Stabilization techniques for pressure recovery applied to POD-Galerkin methods for the incompressible Navier-Stokes equations

Giovanni Stabile, Gianluigi Rozza

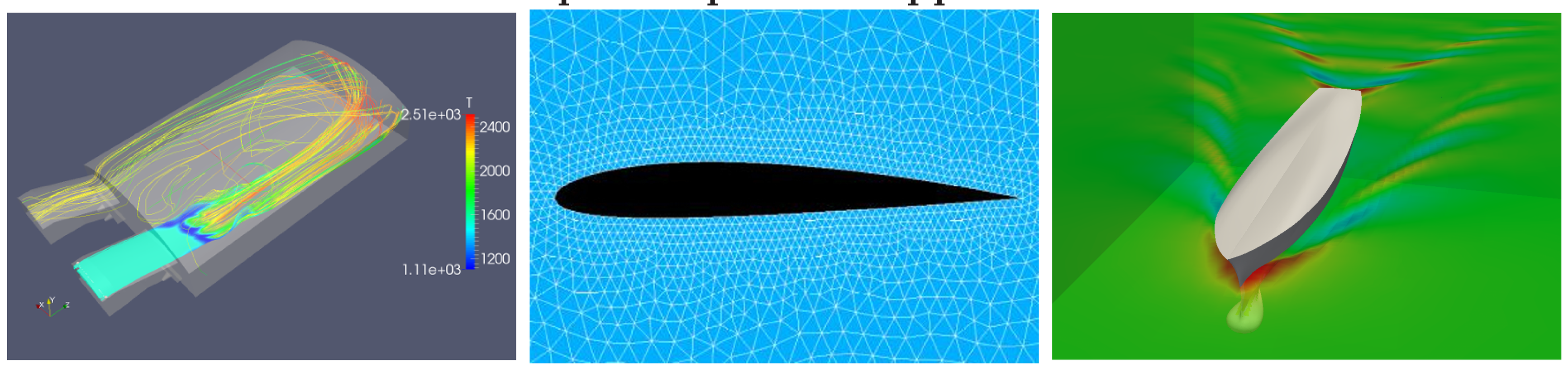
SISSA, International School for Advanced Studies, Mathematics Area, mathLab, Trieste, Italy

Introduction - 1

Motivations

- The numerical resolution of the **incompressible Navier-Stokes** equations is required in many different engineering fields and life sciences (e.g. aeronautical/-naval/civil/mechanical/environmental engineering, hemodynamics.)
- When a large number of different system configurations are in need of being tested (e.g. **uncertainty quantification, optimization**) or a small computational cost is required (e.g. **real-time control**), the numerical resolution of the equations using standard high order discretization techniques (FEM-SEM-FVM-FDM) becomes not feasible. The development of efficient and reliable Reduced Order Models (ROMs) could be a great advantage.
- It is well known that **Galerkin based ROMs** of the incompressible Navier-Stokes equations suffer from **stability issues for what concern the pressure term**.

