

Advanced Mathematical Physics I. Problems

8 October 2026

Lecture 2

1. Consider the equation

$$xu_t + tu_x = 0. \quad (2.1)$$

- (a) Find the characteristic curves and sketch them in the (x, t) plane.
(b) Find the solution with the initial condition

$$u(x, 0) = e^{-x^2}. \quad (2.2)$$

- (c) In which region of the (x, t) plane is the solution uniquely determined by this initial condition?

2. Consider the equation

$$u_t + uu_x = 0. \quad (2.3)$$

- (a) Find a formula for the solution with the initial condition $u(x, 0) = x$.
(b) Let $u(x, 0) = \cos \pi x$. Show that a smooth solution satisfies

$$u = \cos(\pi(x - ut)). \quad (2.4)$$

For which $t > 0$ does the solution exist? At which points x does it break down first?

Lecture 3

1. Let A and B be linear differential operators in one variable x with symbols $a(x, \xi)$ and $b(x, \xi)$.

- (a) Prove that the symbol of $A \circ B$ is

$$\sum_{j \geq 0} \frac{(-i)^j}{j!} \frac{\partial^j a}{\partial \xi^j} \frac{\partial^j b}{\partial x^j}. \quad (3.1)$$

- (b) Let $\text{ord } A = k$ and $\text{ord } B = l$, with principal symbols a_k and b_l . Prove that the commutator $[A, B] = A \circ B - B \circ A$ has order at most $k + l - 1$, and that the terms of degree $k + l - 1$ in ξ of its symbol are

$$-i \left(\frac{\partial a_k}{\partial \xi} \frac{\partial b_l}{\partial x} - \frac{\partial a_k}{\partial x} \frac{\partial b_l}{\partial \xi} \right). \quad (3.2)$$

2. Consider the equations

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0 \quad (3.3)$$

with real constant coefficients. Show that the only ones that are unchanged under all rotations of the (x, y) plane have the form

$$a(u_{xx} + u_{yy}) + bu = 0. \quad (3.4)$$

Lecture 4

1. Solve

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0, \quad u(x, 0) = x^2, \quad u_t(x, 0) = e^x. \quad (4.1)$$

Hint: factor the operator as we did for the wave equation.

2. Let $u(x, t)$ satisfy the wave equation $u_{tt} = u_{xx}$. Prove the identity

$$u(x + h, t + k) + u(x - h, t - k) = u(x + k, t + h) + u(x - k, t - h) \quad (4.2)$$

for all x, t, h, k . Sketch the quadrilateral whose vertices are the arguments in the identity.