A reduced computational framework for optimal control problems

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Abstract

Solving optimal control problems for many different scenarios obtained by varying a set of parameters in the state system is a computationally extensive task. We present a reduced framework for the numerical solution of parametrized PDE-constrained optimization problems. The proposed framework is based on a suitable saddle-point formulation of the optimal control problem and exploits the reduced basis method, leading to a relevant computational reduction with respect to traditional discretization techniques such as the finite element method. This setting is applied to the solution of two problems arising from haemodynamics, dealing with both data reconstruction and data assimilation over domains of variable shape (for which a further geometrical reduction is pursued), which can be recast in a common PDE-constrained optimization formulation.

1. Problem definition and saddle point formulation

Let $\Omega \subset \mathbb{R}^d$ be a spatial domain and $\mathcal{D} \subset \mathbb{R}^p$ be a p-dimensional parameter set. Let $\mathcal{Y}$, $\mathcal{U}$ be two Hilbert spaces for the state and control variables y and u respectively, while $\mathcal{Z} \supset \mathcal{Y}$ shall denote the observation space. We consider the case of a *quadratic cost functional* to be minimized

$$\mathcal{J}(y,u;\boldsymbol{\mu}) = \frac{1}{2} m(y - y_d(\boldsymbol{\mu}), y - y_d(\boldsymbol{\mu}); \boldsymbol{\mu}) + \frac{\alpha}{2} n(u,u;\boldsymbol{\mu}), \quad (1)$$

where $\alpha > 0$ is a given constant and $y_d(\boldsymbol{\mu}) \in \mathcal{Z}$ is a given parameter-dependent observation function. Let $\mathcal{Q} \equiv \mathcal{Y}$ denote the test space, we define the *linear constraint equation*

$$\mathcal{B}(y,u,q;\boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle \qquad \forall q \in \mathcal{Q}, \qquad (2)$$

where the bilinear form $\mathcal{B}(\cdot,\cdot;\boldsymbol{\mu}) : \mathcal{Y} \times \mathcal{U} \times \mathcal{Q} \to \mathbb{R}$ is given by the sum of two contributes

$$\mathcal{B}(y,u;q;\boldsymbol{\mu}) = a(y,q;\boldsymbol{\mu}) - c(u,q;\boldsymbol{\mu});$$

the (weakly) coercive bilinear form $a(\cdot,\cdot;\boldsymbol{\mu})$ represents a linear elliptic operator while the bilinear form $c(\cdot,\cdot;\boldsymbol{\mu})$ expresses the action of the control.

The parametrized optimal control problem ($\text{OCP}_{\boldsymbol{\mu}}$) reads: for any given $\boldsymbol{\mu} \in \mathcal{D}$

$$\min_{y,u} \mathcal{J}(y(\boldsymbol{\mu}), u(\boldsymbol{\mu}); \boldsymbol{\mu}) \quad \text{s.t.} \quad (y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in \mathcal{Y}\times\mathcal{U} \text{ solves (2).}$$

In order to formulate the optimal control problem as a saddle-point problem, we first define the product space $\mathcal{X} = \mathcal{Y} \times \mathcal{U}$ and denote with $x = (y,u) \in X$, $w = (z,v) \in X$ its variables. We can reformulate the $\text{OCP}_{\boldsymbol{\mu}}$ as: given $\boldsymbol{\mu} \in \mathcal{D}$,

$$\begin{cases} \min_x \mathcal{J}(x;\boldsymbol{\mu}) = \frac{1}{2}\mathcal{A}(x,x;\boldsymbol{\mu}) - \langle F(\boldsymbol{\mu}), x\rangle, & \text{s.t.} \\ \mathcal{B}(x,q;\boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle & \forall q \in Q. \end{cases} \qquad (3)$$

where $F(\boldsymbol{\mu}) = m(y_d(\boldsymbol{\mu}), \cdot) \in \mathcal{X}'$ and

$$\mathcal{A}(x,w;\boldsymbol{\mu}) = m(y,z;\boldsymbol{\mu}) + \alpha n(u,v;\boldsymbol{\mu}), \qquad \forall x, w \in \mathcal{X}.$$

