Computational reduction strategies for bifurcation and stability analysis in fluid-dynamics: applications to Coanda effect in cardiac flows

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Introduction

Coanda effect is the tendency of a fluid jet to be attracted to a nearby surface. In cardiology, this effect is responsible for the wall hugging jets in certain cases of mitral valve regurgitation: regurgitant blood flow through a leaky mitral valve sometimes hugs the wall of the left atrium. This makes it difficult to assess the severity of mitral regurgitation using classical color Doppler imaging techniques.

Method

The aim is to study the onset of the Coanda effect in a simplified setting with the Reduced Basis (RB) method. We focus both on the planar and three dimensional case (figure 1). In this work, only the influence of the Reynolds number, the contraction width w_c (for the 2D case), and the channel height (3D case) is considered.

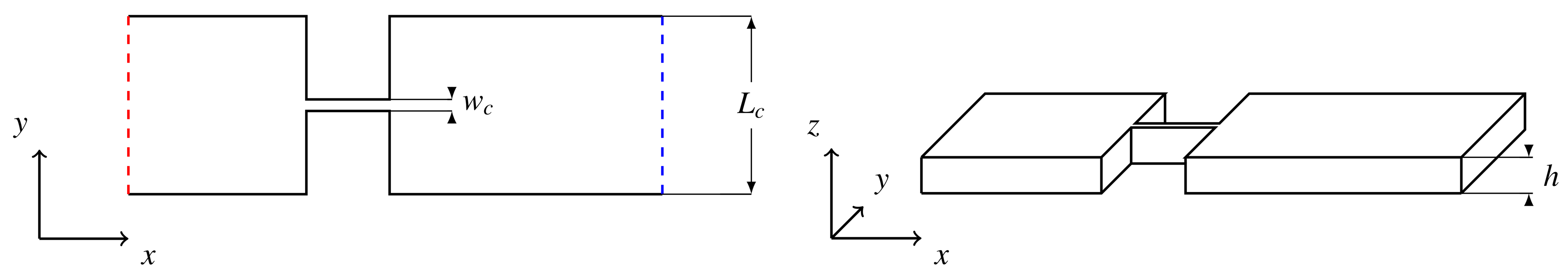

Figure 1: Scheme of the 2D (left) and 3D (right) domains Ω considered in this work. The nomenclature is the same used in [1].

We introduce the following quantities, useful in the characterization of the numerical simulation [1]:

$$\text{aspect ratio } \mathrm{AR} = \frac{h}{w_c}; \quad \text{normalized channel depth } \mathcal{H} = \frac{h}{h + w_c} = \frac{\mathrm{AR}}{\mathrm{AR}+1};$$

$$\text{average horizontal velocity } \langle v_x \rangle = \frac{Q}{w_c h} \text{with } Q \text{ flow rate}; \quad \text{Reynolds number } \mathrm{Re} = \frac{\langle v_x \rangle w_c}{\nu} \frac{2\mathrm{AR}}{\mathrm{AR}+1}.$$

The sampling process is based on a Chebyshev collocation method. After the sampling, a POD procedure is used to compute the velocity basis functions $\{\phi\}_{i=1}^{N}$. If the map adopted for the geometry deformation is used also to map basis functions between the reference and the parametrized domains, it is not necessary to sample the pressure space. This transformation is called *Piola transformation.*

Given a target parameter $\boldsymbol{\mu}^i \in \mathcal{D}$, the approximation problem consists in searching for a couple $(\boldsymbol{u}^N(\boldsymbol{\mu}^i), p^N(\boldsymbol{\mu}^i)) \in \boldsymbol{V}^N \times \boldsymbol{Q}^N$ such that:

$$\left(\frac{\partial \boldsymbol{u}^N(\boldsymbol{\mu}^i)}{\partial t}, \boldsymbol{v}\right)_{\Omega} + \left(\boldsymbol{v}, \boldsymbol{u}^N(\boldsymbol{\mu}^i)\cdot\nabla\boldsymbol{u}^N(\boldsymbol{\mu}^i)\right)_{\Omega} + \nu\left(\boldsymbol{\epsilon}(\boldsymbol{v}), \boldsymbol{\epsilon}(\boldsymbol{u}^N(\boldsymbol{\mu}^i))\right)_{\Omega} - \left(\nabla\cdot\boldsymbol{v}, p^N(\boldsymbol{\mu}^i)\right)_{\Omega} = \langle \boldsymbol{g}, \boldsymbol{v}\rangle_{\Gamma_N} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}^N,$$

$$\left(q, \nabla\cdot\boldsymbol{u}^N(\boldsymbol{\mu}^i)\right)_{\Omega} = 0 \qquad \forall q \in Q^N$$

