Reduced basis methods for parametric bifurcation problems in nonlinear PDEs

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Introduction

The aim of this work is to show the applicability of the Reduced Basis (RB) model reduction in nonlinear systems undergoing bifurcations. Bifurcation analysis, i.e. following the different bifurcating branches and determining the bifurcation points, is a complex computational task. Reduced Order Models (ROM) can reduce the computational burden, enabling fast online evaluation of the solution for arbitrary parameter values. We represent the nonlinear PDE with the parametrized mapping $G : V \times \mathcal{D} \to V'$, so that the weak form reads: given $\lambda \in \mathcal{D}$, find $X(\lambda) \in V$ s.t.

$$g(X(\lambda), Y; \lambda) \doteq \langle G(X(\lambda); \lambda), Y \rangle = 0 , \quad \forall\, Y \in V.$$

We say that $\lambda^* \in \mathbb{R}$ is a bifurcation point for G from the trivial solution, if there is a sequence $(X_n, \lambda_n) \in V \times \mathbb{R}$ with $X_n \neq 0$ and $G(X_n, \lambda_n) = 0$ such that $(X_n, \lambda_n) \to (0, \lambda^*)$. A necessary condition for λ^* to be a bifurcation point for G is that $D_X G(0; \lambda^*)$ is not invertible.
From the algebraic viewpoint the RB method reads: find $\delta \vec{X}_N \in \mathbb{R}^N$ such that

$$\mathbb{J}_N(\vec{X}_N^k(\lambda); \lambda)\delta \vec{X}_N = G_N(\vec{X}_N^k(\lambda); \lambda), \qquad \text{and} \qquad X_N^{k+1} = X_N^k - \delta X_N.$$

Von Kármán equations for plates

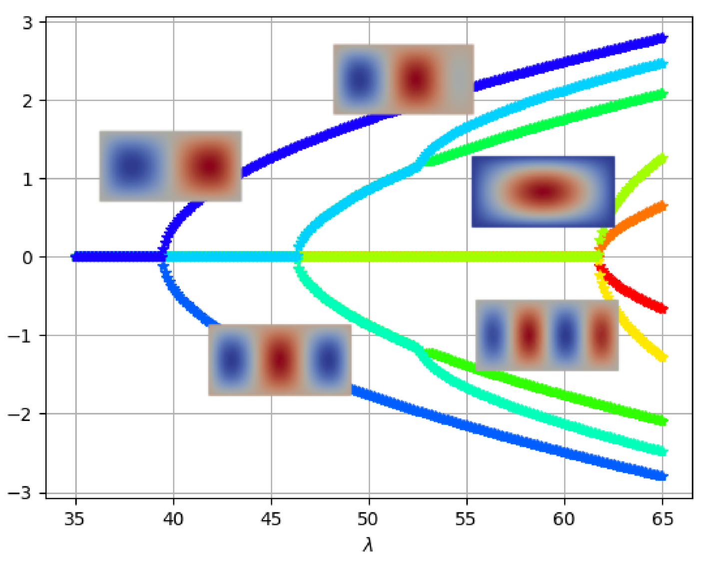
Consider a rectangular plate $\Omega = [0, L] \times [0, 1]$ in its undeformed state, subject to a λ-parametrized external load acting on its edges. The displacement u and the Airy stress potential ϕ satisfy the equations

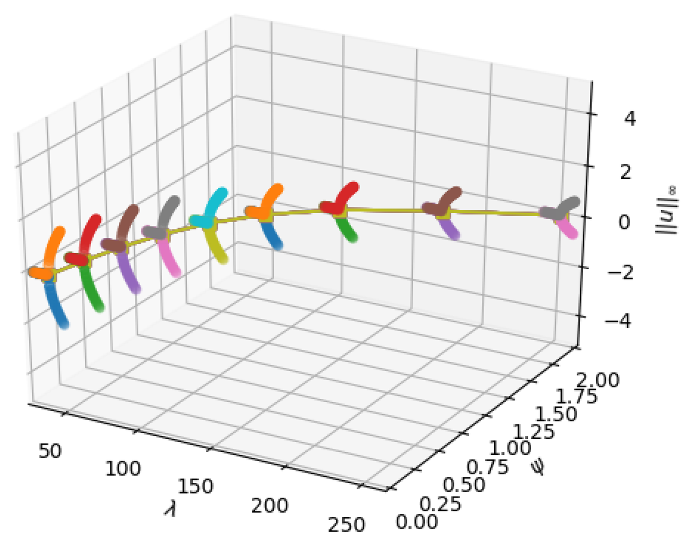
$$\begin{cases} \Delta^2 u = [\lambda h + \phi, u] + f \,, & \text{in } \Omega \\ \Delta^2 \phi = -\,[u, u] \,, & \text{in } \Omega \end{cases} \qquad \begin{cases} u = \Delta u = 0, & \text{in } \partial\Omega \\ \phi = \Delta \phi = 0, & \text{in } \partial\Omega \end{cases}$$

where h and f are some given external forces acting on our plate, while

$$\Delta^2 := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2, \qquad [u, \phi] := \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} ,$$

are the biharmonic operator and the Monge-Ampére bracket, respectively.



