Segregated methods for reduced order models

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Introduction

When dealing with **Parametrized Partial Differential Equations** the computational cost required by a large number of solutions for each new value of the involved parameters may be unaffordably large. To mitigate that, different methods have been studied in order to find solutions in a more effcient way ([1]).
In particular, in our work we exploit a **POD approach** to obtain the basis functions used to project the original problem and to reconstruct an approximate solution manifold. Everything is developed in a **Finite Volume** framework ([2]) where we try to manage segregated approaches also at reduced level.

1) Segregated approaches, FOM level

Incompressible Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{\nabla P}{\rho} = \nu\nabla^2\mathbf{u} \\ \nabla\cdot(\rho\mathbf{u}) = 0 \end{cases}$$

Since the equations are **coupled** in principle it is not possible to solve them separately: they should be treated as a system and iterative block solvers are needed for non-linear systems. Unluckily these methods require the storage of **big matrices** at each step that makes them very unefficient for big meshes.
A good compromise between efficiency and quality of the solution is represented by the **segregated methods** employed in SIMPLE and PISO strategies solvers: momentum and continuity equations are solved iteratively and one at a time exchanging information at the end of each iteration.

2) SIMPLE algorithm

We based our work on the **SIMPLE** algorithm, one of the most spreaded segregated approaches:

1. guess beginning pressure and velocity fields P^* and $\mathbf{u}^*$ respectively and set $P' = P^*$ and $\mathbf{u}' = \mathbf{u}^*$;
2. solve **momentum predictor** step: $\mathbf{u}'' = \mathbf{A}^{-1}\mathbf{H}(\mathbf{u}') - \nabla P'$;
3. solve **pressure correction** step: $\nabla^2 P'' = \mathbf{F}(\mathbf{u}'')$;
4. if convergence check is not satisfied put $\mathbf{u}' = \mathbf{u}''$ and $P' = P''$ and repeat from point 2 on, otherwise stop;

where $\mathbf{A}$ is the diagonal part of the velocity operator in the momentum equation while $\mathbf{H}$ is its extra diagonal counterpart. The pressure correction step is performed by solving the continuity equation in a modified form: a **pressure equation** is obtained by substituting momentum equation into continuity one.

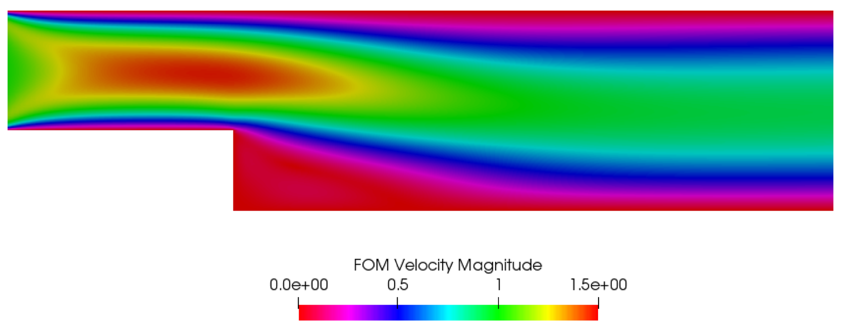
3) Segregated approaches, ROM level

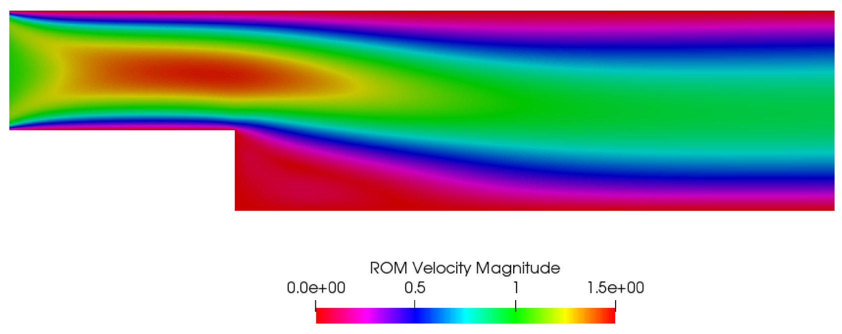
In order to be as coherent as possible with respect to the FOM model, we decided to simulate the SIMPLE algorithm also at the ROM level so that it is possible to project the same equations used to obtain the snapshots:

1. guess beginning discretized pressure and velocity fields P_r^* and $\mathbf{u}_r^*$ respectively, set $P_r' = P_r^*$ and $\mathbf{u}_r' = \mathbf{u}_r^*$, reconstruct the full velocity and pressure fields;
2. project and solve the **momentum equation**;
3. reconstruct the full velocity field;
4. project and solve the **pressure equation**;
5. reconstruct the full pressure field;
6. update **fluxes** to have a **divergence-free** velocity solution;
7. if convergence check is not satisfied repeat from point 2 on, otherwise stop.

