# The effort of increasing Reynolds number in POD-Galerkin Reduced Order Methods: from laminar to turbulent flows

S. Ali[1], S. Hijazi[1], S. Georgaka[2],
F. Ballarin[1], G. Stabile[1], A. Mola[1], and G. Rozza[1]
[1] SISSA, International School for Advanced Studies, Mathematics Area, mathLab, Trieste, Italy
[2] Imperial College London, Dep. of Mech. Eng., London, United Kingdom

## 1) Introduction and motivations

Nowadays the scientific community is facing the demand to simulate fluid dynamics problems with high values of the Reynolds number. High Reynolds number flows can be now simulated accurately using either Stabilized Finite Element or Finite Volume with turbulence modelling. However, in several situations, there is need to perform simulations in a multi-query contest (e.g. optimization, uncertainty quantification) or an extremely reduced computational time is required (real-time control). Therefore, in such situations, the resolution of the governing PDEs using standard discretization techniques may become unaffordable.

In this work we present two methodologies to tackle the increase of Reynolds number in Reduced Order Methods (ROMs) which are based on two different full order discretization techniques. First stabilized Finite Element method [1, 2] or a Variational Multi-Scale (VMS) approach [3], suited to deal with some turbulent patterns, are used for small to moderate Reynolds number. Then Finite Volume method is used for higher Reynolds number.

## 2) Projection based ROMs

For the case of moderate values of Reynolds number we consider the unsteady incompressible Navier-Stokes equations: find $\boldsymbol{u}(x,t)$ and $p(x,t)$, such that

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{0} & \text{in } \Omega\times(0,T)\,, \\ \nabla\cdot\boldsymbol{u} = 0 & \text{in } \Omega\times(0,T) \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega\times(0,T) \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u_0} & \text{in } \Omega \end{cases} \tag{1}$$

where $(0,T)$ with $T>0$ is the time interval of interest, $\Omega$ is bounded domain in $\mathbb{R}^2$, $\boldsymbol{u_0}\in L^2(\Omega)$ and $\nu$ is the viscosity of fluid. We seek the unknown velocity in the Sobolev space $\boldsymbol{V} = \boldsymbol{H}_0^1(\Omega) = \{v \in H^1(\Omega) | v = 0 \text{ on } \partial\Omega\}$, as well as the pressure in $Q = L_0^2(\Omega) = \{q\in L^2(\Omega)| \int_\Omega q d\Omega\}$.

The main assumption of ROM, regardless of the full order discretization technique, is that the system's dynamics and its response into the parameter space is governed by a reduced number of dominant modes. Therefore we can decompose the velocity and pressure into linear combination of global basis functions $\boldsymbol{\varphi}_i(\boldsymbol{x})$ and $\chi_i(\boldsymbol{x})$ (which do not depend on $t$ or the parameter $\mu$, which can be either a geometrical or physical quantity) multiplied by unknown coefficients $a_i(t,\mu)$ and $b_i(t,\mu)$, for velocity and pressure respectively, as follows:

$$\boldsymbol{u}(\boldsymbol{x},t;\mu) \approx \sum_{i=1}^{N_u} a_i(t,\mu)\boldsymbol{\varphi}_i(\boldsymbol{x}), \quad p(\boldsymbol{x},t;\mu) \approx \sum_{i=1}^{N_p} b_i(t,\mu)\chi_i(\boldsymbol{x}).$$

The reduced basis spaces $\boldsymbol{V}_{rb} = \text{span}\{\boldsymbol{\varphi}_i\}_{i=1}^{N_u}$ and $Q_{rb} = \text{span}\{\chi_i\}_{i=1}^{N_p}$ can be obtained either by Reduced Basis method with a greedy approach or using Proper Orthogonal Decomposition (POD) [4].