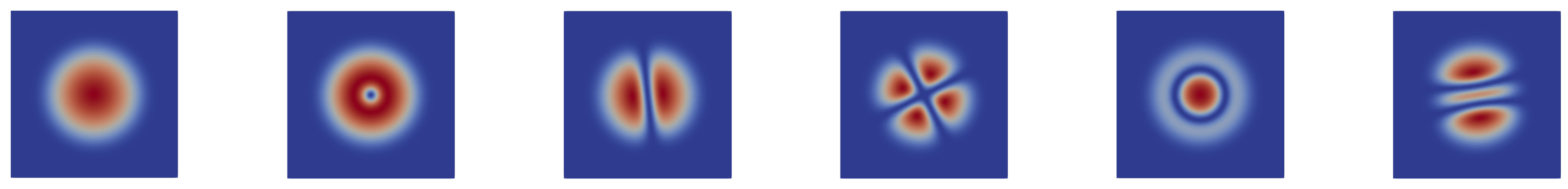


Schrodinger equations in quantum theory — with A. Quaini (Houston)

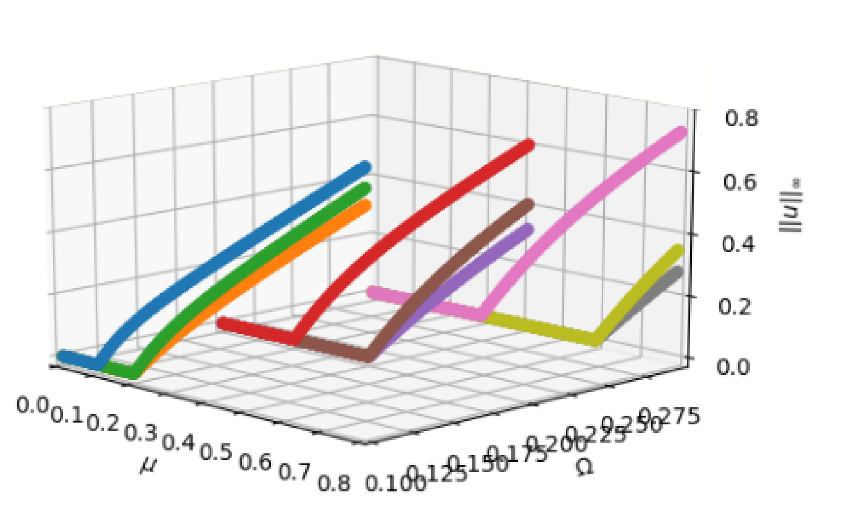
The Gross–Pitaevskii equation, which models certain classes of Bose-Einstein condensates, describes the ground state of a quantum system through the single-particle wave function $\phi : D \to \mathbb{C}$ which satisfies

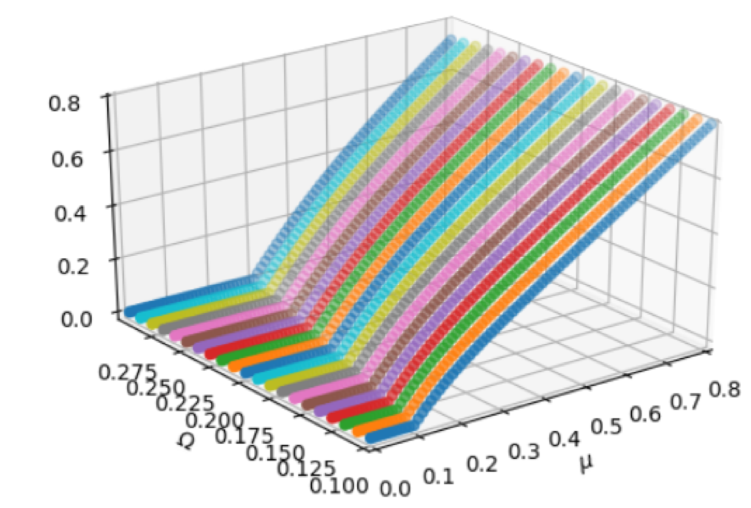
$$-\tfrac{1}{2}\Delta\phi + |\phi|^2\phi + (\tfrac{1}{2}\Omega^2 r^2)\phi - \mu\phi = 0, \quad \text{in } D$$

where Ω is the trap strength and μ is the chemical parameter.



By applying the Empirical Interpolation Method (EIM), it tooks just $t_{RB} = 7$ s for the complete construction of the reduced basis bifurcation diagrams, with respect to the $t_{HF} = 246$ s for the high fidelity one.



