Stochastic optimal Robin boundary control problems constrained by an advection dominated elliptic equation

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1. Introduction

We study stochastic optimal Robin boundary control problems constrained by **an advection dominated** elliptic equation. The advection field and the Robin boundary data are assumed to be **random fields** represented by Karhunen-Loève expansion. **A stochastic saddle point system** is formulated and proved to be equivalent to the first order optimality system for the optimal control problem, based on which we examine the **stochastic regularity**. **Finite element method with SUPG stabilization** in physical space and **sparse grid stochastic collocation (SGSC)** in stochastic space are employed for numerical approximation to the stochastic optimal control problem. **A global error estimate** in both physical and stochastic spaces is obtained and verified by numerical experiments [4].

2. Stochastic optimal Robin boundary control

The stochastic optimal Robin boundary control problem reads : to find $g^* \in \mathcal{G}$ s.t.

$$g^* = \arg\min_{g \in \mathcal{G}} \mathcal{J}(u,g) := \frac{1}{2}||u-u_d||^2_{\mathcal{O}} + \frac{\eta}{2}||g||^2_{\mathcal{G}}, \quad (1)$$

subject to the advection dominated elliptic equation in the weak formulation

$$\mathcal{B}(u,v) = \mathcal{F}(v), \quad \forall v \in L^2(\Gamma; H^1(D)) \quad (2)$$

where the linear functional $\mathcal{F} : L^2(\Gamma; H^1(D)) \to \mathbb{R}$ entails the boundary condition g

$$\mathcal{F}(v) := (f,v)_{\Gamma\times D} + (g,v)_{\Gamma\times\partial D} \equiv \int_\Gamma\int_D fv\rho(y)dxdy + \int_\Gamma\int_{\partial D} gv\rho(y)d\gamma dy \quad (3)$$

and the bilinear form $\mathcal{B} : L^2(\Gamma; H^1(D)) \times L^2(\Gamma; H^1(D)) \to \mathbb{R}$ is defined as

$$\mathcal{B}(u,v) := \int_\Gamma\int_D \big(a\nabla u\cdot\nabla v + (\boldsymbol{b}\cdot\nabla u)v + cuv\big)\rho(y)dxdy + \int_\Gamma\int_{\partial D} kuv\rho(y)d\gamma dy. \quad (4)$$

The first term $||\cdot||_{\mathcal{O}}$ is the measure of the difference between the solution u and the observation u_d, while the second one is regularization with $\eta \geq 0$. The stochastic velocity $\boldsymbol{b} : D\times\Gamma \to \mathbb{R}^d$ is assumed to be a random field represented by Karhunen-Loève expansion on a series of random variables $y_n : \Omega \to \mathbb{R}, 1 \leq n \leq N$

$$\boldsymbol{b}(x,y) = \mathbb{E}[\boldsymbol{b}](x) + \sum_{n=1}^N \sqrt{\lambda_n}\boldsymbol{b}_n(x)y_n(\omega). \quad (5)$$

The first order stochastic optimality system for the problem (1) and (2) is given as

$$\begin{cases} \mathcal{B}(u,\tilde{u}) = \mathcal{F}(\tilde{u}), \quad \forall \tilde{u} \in L^2(\Gamma; H^1(D)), \\ \mathcal{B}(\tilde{p},p) = (u_d - u, \tilde{p}), \quad \forall \tilde{p} \in L^2(\Gamma; H^1(D)), \\ \eta(g,\tilde{g})_{\Gamma\times\partial D} = (p,\tilde{g})_{\Gamma\times\partial D}, \quad \forall \tilde{g} \in L^2(\Gamma; H^{-1/2}(\partial D)), \end{cases} \quad (6)$$

The following proposition establishes the equivalence between the stochastic optimality system and its corresponding stochastic saddle point system [3]

