Data Enhanced Reduced Order Methods for Turbulent Flows

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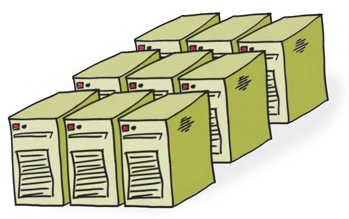
Introduction

The focus of this work is the development of **data-driven reduced order techniques** in CFD context in order to improve the pressure and velocity accuracy of standard reduced order methods. The general framework of Proper Orthogonal Decomposition with Galerkin approach is coupled with a data-driven technique, exploiting the information of full order data to build ***correction/closure* terms**. These terms are added in the reduced order system to reintroduce the contribution of disregarded modes. The technique is applied to the 2D study of the turbulent flow around a cylinder in two different approaches: the **SUP-ROM**, where additional velocity supremizer modes are considered, and the **PPE-ROM**, where the continuity equation in the model is replaced by the Poisson pressure equation.

1. Offline-Online Procedure

OFFLINE PHASE



- **Full Order Model**: Incompressible NSE

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} = -\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla\cdot\nu\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) - \nabla p, \\ \nabla\cdot\mathbf{u} = \mathbf{0}, \\ + \text{ boundary and initial conditions.} \end{cases}$$

- **Case study**: **turbulent flow around a circular cylinder**
- Discretization with **FVM** (*Finite Volume Method*)
- **RANS** (*Reynolds Averaged Navier–Stokes*) approach
- **POD** (*Proper orthogonal Decomposition*) with Galerkin approach

extraction of velocity modes $(\varphi_i)_{i=1}^{N_u^h}$ and pressure modes $(\chi_j)_{i=j}^{N_p^h}$

Library used for POD:

github.com/mathLab/ITHACA-FV
mathlab.github.io/ITHACA-FV

ONLINE PHASE

- Pick a reduced number of modes: $r << N_u^h$, $q << N_p^h$
- Approximated fields: $\boldsymbol{u}_r = \sum_{i=1}^r a_i \boldsymbol{\varphi_i}$, $p_r = \sum_{j=1}^q b_j \chi_j$
- Projection of the equations onto the reduced modes

$\Downarrow$

Dynamical system with
Unknowns: $\mathbf{a} = (a_i)_{i=1}^r$ and $\mathbf{b} = (b_i)_{i=1}^q$

Standard Galerkin-ROM formulation	
SUP-ROM	PPE-ROM
$r = N_u + N_{sup}$, N_{sup} supremizer modes $\begin{cases}\dot{\mathbf{a}} = \mathbf{f}(\mathbf{a},\mathbf{b}),\\ \mathbf{h}(\mathbf{a}) = \mathbf{0}.\end{cases}$	Poisson pressure equation approach $\begin{cases}\dot{\mathbf{a}} = \mathbf{f}(\mathbf{a},\mathbf{b}),\\ \mathbf{h}(\mathbf{a},\mathbf{b}) = \mathbf{0}.\end{cases}$

2. DD-VMS-ROM: the *correction* terms

Motivation: improve the velocity and pressure accuracy to better capture the ***forces***.

How: reintegrating the contribution of the neglected modes with *correction* terms.

Construction of correction terms:

1. build the exact correction $\boldsymbol{\tau}^{\text{exact}}$ from available data;
2. propose an ansatz for the approximated correction term $\boldsymbol{\tau}^{\text{ansatz}}(\mathbf{a},\mathbf{b})$;
3. solve an optimization problem $\min \sum_j ||\boldsymbol{\tau}^{\text{exact}}(t_j) - \boldsymbol{\tau}^{\text{ansatz}}(t_j)||_{L^2}^2$.

Two different types of correction terms:

- $\boldsymbol{\tau}_u(\mathbf{a})$: velocity correction in the momentum equation;
- $\boldsymbol{\tau}_p(\mathbf{a},\mathbf{b})$: ***novel*** pressure correction in the Poisson equation (in the PPE approach).

3. EV-ROM: the turbulence modelling

Motivation: Inclusion of a turbulence model in the ROM.

How: Addition of *reduced eddy viscosity* terms.

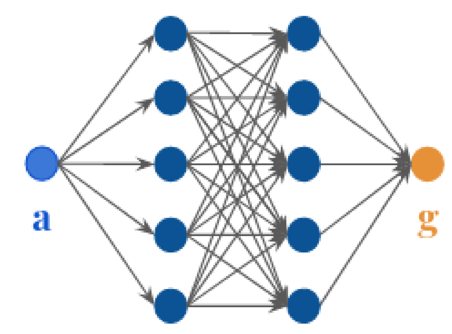
$$\nu_t = \sum_{i=1}^r g_i \eta_i$$

$\mathbf{g} = (g_i)_{i=1}^r$: eddy viscosity coefficients vector,

$(\eta_i)_{i=1}^r$: eddy viscosity modes.

