Data Enhanced Reduced Order Methods for Turbulent Flows
Anna Ivagnes, Giovanni Stabile, Andrea Mola and Gianluigi Rozza
Mathematics Area, mathLab, SISSA, International School of Advanced Studies, Trieste, Italy

Introduction
The focus of this work is the development of data-driven reduced order techniques in CFD context in order to improve the pressure and velocity accuracy of standard reduced order methods. The general framework of Proper Orthogonal Decomposition with Galerkin approach is coupled with a data-driven technique, exploiting the information of full order data to build correction/closure terms. These terms are added in the reduced order system to reintroduce the contribution of disregarded modes. The technique is applied to the 2D study of the turbulent flow around a cylinder in two different approaches: the SUP-ROM, where additional velocity suprimor modes are considered, and the PPE-ROM, where the continuity equation in the model is replaced by the pressure Poisson equation.

1. Offline-Online Procedure

**OFFLINE PHASE**
- Full Order Model: Incompressible NSE
  \[
  \begin{align*}
  \dot{u} - \nabla \cdot (\nabla u + (\nabla u)^T) - \nabla \cdot \tau & = 0 \\
  \nabla \cdot u & = 0 \\
  \end{align*}
  \]
  + boundary and initial conditions.
- Case study: turbulent flow around a circular cylinder
- Discretization with FVM (Finite Volume Method)
- RANS (Reynolds Averaged Navier-Stokes) approach
- POD (Proper orthogonal Decomposition) with Galerkin approach

**ONLINE PHASE**
- Pick a reduced number of modes: \( r << N^u_b \), \( q << N^p_b \)
- Approximated fields: \( \mathbf{u}_i = \sum_{k=1}^r \alpha_{ik} \mathbf{a}_k \), \( \mathbf{p}_i = \sum_{k=1}^q \beta_{ik} \mathbf{b}_k \)
- Projection of the equations onto the reduced modes

\[
Dynamical system with Unknowns: \quad a = (a_i)_{i=1}^r \quad \text{and} \quad b = (b_i)_{i=1}^q
\]

<table>
<thead>
<tr>
<th>Library used for POD:</th>
</tr>
</thead>
<tbody>
<tr>
<td>dd-suprom: github.com/mathLab/ITERCA-FY</td>
</tr>
<tr>
<td>matlab-suprom: github.com/mathLab/ITERCA-FY</td>
</tr>
</tbody>
</table>

2. DD-VMS-ROM: the purely data-driven modeling

Motivation: improve the velocity and pressure accuracy to better capture the *forces*.

How: reintegrating the contribution of the neglected modes with *correction* terms.

Construction of correction terms:
1. build the exact correction \( \tau_{exact} \) from available data;
2. propose an ansatz for the approximated correction term \( \tau_{approx}(\mathbf{a}, \mathbf{b}) \);
3. solve an optimization problem: \( \min_{\tau_{approx}} \| \tau_{approx}(\mathbf{a}, \mathbf{b}) - \tau_{exact} \|_{L^2} \).

Two different types of correction terms:
- \( \tau_u(\mathbf{a}) \): velocity correction in the momentum equation;
- \( \tau_p(\mathbf{a}, \mathbf{b}) \): novel pressure correction in the Poisson equation (in the PPE approach).

3. EV-ROM: the physically-based data-driven modeling

Motivation: Inclusion of a turbulence modeling in the ROM.

How: Addition of reduced eddy viscosity terms.

Construction of \( g \): The eddy viscosity coefficients vector is modeled with regression techniques making use of a fully-connected neural network.

4. Supenrior and Pressure Poisson Approach: numerical results

- \( \tau_u(\mathbf{a}) = \mathbf{a} + a^* \bullet \hat{a} \) quadratic velocity correction term in suprenizer approach;
- \( \tau_p(\mathbf{a}, \mathbf{b}) = \mathbf{b} + b^* \bullet \hat{b} \) quadratic pressure correction term in PPE approach, where \( \mathbf{ab} = (\mathbf{a}, \mathbf{b}) \).

Percentage errors of reduced velocity and pressure fields with respect to full order fields:

\[
\begin{align*}
\epsilon_v(t_j) &= \frac{\|\mathbf{u}^r(t_j) - \mathbf{u}^\text{full}(t_j)\|_{L^2(U)}}{\|\mathbf{u}^\text{full}(t_j)\|_{L^2(U)}} \quad \text{and} \\
\epsilon_p(t_j) &= \frac{\|\mathbf{p}^r(t_j) - \mathbf{p}^\text{full}(t_j)\|_{L^2(U)}}{\|\mathbf{p}^\text{full}(t_j)\|_{L^2(U)}}
\end{align*}
\]

Legend: Results without any data-driven term (--.--); with only turbulence modelling (—-); with only corrections (—.-); with both turbulence and corrections (---); reconstruction error (——-).

5. Graphical results for the PPE approach

- Reduced velocity field
- Reduced pressure field

6. Conclusions and Future Perspectives

Conclusions:
- The velocity correction term improves both the velocity and pressure accuracy, whereas the pressure correction term improves the pressure accuracy in the PPE approach.
- The combination of purely and physically-based data-driven modelling gives the best results and acts as a stabilizer for the error in time.
- The graphical results show a better reconstruction of flow fields, especially nearby the cylinder and it is important in the reconstruction of the *forces fields*.
- Significant reduction in computational cost and time w.r.t. FOM, comparable to the standard ROM. The correction terms are found from a part of the available snapshots and provide a good time extrapolation efficiency.

Outlook:
- The study regards the marginally-resolved modal regime, where the number of modes is enough to represent the underlying dynamics, but the standard ROM yields inaccurate results. Further investigation: different modal regimes.
- Further investigation: more complex computational settings and 3D flows.
- Further investigation: introduction of parameters.

References