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T).$$

From the algorithmic viewpoint, we can set up a bifurcation detection method as follows. Suppose that an initial RB solution $\boldsymbol{u}^N(\boldsymbol{\mu}_i)$ is known for a given value of the parameter $\boldsymbol{\mu}_i$, characterized by a sufficiently small Reynolds number Re_i so that the solution is surely unique. Then, for the 3D case, we proceed as follows. Set $k = 1$ and $\mathrm{Re}_k = \mathrm{Re}^i$, then:

1. Keeping fixed the value of the geometric parameter $\mathcal{H}^i$, increase the value of the Reynolds number by a sufficiently small increment $\Delta\mathrm{Re}$ and set $\boldsymbol{\mu}_{k+1} = (\mathrm{Re}_k + \Delta\mathrm{Re}, \mathcal{H}^i)$.
2. Compute the RB solution $\boldsymbol{u}^N(\boldsymbol{\mu}_{k+1})$ for the new parameter value $\boldsymbol{\mu}_{k+1}$.
3. Compute the Galerkin projection of the linearized operator $\mathcal{L}$ on the RB space $\boldsymbol{V}^N$ to form the matrix $\mathtt{L}(\boldsymbol{\mu}^{i+1})$:
$$\mathtt{L}_{kl}(\boldsymbol{\mu}_{k+1}) = (\boldsymbol{\phi_k}, \mathcal{L}(\boldsymbol{u}^N(\boldsymbol{\mu}_{k+1}))[\boldsymbol{\phi}_l]).$$
4. Compute the eigenvalues of $\mathtt{L}(\boldsymbol{\mu}_{k+1})$ and check if there is one eigenvalue that has changed sign with respect to the previous iteration. If not, set $k = k + 1$ and go back to step 1.

We remark that the above algorithm may be unstable in the sense that in a neighborhood of the bifurcation point it may abruptly switch the approximated solution branch, or fail to converge. To make sure that the approximation is always lying on the correct branch a continuation method may be used [2].

Conclusion

The first results are encouraging: the Reduced Basis method is able to detect correctly the steady state bifurcation points for a wide range of aspect ratio and channel width.

The computational costs of the reduced order method amount to roughly 52.7% of the full order method for a 2D case with only one parameter and 0.5% for the 3D case with two parameters. We expect that such results would provide a further justification of the RB method for this problem with more (mainly geometrical) parameters, and consequently this method could be of interest in the applied medicine community.

References

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Results

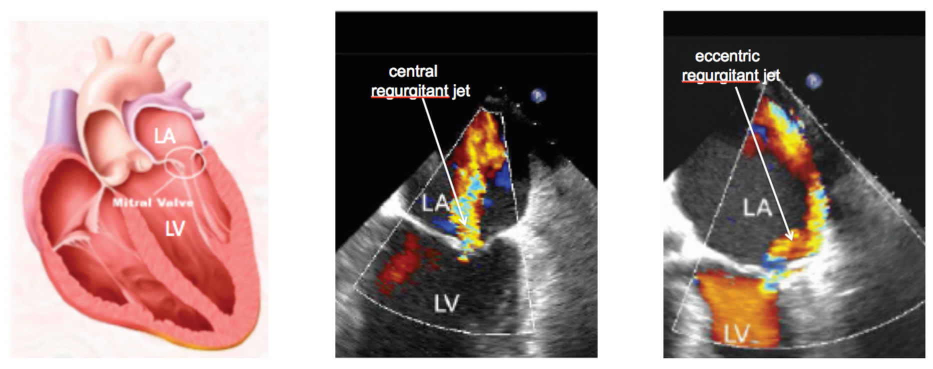

Figure 2: (left) Anatomy of the heart showing the mitral valve; (center) echocardiographic image of central regurgitant jet flowing from the left ventricle (LV) to the left atrium (LA). The colors denote different fluid velocities; (right) echocardiographic image of eccentric regurgitant jet, hugging the walls of the left atrium (LA) known as the Coanda effect.

