A reduced order model for the optimisation-based domain decomposition algorithm for the incompressible Navier-Stokes equations

Ivan Prusak, Monica Nonino, Francesco Ballarin, Gianluigi Rozza

Mathematics Area, mathLab, SISSA, International School of Advanced Studies, Trieste, Italy

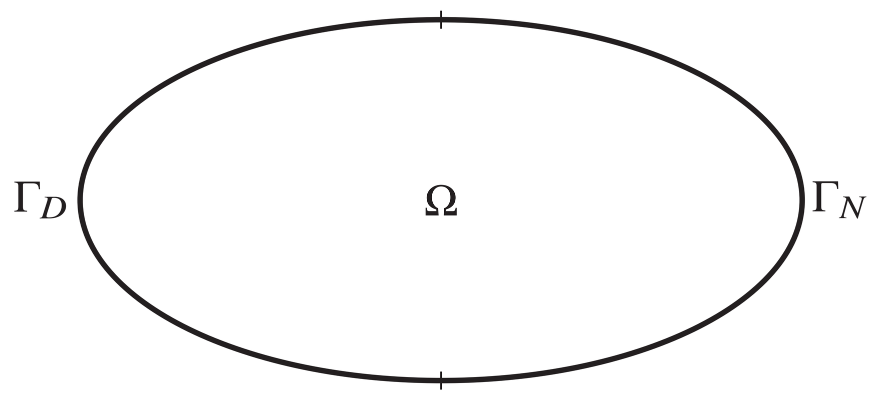
Introduction

The aim of this work is to present a model reduction technique in the framework of optimal control problems for partial differential equations. In particular, we consider an optimisation based domain decomposition algorithm for the incompressible Navier-Stokes equations and propose a reduced–order model for the resulting optimal control problem. The procedure is based on the Proper Orthogonal Decomposition technique and gradient–based optimisation algorithms; the presented methodology is tested on the stationary backward–facing step and lid-driven cavity flow fluid dynamics benchmarks.

1 - Monolithic vs. Domain Decomposition (DD) Formulation

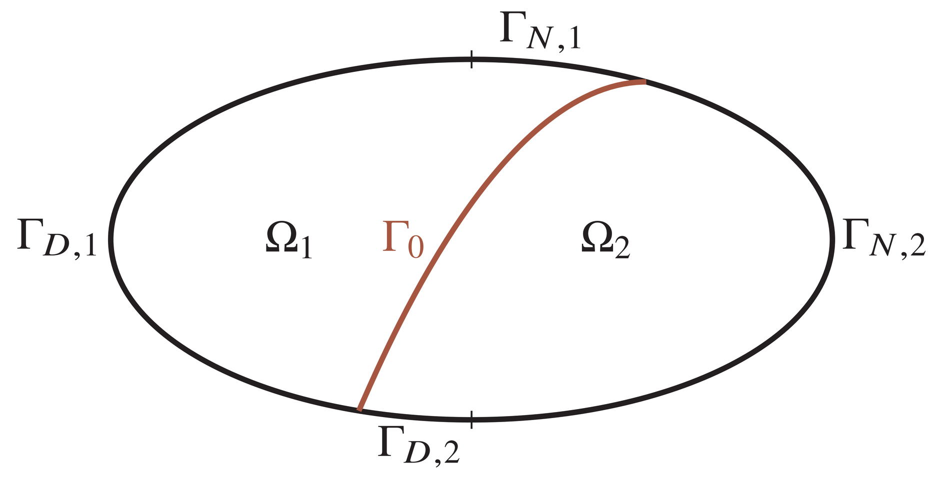
We consider the following stationary boundary valuer problem for the incompressible Navier–Stokes equations: given $f:\Omega\to\mathbb{R}^2$, $u_D:\Gamma_D\to\mathbb{R}^2$, find the velocity field $u:\Omega\to\mathbb{R}^2$ and the pressure $p:\Omega\to\mathbb{R}$ s.t.