The constrained optimization problem (3) falls into the framework of saddle-point problems. The assumptions of Brezzi theorem can be easily verified [4] and therefore, for any $\boldsymbol{\mu} \in \mathcal{D}$, the optimal control problem has a unique solution $x(\boldsymbol{\mu}) \in X$ that can be determined by solving the following saddle-point problem (i.e. the optimality system):

given $\boldsymbol{\mu} \in \mathcal{D}$, find $(x(\boldsymbol{\mu}), p(\boldsymbol{\mu})) \in \mathcal{X} \times \mathcal{Q}$ such that

$$\begin{cases} \mathcal{A}(x(\boldsymbol{\mu}), w;\boldsymbol{\mu}) + \mathcal{B}(w, p(\boldsymbol{\mu});\boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), w\rangle & \forall w \in \mathcal{X}, \\ \mathcal{B}(x(\boldsymbol{\mu}), q;\boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle & \forall q \in \mathcal{Q}, \end{cases}$$

where $p(\boldsymbol{\mu})$ is the Lagrange multiplier (adjoint variable) associated to the constraint.

2. Reduced basis approximation

The RB method gives an efficient way to compute an approximation to the FE *truth* solution $(x_h(\boldsymbol{\mu}), p_h(\boldsymbol{\mu}))$ by considering only a small subspace of the FE space $\mathcal{X}_h \times \mathcal{Q}_h$. We thus take a suitably selected (by a greedy algorithm) set of parameter values $\boldsymbol{\mu}^1, \dots, \boldsymbol{\mu}^N$ ($N \ll N_h$) and the corresponding FE solutions $(x_h(\boldsymbol{\mu}^1), p_h(\boldsymbol{\mu}^1)), \dots, (x_h(\boldsymbol{\mu}^N), p_h(\boldsymbol{\mu}^N))$. The reduced basis control space is given by

$$U_N = \text{span}\{u_h(\boldsymbol{\mu}^n), \quad n = 1, \dots, N\},$$

while, in order to guarantee the stability of the RB approximation, we define the following *aggregated* reduced basis space for the state and adjoint variables

$$\mathcal{Y}_N \equiv \mathcal{Q}_N = \text{span}\{y_h(\boldsymbol{\mu}^n),\ p_h(\boldsymbol{\mu}^n),\ n = 1, \dots, N\}.$$

Let $\mathcal{X}_N = \mathcal{Y}_N \times \mathcal{Q}_N$, the reduced basis approximation reads:

given $\boldsymbol{\mu} \in \mathcal{D}$, find $(x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in \mathcal{X}_N \times \mathcal{Q}_N$ such that

$$\begin{cases} \mathcal{A}(x_N(\boldsymbol{\mu}), w;\boldsymbol{\mu}) + \mathcal{B}(w, p_N(\boldsymbol{\mu});\boldsymbol{\mu}) = \langle F(\boldsymbol{\mu}), w\rangle & \forall w \in \mathcal{X}_N \\ \mathcal{B}(x_N(\boldsymbol{\mu}), q;\boldsymbol{\mu}) = \langle G(\boldsymbol{\mu}), q\rangle & \forall q \in \mathcal{Q}_N. \end{cases}$$

3. Efficiency and reliability

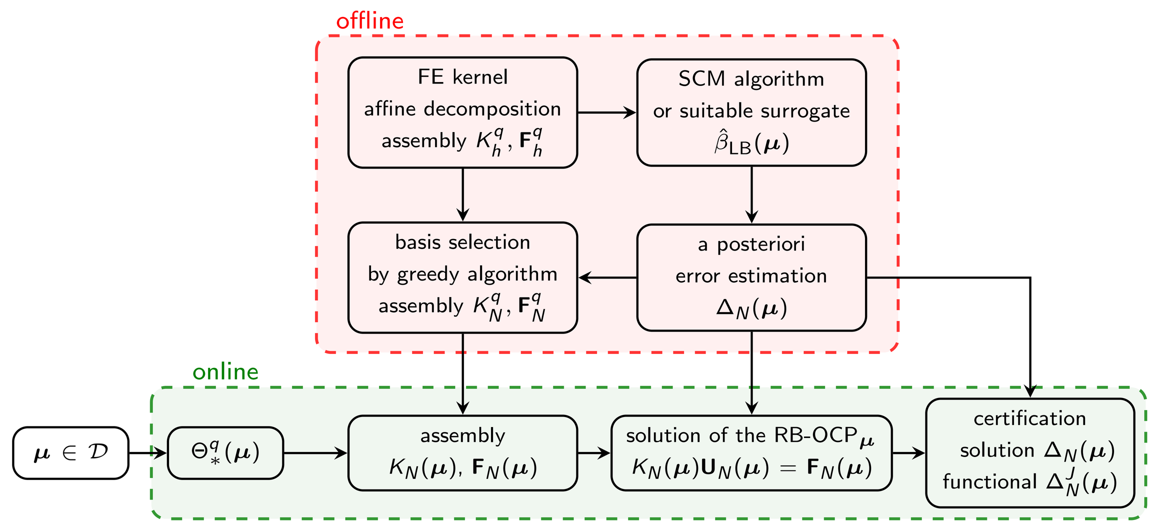
At the algebraic level we obtain the linear system

