# POD-Galerkin Reduced Order Model for the simulation of laminar and turbulent flows around a circular cylinder

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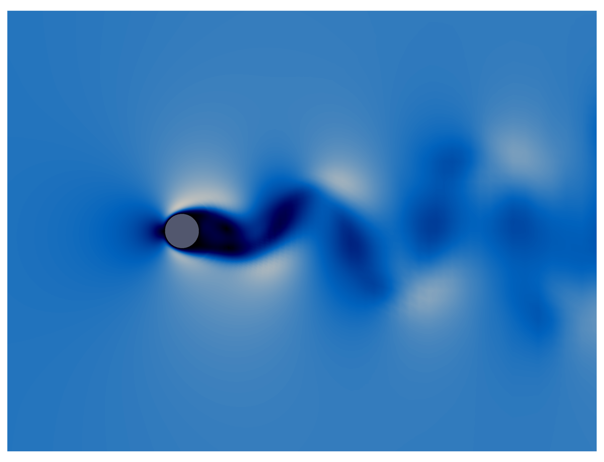
## Introduction - 1

### Motivations

- Vortex-induced vibration is a well known phenomenon in many engineering fields and a reliable ROM is still missing in literature.
- There are interesting non-linear phenomena particularly challenging to be reproduced with a ROM.
- For design purposes several configurations need to be tested and a reliable ROM would dramatically reduce the computational time.



### Methodology-Overview

Development of reduced order methods for the analysis of vortex shedding phenomena around a circular cylinder. The reduction has been performed using the following numerical techiniques:

- High Fidelity simulation of the physical problem trough the finite volume solver OpenFOAM® [4]. (**BOX 2**)
- Collection of the snapshots and construction of the reduced basis space $\mathbb{V}_{rb}$ using a POD [1] approach. (**BOX 3**)
- Projection of the unsteady Navier-Stokes equation onto the reduced basis space $\mathbb{V}_{rb}$ in order to construct the POD-Galerkin dynamical system. [2]. (**BOX 4**)

## High Fidelity problem - 2

### Governing Equations

The physical problem is modelled using the unsteady incompressible Navier-Stokes equations. For low values of the Reynolds number the following system of equations is considered:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nu\nabla^2\boldsymbol{u} + \nabla p = 0 \\ \nabla\cdot\boldsymbol{u} = 0 \end{cases} \quad \text{in } \Omega$$

while for higher values of the Reynolds number a URANS approach with a $k-\omega$ [3] turbulence modelling is used:

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nabla\cdot\left[-p\boldsymbol{I} + (\nu+\nu_t)\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) - \frac{2}{3}k\boldsymbol{I}\right] \\ \nabla\cdot\boldsymbol{u} = 0 \\ \nu_t = f(k,\omega) \\ \text{Transport-Diffusion equation for } k \\ \text{Transport-Diffusion equation for } \omega \end{cases} \quad \text{in } \Omega$$

The space discretization of the equations has beed performed using a finite volume approach

## Construction of $\mathbb{V}_{rb}$ - 3

Assumption: reduced order solution for the velocity, pressure and fluxes fields is given by a linear combination of the bases functions $\boldsymbol{\varphi_i}(\boldsymbol{x})$ (which depends only on space) multiplied by a temporal coefficients $a_i(t)$:

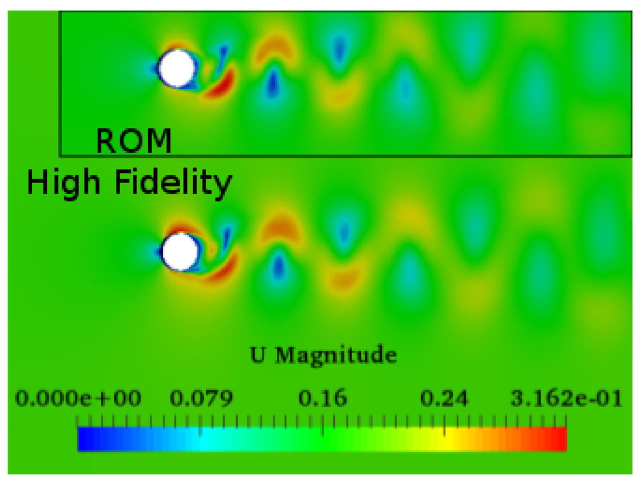
$$\boldsymbol{u}(\boldsymbol{x},t) \approx \boldsymbol{u_r}(\boldsymbol{x},t) = \sum_{i=1}^{N_r} a_i(t)\boldsymbol{\varphi_i}(\boldsymbol{x})$$

The reduced basis space $\mathbb{V}_{rb} = \text{span}(\boldsymbol{\varphi_i})$ is constructed solving the following minimization problem:

$$\mathbb{V}_{rb} = \arg\min \frac{1}{N_s}\sum_{n=1}^{N_s} \|\mathbf{u}_n(\mathbf{x}) - \sum_{i=1}^{N_r}\langle \boldsymbol{u}_n(\boldsymbol{x}), \boldsymbol{\varphi}_i(\boldsymbol{x})\rangle_{L^2}\boldsymbol{\varphi}_i(\boldsymbol{x})\|_{L^2}^2$$

Where $\boldsymbol{u_n}(\boldsymbol{x}, t, u_{in})$ are field solutions (Snapshots) sampled at different inlet velocities and times.