## 3) Stabilized Finite Element RB reduced order model

For low values of Reynolds number, we write the discrete formulation of (1) introducing the stabilization terms as:

$$\begin{cases} \text{Find } \boldsymbol{u}_h(.,t;\mu)\in\boldsymbol{V}_h, p_h(.,t;\mu)\in Q_h; & \\ (\dot{\boldsymbol{u}}_h,\boldsymbol{v}_h) + a(\boldsymbol{u}_h,\boldsymbol{v}_h;\mu) + c(\boldsymbol{u}_h,\boldsymbol{u}_h,\boldsymbol{v}_h;\mu) + b(\boldsymbol{v}_h,p_h;\mu) = \xi_h(\boldsymbol{v}_h;\mu) & \forall \boldsymbol{v}_h\in\boldsymbol{V}_h \\ b(\boldsymbol{u}_h,q_h;\mu) = \phi_h(q_h;\mu) & \forall q_h\in Q_h \end{cases} \tag{2}$$

where $\xi_h(\boldsymbol{v}_h;\mu)$ and $\phi_h(q_h;\mu)$ are the stabilization terms defined as:

$$\begin{aligned} \xi_h(\boldsymbol{v}_h;\mu) &:= \delta\sum_K h_K^2\int_K(\dot{\boldsymbol{u}}_h - \nu\Delta\boldsymbol{u}_h + \boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h, -\nu\gamma\Delta\boldsymbol{v}_h + \boldsymbol{u}_h\cdot\nabla\boldsymbol{v}_h) \\ \phi_h(q_h;\mu) &:= \delta\sum_K h_K^2\int_K(\dot{\boldsymbol{u}}_h - \nu\Delta\boldsymbol{u}_h + \boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h + \nabla p_h, \nabla q_h) \end{aligned} \tag{3}$$

$\forall\ \boldsymbol{v}_h\in\boldsymbol{V}_h$ and $q_h\in Q_h$. We use equal order $\mathbb{P}_k/\mathbb{P}_k$ FE spaces. For $\gamma = 0,1,-1$, the stabilization (3) is respectively known as Streamline Upwind Petrov Galerkin (SUPG), Galerkin least-squares (GLS), Douglas-Wang (DW). We then project (2) onto the reduced basis generated by a greedy method, either with or without supremizer enrichment.

## 4) Finite Volume POD-Galerkin reduced order model

For higher values of the Reynolds number, the problem is modelled through URANS equations with a $k-\omega$ turbulence modelling:

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\cdot\left[-p\mathbf{I} + (\nu+\nu_t)\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) - \frac{2}{3}k\mathbf{I}\right] & \\ \nabla\cdot\boldsymbol{u} = 0 & \\ \nu_t = f(k,\omega) & \text{in } \Omega\times(0,T) \\ \text{Transport-Diffusion equation for } k & \\ \text{Transport-Diffusion equation for } \omega & \end{cases}$$

The previous decomposition assumption is extended to the turbulent viscosity field $\nu_t$:

$$\nu_t(\boldsymbol{x},t;\mu) \approx \nu_{t,r}(\boldsymbol{x},t;\mu) = \sum_{i=1}^{N_{\nu_t}} d_i(t,\mu)\eta_i(\boldsymbol{x})$$

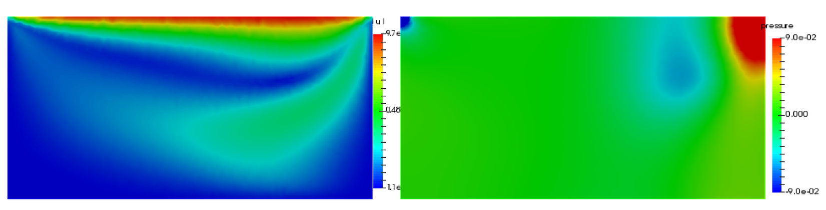
Afterwards, the momentum equation is projected onto the spatial bases of velocity, as well as the continuity equations onto the spatial bases for pressure. In contrast, $k-\omega$ transport-diffusion equations are not used in the projection procedure. The coefficients $\boldsymbol{d} = [d_i]_{i=1}^{N_{\nu_t}}$ are computed using a radial basis interpolation. The resulting system is thus:

$$\begin{cases} (\boldsymbol{B} + \boldsymbol{B_T})\boldsymbol{a} - \boldsymbol{a^T C a} + \boldsymbol{d^T}(\boldsymbol{C_{T1}} + \boldsymbol{C_{T2}})\boldsymbol{a} - \boldsymbol{K b} = \dot{\boldsymbol{a}} \\ \boldsymbol{P a} = \boldsymbol{0} \end{cases} \tag{4}$$