$$\underbrace{\begin{pmatrix} A_N(\boldsymbol{\mu}) & B_N^T(\boldsymbol{\mu}) \\ B_N(\boldsymbol{\mu}) & 0 \end{pmatrix}}_{K_N(\boldsymbol{\mu})} \begin{pmatrix} \mathbf{x}_N(\boldsymbol{\mu}) \\ \mathbf{p}_N(\boldsymbol{\mu}) \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{F}_N(\boldsymbol{\mu}) \\ \mathbf{G}_N(\boldsymbol{\mu}) \end{pmatrix}}_{\mathbf{f}_N(\boldsymbol{\mu})}. \qquad (4)$$

Thanks to the affine assumption, we can write

$$K_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_k} \Theta_k^q(\boldsymbol{\mu}) K_N^q, \qquad \mathbf{f}_N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \Theta_f^q(\boldsymbol{\mu}) f_N^q,$$

where K_N^q and f_N^q are $\boldsymbol{\mu}$-independent, and we can therefore provide the usual Offline-Online computational decomposition.



Recasting the problem in the general Babuška framework [3] we can provide an efficient and rigorous a posteriori error estimate on the solution variables (as well as on the cost functional [2]):

$$\|(x_h(\boldsymbol{\mu}), p_h(\boldsymbol{\mu})) - (x_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}))\|_{\mathcal{X}\times\mathcal{Q}} \leq \frac{\|r(\cdot;\boldsymbol{\mu})\|}{\hat{\beta}_{LB}(\boldsymbol{\mu})} = \Delta_N(\boldsymbol{\mu})$$

where $0 < \hat{\beta}_{LB}(\boldsymbol{\mu}) \leq \hat{\beta}_h(\boldsymbol{\mu})$ is a *lower bound* of the inf-sup constant of the optimality system.

4. A surface reconstruction problem

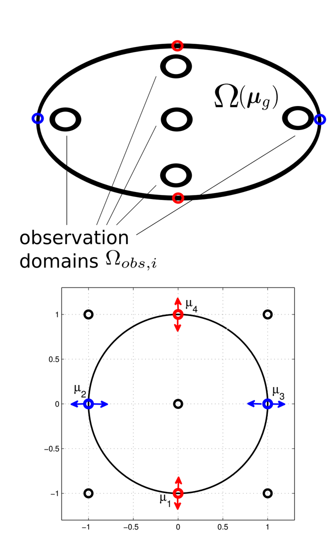
We wish to reconstruct, from areal data provided by eco-dopplers measurements, the blood velocity field in a section of a carotid artery. The problem can be seen as a problem of surface reconstruction starting from scattered data, and it turns out [5] that it can be modeled as a minimization problem for a suitable PDE-penalized least-square cost functional:

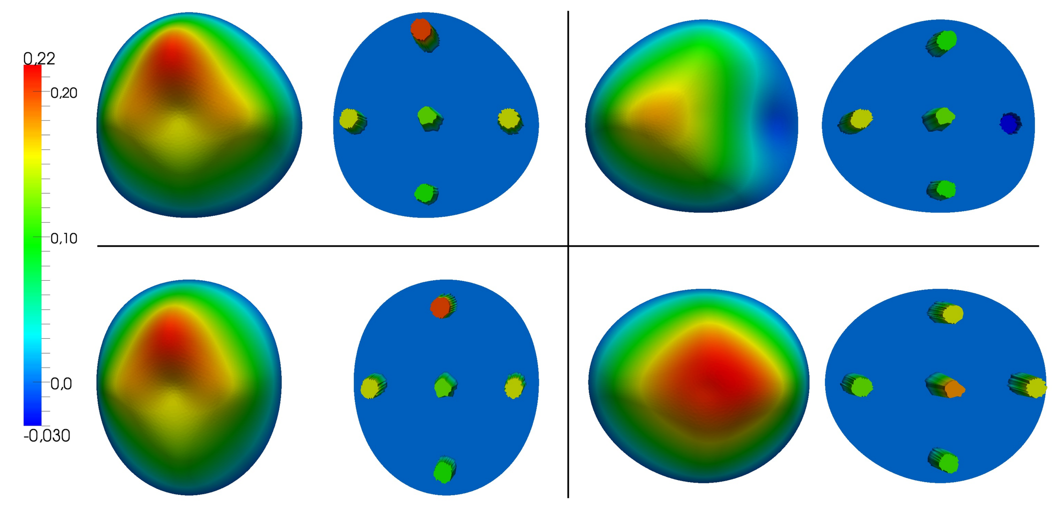
$$\min_{y,u} \mathcal{J}(y,u;\boldsymbol{\mu}) = \frac{1}{2}\sum_{i=1}^{5} \int_{\Omega_{obs,i}} |y(\boldsymbol{\mu}) - z_i|^2 d\Omega + \frac{\alpha}{2}\|u(\boldsymbol{\mu})\|^2_{L^2}$$

