Reduced Order Methods Applied to Nonlinear Time Dependent Optimal Flow Control Problems in Environmental Marine Sciences and Engineering

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Motivations in Environmental Sciences

Problems dealing with **environmental** marine **sciences** must be run many times for different **physical and/or geometrical** parametric configurations ($\boldsymbol{\mu} \in \mathcal{D} \subset \mathbb{R}^p$) in order to describe several natural phenomena. Moreover, they are usually connect to very **time consuming activities** such as **data assimilation** and **inverse problems** governed by parametrized partial differential equations (PDE($\boldsymbol{\mu}$)s).

Reduced Order Modelling $\rightarrow$ **fast** and **reliable** tool needed in order to **manage rapidly** and **efficiently** different (potentially dangerous) **situations** thanks to numerical simulations.

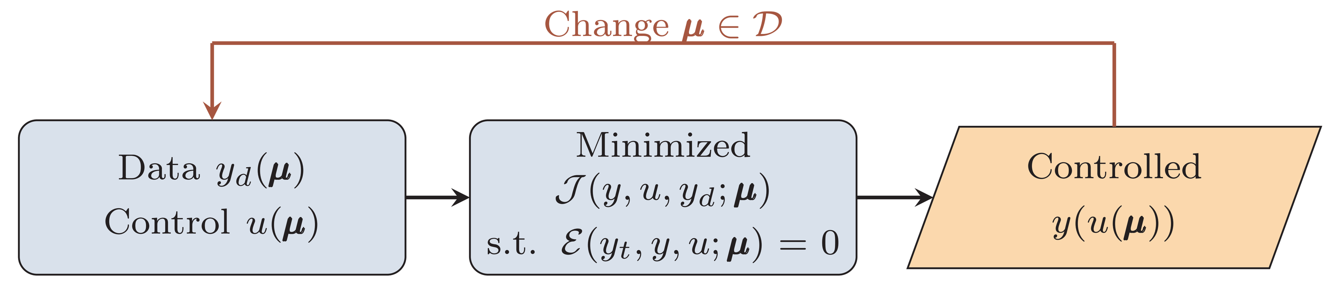
Problem Formulation: Parametrized Optimal Flow Control Problems (OFCP($\boldsymbol{\mu}$)s)

Given $\boldsymbol{\mu} \in \mathcal{D}$, a space-time domain $Q = \Omega \times [0, T]$, find the *state-control* variable $x := (y, u) \in X := Y \times U$ (suitable Hilbert spaces) which solves the PDE($\boldsymbol{\mu}$) $\mathcal{E}(y_t, y, u; \boldsymbol{\mu}) = 0$ and minimizes:

$$\mathcal{J}(y, u; \boldsymbol{\mu}) = \frac{1}{2}\|y(\boldsymbol{\mu}) - y_d(\boldsymbol{\mu})\|^2_{Y(\Omega_{\text{OBS}})} + \frac{\alpha}{2}\|u(\boldsymbol{\mu})\|^2_{U(\Omega_u)} \qquad (1)$$

- $\Omega_{\text{OBS}} \subseteq \Omega =$ *observation domain*, $\Omega_u \subseteq \bar{\Omega} =$ *control domain*,
- $y_d(\boldsymbol{\mu}) \in Y(\Omega_{\text{OBS}}) =$ data, $\alpha \in (0, 1] =$ penalization parameter.

Solved with **Lagrangian Approach** ($w \in Y =$ *adjoint variable*).



From Truth Problem to Reduced Order Model (ROM)

Truth Problem [1,2]: Spatial Discretization (dim $\mathcal{N}_\Omega$) + Time Discretization ($dim = N_t$) $\rightarrow$ **Unfeasible** for **many-query context** and **real-time context**.
Goal: to achieve the **accuracy** of a high fidelity approximation but at greatly **reduced cost** of a **low order model**

Idea: $x(\boldsymbol{\mu}) \xrightarrow{\text{truth problem(dim}=\mathcal{N})} x^{\mathcal{N}}(\boldsymbol{\mu}) \xrightarrow[\|x(\boldsymbol{\mu}) - x^{\mathcal{N}}(\boldsymbol{\mu})\| \to 0]{\text{ROM (dim } N \ll \mathcal{N})} x_N(\boldsymbol{\mu})$

Algorithm: Apply Proper Orthogonal Decomposition (**POD**) to the solution to the solution $\delta^{\mathcal{N}} = y^{\mathcal{N}}, u^{\mathcal{N}}, w^{\mathcal{N}}$ which contains information about **all the time instances** [4] $\rightarrow$ $POD(\delta^{\mathcal{N}}) = POD([\delta_1^{\mathcal{N}_\Omega}, \dots, \delta_{N_t}^{\mathcal{N}_\Omega}])$ $\rightarrow$ **Galerkin Projection**.

