Artificial neural network for bifurcating phenomena modelled by nonlinear parametrized PDEs

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Goal: Develop an integrated **NN** framework to deal with bifurcating **nonlinear PDEs**, overcoming intrusive **EIM/DEIM** strategies.

How: POD-NN approach which combines ROMs and non-intrusive learning of reduced coefficients.

Why: Investigate efficiently complex **bifurcating** behaviour in a **real-time** context.

Approximating nonlinear parametric PDEs

$$\text{Given } \mu \in \mathcal{P} \subset \mathbb{R}^P, \text{ seek } X \in \mathbb{X} \text{ such that}$$

$$G(X;\mu) = 0$$

Complex nonlinear PDEs can exhibit a bifurcating behaviour, i.e. a sudden change in solution stability properties, usually linked to non-uniqueness issues and singularities.

HF **RB**

$$G_{\mathcal{N}}(X_{\mathcal{N}};\mu) \qquad N \ll \mathcal{N} \qquad G_N(X_N;\mu)$$

$$X_N(\mu) = \sum_{n=1}^{N} X_N^{(n)}(\mu)\Xi_n$$

$$\approx \qquad \pi : \mathcal{P} \subset \mathbb{R}^P \longrightarrow \mathbb{R}^N, \quad \mu \longmapsto \mathbb{V}^T X_{\mathcal{N}}(\mu)$$

POD-NN

$$X_{NN}(\mu) = \sum_{n=1}^{N} \pi^{(n)}(\mu)\Xi_n$$

Navier-Stokes equations

$$\begin{cases} u \cdot \nabla u - \mu\Delta u + \nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega. \end{cases}$$

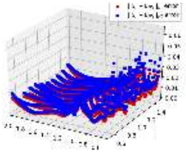
- Model a viscous, steady and incompressible flow.
- Exist a critical viscosity value at which bifurcates.
- Costly HF approximation, need for reduction.

Further applications:
- **Von Kármán equation for buckling plate;**
- **Gross-Pitaevskii model for Bose-Einstein condensates;**
- **Driving bifurcation by optimal control problem.**

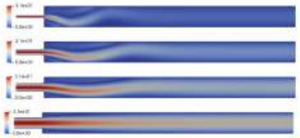
Flow in a channel: the Coanda effect

Multi-parameter application:
P = 2, N = 50, N_{train} = 200 · 6
2 layers, 15 neurons, mini-batch
geom. parametrized inlet width w

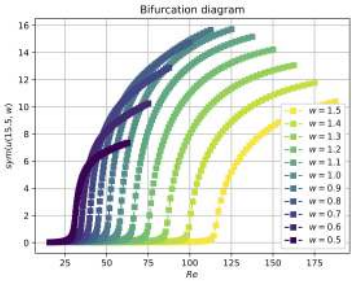
$\mathcal{P} = [0.5, 2] \times [0.5, 2]$
Speed-up NN = 1.e+6
Speed-up RB = 1.5



Low viscosity fluid tends to be attracted to a nearby surface, due to eddies which cause a wall-hugging behaviour



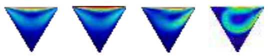
Mean HF-ANN error in $H^1(\Omega)$ = 1.e-2
Max HF-ANN error in $H^1(\Omega)$ = 6.e-2

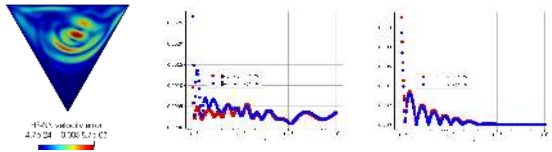


Triangular cavity flow

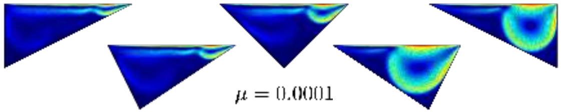
Benchmark application:
P = 1, N = 36, N_{train} = 400
3 layers, 15 neurons, mini-batch
log-equispaced points

$\mathcal{P} = [1, 0.001]$
$\mu \in \{1, 0.1, 0.01, 0.001\}$
Low viscosity causes the vortex to attach at the top-right vertex.





Towards geometrical parametrization and bifurcating behaviour

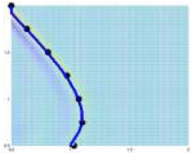


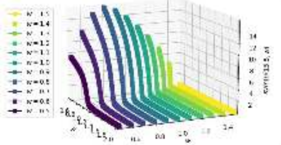
$\mu = 0.0001$

Reduced manifold based bifurcation diagram

We aim at efficiently reconstruct a bifurcation diagram, where the output is entirely based on the reduced coefficients which appears in the reduced basis expansion.

The idea is to take advantage of the non-smoothness of the manifold, constructing a detection tool that is able to track the critical points.





References

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[2] F. Pichi and G. Rozza. Reduced basis approaches for parametrized bifurcation problems held by non-linear Von Kármán equations. Journal of Scientific Computing, 339:667–672, 2019.
[3] F. Pichi, F. Ballarin, J. Hesthaven, and G. Rozza. Artificial neural network for bifurcating phenomena modelled by nonlinear parametrized PDEs. In preparation, 2020.