Proposition 1 *The optimality system (6) is equivalent to the following saddle point problem : to find* $\underline{u} = (u,g) \in L^2(\Gamma; Z^1(D)) := L^2(\Gamma; H^1(D)) \times L^2(\Gamma; H^{-1/2}(\partial D))$*, s.t.*

$$\begin{cases} \mathcal{A}(\underline{u},\underline{v}) + \mathcal{B}(\underline{v},p) = (\underline{u_d},\underline{v})_{\Gamma\times D}, \quad \forall \underline{v} \in L^2(\Gamma; Z^1(D)), \\ \mathcal{B}(\underline{u},q) = (f,q)_{\Gamma\times D}, \quad \forall q \in L^2(\Gamma; H^1(D)), \end{cases} \quad (7)$$

where $\mathcal{A}(\underline{u},\underline{v}) := (u,v)_{\Gamma\times D} + \eta(g,h)_{\Gamma\times\partial D}$ *is continuous and coercive and* $\mathcal{B}(\underline{u},q) := \mathcal{B}(u,q) - (g,q)_{\Gamma\times\partial D}$ *is continuous and satisfies the compatibility condition. Therefore, there exists a unique solution* $(\underline{u},p)$ *to (7) by Brezzi's theorem, satisfying*

$$\begin{aligned} ||\underline{u}||_{L^2(\Gamma;Z^1(D))} &\leq \alpha_1||u_d||_{L^2(\Gamma;L^2(D))} + \beta_1||f||_{L^2(D)}, \\ ||p||_{L^2(\Gamma;H^1(D))} &\leq \alpha_2||u_d||_{L^2(\Gamma;L^2(D))} + \beta_2||f||_{L^2(D)}. \end{aligned} \quad (8)$$

3. Regularity in high dimensional stochastic space

Theorem 1 [4] *Holding the assumptions in Proposition 1 and the Karhunen-Loève expansion for the random field* $\mathbf{b}$ *in (5), we have for* $\forall\,\nu = (\nu_1,\dots,\nu_N) \in \mathbb{N}^N$

$$\begin{aligned} ||\partial_y^\nu \underline{u}(y)||_{Z^1(D)} &\leq \sum_{0\preceq\mu\preceq\nu} C^{\underline{u},u_d}_{\nu-\mu}|\nu-\mu|!||\boldsymbol{b}||^{\nu-\mu}_{(L^\infty(D))^d}||\partial_y^\mu u_d(y)||_{L^2(D)} + C^{\underline{u},f}_\nu|\nu|!||\boldsymbol{b}||^\nu_{(L^\infty(D))^d}||f||_{L^2(D)}, \\ ||\partial_y^\nu p(y)||_{H^1(D)} &\leq \sum_{0\preceq\mu\preceq\nu} C^{p,u_d}_{\nu-\mu}|\nu-\mu|!||\boldsymbol{b}||^{\nu-\mu}_{(L^\infty(D))^d}||\partial_y^\mu u_d(y)||_{L^2(D)} + C^{p,f}_\nu|\nu|!||\boldsymbol{b}||^\nu_{(L^\infty(D))^d}||f||_{L^2(D)}, \end{aligned} \quad (9)$$

where $\mu \preceq \nu$ *implies* $\mu_n \leq \nu_n, \forall n = 1,2,\dots,N$*, and the constant* $C^{\underline{u},u_d}_{\nu-\mu} = C^{\underline{u},u_d}_{\nu-\mu}(\alpha_1,\alpha_2,\beta_1,\beta_2)$ *is the sum of* $2^{|\nu-\mu|}$ *basic elements in the form of* $\alpha_1^{n_1}\alpha_2^{n_2}\beta_1^{m_1}\beta_2^{m_2}$ *s.t* $n_1+n_2+m_1+m_2 = |\nu-\mu|+1$*. The same expression holds for* $C^{p,u_d}_{\nu-\mu}, C^{\underline{u},f}_\nu, C^{p,f}_\nu$*.*

$$\text{If } 2C\sum_n^N ||\boldsymbol{b}_n||_{(L^\infty(D))^d}|y_n - \bar{y}_n| < 1 \text{ and } \sum_\mu (|\mu|/\mu!)|y-\bar{y}|^\mu||\partial_y^\mu u_d(\bar{y})||_{L^2(D)} \leq \infty, \quad (10)$$