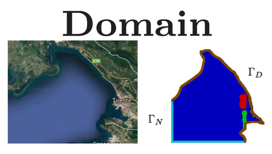
Steady Case: $Q = \Omega$, $y_t = 0$ and $N_t = 1 \rightarrow$ **Standard POD applied to the different variables.**

Pollutant Control in the Gulf of Trieste [3]

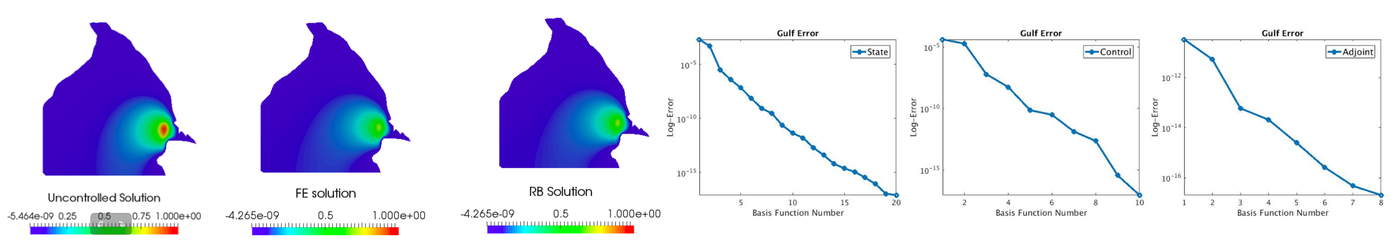
Motivations: *monitor*, *manage* and predict dangerous marine phenomena in a *fast way* in order to set up an environmental plan of action.
Aim: pollutant loss $y \in H^1_{\Gamma_D}(\Omega)$ under a safeguard threshold y_d.
Given $\boldsymbol{\mu} \in [0.5, 1] \times [-1, 1] \times [-1, 1]$, find $(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in H^1_{\Gamma_D}(\Omega) \times L^2(\Omega_u)$ which minimizes (1) and solves

$$\begin{cases} \mu_1 \Delta y + [\mu_2, \mu_3] \cdot \nabla y = u\chi_{\Omega_u} & \text{in } \Omega \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \Gamma_N \\ y = 0 & \text{on } \Gamma_D \end{cases}$$

Domain



$\Gamma_D =$ coasts, $\Gamma_N =$ Adriatic Sea, $\Omega_{OBS} =$ Natural area of Miramare; $\Omega_u =$ Source of pollutant, $N_{\max}$: 100, α: 10^{-5}, μ_1: diffusivity, μ_2, μ_3: advection (wind action).



N	Speedup
5	364
10	350
15	317

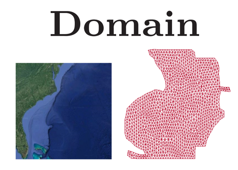
Results and Relative Errors $\sim 10^{-15}$

Nonlinear Oceanographic Solution Tracking [3]

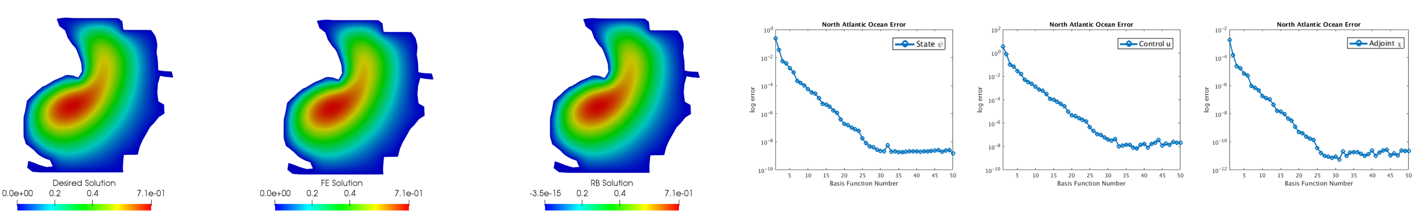
Motivations: unify *standard model* and *data* giving more reliable simulations *as quickly as possible.*
Aim: make the state the most similar to a given data y_d (**Gulf Stream Dynamic**).
Given $\boldsymbol{\mu} \in [10^{-4}, 1] \times [10^{-4}, 1] \times [10^{-4}, 0.045^2]$., find $(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in H^1_0(\Omega) \times L^2(\Omega)$ which minimize (1) and solves

$$\begin{cases} \dfrac{\partial y}{\partial x_0} + \mu_1 \Delta y - \mu_2 \Delta^2 y + \mu_3 \left(\dfrac{\partial y}{\partial x_0}\dfrac{\partial \Delta y}{\partial x_1} - \dfrac{\partial \Delta y}{\partial x_0}\dfrac{\partial y}{\partial x_1} \right) = u & \text{in } \Omega \\ y = 0, & \text{on } \partial\Omega, \\ \Delta y = 0, & \text{on } \partial\Omega. \end{cases}$$

Domain



Streamline Formulation: $y =$ streamfunction, $\Delta y = -$vorticity. The velocity solution of Navier-Stokes Equation + Earth rotation effect $\boldsymbol{v}$ is $(v_1, v_2) = (y_{x_1}, -y_{x_0})$.
μ_1, μ_2: diffusivity, μ_3: advection, $N_{\max}$: 100, α: 10^{-5}.