$$\text{s.t.} \quad \begin{cases} -\Delta y(\boldsymbol{\mu}) = u(\boldsymbol{\mu}) & \text{in } \Omega(\boldsymbol{\mu}_g) \\ y(\boldsymbol{\mu}) = 0 & \text{on } \partial\Omega(\boldsymbol{\mu}_g) \end{cases}$$

- Geometrical parametrization: Free Form Deformation $\boldsymbol{\mu}_g \in (-0.15, 0.15)^4$ displacements of the control points • •
- Parametrized observation values: $\mu^i_{obs} = z_i$, $1 \leq i \leq 5$

Number of FE dof N_h	$3.3 \cdot 10^4$
Regularization constant α	10^{-4}
Number of parameters P	9
Number of RB functions N	42
Affine components Q_k	53
Linear system dimension red.	160:1
RB solution t_{RB}^{online}(s)	13*ms*
Speedup	100x



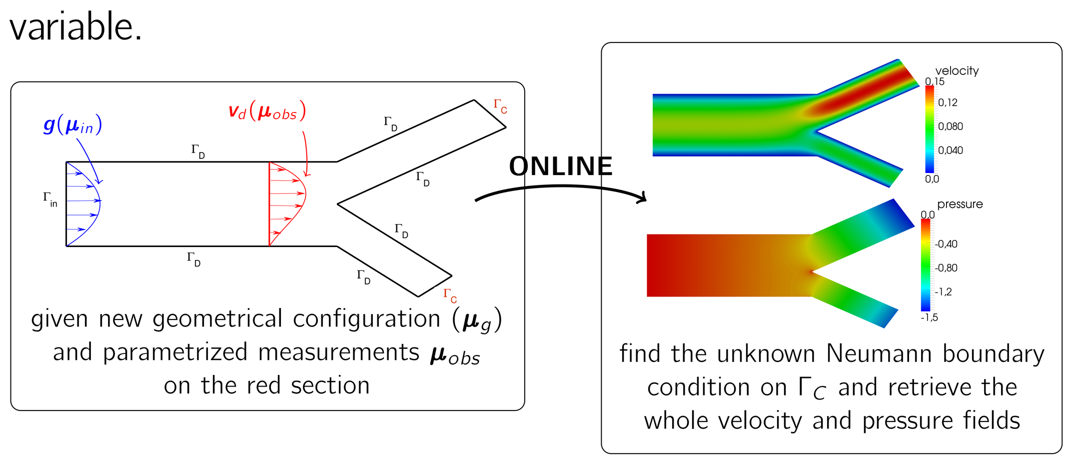


In each box we report the observation values z_i on the right and the reconstructed surface y on the left.

5. Stokes constraint: a boundary inverse problem

The methodology has been extended to treat $\text{OCP}_{\boldsymbol{\mu}}$ with Stokes constraints. The stability of the RB approximation can be fulfilled by introducing suitable supremizer operators [3] and by defining suitable aggregated spaces for state and adjoint variables [1].

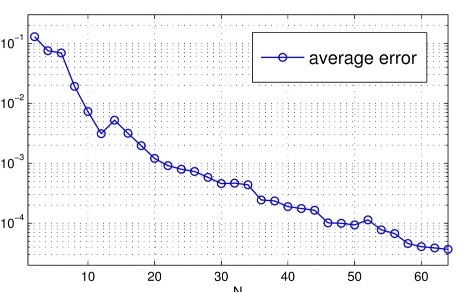
As a possible application, we consider an inverse boundary problem in haemodynamics (inspired by the work in [6]) where the state equation models the blood flow (supposed to obey the Stokes equations) in a parametrized arterial bifurcation and we suppose to have a measured velocity profile on the a transverse section, but not the Neumann flux on Γ_C that will be our control variable.

