Data-driven methods for delay-differential equations

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Introduction

We present an investigation of data-driven techniques in the study and analysis of dynamical systems, and in particular in the study of dynamical systems originated from delay-differential equations (DDEs). DDEs are a generalization of ODEs in which the derivative of the state variables depends also on the past values of the state (Box 1). The techniques of interest are Dynamic Mode Decomposition (Box 2) - a decomposition method - and the SINDy algoritm (Box 3) - an algorithm which is used to discover non-linear dynamics in dynamical systems. We applied the DMD to DDEs to decompose time series originated from DDEs in linear modes and exploit this decomposition to identify the most important modes to reconstruct the state of the system and to predict its future values (Box 4). We then tried to improve the future prediction applying the SINDy algorithm in a multy-fidelity context, where the low fidelity data are the modes identifyied by the DMD (Box 5).

1 - Delay-Differential Equations [3]

A generic DDE is a differential equation for $y(t) \in \mathbb{R}^n$ of the form

$$y'(t) = f(t, y_t(\cdot))$$

where $y_t(\cdot)$ denotes the map $s \mapsto y(s)$ for all $s \leq t$. Among this general class of functional differential equations we concentrate on the autonomous case with a single fixed delay $\tau > 0$, i.e.

$$y'(t) = f(y(t), y(t-\tau)).$$

A very famous DDE is the Mackey-Glass equations

$$y'(t) = \beta \frac{y(t-\tau)}{1 + y(t-\tau)^n} - \gamma y(t) \qquad \gamma, \beta, n > 0,$$

which describes some processes in physiology and cell production.
DDEs lie in the middle between ODEs and PDEs: they describe an infinite dimensional object, admit a finite dimensional formulation and can exhibit chaotic behaviour even in the scalar case.

2 - Dynamic Mode Decomposition (DMD) [2]

With the DMD one want to approximate a dynamical system with a discrete linear one governed by the equation

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n.$$

The main objectives of this method are identifying the most relevant spatio-temporal modes of the system and use them to perform a low-rank reconstruction of the state of the system and a prediction of the future state.
To do this one operates as follows:

- creation of two matrices: snapshots $\mathbf{X}$ and time-shifted snapshots $\mathbf{Y}$ where we want $\mathbf{Y} = \mathbf{A}\mathbf{X}$;
- using SVD of $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$ to obtain $\mathbf{A}$, or its reduction to a certain rank $\tilde{\mathbf{A}} = \mathbf{U}^*\mathbf{Y}\mathbf{V}\mathbf{\Sigma}^{-1}$;
- using the eigendecomposition of $\mathbf{A}$, i.e. $\tilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\mathbf{\Lambda}$ to compute the modes $\mathbf{\Phi} = \mathbf{Y}\mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{W}$;
- using the modes to recontruct and forecast $\mathbf{x}_k = \mathbf{\Phi}\mathbf{\Lambda}^k\mathbf{b}$.

3 - Sparse Identification of Nonlinear Dynamics [1]

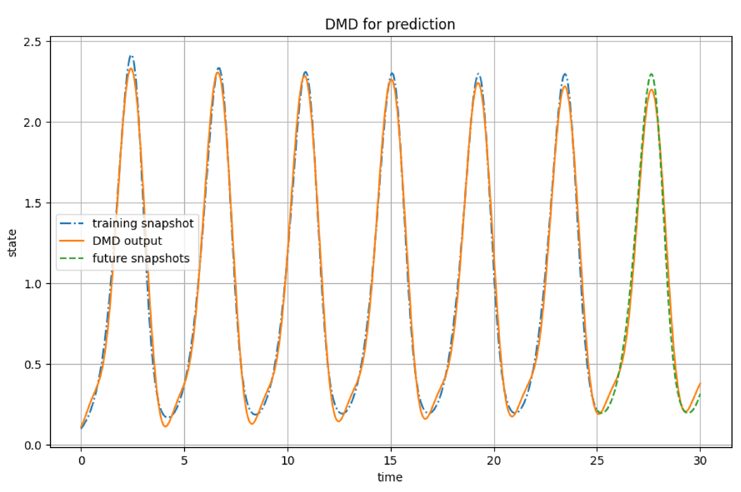
SINDy is an algorithm used to identify non-linear dynamics through a sparsity criterium using data collected from a dynamical system.
This is done proceeding as follows:

- collect the measurements of the state variable and its derivative in two matrices $\mathbf{X}$ and $\mathbf{X}'$;
- select the functions of the state variable to be used in the reconstruction of the derivative $\Theta(\mathbf{X})$;
- impose an equation of the form $\mathbf{X}' \approx \Theta(\mathbf{X})\Xi$, where Ξ is a coefficients matrix;
- solve the equation promoting the sparsity of the matrix Ξ.