Construction of g: The eddy viscosity coefficients vector is modelled with *regression techniques* making use of a fully-connected neural network:

$$\mathbf{g} = \mathrm{f}(\mathbf{a})$$



4. Supremizer enrichment and Poisson Pressure approach: numerical results

- $\boldsymbol{\tau}_u(\mathbf{a}) = \tilde{A}\mathbf{a} + \mathbf{a}^T\tilde{B}\mathbf{a}$ quadratic velocity correction term in *supremizer* approach ;
- $(\boldsymbol{\tau}_u(\mathbf{a}), \boldsymbol{\tau}_p(\mathbf{a},\mathbf{b})) = \tilde{I}_A\mathbf{ab} + \mathbf{ab}^T\tilde{I}_B\mathbf{ab}$ quadratic pressure correction term in *PPE* approach, where $\mathbf{ab} = (\mathbf{a},\mathbf{b})$.

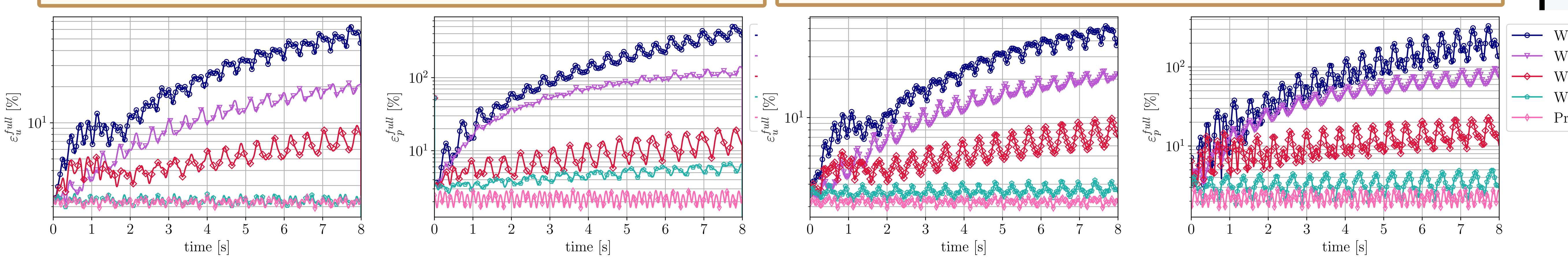
DD-SUP-ROM	
$\begin{cases}\dot{\mathbf{a}} = \mathbf{f}(\mathbf{a},\mathbf{b},\mathbf{g}) + \tau_u(\mathbf{a}),\\ \mathbf{h}(\mathbf{a}) = \mathbf{0}.\end{cases}$	Addition of N_{sup} **velocity modes** to fulfill the *inf-sup condition*: $\mathbf{a} = (a_i)_{i=1}^{N_u+N_{sup}}$

DD-PPE-ROM	
$\begin{cases}\dot{\mathbf{a}} = \mathbf{f}(\mathbf{a},\mathbf{b},\mathbf{g}) + \tau_u(\mathbf{a}),\\ \mathbf{h}(\mathbf{a},\mathbf{b}) + \tau_p(\mathbf{a},\mathbf{b}) = \mathbf{0}.\end{cases}$	Replacement of the continuity equation with **pressure Poisson equation** at the reduced order level.

Percentage errors of reduced velocity and pressure fields with respect to full order fields:

$$\varepsilon_u(t_j) = \frac{||\mathbf{u}_r(\mathbf{x},t_j) - \mathbf{u}_{\text{FOM}}(\mathbf{x},t_j)||_{L^2(\Omega)}}{||\mathbf{u}_{\text{FOM}}(\mathbf{x},t_j)||_{L^2(\Omega)}},$$