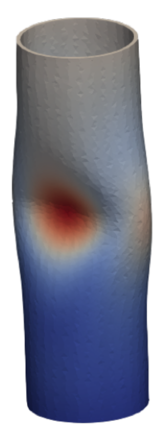


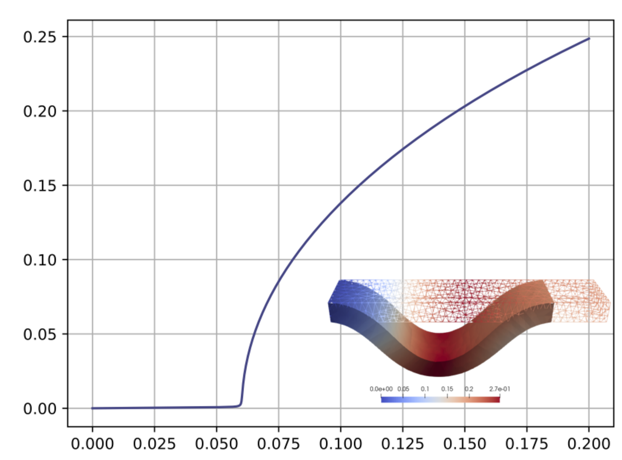
Hyperelastic beam in 2D and 3D cases — with A. Patera (MIT)

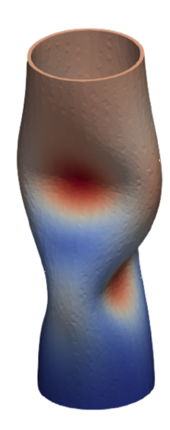
To study the deformation of an hyperelastic beam, we can transform it in a minimization problem, defining the potential energy as

$$\Pi(\mathbf{u}) = \int_\Omega \psi(\mathbf{u})\, \mathrm{dx} - \int_\Omega \mathbf{B} \cdot \mathbf{u}\, \mathrm{dx} - \int_{\partial\Omega} \mathbf{T} \cdot \mathbf{u}\, \mathrm{ds},$$

where $\mathbf{u}$ is the in plane displacement and $\psi(\mathbf{u})$ is the strain energy function, which becomes $\psi(\mathbf{u}) = \frac{\mu}{2}(I_c - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2$ with Neo-hookean law.





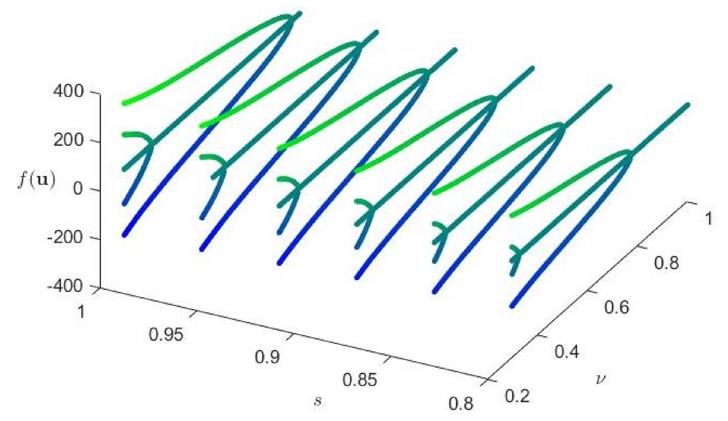


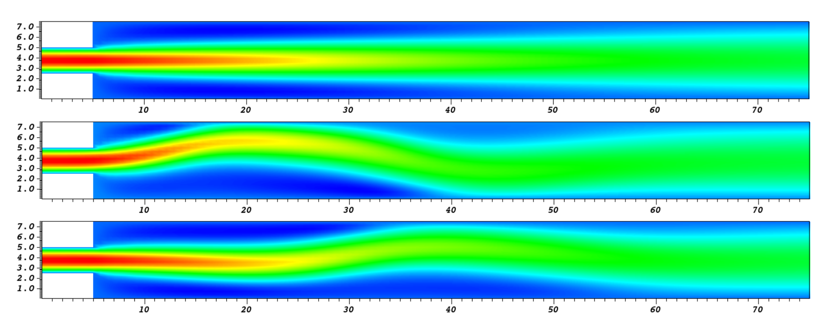
Navier-Stokes equations in CFD — with M. Hess (SISSA)

The test case is the study of a viscous, steady and incompressible flow in a planar straight channel with a narrow inlet, described by N-S equations

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla p = \mathbf{0}, & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \end{cases}$$

where $\mathbf{u}$ is the velocity, p the pressure and ν the kinematic viscosity.
The multi-parameter case is obtained by varying the inlet BC magnitude.





References

[1] F. Pichi, A. Quaini, and G. Rozza. Reduced technique in bifurcating phenomena: application to the Gross-Pitaevskii equation. *In preparation*, 2019.

[2] F. Pichi and G. Rozza. Reduced basis approaches for parametrized bifurcation problems held by nonlinear Von Kármán equations. *Journal of Scientific Computing*, 2019, in press. doi: 10.1007/s10915-019-01003-3.

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