$$\begin{aligned} -\nu\Delta u + (u\cdot\nabla)u + \nabla p &= f \quad \text{in} \quad \Omega,\\ -\mathrm{div} u &= 0 \quad \text{in} \quad \Omega,\\ u &= u_D \quad \text{on} \quad \Gamma_D,\\ \nu\frac{\partial u}{\partial n} - pn &= 0 \quad \text{on} \quad \Gamma_N, \end{aligned}$$



The DD formulation reads as follows: for $i=1,2$, given $f_i:\Omega_i\to\mathbb{R}^2$ and $u_{i,D}:\Gamma_{i,D}\to\mathbb{R}^2$, find $u_i:\Omega_i\to\mathbb{R}^2$, $p_i:\Omega_i\to\mathbb{R}$ s.t. for some $g:\Gamma_0\to\mathbb{R}^2$

$$\begin{aligned} -\nu\Delta u_i + (u_i\cdot\nabla)u_i + \nabla p_i &= f_i \quad \text{in} \quad \Omega_i,\\ -\mathrm{div} u_i &= 0 \quad \text{in} \quad \Omega_i,\\ u_i &= u_{i,D} \quad \text{on} \quad \Gamma_{i,D},\\ \nu\frac{\partial u_i}{\partial n_i} - p_i n_i &= 0 \quad \text{on} \quad \Gamma_{i,N},\\ \nu\frac{\partial u_i}{\partial n_i} - p_i n_i &= (-1)^{i+1} g \quad \text{on} \quad \Gamma_0. \end{aligned}$$



For any g the solution to the monolithic problem is not the same as the solution to the DD problem, but there exists a choice for g, $g=\left(\nu\frac{\partial u_1}{\partial n_1}-p_1n_1\right)|_{\Gamma_0} = -\left(\nu\frac{\partial u_2}{\partial n_2}-p_2n_2\right)|_{\Gamma_0}$, such that the solutions coincide on the corresponding subdomains. Therefore, we must find such a g, so that u_1 is as close as possible to u_2 on the interface Γ_0. One way to accomplish this is to minimise the functional

$$\mathcal{J}_\gamma(u_1,u_2;g) =: \frac{1}{2}\int_{\Gamma_0}|u_1-u_2|^2\,d\Gamma + \frac{\gamma}{2}\int_{\Gamma_0}|g|^2\,d\Gamma,$$

. Thus we face an optimisation problem under PDE constraints: minimise the functional $\mathcal{J}_\gamma$over a suitable function g subject to DD-equations.

2 - Iterative Optimisation Algorithm

Choose $g^{(0)}$, α. For n=0,1,2,... until convergence

1. Determine $u_1^{(n)}, u_2^{(n)}$ by solving the state equations

$$\begin{aligned} -\nu\Delta u_i^{(n)} + \left(u_i^{(n)}\cdot\nabla\right)u_i^{(n)} + \nabla p_i^{(n)} &= f_i \quad \text{in} \quad \Omega_i\\ -\mathrm{div} u_i^{(n)} &= 0 \quad \text{in} \quad \Omega_i\\ \nu\frac{\partial u_i^{(n)}}{\partial n_i} - p_i^{(n)} n_i &= (-1)^{i+1} g^{(n)} \quad \text{on} \quad \Gamma_0 \end{aligned}$$

2. Determine $\xi_1^{(n)}, \xi_2^{(n)}$ by solving the adjoint equations

$$\begin{aligned} -\nu\Delta\xi_i^{(n)} + \left(\nabla u_i^{(n)}\right)^T\xi_i^{(n)} - \left(u_i^{(n)}\cdot\nabla\right)\xi_i^{(n)} + \nabla\lambda_i^{(n)} &= 0 \quad \text{in} \quad \Omega_i\\ -\mathrm{div}\xi_i^{(n)} &= 0 \quad \text{in} \quad \Omega_i\\ \nu\frac{\partial\xi_i^{(n)}}{\partial n_i} - \lambda_i^{(n)} n_i &= (-1)^{i+1}\left(u_1^{(n)}-u_2^{(n)}\right) \quad \text{on} \quad \Gamma_0 \end{aligned}$$