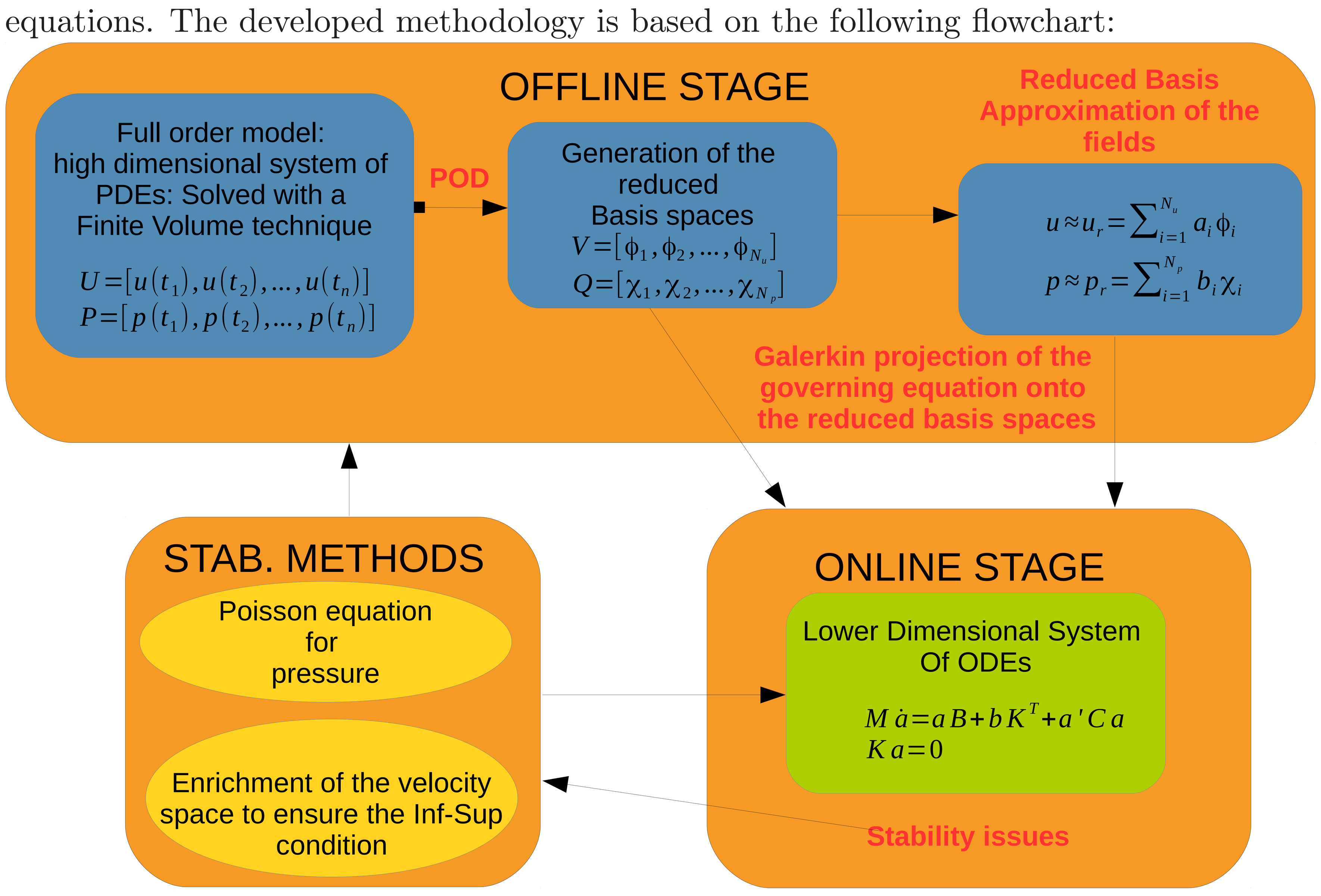
Examples of possible applications



Industrial Engineering — Aeronautical Engineering — Naval Engineering

Methodology-Overview

Development and comparison of different stabilization techniques for the recovery of the pressure term in the contest POD-Galerkin ROMs of the incompressible Navier-Stokes equations. The developed methodology is based on the following flowchart:



- High Fidelity simulation of the physical problem trough the **finite volume** solver **OpenFOAM®**. (**BOX 2**)
- Collection of the **snapshots** and construction of the reduced basis space $\mathbb{V}_{rb}$ using a **POD** [1] approach. (**BOX 3**)
- Projection of the unsteady Navier-Stokes equation onto the reduced basis space $\mathbb{V}_{rb}$ in order to construct the **POD-Galerkin dynamical system**. [2, 3]. (**BOX 4**)
- Development and comparison of different stabilization techniques **for the pressure term.** [4] (**BOX 4**)

Construction of $\mathbb{V}_{rb}$ - 3

The reduced order space V_u and Q_p are constructed using a **SVD** on the snapshots matrices of **velocity** and **pressure**:

$$\mathcal{U}' = \mathcal{W}^u\Sigma^u\mathcal{V}^{uT}, \quad \mathcal{W}^p = [\phi_1, \phi_2, ..., \phi_n], \quad \Sigma^u_{ii} = \lambda^u_i \qquad (2)$$

$$\mathcal{P} = \mathcal{W}^p\Sigma^p\mathcal{V}^{pT}, \quad \mathcal{W}^p = [\chi_1, \chi_2, ..., \chi_n], \quad \Sigma^p_{ii} = \lambda^p_i \qquad (3)$$

