Itroduction to cluster algebras and varieties

Lecture 1

(Preliminary) Examples

Somos sequence

- **Def** Somos-4 sequence $\quad z_1 = z_2 = z_3 = z_4 = 1$

$$z_{k+4} z_k = z_{k+3} z_{k+1} + z_{k+2}^2$$

First terms are $1, 1, 1, 1, 2, 3, 7, 23, 59, 314$

Theorem All z_k are integer numbers

- **Def** Somos-5 sequence $\quad z_1 = \ldots = z_5 = 1$

$$z_{k+5} z_k = z_{k+4} z_{k+1} + z_{k+3} z_{k+2}$$

$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217,$

Theorem All z_k are integer numbers

- More simple example $z_1 = z_2 = 1$

$$z_{k+2} z_k = z_{k+1}^2 + 1$$

— $1, 1, 2, 5, 13, 34, 89, 233$

$0, 1, 3, 8, 21, 55, 144$

(half of) Fibonacci sequence

∃ combinatorial proof z_k = # perfect matchings

∃ similar proof for Somos sequence

- $$z_{k+4} z_k = z_{k+3} z_{k+1} + z_{k+2}^2$$

Change initial values: $3, 1, 1, 1, \frac{2}{3}, \frac{5}{3}, \frac{19}{9}, \frac{113}{27}, \frac{463}{27}, \frac{\quad}{243}$

Laurent property: $\left(a, b, c, d, \frac{c^2+bd}{a}, \frac{c^2+bcd+ad^2}{ab}, \frac{\quad}{a^2bc}, \frac{\quad}{a^3b^2cd}, \frac{\quad}{a^3b^3c^2d}, \ldots\right)$

- <u>Def</u> Laurent polynom. on $x_1, \ldots, x_n$ —

$$P = \sum_{i_1 \ldots i_n \in \mathbb{Z}} a_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

$\mathbb{C}[x_1^{\pm 1}, \ldots, x^{\pm 1}]$ - space of Laurent polynomials

- <u>Theorem</u> z_n are Laurent polynomials on z_1, z_2, z_3, z_4 with integer coefficients.

Configuration spaces

- $A_n = \left\{ \begin{pmatrix} x_1 & & x_n \\ y_1 & & y_n \end{pmatrix} \right\} / SL_2$

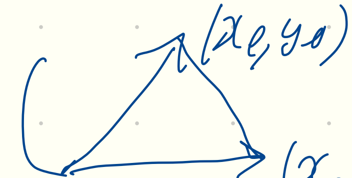
  $\{$ maps from $\{1, .., n\} \to \mathbb{C}^2 \} / SL_2$

  $\mathbb{C}[A_n] = \mathbb{C}[x_1, x_2, \dots, y_{n-1}, y_n]^{SL_2}$

Remark Sometimes we will change the base field (e.g. $\mathbb{R}$)

- **Th** $\mathbb{C}[A_n]$ generated by $p_{ab} = \begin{vmatrix} x_a & x_b \\ y_a & y_b \end{vmatrix}$, $a<b$ as an algebra

Pf p_{ab} – is SL_2 invariant

(geometrically p_{ab} = Area(

))

$GL_n \curvearrowright \mathbb{C}[A_n] \Rightarrow \mathbb{C}[A_n] = \bigoplus V_\lambda$ - reps of GL_n

Any V_λ is generated by h.w. F_λ - invariant under the $U = \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\} \subset GL_n$

Using $U \times SL_2$ any (on open subset) $\begin{pmatrix} x_1 & & x_n \\ y_1 & & y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_1 & x_2 & 0 & & 0 \\ y_1 & y_2 & 0 & \dots & 0 \end{pmatrix}$

Hence F_λ is determined by the values on the set $\left\{\begin{pmatrix} x_1 & x_2 & 0 & & 0 \\ y_1 & y_2 & 0 & \dots & 0 \end{pmatrix}\right\}$

using $SL_2 \rightsquigarrow \left\{\begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & z & 0 & \dots & 0 \end{pmatrix}\right\}$

Hence $F_\lambda = z^d$ for some d, $\Rightarrow F_\lambda = P_{12}^d$

Hence V_λ - spanned by $P_{a_1 b_1} \cdot \ldots \cdot P_{a_d b_d}$ $\square$

- $Gr(2,n) = \left\{ \begin{pmatrix} x_1 & & x_n \\ y_1 & & y_n \end{pmatrix} \middle| \ \text{rank} = 2 \right\} / GL_2$