3. Update $g^{(n+1)}$ by setting

$$g^{(n+1)} := g^{(n)} - \alpha\frac{d\mathcal{J}_\gamma}{dg}\left(u_1^{(n)}, u_2^{(n)}; g^{(n)}\right)$$

$$g^{(n+1)} := g^{(n)} - \alpha\left(\gamma g^{(n)} + (\xi_1^{(n)}-\xi_2^{(n)})|_{\Gamma_0}\right)$$

In practice, the typical methods used to solve problems like the one considered in this paper are Broyden–Fletcher–Goldfarb–Shanno (BFGS) and Newton Conjugate Gradient (CG) algorithms which tend to show much faster convergence and higher efficiency with respect to the steepest-decent algorithm.

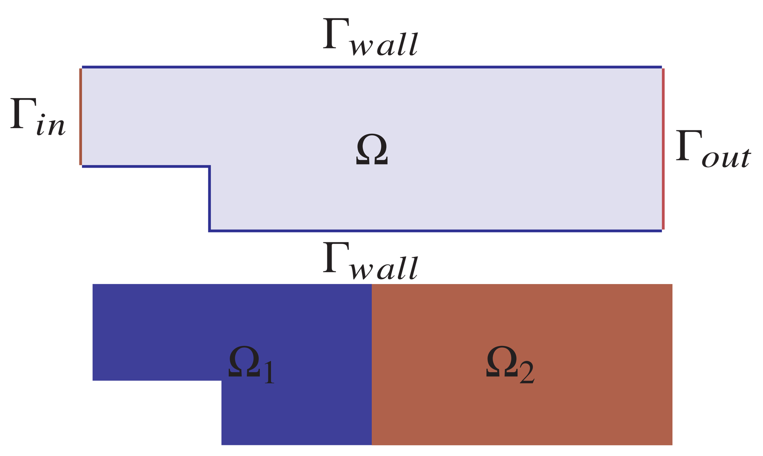
3 - FEM discretisation

- FEM spaces $V_{i,h}\subset H^1(\Omega_i)$, $\quad Q_{i,h}\subset L^2(\Omega_i)$, $\quad X_h\subset L^2(\Gamma_0)$
- Inf-sup conditions $\inf_{q_{i,h}\in Q_{i,h}\setminus\{0\}}\sup_{v_{i,h}\in V_{i,h}\setminus\{0\}}\frac{b_i(v_{i,h},q_{i,h})}{||v_{i,h}||_{V_{i,h}}||q_{i,h}||_{Q_{i,h}}}\geq c_i>0$
- Minimise discretized functional $\mathcal{J}_{\gamma,h}(u_{1,h},u_{2,h};g_h) := \frac{1}{2}\int_{\Gamma_0}|u_{1,h}-u_{2,h}|^2\,d\Gamma + \frac{\gamma}{2}\int_{\Gamma_0}|g_h|^2\,d\Gamma$ subject to Galerkin-projection of the state equations

