Optimisation-Based FEM/ROM Couplings in CFD: Exploring Diverse Scenarios

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Introduction

This work aims to present an optimisation–based framework for coupling different discretisation models, such as Finite Element (FEM) and ROM for separate subcomponents. In particular, we consider an optimisation–based DD model on two non–overlapping subdomains where the coupling on the common interface is performed by introducing a control variable representing a normal flux. Gradient–based optimisation algorithms are used to construct an iterative procedure to fully decouple the subdomain state solutions as well as to locally generate ROMs on each subdomain. Then, we consider FEM or ROM discretisation models for each of the DD problem components, namely, the adjoint state–1–state2–control. We perform numerical tests on the backward–forward step Navier–Stokes problem to investigate the efficacy of the presented coupled treatments in terms of optimisation iterations and relative errors.

1 - Monolithic vs. Domain Decomposition (DD) FEM Formulation

We start with introducing high–fidelity monolithic model based on FEM discretisation in space and explicit Euler scheme in time of the non–stationary incompressible Navier–Stokes equations:

\[
\frac{\partial u^n}{\partial t} - \nu \Delta u^n + (u^n \cdot \nabla) u^n + f = 0, \quad \text{in } \Omega, \quad \text{on } \Gamma_D, \\
\n\n\text{subject to the variational problem:} \\
\int_{\Omega} \left( \frac{\partial u^n}{\partial t} - \nu \Delta u^n + (u^n \cdot \nabla) u^n + f \right) \cdot v \, \text{d}x = 0 \quad \forall v \in V_h, \\
\int_{\Gamma_D} u^n \cdot n \, \text{d}s = g_D \quad \forall \Gamma_D, \\
\int_{\Gamma_N} n \cdot (u^n ) \, \text{d}s = 0 \quad \forall \Gamma_N.
\]

The DD conditions: \( u^{n^h}_{1,h} = u^{n^h}_{2,h} \) and \( \frac{\partial u^{n^h}_{1,h}}{\partial t} + \frac{\partial u^{n^h}_{2,h}}{\partial t} = \frac{\partial u^{n^h}_{1,h}}{\partial t} \) in \( \Omega_h \).

The optimisation–based DD formulation reads as follows: for \( n \geq 1 \)

\[
\begin{align*}
\frac{\partial u_{1,h}^n}{\partial t} - \nu \Delta u_{1,h}^n + (u_{1,h}^n \cdot \nabla) u_{1,h}^n + f_{1,h} & = 0, \quad \text{in } \Omega_1, \\
\frac{\partial u_{2,h}^n}{\partial t} - \nu \Delta u_{2,h}^n + (u_{2,h}^n \cdot \nabla) u_{2,h}^n + f_{2,h} & = 0, \quad \text{in } \Omega_2, \\
\frac{\partial u_{1,h}^n}{\partial t} + \frac{\partial u_{2,h}^n}{\partial t} & = 0, \quad \text{on } \Gamma_D, \\
\int_{\Gamma_N} n \cdot (u_{1,h}^n ) \, \text{d}s & = 0, \quad \forall \Gamma_N, \\
\int_{\Gamma_D} u_{1,h}^n \cdot n \, \text{d}s & = g_D, \quad \forall \Gamma_D, \\
\int_{\Gamma_N} n \cdot (u_{2,h}^n ) \, \text{d}s & = 0, \quad \forall \Gamma_N.
\end{align*}
\]

2 - Adjoint system and the gradient expression

Adjoint problem:

\[
m(\eta, \xi, \xi_h) = \frac{\partial}{\partial \eta} \left( \frac{\partial}{\partial \xi_h} \right) + c_1(\eta, \xi, \xi_h, \xi), \quad c_1(\eta, \xi, \xi_h, \xi) = c(\eta, \xi, \xi_h, \xi),
\]

Optimality system:

\[
\left. \frac{dJ}{d\xi} \right|_{\xi} = \xi_h, \quad \xi \in \mathbb{X}, \quad \mathbb{X} = \mathcal{H}^1(\Omega), \quad \mathcal{H}^1(\Gamma_D), \quad \mathcal{H}^1(\Gamma_N), \quad \mathcal{H}^1(\Gamma_N).
\]

3 - ROM setting

- POD–compression in terms of time and physical parameters
- POD–Galerkin ROM
- POD–NN ROM

4 - FEM–ROM couplings

- FEM–FEM–FEM (FFF) coupling
- FEM–ROM–FEM (FRF) coupling
- ROM–ROM–ROM (RRR) coupling

5 - Numerical results

6 - Computational science and engineering softwares

Reference