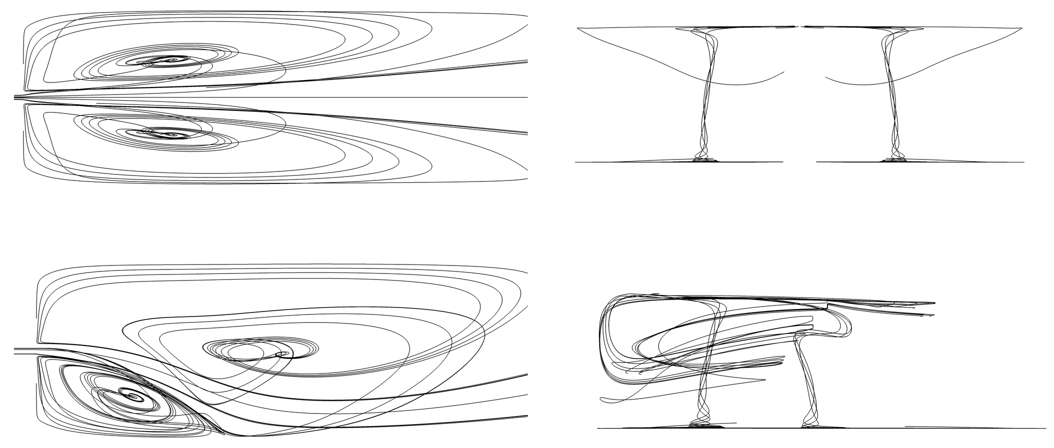

Figure 3: 3D case: streamlines on the xy-plane (left) and yz-plane (right) for$\lambda = 15.4$, $\mathcal{H} = 0.9517$, and $\mathrm{Re} = 76.82$. (a) and (b) unstable solution, (c) and (d) stable solution. The projection on the yz-plane for symmetry reasons shows only half of the geometry.

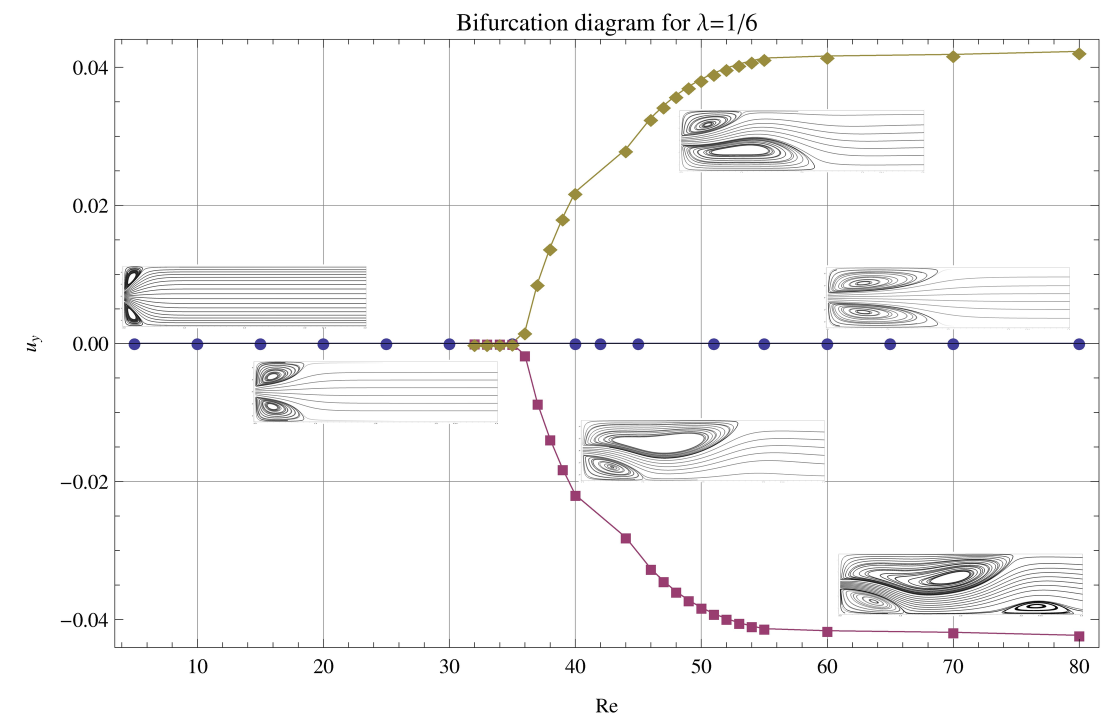

Figure 4: Bifurcation diagram for the 2D case as reconstructed by the Reduced Order Method.

