Reduced Order Models for incompressible turbulent Navier-Stokes flows in a finite volume environment

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Introduction

When dealing with Parametrized Partial Differential Equations the computational cost required by a large number of solutions for each new value of the involved parameters may be unaffordably large. To mitigate that, different methods have been studied in order to find solutions in a more efficient way ([1]).
In our work we exploit a POD approach to obtain the basis functions used to project the original problem and to reconstruct an approximate solution manifold. We focus on a new reduced segregated approach for laminar flows and on a mixed projection-data driven technique for turbulent flows, in a Finite Volume framework ([3]).

Segregated methods: the reduced SIMPLE algorithm

Incompressible Navier-Stokes equations:

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{\nabla P}{\rho} = \nu\nabla^2\mathbf{u} \\ \nabla\cdot(\rho\mathbf{u}) = 0 \end{cases}$$

Since the equations are coupled and block solvers require big matrices to be stored at each step, segregated methods employed, e.g., in SIMPLE solver, are to be preferred: momentum and continuity equations are solved iteratively and one at a time. In order to be as coherent as possible, we developed a segregated algorithm also at the reduced level.

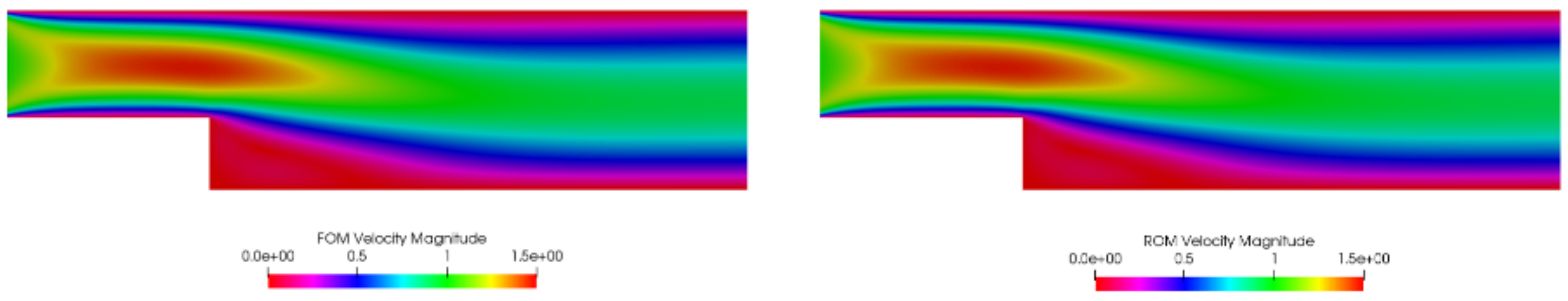
The Reduced SIMPLE Algorithm

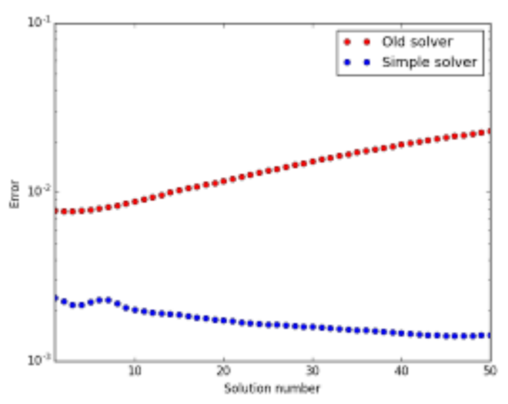
Input: Tentative value of the velocity and pressure coefficient vectors $\boldsymbol{a}_0^u$ and $\boldsymbol{a}_0^p$, $k=1$, tolerance tol, $res = tol + 1$.