We can **truncate** the dimension of the reduced basis space looking at the eigenvalues and we can finally construct the reduced basis spaces for the **Galerkin projection**:

$$\mathbb{V}_{N_u} = \text{span}(\phi_1, \phi_2, ..., \phi_{N_u}) \qquad \mathbb{Q}_{N_p} = \text{span}(\chi_1, \chi_2, ..., \chi_{N_p})$$

High Fidelity problem - 2

Governing Equations

The physical problem is modelled using the **unsteady incompressible Navier-Stokes equations**. The considered system of PDEs are the **unsteady parametrized incompressible Navier Stokes Equations**. The space discretization of the equations has beed performed using a **finite volume** approach.

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot\nu\nabla\boldsymbol{u} = -\nabla p & \text{in } \Omega \\ \nabla\cdot\boldsymbol{u} = \boldsymbol{0} & \text{in } \Omega \\ \boldsymbol{u} = \overline{\boldsymbol{u}}(\mu) & \text{on } \partial\Omega_{,in} \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega_{,0} \\ (\nu\nabla\boldsymbol{u} - p\boldsymbol{I})\boldsymbol{n} = 0 & \text{on } \partial\Omega_{,out} \end{cases} \qquad (1)$$

The POD-Galerkin Dynamical system - 4

The Galerkin Projection

Performing a standard Galerkin projection of the governing equations onto the POD spaces of velocity and pressure and approximating the fields with the POD spaces one obtains:

$$\begin{cases} \dot{\boldsymbol{a}} = \mu\boldsymbol{B}\boldsymbol{a} - \boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} - \cancel{\boldsymbol{K}\boldsymbol{b}} \\ \cancel{\boldsymbol{K}^T\boldsymbol{a}} = 0 \end{cases} \qquad (4)$$

Due to the **divergence-free** property of the velocity modes, the terms with red strikethrough are in most of the cases numerically zero so the resulting system suffer from **stability issues**. Normally only the first equation is solved and only the velocity field is recovered.

The Poisson equation for pressure

One possible way to reconstruct the pressure is to exploit a **Poisson equation for pressure** obtained taking the divergence of the momentum equation and exploiting the **divergence-free** constraint. The momentum equaton is then projected onto the POD velocity space and the Poisson equation for pressure is projected onto the POD pressure space:

$$\begin{cases} (\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla\cdot\nu\nabla\boldsymbol{u} + \nabla p, \boldsymbol{\phi})_{(L^2(\Omega)} = \boldsymbol{0} & \forall\boldsymbol{\phi}\in\mathbb{V}_{N_u} \\ (\Delta p + \nabla\cdot((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}), \boldsymbol{\chi})_{(L^2(\Omega)} = \boldsymbol{0} & \forall\boldsymbol{\chi}\in\mathbb{Q}_{N_p} \end{cases} \qquad (5)$$

$$\begin{cases} \dot{\boldsymbol{a}} = \boldsymbol{B}\boldsymbol{a} - \boldsymbol{a}^T\boldsymbol{C}\boldsymbol{a} - \boldsymbol{K}\boldsymbol{b} \\ \boldsymbol{b} = \boldsymbol{D}^{-1}(\boldsymbol{a}^T\boldsymbol{G}\boldsymbol{a}) \end{cases} \qquad (6)$$

The Supremizer Stabilization

We know that in a Galerkin approach to ensure the solvability and stability of the problem the reduced basis spaces must fulfill the LBB parametrized **inf-sup** condition.

