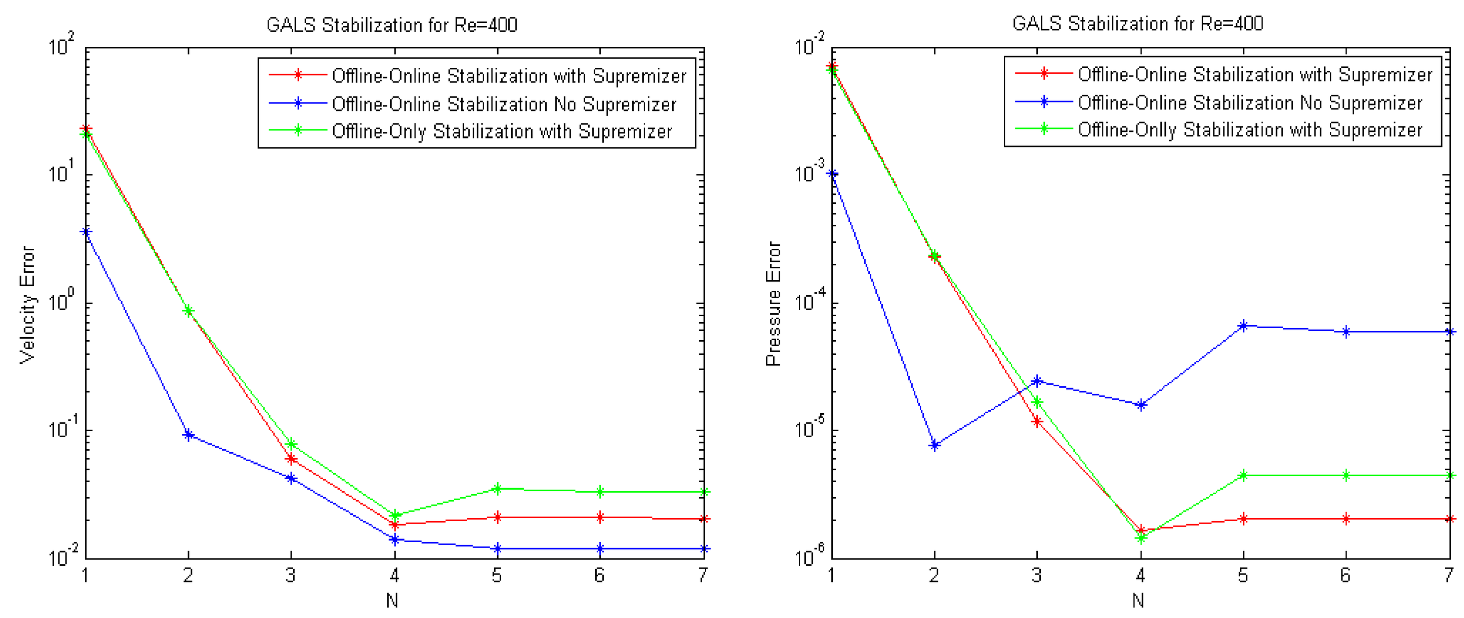
Stokes Problem: RB Stabilization (Geometrical Parameter)



Offline-Online stabilization (FH) with/without supremizer: velocity (left) and pressure (right). Parameter ranges: $\nu \in [10, 1000]$ and L$\in [0.56, 3]$.

- Offline-Only stabilization with/without supremizer is not working

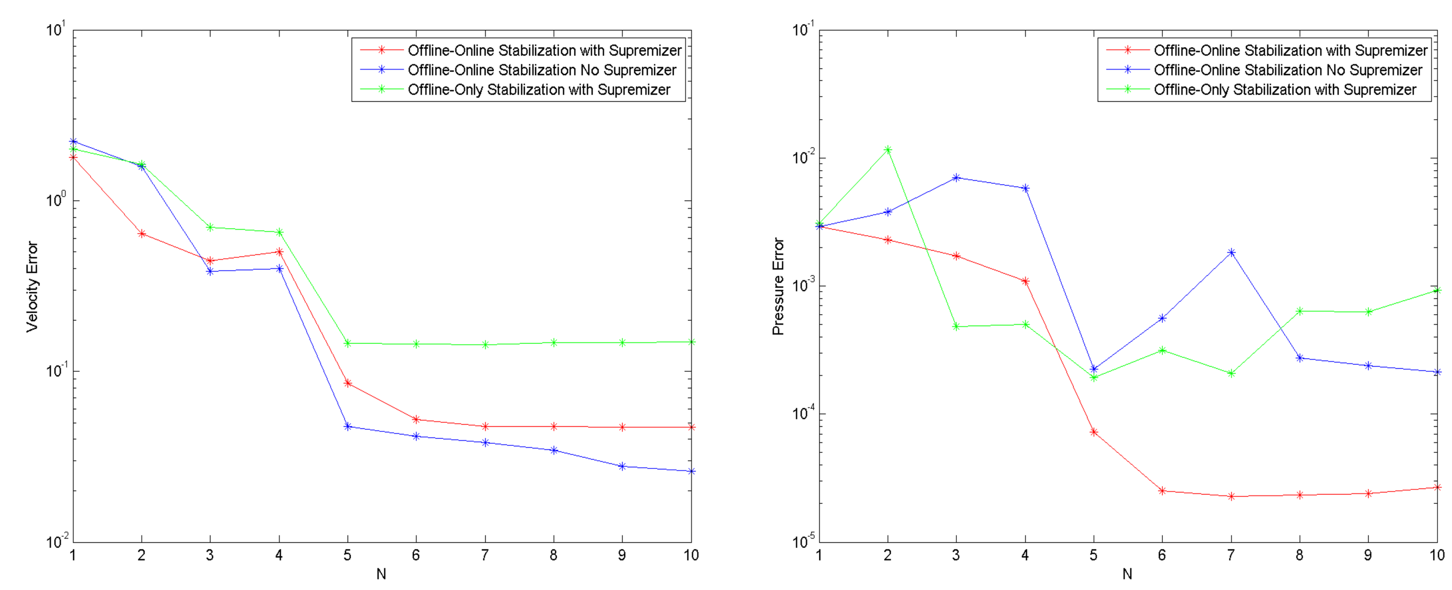
Navier-Stokes Problem: RB Stabilization (Physical Parameter)



Comparison between various options of stabilization (GALS), velocity (left) and pressure (right) Parameter ranges: Re$\in [10, 500]$.

- Offline-Only stabilization without supremizer is not working

Navier Stokes: RB Stabilization (Geometrical Parameter)



Comparison between various options of stabilization (GALS), velocity (left) and pressure (right). Parameters range Offline: Re$\in [10, 500]$, L$\in [0.5, 3]$, Online: Re=100, L=1.15, RB dimension N=10.

- Offline-Only stabilization without supremizer is not working

Summary

- Offline-Online stabilization for the physical and geometrical parameterization of steady Stokes and Navier-Stokes problem are presented
- offline-online stabilization is sufficient for velocity which is polluted a bit by supremizer, but it guarantees a good pressure approximation.

References

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[2] P. Pacciarini and G. Rozza. Stabilized reduced basis method for parametrized scalar advection-diffusion problems at higher Peclet number: roles of the boundary layers and inner fronts. In *Proceedings of the jointly organized WCCM XI, ECCM V, ECFD VI*, pages 5614–5624, 2014.

[3] P. Pacciarini and G. Rozza. Reduced basis approximation of parametrized advection-diffusion pdes with high péclet number. In *Numerical Mathematics and Advanced Applications - ENUMATH 2013*, pages 419–426. Springer International Publishing, 2015.

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