1: Reconstruct the first attempt full velocity and pressure fields: $\boldsymbol{u}_0 = \boldsymbol{L}_u\boldsymbol{a}_0^u$, $\boldsymbol{p}_0 = \boldsymbol{L}_p\boldsymbol{a}_0^p$,
2: **while** $res > tol$ **do**
3: assemble reduced momentum equation $\boldsymbol{A}_u^r = \boldsymbol{L}_u^T\boldsymbol{A}_u(\boldsymbol{u}_{k-1},\boldsymbol{\mu})\boldsymbol{L}_u$; $\boldsymbol{b}_u^r = \boldsymbol{L}_u^T\boldsymbol{b}_u(\boldsymbol{\mu}, p_{k-1})$;
4: compute reduced velocity residual $\rightarrow r_u = |\boldsymbol{A}_u^r\boldsymbol{a}_{k-1}^u - \boldsymbol{b}_u^r|$;
5: solve $\boldsymbol{A}_u^r\boldsymbol{a}_k^u = \boldsymbol{b}_u^r$ and reconstruct $\boldsymbol{u}_k = \boldsymbol{L}_u\boldsymbol{a}_k^u$;
6: assemble reduced pressure equation $\boldsymbol{A}_p^r = \boldsymbol{L}_p^T\boldsymbol{A}_p(\boldsymbol{\mu})\boldsymbol{L}_p$; $\boldsymbol{b}_p^r = \boldsymbol{L}_p^T\boldsymbol{b}_p(\boldsymbol{\mu},\boldsymbol{u}_k)$;
7: compute reduced pressure residual $\rightarrow r_p = |\boldsymbol{A}_p^r\boldsymbol{a}_{k-1}^p - \boldsymbol{b}_p^r|$;
8: solve $\boldsymbol{A}_p^r\boldsymbol{a}_k^p = \boldsymbol{b}_p^r$ and reconstruct $\boldsymbol{p}_k = \boldsymbol{L}_p\boldsymbol{a}_k^p$;
9: $res = max(r_u, r_p)$, $k = k+1$;
10: **end while**

Segregated methods: a parametrized viscosity problem

The geometry is represented by a back step and the parameter we have choosen is the viscosity $\mu \in [0.01, 1]$. 50 snapshots have been solved to apply the POD.