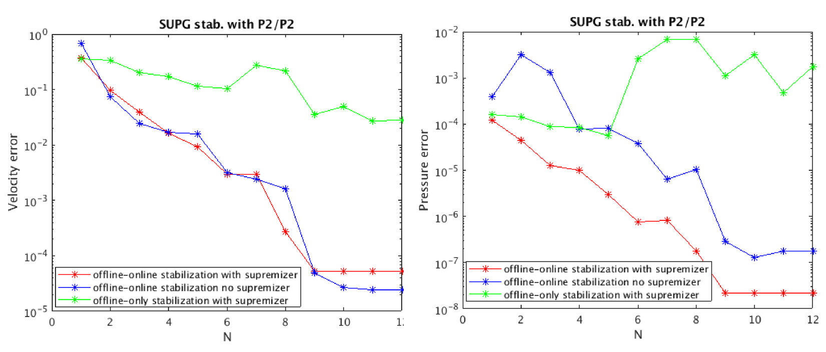
where $\boldsymbol{a}$ and $\boldsymbol{b}$ represent the unknown vectors of coefficients. In order to avoid pressure instabilities given by spurious pressure modes we employ a supremizer stabilization approach [5].

## 5) Numerical Results

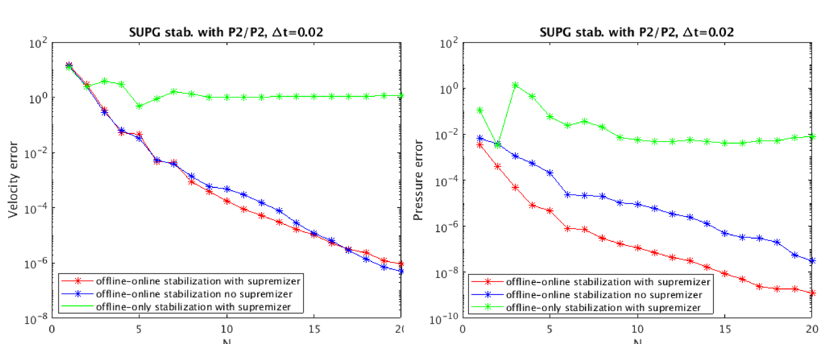
### FE Based ROM Results



**Steady Navier-Stokes problem:** SUPG-Stabilized RB solution for Velocity (left) and Pressure (right): Re=120 (online), L=2 (geometrical parameter), T=4885s (Offline), T=242s (Online), FE dim $\mathcal{N}$=44091, RB dim N=18 (with supremizer), N=12 (no supremizer).



**Steady Navier-Stokes problem:** Error comparison for velocity (left) and pressure (right). Parameters range $Re\in[100,200], L\in[0.5,3]$.



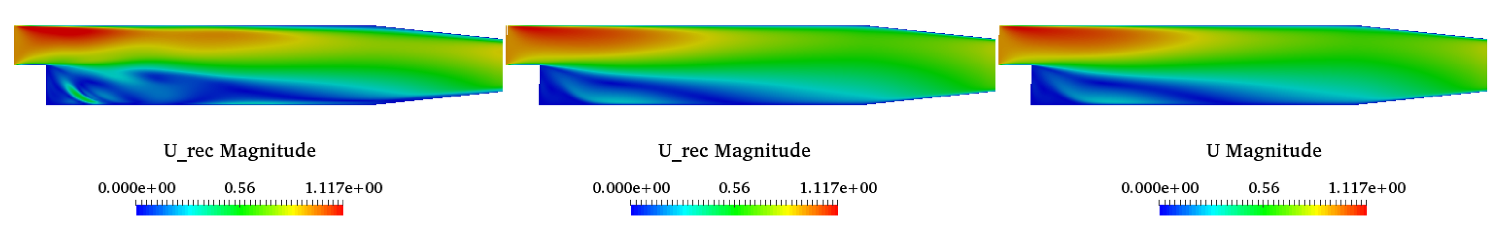
**Unsteady Navier-Stokes problem:** $L^2$ error in time for velocity (left) and pressure (right). Parameters range $Re\in[100,200]$, Online: $Re=120$, RB dimension $N=20$.