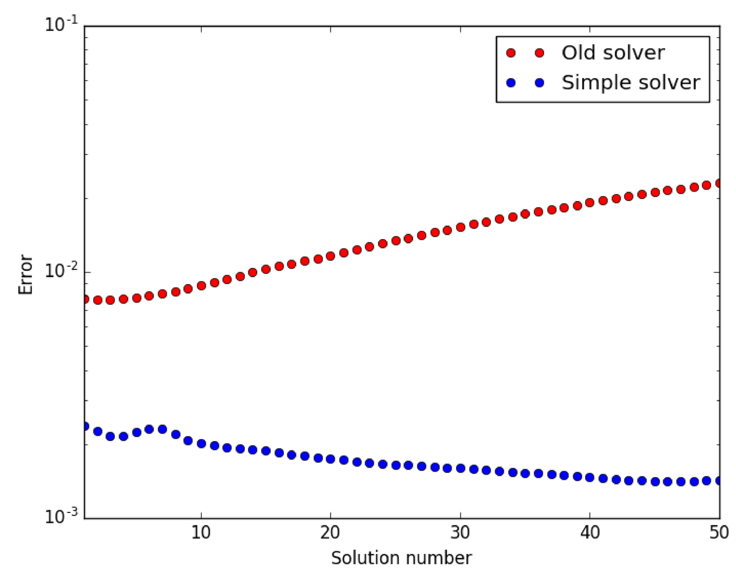
4) Applications: parametrized viscosity problem

We have applied the ROM SIMPLE algorithm to an incompressible Navier-Stokes problem where the geometry is represented by a back step and the parameter we have choosen is the **viscosity** $\mu \in [0.01, 1]$. 50 snapshots have been solved to apply the POD.





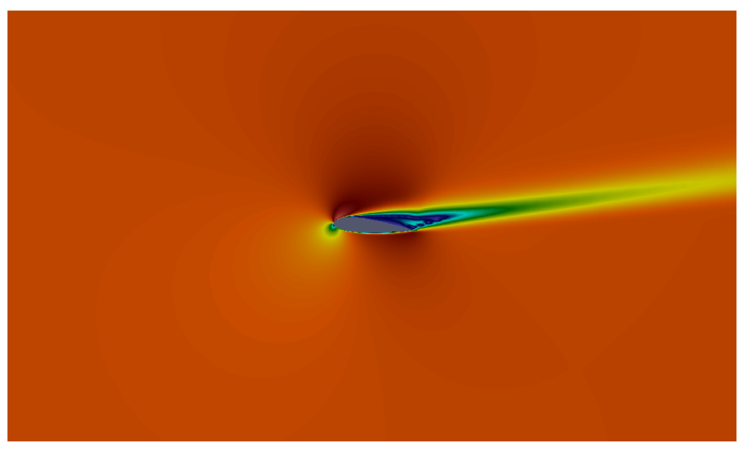
In the left figure we can see the full order velocity solution for $\mu = 0.024$ while in the right one its reduced counterpart is represented for the same value of the parameter. It has been obtained by projecting the problem with only 10 basis functions for both velocity and pressure fields.

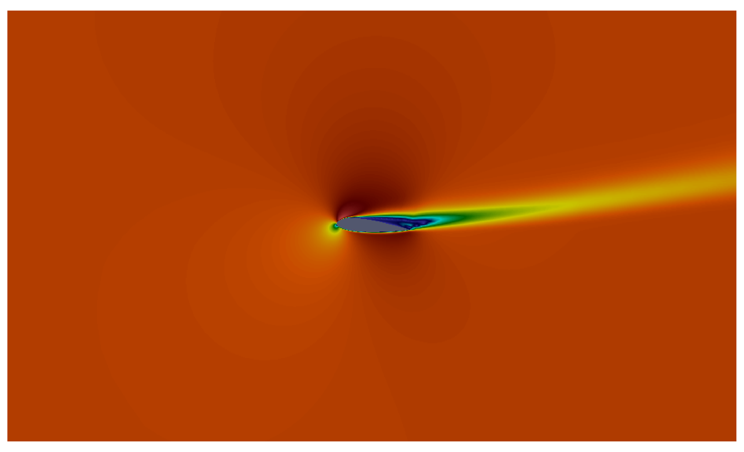


In the plot on the left it is reported a comparison between the L^2 norm of the error for each value of the parameter in the training set for the old block ROM solver (red) and for the SIMPLE ROM solver (blue).