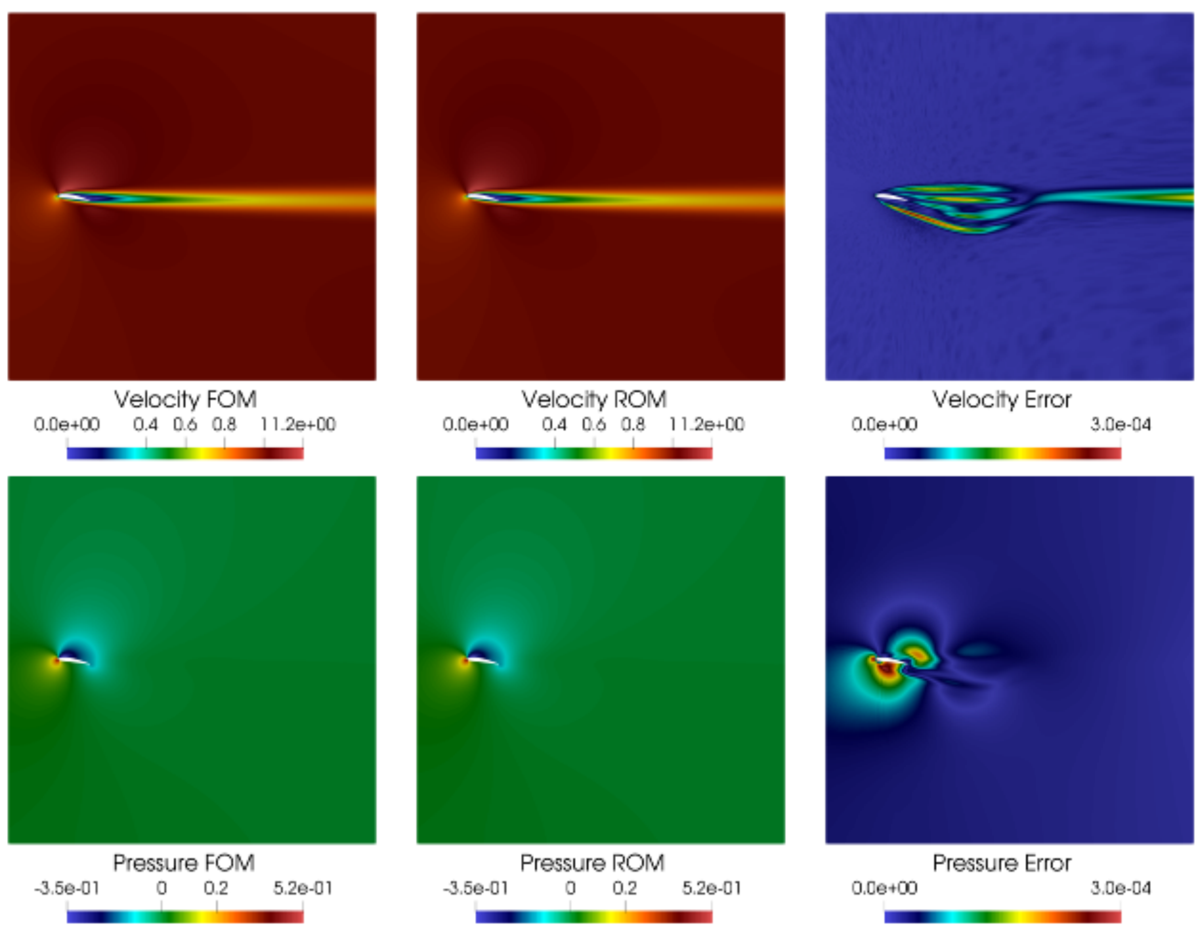




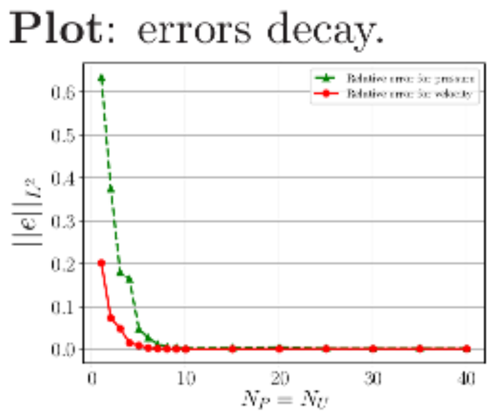
Left figure: full order velocity solution for $\mu = 0.024$. **Right figure**: reduced counterpart for the same value of the parameter. Only 10 basis functions have been used during projection for both velocity and pressure fields. **Bottom figure**: comparison between the L^2 norm of the error for each value of the parameter in the training set for the old block ROM solver (red) and for the SIMPLE ROM solver (blue).

Segregated methods: a parametrized geometry problem

One of the in-progress works in the group is relative to geometrical parametrization and hyper-reduction ([4]). In this case the angle of attack of an airfoil has been parametrized: $\alpha \in [-10, 10]$. 100 snapshots have been solved, 40 basis functions have been employed for the projection. The modified mesh is obtained by the use of radial basis functions interpolation.



Top figures: from left to right, FOM and ROM velocity solutions and velocity error magnitude for $\alpha = 9.15$.
Bottom figures: from left to right, FOM and ROM pressure solutions and pressure error for $\alpha = 9.15$.
Plot: errors decay.



Turbulent flows: mixing projection and data-driven

The starting point in developing the ROM is the usual decomposition of the fields into a sum of global spatial modes multiplied by temporal coefficients:

$$\overline{\boldsymbol{u}}(\boldsymbol{x},t;\boldsymbol{\mu}) \approx \sum_{i=1}^{N_u} a_i(t;\boldsymbol{\mu})\boldsymbol{\phi}_i(\boldsymbol{x}), \quad \overline{p}(\boldsymbol{x},t;\boldsymbol{\mu}) \approx \sum_{i=1}^{N_p} b_i(t;\boldsymbol{\mu})\chi_i(\boldsymbol{x}),$$
$$\nu_t(\boldsymbol{x},t;\boldsymbol{\mu}) \approx \sum_{i=1}^{N_{\nu_t}} g_i(t,\boldsymbol{\mu})\eta_i(\boldsymbol{x}).$$

The reduced basis $\boldsymbol{\phi}_i(\boldsymbol{x}), \chi_i(\boldsymbol{x})$ and $\eta_i(\boldsymbol{x})$ are computed by means of Proper Orthogonal Decomposition (POD), which is implemented using the snapshots method. The velocity snapshots matrix $\boldsymbol{\mathcal{S}}_{\boldsymbol{u}}$ is given by:

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{u}} = \{\overline{\boldsymbol{u}}(\boldsymbol{x},t_1;\boldsymbol{\mu}_1),...,\overline{\boldsymbol{u}}(\boldsymbol{x},t_{N_T};\boldsymbol{\mu}_M)\} \in \mathbb{R}^{N_u^h \times N_s}.$$

We proceed to the projection step of the momentum equation of RANS. This step will give the following dynamical system with the unknowns being the vectors of coefficients $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{M}\dot{\boldsymbol{a}} = \nu(\boldsymbol{B}+\boldsymbol{B_T})\boldsymbol{a} - \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{C}\boldsymbol{a} + \boldsymbol{g}^{\boldsymbol{T}}(\boldsymbol{C_{T1}}+\boldsymbol{C_{T2}})\boldsymbol{a} - \boldsymbol{H}\boldsymbol{b}.$$