SUPG stabilization method for RB stability at low & moderate Reynolds number is applied to lid driven cavity flow. We summarize the main outcomes as follows:

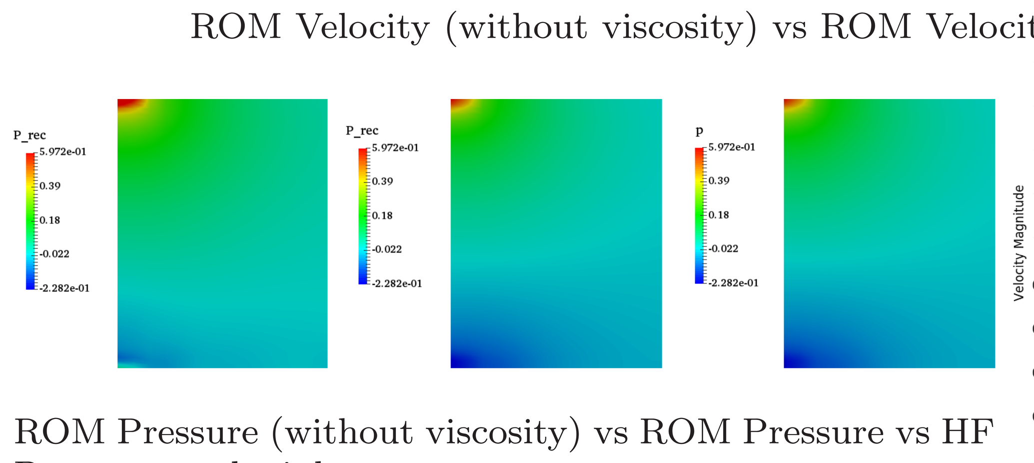
- we need to stabilize both offline and online stages for $\mathbb{P}_1/\mathbb{P}_1$ or $\mathbb{P}_2/\mathbb{P}_2$,
- no need to add supremizers to velocity space,
- offline-only stabilization is not consistent.

### FV Based ROM Results

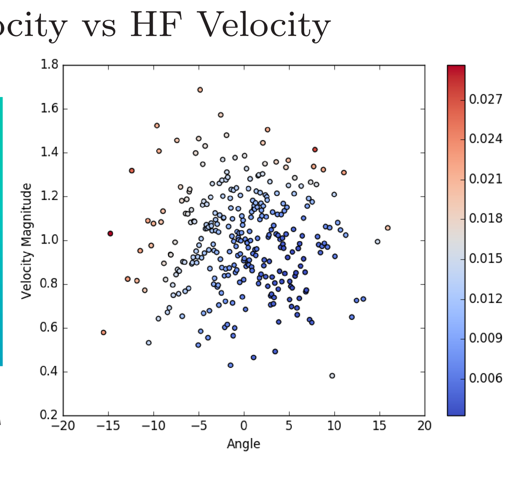
The presented results are for a backstep benchmark in a steady state setting. The parameters in this case are the magnitude of the velocity at the inlet and its inclination with respect to the inlet. In the offline stage we sampled randomly the parameters using latin hypercube sampling approach. The Re $\in$ [3816,16866] with a mean value equal to 10000. The ROM is constructed with 7 modes for velocity, pressure and supremizer and with 5 modes for the turbulent viscosity field. The fields computed with the ROM of (4) are compared with those obtained without considering the eddy viscosity's contribution at the reduced order level.



ROM Velocity (without viscosity) vs ROM Velocity vs HF Velocity



ROM Pressure (without viscosity) vs ROM Pressure vs HF Pressure at the inlet



Rel. $L^2$ error for velocity

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