N	Speedup
5	8
15	7
20	6

Results and Relative Errors $\sim 10^{-9}$

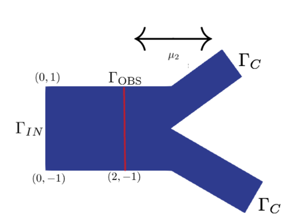
Time Dependent Problem: River Bed [5]

Aim: recover $y_d = [\mu_3(8(y^3 - y^2 - y + 1) + 2(-y^3 - y^2 + y + 1)), 0]$ in Ω_{OBS} with a Neumann control. Given $\boldsymbol{\mu} \in [1/20, 1/6] \times [1, 2] \times [1, 3]$, find $(y(\boldsymbol{\mu}), p(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in L^2(0, T; [H^1_{\Gamma_D}(\Omega)]^2) \times L^2(0, T; L^2(\Omega)) \times L^2(0, T; [L^2(\Omega)]^2)$ which solves

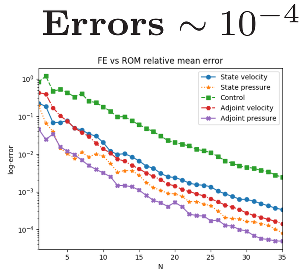
$$\min_{(y,p,u) \in X} \frac{1}{2}\int_0^T \int_{\Gamma_{OBS}} (y - y_d(\mu_3))^2 ds dt + \frac{\alpha_1}{2}\int_0^T \int_{\Gamma_C} u^2 ds dt + \frac{\alpha_2}{2}\int_0^T \int_{\Gamma_C} |\nabla u \cdot t|^2 ds dt$$

$$s.t. \begin{cases} y_t - \mu_1 \Delta y + \nabla p = 0 & \text{in } \Omega(\mu_2) \times [0, 1], \\ \text{div}(y) = 0 & \text{in } \Omega(\mu_2) \times [0, 1], \\ y = g & \text{on } \Gamma_{IN}(\mu_2) \times [0, 1], \\ y = 0 & \text{on } \Gamma_D(\mu_2) \times [0, 1], \\ -pn + \nabla y \cdot n = u & \text{on } \Gamma_C(\mu_2) \times [0, 1], \\ y(0) = y_0 & \text{in } \Omega(\mu_2) \times \{0\}, \end{cases}$$

Domain

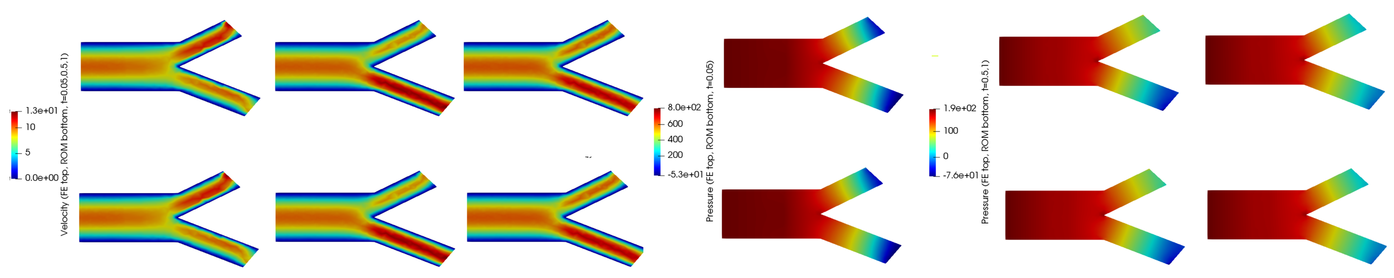


Errors $\sim 10^{-4}$



N	Speedup
20	47579
25	34335
30	22477
35	17420

Results: velocity and pressure



μ_1: diffusivity, μ_2: length of Ω_2 $N_{\max}$: 70, N_t: 20, α_1, α_2: $10^{-3}, 10^{-4}$

References

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[4] M. Strazzullo, F. Ballarin, and G. Rozza, "POD-Galerkin based Model Order Reduction for Parametrized Time Dependent Linear Quadratic Optimal Control Problems", *In preparation.*
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Acknowledgements

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