In order to be able to use the continuity equations we used the supremizer enrichment method which allows to project the continuity equation onto the pressure modes, the resulting system is the following

$$\begin{cases} \boldsymbol{M}\dot{\boldsymbol{a}} = \nu(\boldsymbol{B}+\boldsymbol{B_T})\boldsymbol{a} - \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{C}\boldsymbol{a} + \boldsymbol{g}^{\boldsymbol{T}}(\boldsymbol{C_{T1}}+\boldsymbol{C_{T2}})\boldsymbol{a} - \boldsymbol{H}\boldsymbol{b}, \\ \boldsymbol{P}\boldsymbol{a} = \boldsymbol{0}. \end{cases} \tag{1}$$

Interpolation using Radial Basis Functions (RBF) is used to approximate the value of the eddy viscoisty coefficients vector $\boldsymbol{g}$ [2]. First one can notice the following:

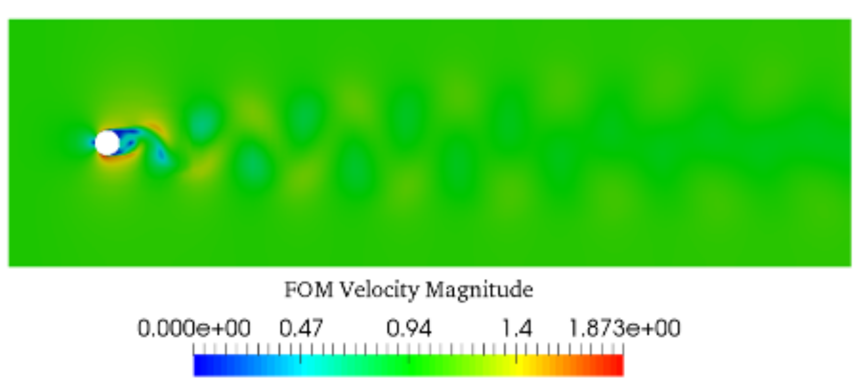
$$\nu_t = f(\overline{\boldsymbol{u}}) \rightarrow \boldsymbol{g} = \tilde{f}(\boldsymbol{a})$$

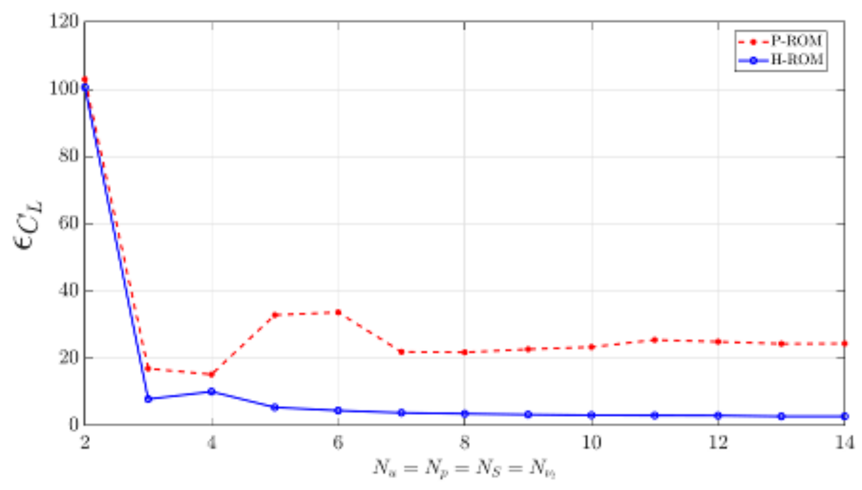
The interpolation using RBF functions is based on the following formula :

$$g_L(\boldsymbol{a}) \approx G_L(\boldsymbol{a}) = \sum_{j=1}^{N_s} w_{L,j}\zeta_{L,j}(\|\boldsymbol{a} - \boldsymbol{a}_{L_2}^j\|_{\mathbb{R}^{N_u}}), \quad \text{for} \quad L = 1,2,...,N_{\nu_t},$$

