Reduced Order Modelling for Data Assimilation in Parametrized Optimal Control Framework

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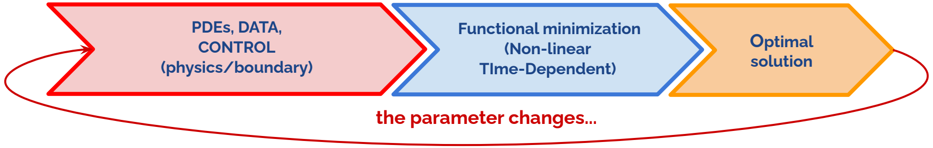
MOTIVATIONS

Parametrized Optimal Flow Control Problems are suited for several field of applied mathematics, such as:

- data assimilation inverse problems
- forecasting models

PDEs ⇄ **DATA**



STRATEGY (FROM HIGH-FIDELITY TO REDUCED MODEL)

Problem: solve several parametric instances of

$$\min_{y\in\mathbb{Y},u\in\mathbb{U}} \frac{1}{2}||y - y_{\rm d}||^2_{\mathbb{Y}} + \frac{\alpha}{2}||u||^2_{\mathbb{U}} \quad \text{subject to} \quad \mathcal{E}(y,u;\boldsymbol{\mu}) = 0$$

Space-Time ($\mathrm{DIM} = \mathcal{N} = N_h + N_t$) Lagrangian Formulation

$$(\mathrm{DIM} = 3\mathcal{N}) \begin{cases} D_y\mathcal{L}((y,u,p);y_{\rm d},\boldsymbol{\mu})[\omega] = 0 & \forall \omega \in \mathbb{Y}, \\ D_u\mathcal{L}((y,u,p);y_{\rm d},\boldsymbol{\mu})[\kappa] = 0 & \forall \kappa \in \mathbb{U}, \\ D_p\mathcal{L}((y,u,p);y_{\rm d},\boldsymbol{\mu})[\zeta] = 0 & \forall \zeta \in \mathbb{Y}. \end{cases}$$

- General Framework (linear, non-linear, time-dependent)
- Three equations
- High - Dimensionality

Proper Orthogonal Decomposition (**POD**) for **Space-Time** Optimal Control Problems [1]

APPLICATION: CONTROL FOR COASTAL WATER HEIGHT (SWE) [2]

Minimisation of

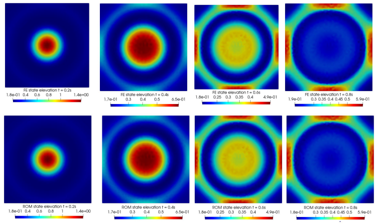
$$\frac{1}{2}\int_0^T\int_\Omega (h - h_d(\mu_3))^2 dxdt + \frac{1}{2}\int_0^T\int_\Omega (\boldsymbol{v} - \boldsymbol{v}_d(\mu_3))^2 dxdt + \frac{\alpha}{2}\int_0^T\int_\Omega \boldsymbol{u}^2 dxdt$$

constrained to

$$\begin{aligned} &\boldsymbol{v}_t + \mu_1\Delta\boldsymbol{v} + \mu_2(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + g\nabla\eta - \boldsymbol{u} = 0 && \text{in } \Omega\times[0,T], \\ &h_t + \mathrm{div}(\eta\boldsymbol{v}) = 0 && \text{in } \Omega\times[0,T], \\ &\boldsymbol{v} = \boldsymbol{v}_0 && \text{on } \Omega\times\{0\}, \\ &h = h_0 && \text{on } \Omega\times\{0\}, \\ &\boldsymbol{v} = \boldsymbol{0} && \text{on } \partial\Omega\times[0,T]. \end{aligned}$$

Errors ~ 1e-4, Speedup ~ 30

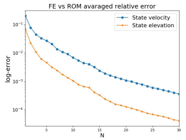
ROM vs FE dim = 270 vs 94'016.

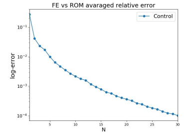


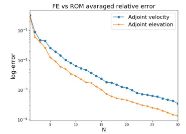
Water height solution for

$$\boldsymbol{\mu} = (0.1, 0.5, 1)$$

$$[0,T] = [0, 0.8], \quad \Omega = [0,10]\times[0,10], \quad \alpha = 10^{-5}, \quad \Delta t = 0.1.$$

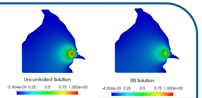




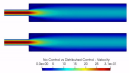


OTHER APPLICATIONS

Pollutant control in the Gulf of Trieste (Uncertainty Quantification) [3].



Driving bifurcations phenomena through optimal control (F. Pichi Poster on ROM and NN for Bifurcations) [4].



WHY NEURAL NETWORKS?

1) To correct data which are difficult to interpret, scattered.
2)Faster online reduced solver
3) Analysis (pre-process and post-process phase)

Acknowledgements

We acknowledge the support by European Union Funding for Research and Innovation -- Horizon 2020 Program -- in the framework of European Research Council Executive Agency: Consolidator Grant H2020 ERC CoG 2015 AROMA-CFD project 681447 ``Advanced Reduced Order Methods with Applications in Computational Fluid Dynamics''. We also acknowledge the PRIN 2017 ``Numerical Analysis for Full and Reduced Order Methods for the efficient and accurate solution of complex systems governed by Partial Differential Equations'' (NA-FROM-PDEs) and the INDAM-GNCS project ``Tecniche Numeriche Avanzate per Applicazioni Industriali''. The computations in this work have been performed with RBniCS library, developed at SISSA mathLab, which is an implementation in FEniCS of several reduced order modelling techniques; we acknowledge developers and contributors to both libraries.

References

[1] M. Strazzullo, F. Ballarin, and G. Rozza, "POD–Galerkin Model Order Reduction for Parametrized Time Dependent Linear Quadratić Optimal Control Problems in Saddle Point Formulation", *Journal of Scientific Computing*, 83(3), pp. 55, 2020.
[2] M, Strazzullo, F. Ballarin, G. Rozza, POD-Galerkin Model Order Reduction for Parametrized Nonlinear Time Dependent Optimal Flow Control: an Application to Shallow Water Equations. *Submitted.*
[3] G. Carere, M. Strazzullo, F. Ballarin, G. Rozza, R. Stevenson. Weighted POD-reduction for parametrized PDE-constrained Optimal Control Problems with random inputs and its applications to environmental sciences, *In preparation.*
[4] F.Pichi, M.Strazzullo, F.Ballarin, and G. Rozza "Driving bifurcating parametrized nonlinear PDEs by optimal control strategies: application to Navier-Stokes equations and model reduction.", *In preparation.*