we have a Taylor expansion of $(\underline{u},p)$ *around* $\bar{y} \in \Gamma$ *so that* $(\underline{u},p)$ *is analytic in the set* $\Sigma = \big\{y \in \mathbb{R}^N : \exists\,\bar{y} \in \Gamma \text{ s.t } (10) \text{ holds}\big\}$*, and we define* $\Sigma(\Gamma;\tau) := \{z \in \mathbb{C} : dist(z,\Gamma) \leq \tau\} \subset \Sigma$ *for the largest possible vector* $\tau = (\tau_1,\dots,\tau_N)$*. Note* $C = \max(\alpha_1,\alpha_2,\beta_1,\beta_2)$*.*

4. Error estimate for SUPG + SGSC approximation

We use the SUPG stabilized finite element approximation in physical space [1]

$$\begin{cases} \mathcal{B}(u_h(y),v_h) + \sum_{K\in\mathcal{T}_h}\delta_K h_K\left(Lu_h(y),(\boldsymbol{b}\cdot\nabla)v_h/|\boldsymbol{b}|\right) \\ = \dfrac{1}{\alpha}(p_h(y),v_h) + (f,v_h) + \sum_{K\in\mathcal{T}_h}\delta_K h_K\left(f,(\boldsymbol{b}\cdot\nabla)v_h/|\boldsymbol{b}|\right) \forall\, v_h \in X_h^k, \\ \mathcal{B}(v_h,p_h(y)) + \sum_{K\in\mathcal{T}_h}\delta_K h_K\left(L'p_h(y),(\boldsymbol{b}\cdot\nabla)v_h/|\boldsymbol{b}|\right) \\ = (u_d(y)-u_h(y),v_h) + \sum_{K\in\mathcal{T}_h}\delta_K h_K\left(u_d(y)-u_h(y),(\boldsymbol{b}\cdot\nabla)v_h/|\boldsymbol{b}|\right) \forall\, v_h \in X_h^k, \end{cases} \quad (11)$$

where the finite element space $X_h^k := \big\{v_h \in C(\bar{D})\big|v_{h|K} \in \mathbb{P}_k \forall\, K \in \mathcal{T}_h\big\}$, $k \geq 0$ and L, L' are the elliptic operators for the state and adjoint problems. In stochastic space, we apply sparse grid stochastic collocation approximation with $w \in \mathbb{R}^N_+$ [2]

$$\mathcal{S}_q^w u(y) = \sum_{i\in X_w(q,N)}\left(\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_N}\right)u(y) = \mathcal{S}_{q-1}u(y) + \sum_{|i|=q}\left(\Delta^{i_1}\otimes\cdots\otimes\Delta^{i_N}\right)u(y), \quad (12)$$

where $X_w(q,N) := \Big\{i \in N_+^N, i \geq 1 : \sum_{n=1}^N (i_n - 1)w_n \leq \min(w)q\Big\}$ and $\Delta^{i_n} = \mathcal{U}^{i_n} - \mathcal{U}^{i_n-1}$ is defined as the difference of 1-D interpolation operator $\mathcal{U}^{i_n}, 1 \leq n \leq N$.

Theorem 2 [4] *With* δ_K *small and Proposition 1 and Theorem 1 holding, we have the global error estimate for the combination of SUPG and SGSC approximation*

$$\begin{aligned} &||u-u_{h,q}||_{L^2(\Gamma;V(D))} + ||g-g_{h,q}||_{L^2(\Gamma;L^2(\partial D))} + ||p-p_{h,q}||_{L^2(\Gamma;V(D))} \\ &\leq C_s N_q^{-r(w)} + C_p(a_{\min}^{1/2} + h^{1/2})h^s||(|u|_{s+1} + |p|_{s+1} + h|g|_{s+1/2})||_{L^2(\Gamma)}, \end{aligned} \quad (13)$$

where the norm $||v||_V^2 = a_{\min}|v|_1 + ||v||^2_{L^2(D)} + \sum_{K\in\mathcal{T}_h}\delta_K||\boldsymbol{b}\cdot\nabla v||^2_{L^2(K)}$ *and* $|v|_s, s \geq 1$ *is the semi-norm in the Hilbert space* $H^s(D), s \geq 1$*.* N_q *is the number of collocation nodes.* C_s, C_p *are approximation constants.* $r(w)$ *is convergence rate on weight* w*.*