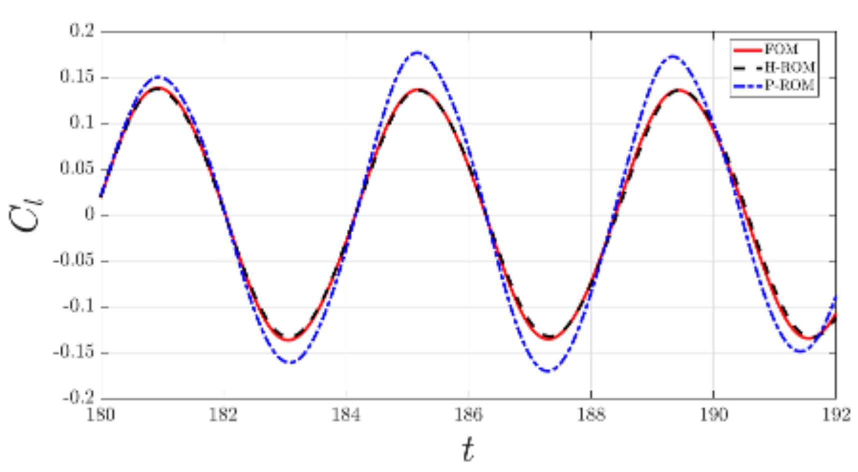
Turbulent flows: results for a circular cylinder

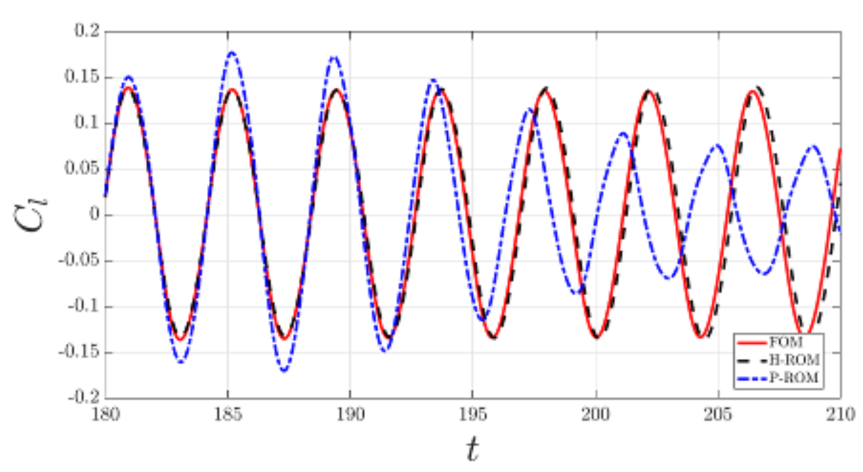
The presented results are for the benchmark case of the flow around a circular cylinder in unsteady state setting. The case is without parameterization so the reduction is done on time (both reproduction of the snapshots and extrapolation in time). The Reynolds number is equal to 10000. The results shown are for the lift coefficient C_l. The FOM results are compared to those of both the Hybrid ROM (H-ROM) and the Projection ROM (P-ROM) which is based on solving (1) neglecting the turbulent term in RANS equations. A quantitative convergence analysis is shown for the decay of the error with the increase of the number of modes used in the H-ROM.





Left figure: veloctiy magnitude FOM field. **Right figure**: lift coefficients L^2 error curve (reproduction case).





Left figure: lift coefficients curves for the FOM, the H-ROM and the P-ROM (reproduction case), 8 modes used. **Right figure**: lift coefficients curves for the FOM, the H-ROM and the P-ROM (extrapolation case), 8 modes used.

References

[1] J. S. Hesthaven, G. Rozza, and B. Stamm. *Certified Reduced Basis Methods for Parametrized Partial Differential Equations.* MS&A Series. Springer International Publishing, 2016.
[2] S. Hijazi, G. Stabile, A. Mola, and G. Rozza. Data-Driven POD–Galerkin Reduced Order Model for turbulent flows. *arXiv: https://arxiv.org/abs/1907.09909*, 2019.
[3] G. Stabile and G. Rozza. Finite volume POD-Galerkin stabilised reduced order methods for the parametrised incompressible Navier-Stokes equations. *Computers & Fluids*, 173:273–284, 2018.
[4] G. Stabile, M. Zancanaro, and G. Rozza. Efficient Geometrical parametrization for Finite-Volume based Reduced Order Methods. *arXiv: https://arxiv.org/abs/1901.06373*, 2019.

Utilities

ITHACA-FV

Poster

Acknowledgements

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