A Numerical Proof of Shell Model Turbulence Closure

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Introduction and Motivations

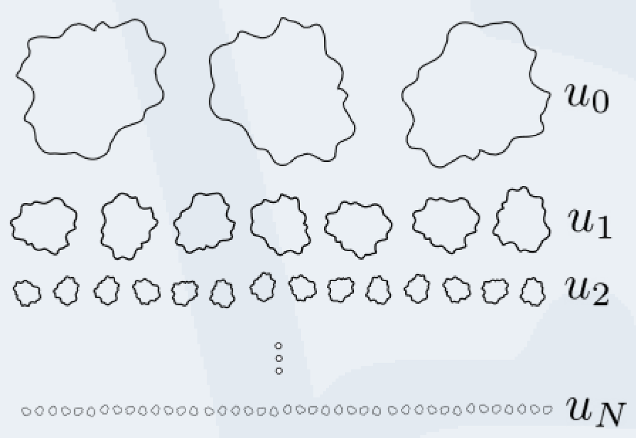
In this work we define a **Subgrid Closure** model that, employed in a **Large Eddy Simulation** approach, exhibits correct scaling laws in **high order Structure Functions**, encompassing intermittent effects and energy cascade dynamics.

Due to the massive amount of data needed to reach converged statistics of high order statistical moments, we consider the setting of **Shell Models** of Turbulence [1].

Our method employs a custom-made **Deep Learning** architecture comprising a **Runge-Kutta** integration scheme for the large scales of turbulence, augmented with a **Recurrent Artificial Neural Network**.

Shell Models of Turbulence

Shell models mimic the dynamics of **Homogeneous Isotropic Turbulence in Fourier space** via a (small) number of scalars u_n, $n = 0, 1, \dots, N$, whose magnitude represents the energy of fluctuations at representative logarithmically equispaced spatial scales (wavelength $k_n = k_0 \lambda^n$).



The governing equations read:

$$\frac{du_n}{dt} = \epsilon \delta_{n,0} - \nu k_n^2 u_n + F(u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2})$$

where $F(\cdot)$ is a nonlinear coupling between the shells, mimicking the convective term of NSE.

A Large Eddy Simulation (LES) consists in evolving the large scales $u^<$ above an (arbitrary) **cutoff scale** $N_{cut} \ll N$ independently from the **small** (unresolved) **scales** $u^>$:

$$\boldsymbol{u} = \{u_n\}_{n=0}^N = \begin{bmatrix} u^< \\ u^> \end{bmatrix} = \begin{bmatrix} \{u_n\}_{n=0}^{N_{cut}} \\ \{u_n\}_{n=N_{cut}}^{N} \end{bmatrix}$$

The numerical values for the experiments:

- $\nu = 10^{-12}$ ($Re \approx 10^{12}$);
- $N_{cut} = 15$, $N = 40$.