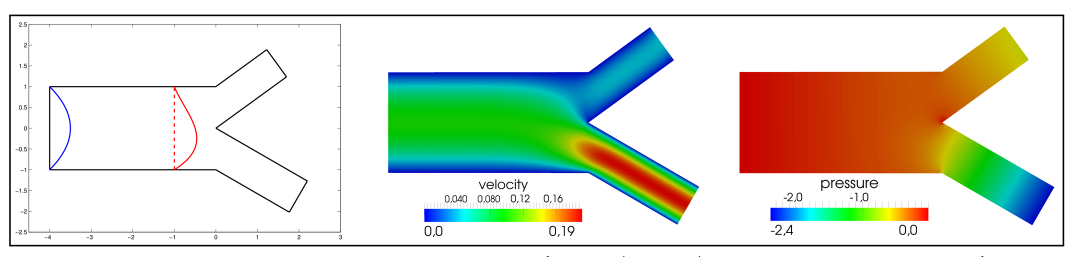


$$\min_{\boldsymbol{v},\pi,\boldsymbol{u}} \mathcal{J}(\boldsymbol{v},\pi;\boldsymbol{u};\boldsymbol{\mu}) = \frac{1}{2}\int_{\Gamma_{obs}} |\boldsymbol{v} - \boldsymbol{v}_d(\boldsymbol{\mu}_{obs})|^2\, d\Gamma + \frac{\alpha_1}{2}\int_{\Gamma_C} |\nabla \boldsymbol{u}\cdot\mathbf{t}|^2 d\Gamma + \frac{\alpha_2}{2}\int_{\Gamma_C} |\boldsymbol{u}|^2 d\Gamma$$

$$\text{s.t.} \quad \begin{cases} -\nu\Delta\boldsymbol{v} + \nabla\pi = 0 & \text{in } \Omega(\boldsymbol{\mu}) \\ \text{div}\, \boldsymbol{v} = 0 & \text{in } \Omega(\boldsymbol{\mu}) \\ \boldsymbol{v} = 0 & \text{on } \Gamma_D(\boldsymbol{\mu}) \\ \boldsymbol{v} = \boldsymbol{g}(\boldsymbol{\mu}_{in}) & \text{on } \Gamma_{in}(\boldsymbol{\mu}) \\ -\pi\boldsymbol{n} + \nu\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}} = \boldsymbol{u} & \text{on } \Gamma_C(\boldsymbol{\mu}). \end{cases}$$

Number of FE dof N_h	$4.6 \cdot 10^4$
Number of parameters P	8
Number of RB functions N	65
Affine components Q_k	71
Linear system dimension red.	60:1





Representative solution for $\boldsymbol{\mu} = (1, \pi/5, \pi/6, 1, 1.7, 2.2, 0.8, 1)$.

References

[1] G. Rozza, A. Manzoni, and F. Negri, *Reduced strategies for PDE-constrained optimization problems in haemodynamics*, in Proceedings of ECCOMAS 2012, Vienna, 10-14 Sept. 2012.

[2] F. Negri, G. Rozza, A. Manzoni, and A. Quarteroni, *Reduced basis method for parametrized elliptic optimal control problems.* Submitted, 2012.

[3] G. Rozza, D. Huynh, and A. Manzoni, *Reduced basis approximation and a posteriori error estimation for Stokes flows in parametrized geometries: roles of the inf-sup stability constants*, Tech. Rep. 22.2010, MATHICSE. Submitted.

[4] M. D. Gunzburger and P. B. Bochev, *Least-Squares Finite Element Methods*, Springer, 2009.

[5] L. Azzimonti, M. Domanin, L. M. Sangalli, and P. Secchi, *Surface estimation via spatial spline models with PDE penalization*, in Proceedings of S.Co.2011 Conference, Padova, 2011.

[6] M. D'Elia, L. Mirabella, T. Passerini, M. Perego, M. Piccinelli, C. Vergara, and A. Veneziani, *Applications of Variational Data Assimilation in Computational Hemodynamics*, in Modelling of Physiological Flows, D. Ambrosi, A. Quarteroni, and G. Rozza, eds., vol. 5 of MS&A Series, Springer, 2011.

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