Reduced Order Models for incompressible turbulent Navier-Stokes flows in a finite volume environment

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Introduction

When dealing with Parametrized Partial Differential Equations the computational cost required by a large number of solutions for each new value of the involved parameters may be unfeasibly large. To mitigate that, different methods have been studied in order to find solutions in a more efficient way [1].

In our work we exploit a POD approach to obtain the basis functions used to project the original problem and to reconstruct an approximate solution manifold. We focus on a new reduced segregated approach for laminar flows and on a mixed projection-data driven technique for turbulent flows, in a Finite Volume framework [3].

Segregated methods: the reduced SIMPLE algorithm

Incompressible Navier-Stokes equations:

\[
\begin{align*}
- \nabla P + \nabla \cdot \mathbf{u} &= 0 \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]

Since the equations are coupled and block solves require big matrices to be stored at each step, segregated methods employed, e.g., in SIMPLEx, are preferred: momentum and continuity equations are solved iteratively and one at a time. In order to be as coherent as possible, we developed a segregated algorithm also at the reduced level.

The Reduced SIMPLE Algorithm

1. Reconstruct the first attempt full velocity and pressure fields: \( \mathbf{v}_0 = \mathbf{V}_0, p_0 = \mathbf{P}_0 \).
2. while residual toler. do
3. assemble reduced momentum equation:
   \[ \mathbf{A}_v \mathbf{u} + \mathbf{b}_v = \mathbf{L}_v \mathbf{b}_p \]
4. solve \( \mathbf{A}_v \mathbf{u} = \mathbf{b}_v \) and reconstruct \( \mathbf{u} = \mathbf{V}_0 \).
5. assemble reduced pressure equation:
   \[ \mathbf{A}_p \mathbf{p} = \mathbf{L}_p \mathbf{b}_p \]
6. solve \( \mathbf{A}_p \mathbf{p} = \mathbf{b}_p \) and reconstruct \( p = \mathbf{P}_0 \).
7. end while.

Segregated methods: a parametrized viscosity problem

The geometry is represented by a back step and the parameter we have chosen is the viscosity \( \mu \in [0.01, 1] \). 50 snapshots have been solved to apply the POD.

Turbulent flows: mixing projection and data-driven

The starting point in developing the ROM is the usual decomposition of the fields into a sum of global spatial modes multiplied by temporal coefficients:

\[
\begin{align*}
\mathbf{u}(x, t; \mu) &\approx \sum_{i=1}^{N_m} b_i(t; \mu) \phi_i(x), \\
p(x, t; \mu) &\approx \sum_{i=1}^{N_m} b_i(t; \mu) \psi_i(x)
\end{align*}
\]

The reduced basis \( \phi_i(x), \psi_i(x) \) and \( \phi_i(x) \) are computed by means of Proper Orthogonal Decomposition (POD), which is implemented using the snapshots method. The velocity snapshots matrix \( \mathbf{S}_v \) is given by:

\[
\mathbf{S}_v = \{ \mathbf{u}(x, t_1; \mu_1), \ldots, \mathbf{u}(x, t_{N_t}; \mu_{N_t}) \} \in \mathbb{R}^{N_t \times N_m}
\]

We proceed to the projection step of the momentum equation of RANS. This step will give the following dynamical system with the unknowns being the vectors of coefficients and \( b \):

\[
\begin{align*}
\frac{d}{dt} b &+ \mathbf{A} b = \mathbf{F} b \quad \text{with} \quad \mathbf{A} = \mathbf{M}^{-1} \mathbf{F} \equiv \mathbf{M}^{-1} \mathbf{S}_v \mathbf{G} \mathbf{S}_v^T, \\
\mathbf{F} &\equiv \mathbf{M}^{-1} \mathbf{S}_v \mathbf{G} \mathbf{S}_v^T
\end{align*}
\]

The interplay between the eddy viscosity coefficients vector \( \mathbf{a} \) and \( \mathbf{b} \) is an open question. It is used to approximate the value of the eddy viscosity coefficients vector \( \mathbf{a} \) [2]. First one can notice the following:

\[
\mathbf{a} = \mathbf{f}^{-1}(\mu) \rightarrow g(\mu) = \mathbf{f}(\mathbf{a})
\]

The interpolation using RBF functions is based on the following formula:

\[
g_L(\mu) \approx \sum_{i=1}^{N} \mathbf{g}_i(\mu, \{ \mathbf{a} - a_i^{(L)} \}) \quad \text{for} \quad L = 1, 2, ..., N_m.
\]

Turbulent flows: results: for a circular cylinder

The presented results are for the benchmark case of the flow around a circular cylinder in an unsteady state setting. The case is without parametrization so the reduction is done on time (both reproduction of the snapshots and extrapolation in time). The Reynolds number is equal to 1000. The results shown are for the lift coefficient \( C_l \). The FOM results are compared to those of both the Hybrid ROM (H-ROM) and the Projection ROM (P-ROM) which is based on solving (1) neglecting the turbulent term in RANS equations. A quantitative convergence analysis is shown for the decay of the error with the increase of the number of modes used in the H-ROM.

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