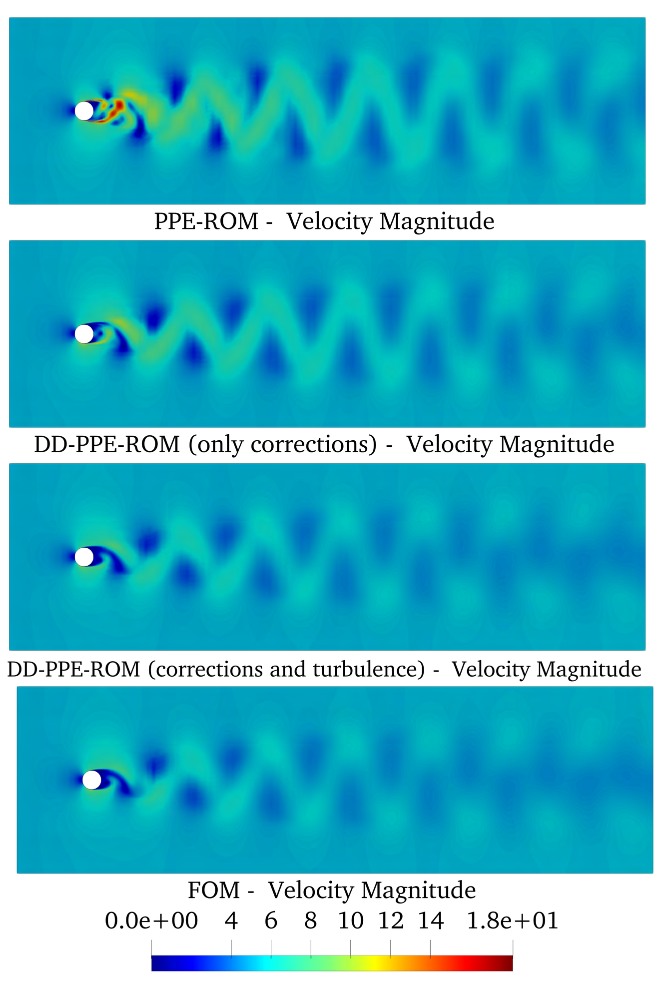
$$\varepsilon_p(t_j) = \frac{||p_r(\mathbf{x},t_j) - p_{\text{FOM}}(\mathbf{x},t_j)||_{L^2(\Omega)}}{||p_{\text{FOM}}(\mathbf{x},t_j)||_{L^2(\Omega)}}.$$

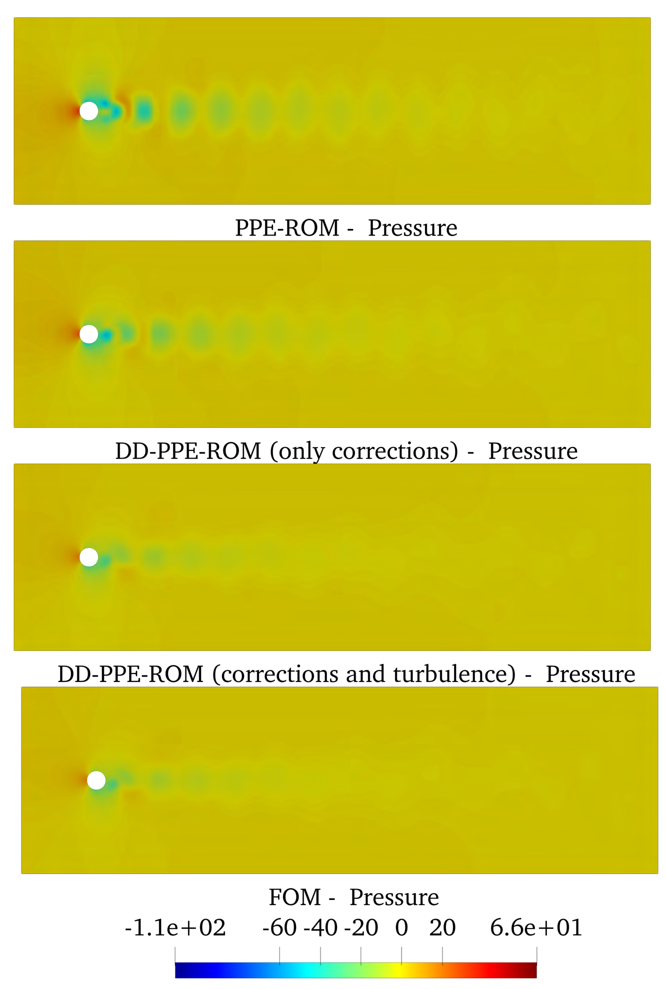


Legend: Results without any data-driven term (—o—); with only turbulence modelling (—▽—); with only corrections (—o—); with both turbulence modelling and corrections (—□—); reconstruction error (—◇—).

5. Graphical results for the PPE approach

Reduced velocity field | **Reduced pressure field**





6. Conclusions and Future Perspectives

Conclusions:

- The **velocity correction** term improves both the velocity and pressure accuracy, whereas the **pressure correction** term improves the pressure accuracy in the PPE approach.
- The **combination** of correction terms and eddy viscosity modelling gives the best results and acts as a **stabilizer** for the error in time.
- The graphical results show a better reconstruction of flow fields, especially *nearby the cylinder* and it is important in the reconstruction of the **forces fields**.
- Significant reduction in **computational cost and time** w.r.t. FOM, comparable to the standard ROM. The correction terms are found from a part of the available snapshots and provide a good **time extrapolation efficiency**.

Outlooks:

- The study regards the **marginally-resolved** modal regime, where the number of modes is enough to represent the underlying dynamics, but the standard ROM yields inaccurate results. *Further investigation*: different modal regimes.
- *Further investigation*: more complex computational settings and 3D flows.
- *Further investigation*: introduction of parameters.

References

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[2] A. Ivagnes. *Data Enhanced Reduced Order Methods for Turbulent Flows.* PhD thesis, Politecnico di Torino, 2021.

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