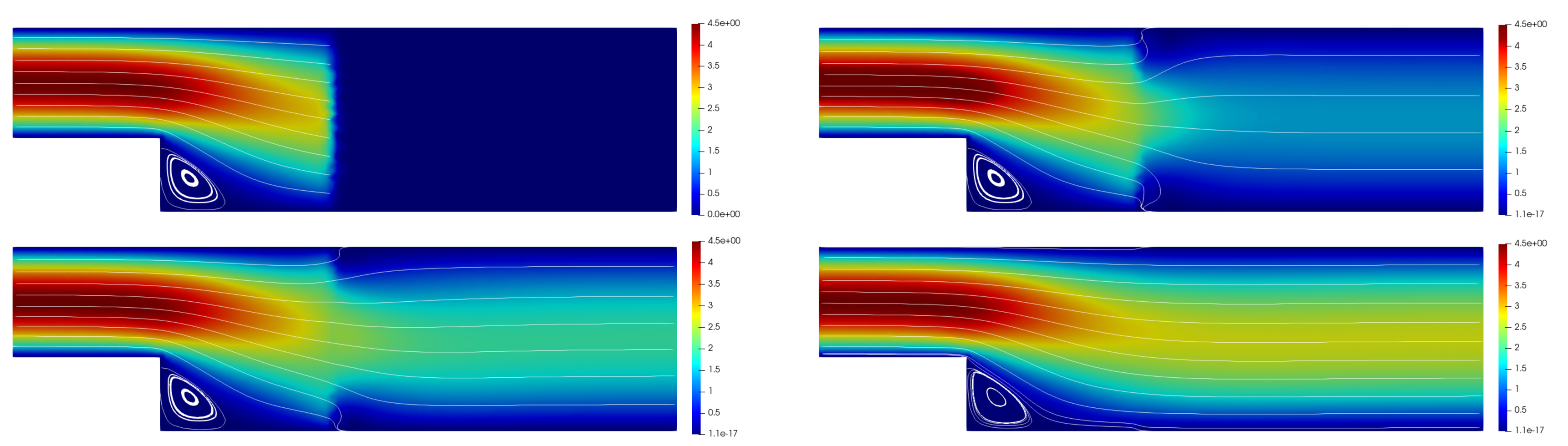
4A - Reduced Order Model. Offline stage

- Consider parametrised Navier–Stokes equations $\mu\in\mathbb{R}^P$
- Sample the parameter space $\{\mu_1,\dots,\mu_{N_{max}}\}$
- Solve FE optimisation problem of each parameter in the training set and store the snapshots
- Perform POD-compression for each component $u_1, p_1, u_2, p_2, \xi_1, \xi_2, g$ separately
- Construct reduced spaces $V_{i,N}\subset V_i,h, Q_{i,N}\subset Q_{i,h},\ i=1,2, X_N\subset X_h$

5A - Numerical results



- Parabolic inlet profile on Γ_{in} with max. velocity $\bar{U}$
- Two physical parameters considered: ν and $\bar{U}$
- Optimisation method: L-BFGS-B



Velocity profiles at iterations 0, 5, 10, 40 (left to right, top to bottom).

4B - Reduced Order Model. Online stage

- Galerkin projection of the state and adjoint eqiations onto the reduced spaces
- Usually the dimensions of the ROM problem are much lower than of the corresponding FEM problems
- Minimise discretized functional $\mathcal{J}_{\gamma,N}(u_{1,N},u_{2,N};g_N) := \frac{1}{2}\int_{\Gamma_0}|u_{1,N}-u_{2,N}|^2\,d\Gamma + \frac{\gamma}{2}\int_{\Gamma_0}|g_N|^2\,d\Gamma$ subject to Galerkin-projection of the state equations onto the RB spaces
- The optimisation problem of much smaller dimension: dim = # RB functions for the control

6 - Computational science and engineering softwares

MULTIPHENICS
https://mathlab.sissa.it/multiphenics/

multiphenics is a python library that aims at providing tools in FEniCS for an easy prototyping of multiphysics problems on conforming meshes.

RBNICS
github.com/mathLab/rbnics

The RBniCS Project contains an implementation in FEniCS of several reduced order modelling techniques for parametrized problems.

5B - Numerical results

- Reduction of the state nonlinear equation dimension: FEM - 27,890 vs. ROM - 10
- Reduction of the optimisation problem dimension: FEM - 130 vs. ROM - 10
- Reduction in terms of #iterations: FEM - 40 vs. ROM - 10
- Enhanced stability of ROM w.r.t. FOM (FOM optimisation process is very sensitive to the initial approximation)

References and Acknowledgements

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