## Results - 5




Comparison between velocity fields at Re=6000 with 20 modes. At the top is the reduced basis case lower at the bottom is the high fidelity one.

The error in $L^2-$norm for the ROM lift coefficient, $Re = 100$ for different number of modes used.

| Number of modes | Relative Error |
|---|---|
| 2 | 1.0014363 |
| 3 | 0.0685620 |
| 5 | 0.0886061 |
| 8 | 0.0660731 |
| 10 | 0.0054784 |
| 13 | 0.0040657 |

## The POD-Galerkin Dynamical system - 4

### Laminar Case

Using a finite volume discretization it appears inside the discretized equation also the mass fluxes over the faces of the cells. It is then made the assumption that the velocity field, the mass flux field and the pressure field can be decomposed as:

$$\begin{pmatrix}\mathbf{u}(\mathbf{x},t)\\ F(\mathbf{x},t)\\ p(\mathbf{x},t)\end{pmatrix} \approx \begin{pmatrix}\mathbf{u}_r(\mathbf{x},t)\\ F_r(\mathbf{x},t)\\ p_r(\mathbf{x},t)\end{pmatrix} = \sum_{i=1}^{N_r} a_i(t)\begin{pmatrix}\boldsymbol{\varphi}_i(\mathbf{x})\\ \psi_i(\mathbf{x})\\ \chi_i(\mathbf{x})\end{pmatrix}$$

The governing equations are then projected onto the spatial bases and the original fields are replaced with the approximated fields. This operation generates the POD-Galerkin system:

$$\frac{da_i(t)}{dt} = \nu\sum_{i=1}^{N_r} B_{ji}a_i(t) - \sum_{k=1}^{N_r}\sum_{i=1}^{N_r} C_{jki}a_k(t)a_i(t) - \sum_{i=1}^{N_r} A_{ji}a_i(t)$$

Where $B$, $C$ and $A$ read:

$$B_{ji} = (\boldsymbol{\varphi}_j, \Delta\boldsymbol{\varphi}_i)_{L^2}\ ;\ C_{jki} = (\boldsymbol{\varphi}_j, \nabla\cdot(\psi_k\boldsymbol{\varphi}_i))_{L^2}\ ;\ A_{ji} = (\boldsymbol{\varphi}_j, \nabla\chi_i)_{L^2}$$

and the dynamical system can be rewritten as:

$$\dot{\boldsymbol{a}} = \nu\mathbf{Ba} - \mathbf{a^TCa} - \mathbf{Aa}$$

### Turbulent Case

For the turbulent the approximated eddy viscosity is written as:

$$\nu_t(\boldsymbol{x},t) \approx \nu_{t,r}(\boldsymbol{x},t) = \sum_{i=1}^{N_r} a_i(t)\boldsymbol{\phi_i}(\boldsymbol{x})$$

that leads to the following dynamical system:

$$\dot{\boldsymbol{a}} = \nu(\boldsymbol{B}+\boldsymbol{BT})\boldsymbol{a} - \boldsymbol{a}^T(\boldsymbol{C} - \boldsymbol{CT_1} - \boldsymbol{CT_2})\boldsymbol{a} - \boldsymbol{Aa} + \tau(\boldsymbol{u_{bc}D} - \boldsymbol{Ea})$$

where it is introduced also the effect of different inlet velocities $\boldsymbol{u_{BC}}$ with a penalization factor $\tau$. The additional terms are still obtained with Galerkin projection onto $\mathbb{V}_{rb}$ and read:

$$D_j = \langle\boldsymbol{\varphi_j}\rangle_{L^2,\partial\Omega}\ ;\ E_{ji} = \langle\boldsymbol{\varphi}_j, \boldsymbol{\varphi}_i\rangle_{L^2,\partial\Omega}\ ;\ BT_{ji} = \langle\boldsymbol{\varphi_j}, \nabla\cdot(\nabla\boldsymbol{\varphi_i^T})\rangle_{L^2}$$
$$CT_{1jki} = \langle\boldsymbol{\varphi_j}, \phi_k\Delta\boldsymbol{\varphi_i}\rangle_{L^2}\ ;\ CT_{2jki} = \langle\boldsymbol{\varphi_j}, \nabla\cdot\phi_k(\nabla\boldsymbol{\varphi_i^T})\rangle_{L^2}$$

## References

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