Optimal transport between fibers: Riemann surfaces

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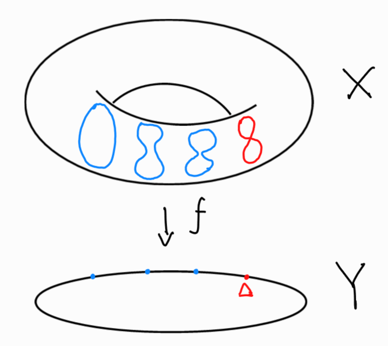
Given an open holomorphic map $f\colon X \to Y$, is there a natural structure on Y, seen as the space of fibers of f within the geometric space X? And what if I tell you that this structure:

- recovers the classical *generalized Weil–Petersson* metric for moduli spaces;
- *and* can potentially enhance the classical homotopy continuation algorithm for the numerical solution of polynomial systems?

Feel free to contact me in person or at my email address: lcecchi@sissa.it



$$\begin{array}{c}\{(p,z) \in P_{n,d} \times \mathbb{CP}^n \mid p(z) = 0\} \\ \big\downarrow f \\ P_{n,d}\end{array}$$

An example of $f\colon X \to Y$, where $P_{n,d}$ is the space of homogeneous polynomials in $z_0,\ldots,z_n$ of degree d

📖 The recipe 📖

- A compact Kähler manifold (X, ω) with a holomorphic, open map f onto a complex manifold Y
- The 2-Wasserstein distance from optimal transport theory. Roughly speaking, given two probability measures, we set their (squared) distance to be the total energy necessary to transport one onto each other in the optimal way:
$$W_2(\nu_0, \nu_1) := \inf_{\pi \in \Pi(\nu_0,\nu_1)} \left\{ \int_{X\times X} d(x,y)^2 \, d\pi(x,y) \right\}^{\frac{1}{2}}$$
- A theorem of Stoll (1966): the fiber integral
$$I(y) := \int_{f^{-1}(y)} \nu_f \, \chi$$
of any form χ is continuous at singular values too, if we keep track of the multiplicity ν_f of f
- Hence, we can embed continuously Y into the space of probability measures on X, by duality:
$$\mu(y)(\varphi) := \frac{1}{\operatorname{vol} f^{-1}(y)} \int_{f^{-1}(y)} \varphi(x)\, \nu_f(x)\, \frac{\omega^k}{k!}(x)$$
- Finally, we consider the *intrinsic* distance d_f on $Y \cong \mu(Y) \subset (\mathcal{P}(X), W_2)$

🔎 The theorems 🔎

Theorem 1. *Let $f\colon (X,\omega) \to Y$ be an open holomorphic map between compact, connected, complex manifolds. Then, the space (Y, d_f) is a complete, connected, geodesic metric space. The metric speed of curves in $Y \setminus \Delta$ can be computed through the Riemannian metric associated to ω_f, where*
$$\omega_f := \frac{1}{\operatorname{vol}(X_y)} f_* \left(\frac{\omega^{k+1}}{(k+1)!} \right)$$
is a Kähler form.

This form is related to the *generalized Weil–Petersson* metrics studied by Fujiki and Schumacher. When X and Y are Riemann surfaces, we are able to establish further properties.

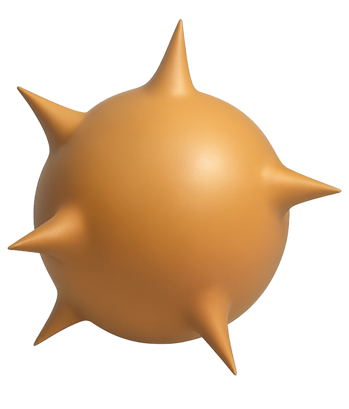
Theorem 2. *Let $f\colon (X,\omega) \to Y$ be an open holomorphic map between compact Riemann surfaces. The metric space (Y, d_f) is compact, and $Y \setminus \Delta$ is a (geodesically) convex subset of Y.*

Intuitively, this structure blows up near Δ at a “favorable” rate, as stated in the theorem: slow enough to maintain compactness and fast enough to ensure convexity.

💡 The idea: Monge—Mather 💡

Conceptually, the convexity property of the distance d_f is due to the Monge–Mather shortening principle. Intuitively, it states the following: along a Wasserstein geodesic, distinct points belonging to the support of μ_t cannot merge at intermediate times. In our setting, given a geodesic y_t with respect to d_f, it means that distinct points in a fiber $f^{-1}(y_t)$ cannot merge at any intermediate time, hence the fiber itself cannot become singular.

Although the classical statement cannot be directly applied since we are considering *constrained* Wasserstein geodesics, a possible proof strategy would be to “average” the metric on X through f, so that “extrinsic” and “constrained” geodesics on X would agree. This procedure, however, creates singularities where the new Riemannian metric on X blows up to infinity at some points. . .



The space (Y, d_f) looks something like this: in particular, you can notice its *cone* singularities at points in the discriminant. Their total angle is given by $2\pi/M$, where M is the maximum local multiplicity of f at points in the singular fiber!

🤔 Follow-up questions 🤔

1) What is the complexity of this variation of the homotopy continuation algorithm?

2) Does a generalized Monge—Mather theorem for extended metrics (that is, they can take ∞ as a value) hold?

3) Is (Y, d_f) the metric completion of $(Y \setminus \Delta, \omega_f)$?

4) Does Theorem 2 hold in general? Preliminary investigations led to interesting singular Riemannian metrics, such as
$$ds^2 = -\frac{\log|z|}{1-|z|^2}\, |dz|^2$$

📑 The application: polynomial systems 📑

The convexity property in Theorem 2 has potential applications in numerical algebraic geometry, where the primary computational method for solving systems of polynomial equations is *homotopy continuation*. Roughly speaking, one first constructs a system of polynomial equations that has a similar structure to the given system and whose solutions are known. Then a homotopy (for example, a linear homotopy) is built between the two systems. Finally, we *follow* the evolution of these known solutions along the homotopy, typically using Newton’s method. The complexity of this algorithm is quantified by the *condition number*, given by the inverse of the distance from the *discriminant* locus.

Our result, applied to a generic *pencil* of systems $\{sP_0 + tP_1\}$, suggests an alternative algorithmic strategy: rather than employing a standard random linear homotopy, one can follow a geodesic in the Wasserstein metric. This path is theoretically guaranteed to remain at a safe distance from $\Delta(f) = \Delta \cap L$, and consequently from Δ itself. However, this approach presents a trade-off: the improved conditioning of the homotopy path is offset by the computational cost of constructing the geodesic. Further investigation is required to assess this balance in practice.

(we thank Simon Telen for useful discussions on this topic!)