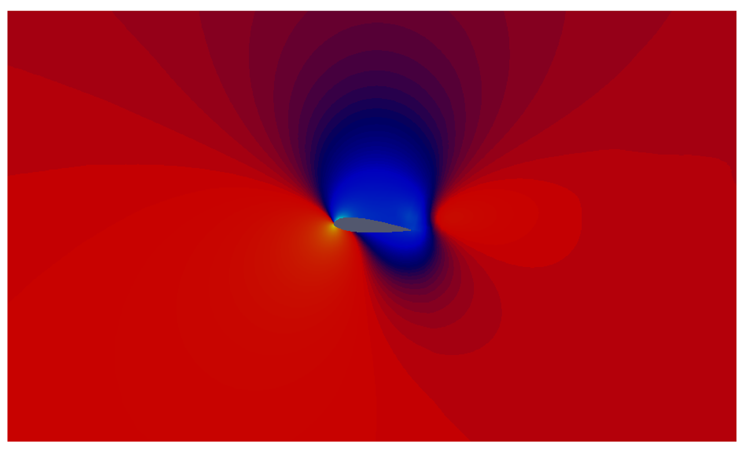
5) Applications: parametrized geometry problem

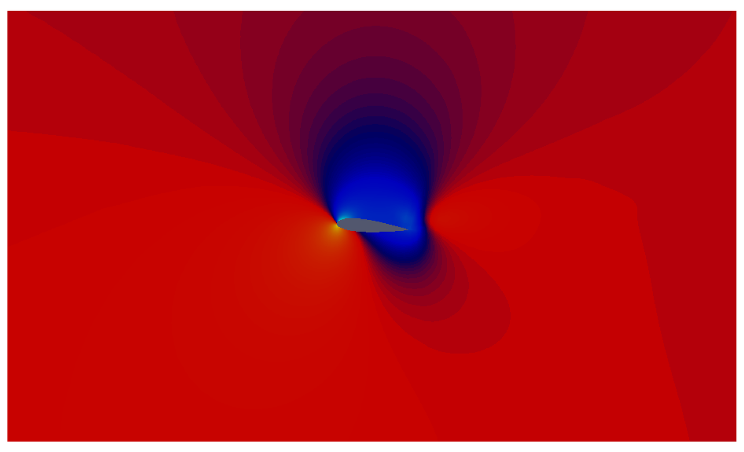
One of the in-progress works in the group is relative to **geometrical parametrization** and **hyper-reduction** ([3]) In this case the **angle of attack** of an airfoil has been parametrized. We fixed $\alpha \in [-5, 5]$ and solved 100 snapshots to project the problem. To compare the different snaphots it has been necessary to move the mesh in the right way: starting from a reference one, the modified one is obtained by the use of radial basis functions interpolation. In this way it is possible to compare the solutions: all the meshes have the same number of cells.





As we can see from the figures before, the FOM (left) and ROM (right) velocity solutions are pretty much the same.



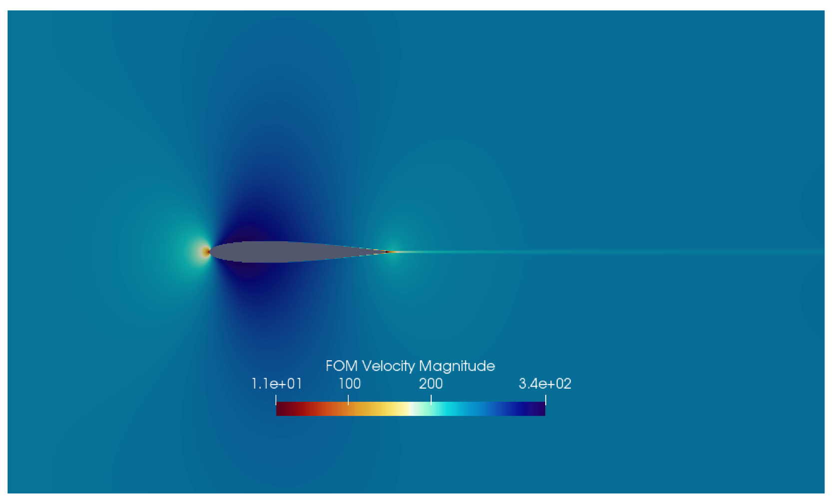


Also FOM (left) and ROM (right) pressure solutions are comparable.

6) Applications: compressible flows

Future work in the group will be devoted to compressible flows. In this case continuum Navier-Stokes equations are needed:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0 \\ \frac{\partial \rho\mathbf{u}}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) - \nabla\cdot[\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)] = \rho\mathbf{g} - \nabla\left(P + \frac{2}{3}\mu\nabla\cdot\mathbf{u}\right) \\ \frac{\partial \rho e}{\partial t} + \nabla\cdot(\rho e\mathbf{u}\mathbf{u}) - \nabla\cdot(k\nabla T) = \rho\mathbf{g}\cdot\mathbf{u} - \nabla\cdot\left(\frac{2}{3}\mu(\nabla\cdot\mathbf{u})\mathbf{u}\right) + \nabla\cdot[\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\cdot\mathbf{u}] \end{cases}$$



In this last figure it is shown the velocity field obtained by a **compressible FOM SIMPLE** algorithm where state and energy equations have been added with respect to the incompressible model. The work for the reduced part is still in progress.

References

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