**Introduction**

The Active Subspace property (AS) proposed by Trent Russi and developed by Paul Constantine [1] is employed in supervised dimensional reduction techniques. Dimension reduction in parameter studies confers benefits in a great number of engineering applications [4]. One of the main reasons behind it, is that parameter studies and scientific computing tasks are affected by the curse of dimensionality: the methods involved scale exponentially with the dimension of the parameters space making it unfeasible to be applied unless increasing the computational budget.

**Random Fourier Features**

Thanks to Bochner theorem and Monte Carlo method we can approximate the RBF-ARD kernel

\[
    k(x, x') = \sigma_f \exp \left( -\frac{(x - x')^2}{2} \right)
\]

with Random Fourier features [3]

\[
    k(x, x') = \int_{\mathbb{R}^m} e^{i\omega^T(x-x')}\lambda(\omega) \, d\omega \approx \langle z(x), z(x') \rangle
\]

where \( n \) is the number of random Fourier features, \( \mathbf{b} \in \mathbb{R}^n \) is a uniform sampled bias and \( \mathbf{W} \in \mathbb{R}^{n \times m} \) is the projection matrix sampled from the spectral measure \( \lambda \) obtained from the Bochner theorem

\[
    \lambda(\omega) = \mathcal{N}(0, \Lambda^{-1})
\]

The feature map \( \phi: \chi \subset \mathbb{R}^m \rightarrow \mathcal{H} \) from the input space to the Reproducing Kernel Hilbert Space can be approximated with \( z(x) \).

**The Active Subspace Property**

Consider a Lipschitz continuous, differentiable and square-integrable function, its gradient vector and a sampling density

\[
    f : \chi \subset \mathbb{R}^m \rightarrow \mathbb{R}, \quad \nabla f(x) \in \mathbb{R}^m, \quad \rho : \mathbb{R}^m \rightarrow \mathbb{R}_+
\]

Take the uncentered covariance matrix of the gradient, evaluate its approximation with Monte Carlo and partition its eigendecomposition,

\[
    C = \int (\nabla_x f)(\nabla_x f)^T \rho \, dx \approx \frac{1}{M} \sum_{i=1}^{M} (\nabla_x f_i)(\nabla_x f_i)^T = \mathbf{W}\mathbf{A}\mathbf{W}^T
\]

\[
    \Lambda = \begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2 & \end{bmatrix}, \quad \mathbf{W} = [\mathbf{W}_1 \mid \mathbf{W}_2], \quad \mathbf{W}_1 \in \mathbb{R}^{m \times l}
\]

where \( l \) is the dimension of the Active Subspace. Then the input data can be decomposed as

\[
    x = \mathbf{W}\mathbf{W}^T x = \mathbf{W}_1^T x + \mathbf{W}_2^T x = \mathbf{W}_1 y + \mathbf{W}_2 z.
\]

**Non-linear Active Subspaces**

The nonlinear extension (NAS) of Active Subspaces [2] is obtained applying the usual procedure to the new simulation map \( \tilde{f} : \phi(\chi) \subset \mathcal{H} \rightarrow \mathbb{R} \). The gradient is approximated with the chain rule as solution of the over-determined system

\[
    \frac{\partial \tilde{f}}{\partial z} = \left( \frac{\partial \tilde{f}}{\partial x} \right)^T \left( \frac{\partial f}{\partial x} \right)
\]

\[
    \frac{\partial z}{\partial x} = -\frac{\sqrt{2}}{n} \sigma_f \sin \left( \sum_{j=1}^{m} W_{ij} x_j + b_i \right) W_{id}
\]

where \( \left( \frac{\partial \tilde{f}}{\partial x} \right)^T \approx \frac{\partial z}{\partial x} \left( \frac{\partial \tilde{f}}{\partial x} \frac{\partial \tilde{f}}{\partial x} \right)^{-1} \) is the pseudo inverse matrix.

**Application to Gaussian Process regression**

The following Gaussian Process regressions (GPR) are the result of AS and NAS procedures applied to test cases with simulation functions given by the piston function from the benchmarks of Constantine work and some hypersurfaces of revolution in \( \mathbb{R}^3 \) with different generatrices and input space \([0, 2]^8\).

The piston function \( C \) models the cycle time of a piston within a cylinder

\[
    C = 2\pi \sqrt{\frac{W}{k + S^2 + \frac{P_{amb}}{\rho_{atm}}}}
\]

The parameters involved are the Piston Weight(W), the piston Surface Area(S), the initial Gas Volume(V0), the spring Coefficient(k), the atmospheric Pressure(P0), the ambient Temperature(T0), and the filling Gas Temperature(Tf). They have different ranges and are sampled uniformly. The chosen dimension of the Active Subspace for NAS is estimated over 10 runs, the associated variance is reported.

<table>
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<tr>
<th>Function</th>
<th>RRMSE</th>
<th>AS RRMSE</th>
<th>NAS RRMSE</th>
<th>var NAS</th>
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</tbody>
</table>

M=1500 n=1000, the chosen dimension of the Active Subspace for the GPR is \( l = 1 \), the relative root mean square error (RRMSE) of NAS is estimated over 10 runs, the associated variance is reported.

**References**


Acknowledgements

This work was partially supported by European Union Funding for Research and Innovation — Horizon 2020 Program — in the framework of European Research Council Executive Agency: H2020 ERC CoG 2015 AROMA-CFD project 681447 “Advanced Reduced Order Methods with Applications in Computational Fluid Dynamics” P.I. Gianluigi Rozza. We acknowledge also FARE-X-AROMA-CFD MIUR project to support excellence in research for ERC grantee in Italy.