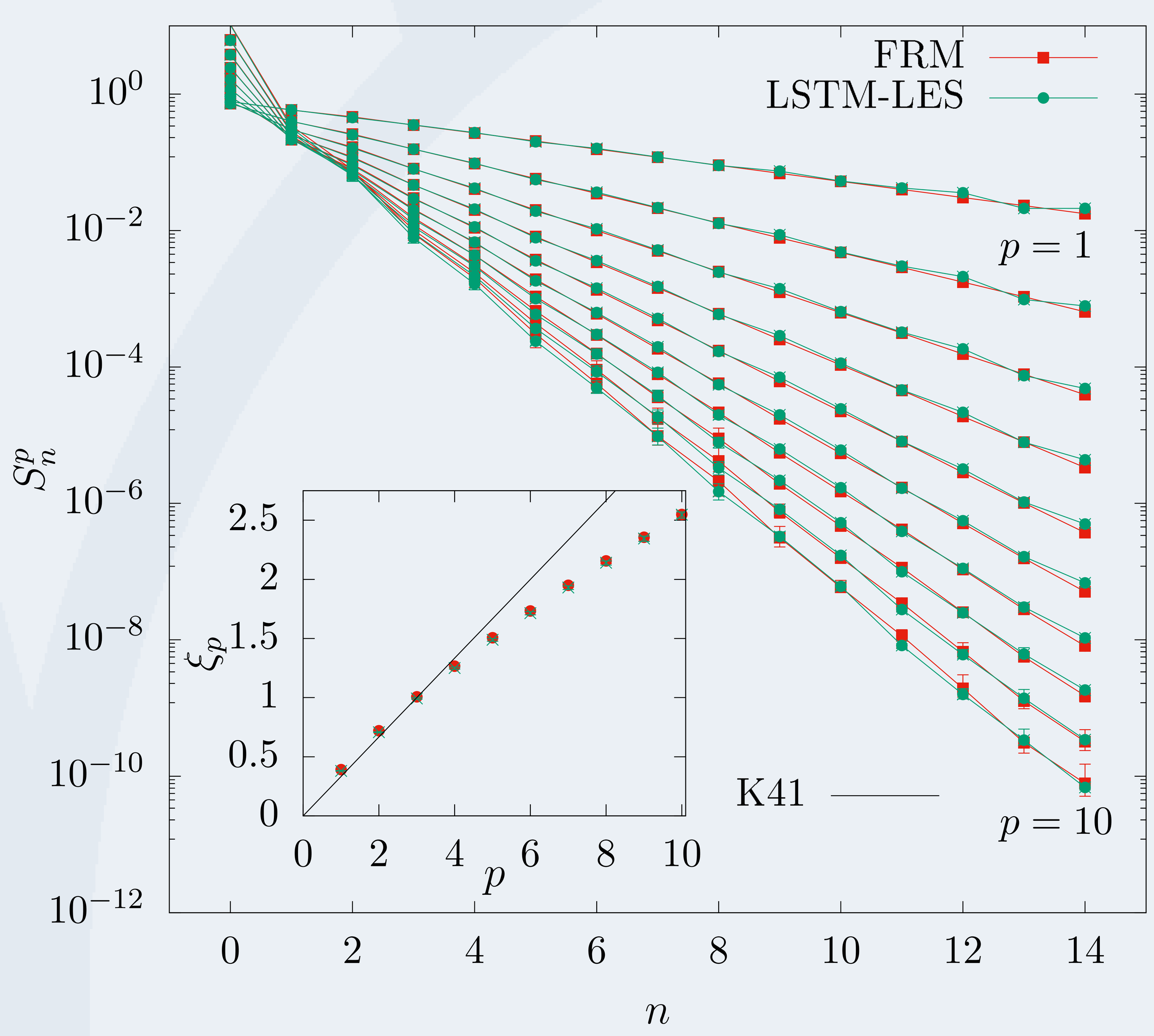
References

[1] L. Biferale. Shell models of energy cascade in turbulence. *Annu. Rev. Fluid Mech.*, 35(1):441–468, 2003.

[2] L. Biferale, A. A. Mailybaev, and G. Parisi. Optimal subgrid scheme for shell models of turbulence. *Phys. Rev. E*, 95(4), Apr 2017.

[3] G. Ortali, A. Corbetta, G. Rozza, and F. Toschi. A numerical proof of shell model turbulence closure. *arXiv:2202.09289*.

Results (1): Eulerian Structure Functions



The the p^{th} order **Eulerian structure functions**, with $p = 1, \cdots, 10$, can be computed as:

$$S_n^p = \langle |u_n|^p \rangle. \quad (1)$$

On the left, results for the Fully Resolved Model (**FRM**) and our **LSTM-LES** model (shell index n on the x-axis).

In the inset, the values of the **anomalous exponents**, ξ_p, $S_n^p \propto k_n^{-\xi_p}$, with the predictions from the **K41 theory**.

Find in [3] results for high order **Lagrangian structure functions**, with specifications on **statistical error bars**.