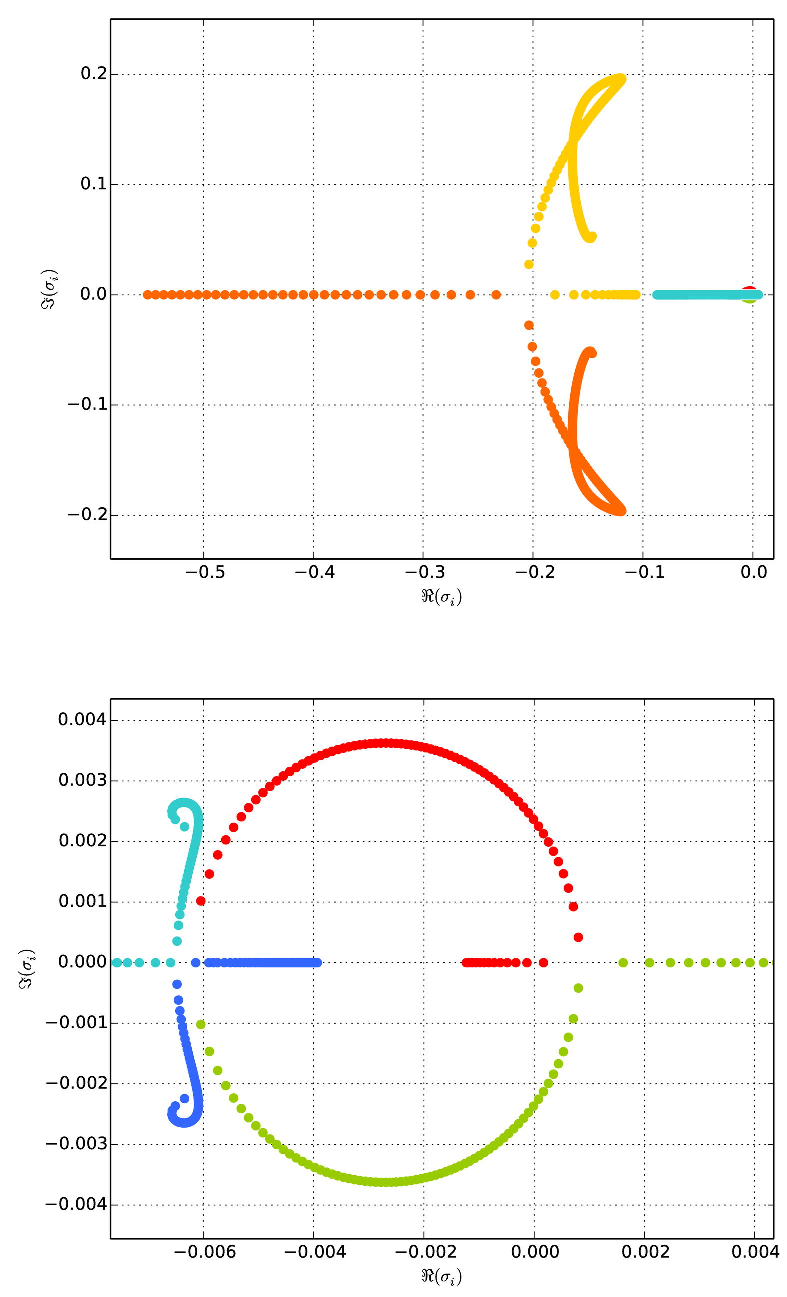

Figure 5: Path of the eigenvalues in the complex plane for the 3D case. The eigenvalue in red crossing the imaginary axis is responsible for the bifurcation point.

Acknowledgements

We acknowledge the funding provided by COST EU-MORNET network and by the INDAM-GNCS group.