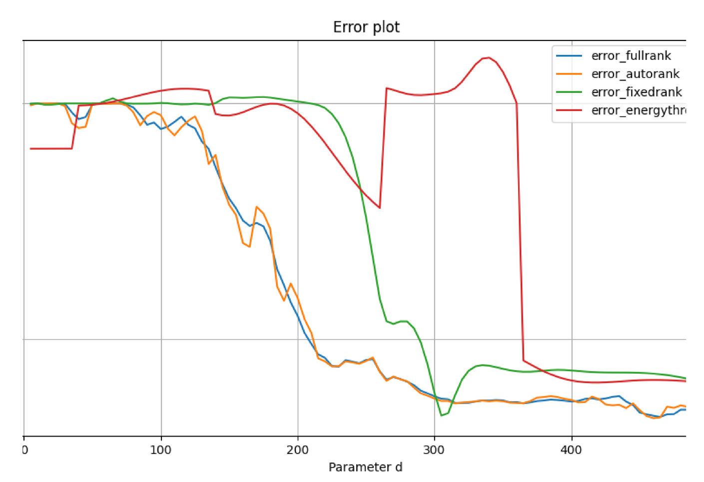
The condition of sparsity in the coefficients is promoted using an L1 regularization.

4 - DMD applied to DDEs

We applied the DMD to time series obtained from delay-differential equations. The time series onsidered were characterized by different behaviours: asymptotically stable, stable, unstable, periodic and chaotic. We were interested both in the recontruction and in the prediction of the state.

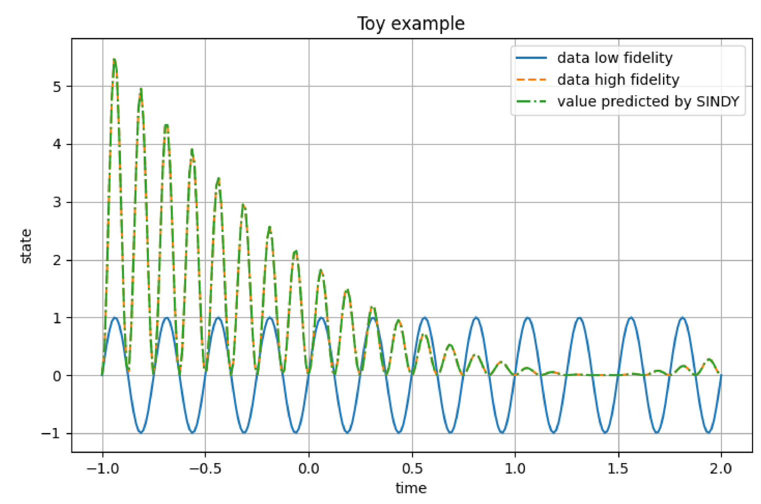


We perform a comparative analysis of the error with respect to different criteria used in the selection of the rank of the approximation.

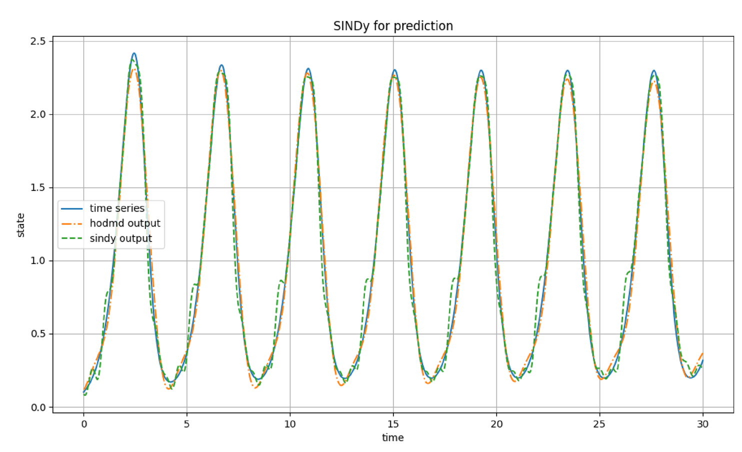


5 - SINDy in a Multy-Fidelity Context

We applied SINDy in a multy-fidelity context: we used it to find possible relations between high-fidelity and low-fidelity. We used then the relation provided by the algorithm to extrapolate the value of the high-fidelity data when just the low-fidelity data are known. We tested this approach with toy examples from the literature.



We tested this approach also on time series coming from dynamical systems, in particular from DDEs. We used as high-fidelity data the training values of the time series, and as low fidelity data the DMD modes.



6 - Possible future developments

Possible future developments of this methodology include the applications to the following examples:

- non scalar DDEs with low dimension;
- non scalar DDEs with high dimension;
- partial delay differential equations.

4 - Computational science and engineering softwares

PyDMD
`github.com/mathLab/PyDMD`
`mathlab.github.io/PyDMD`

PyDMD is a Python package that uses Dynamic Mode Decomposition for a data-driven model simplification based on spatiotemporal coherent structures.

PySINDy
`github.com/dynamicslab/pysindy`

PySINDy is a sparse regression package with several implementations for the Sparse Identification of Nonlinear Dynamical systems (SINDy) method introduced in [1].

References

[1] S. L. Brunton, J. L. Proctor, and J. N. Kutz. Discovering governing equations from data by sparse identification of nonlinear dynamical systems. *Proceedings of the National Academy of Sciences*, 113(15):3932–3937, 2016.

[2] N. Demo, M. Tezzele, and G. Rozza. PyDMD: Python Dynamic Mode Decomposition. *The Journal of Open Source Software*, 3(22):530, 2018.

[3] O. Diekmann, S. A. Verduyn Lunel, S. M.and Gils, and W. H.-O. *Delay equations: functional-, complex-, and nonlinear analysis.* Applied mathematical sciences 110. Springer, New York, 1995.