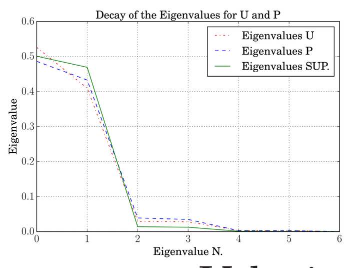
$$\inf_{q\in Q}\sup_{\boldsymbol{v}\in V}\frac{b(q,\boldsymbol{v};\mu)}{\|q\|_Q\|\boldsymbol{v}\|_V} = \beta(\mu) > 0 \qquad (7) \qquad b(q,\boldsymbol{v}) = \int_\Omega q\nabla\cdot\boldsymbol{v}d\boldsymbol{x} \qquad (8)$$

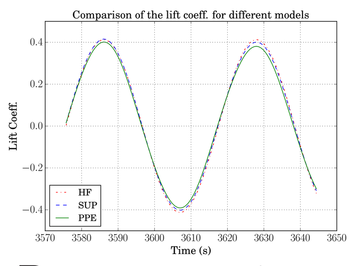
In order to fulfil this condition at reduced order level a **supremizer problem** is solved, and the velocity space is enriched with the additional modes obtained applying the POD onto the supremizer solutions.

$$\begin{cases} \Delta\boldsymbol{s} = -\nabla p & \text{in } \Omega \\ \boldsymbol{s} = \boldsymbol{0} & \text{on } \partial\Omega \end{cases} \qquad (9) \qquad \tilde{V}_u = \text{span}\{\phi_1, ..., \phi_{N_u}\} \oplus \text{span}\{\psi_1, .., \psi_{N_s}\} \qquad (10)$$

Results and Outlooks - 5

The methodology is tested studying the laminar flow (RE= 100) around a **circular cylinder**.

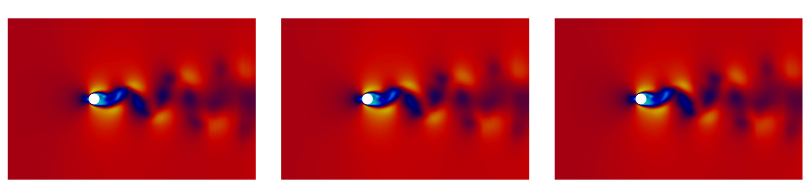




	ε_U	ε_p	comp. t	SU
NO stab ($4\phi, 4\chi$)	NaN	NaN	NaN	NaN
NO stab (4ϕ)	1.25%	-	1.80 s	823
with sup. ($4\phi, 4\chi, 4s$)	1.95%	0.67%	21.73 s	68
with PPE ($4\phi, 4\chi$)	1.25%	0.51%	2.93 s	506

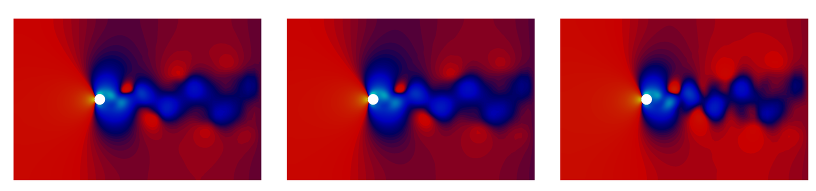
Velocity Reconstruction

HF, $t = 20\Delta t$ PPE, $t = 20\Delta t$ SUP, $t = 20\Delta t$



Pressure Reconstruction

HF, $t = 20\Delta t$ PPE, $t = 20\Delta t$ SUP, $t = 20\Delta t$



Outlook

As future outlooks it would be interesting to test the presented methodologies for **UQ problems**, to investigate the stability for **higher values of the Reynolds number** and to test the stability for **long time integrations**.

References

[1] J. S. Hesthaven, G. Rozza, and B. Stamm, *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. Springer, 2016.

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[3] S. Lorenzi, A. Cammi, L. Luzzi, and G. Rozza, "POD-Galerkin method for finite volume approximation of Navier-Stokes and RANS equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 311, pp. 151 – 179, 2016.

[4] G. Stabile and G. Rozza, "Stabilized Reduced order POD-Galerkin techniques for finite volume approximation of the parametrized Navier–Stokes equations," *submitted*, 2017.

Acknowledgements

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