Results (2): Pdf of Energy Fluxes

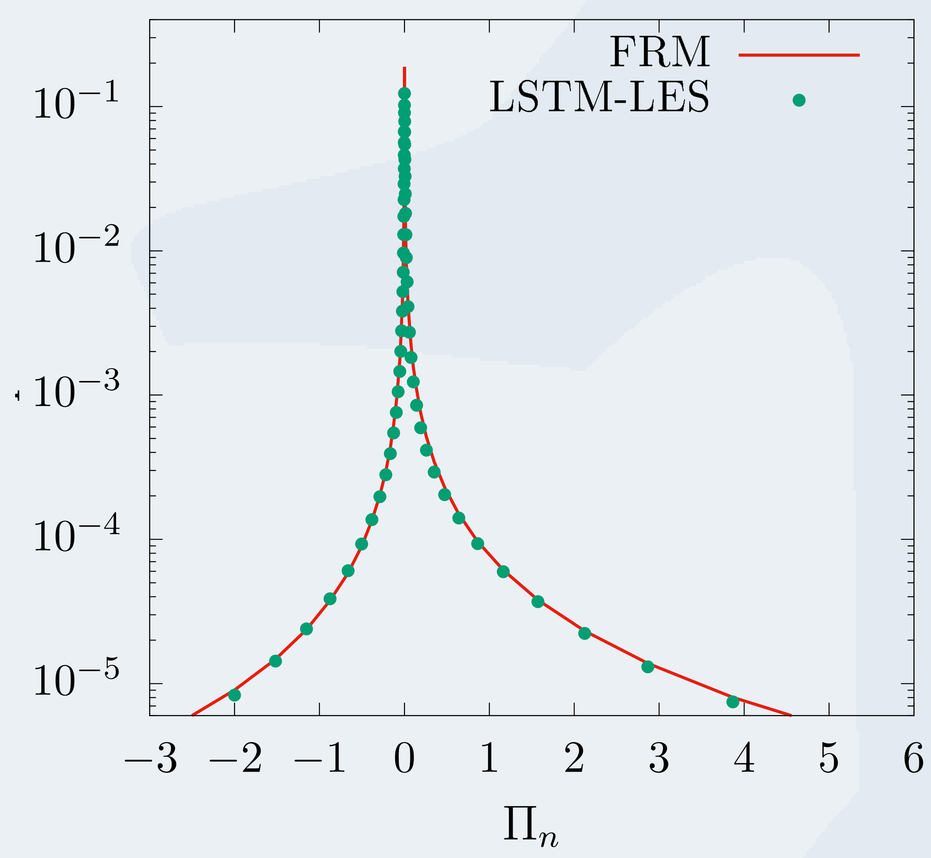
The **convective fluxes** at shell N_{cut} can be computed as [2]:

$$\Pi_{N_{cut}} = \frac{d}{dt} \sum_{n=0}^{N_{cut}} |u_n(t)|^2$$

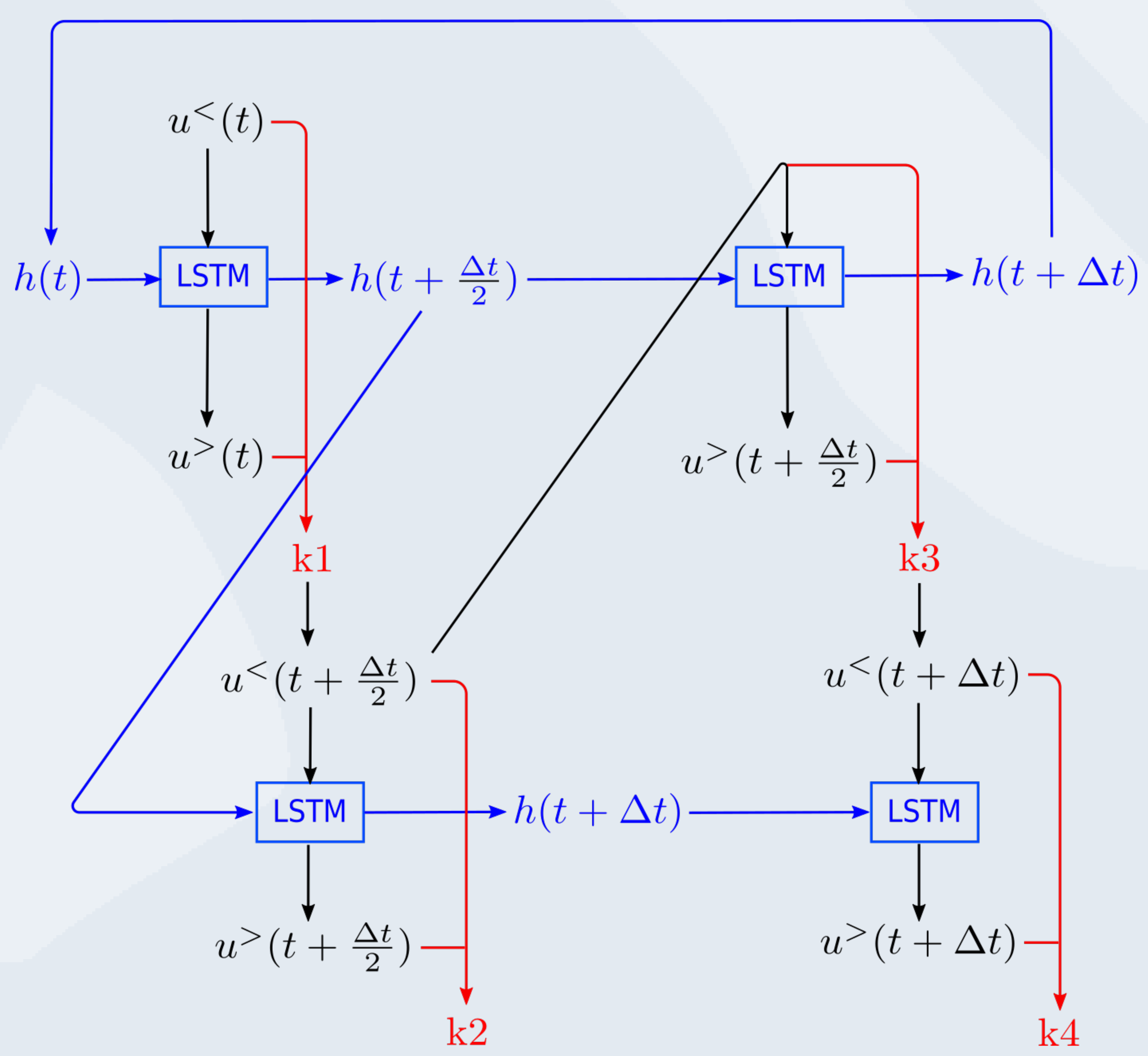
On the right, the probability distribution function for both Fully Resolved Model (**FRM**) and our **LSTM-LES**.

Negative values of the flux correspond to **backscatter** events, correctly reproduced by our model.

Find in [3] results for **pdf of shell variables**.



Methodology: LSTM-LES



Our method (**LSTM-LES**) consists of a modification of a Runge-Kutta integration scheme for the large scales:

$$u^<(t + \Delta t) = u^<(t) + \frac{\Delta t}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)$$

The terms $\mathbf{k}_1$,$\mathbf{k}_2$,$\mathbf{k}_3$ and $\mathbf{k}_4$ are computed by considering:

- the large scales $u^<$;
- the small scales $u^>$, computed by a Long-Short Term Memory (**LSTM**) Artificial Neural Network, taking as input the large scales $u^<$ and a memory term h.

Conclusion and Outlook

This work shows the capability of Machine Learning to capture complex multiscale dynamics and reproduce complex **multi-scale and multi-time non-gaussian behaviors**, opening up the possibility to use such methods to tackle turbulence modelling in Navier-Stokes Equations.