5. Numerical experiments

Set $D = [0,1]^2$, $a = 0.01, c = 1, k = 1, f = 0.1$, the stochastic velocity $\boldsymbol{b} = (b_{x_1}, b_{x_2})$ with $b_{x_2} = 0$ and $\mathbb{E}[b_{x_1}](x) = x_2(1-x_2)$, $\mathbb{C}ov[b_{x_1}](x,x') = (x_2(1-x_2)/10)^2\exp(-(x_1-x_1')^2/L^2)$. The observation u_d is computed by setting $\mathbb{E}[g](x) = 1, \mathbb{C}ov[g](x,x') = \exp(-|x-x'|^2/L^2)/4$. Kahunen-Loève expansion is applied to represent $\boldsymbol{b}$ and g, with eigenvalues decaying in Figure 1 for different L. The error (13) is computed with different h and different q for isotropic and anisotropic sparse grids, see numerical results in Figure 1, which is in good agreement with the theoretical results in Theorem 2.

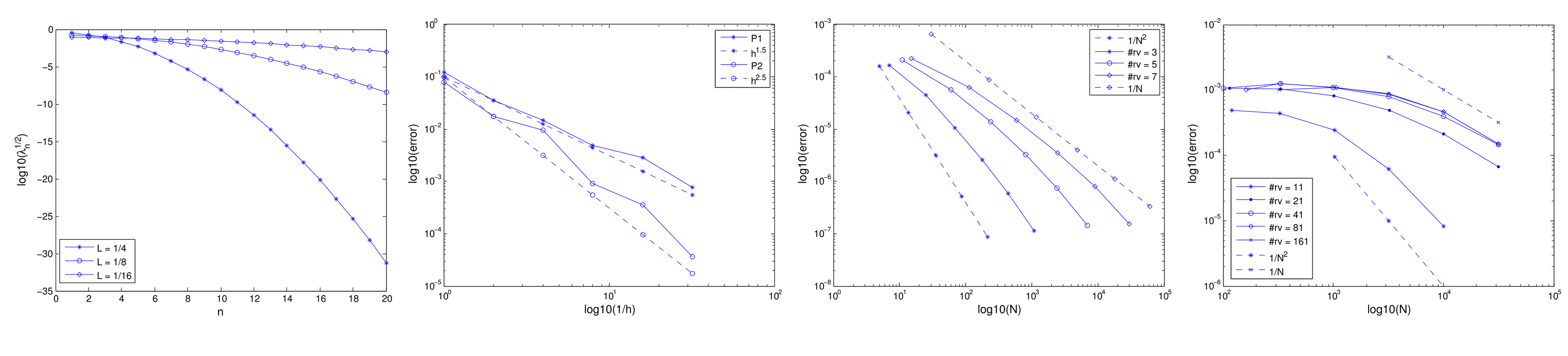

Figure 1: *1. Decay of eigenvalues for different correlation length* $L = 1/4, 1/8, 1/16$ *; 2. convergence rate for stabilized finite element approximation in* X_h^1 *and* X_h^2 *; convergence rate for 3. isotropic sparse grid with* $3,5,7$ *random variables,* $L = 1/4$*, and for 4. anisotropic sparse grid with* $11,21,41,81,161$ *random variables,* $L = 1/16$*.*

6. Concluding remarks

We established the equivalence between saddle point system and optimality system for stochastic optimal Robin boundary control problem, based on which we obtained the stochastic regularity. Finite element approximation with SUPG stabilization and sparse grid stochastic collocation approximation were successfully applied with a global error estimate derived and verified by numerical experiments.

7. Reference

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