  $\parallel$

  $\{ U \subset \mathbb{C}^n \mid \dim U = 2 \}$ $\quad U = \langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle$

- Plucker embedding $Gr(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$

  $$U \mapsto [(x_1 \dots, x_n) \wedge (y_1 \dots, y_n)]$$
  $$= \sum P_{ab}\, e_a \wedge e_b$$

Hence $\hat{Gr}(2,n) \hookrightarrow \mathbb{C}^{\binom{n}{2}}$

- <u>Claim</u> $\mathbb{C}[A_n] = \mathbb{C}[\hat{Gr}(2,n)] =$

  $=$ homogenious functions on $Gr(2,n)$

- Dimension

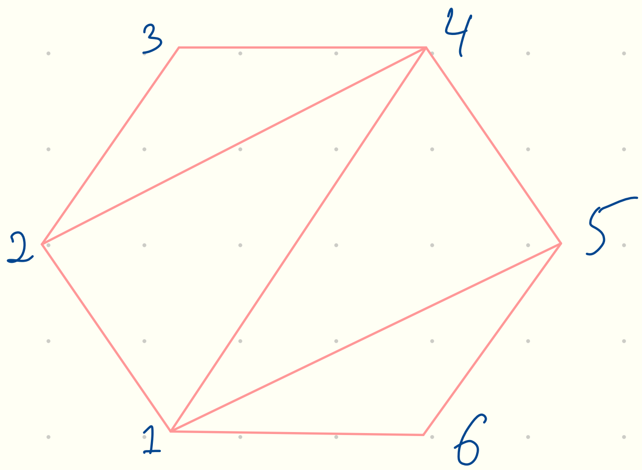
  $\dim A_n = 2n-3$

  $\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & * & \dots & * \\ 0 & P_{12} & * & \dots & * \end{pmatrix}$

  $\dim Gr(2,n) = 2(n-2)$

  $\# P_{ab} = \binom{n}{2} \Rightarrow$ many relations

- Triangulation



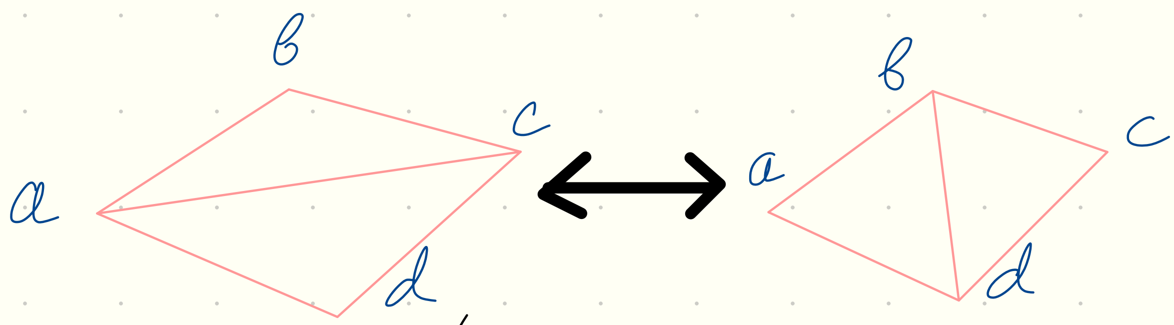
$n = 6$

Edge a — b $\rightsquigarrow P_{ab}$

$\rightsquigarrow \{P_{12}, P_{23}, P_{34}, P_{45}, P_{56}, P_{61}, P_{24}, P_{14}, P_{15}\}$

$2n-3 = 9$ elements

- Flip



$a<b<c<d$

<u>Lemma</u> Any two triangulations are related by sequence of flips

<u>Lemma</u> $P_{ac}P_{bd} = P_{ab}P_{cd} + P_{ad}P_{bc}$ Plücker relation

Remark "old" · "new" = binom

Coroll P_{ab} for given triangulation are local coordinates on A_n

Hence P_{ab} algebraically independent

Pf Use Lemma, Lemma + any diagonal belongs to some triangulation $\square$

Th $\mathbb{C}[A_n] = \mathbb{C}[P_{ab}]/I$, where I is generated by Plücker relations

Def Cluster monomial $\prod P_{a_i b_i}$ where $\exists$ triang. T s.t $\forall$ $P_{a_i b_i}$ belong to T.

Ex $P_{12}^2 P_{13} P_{14}$ – cluster $\qquad P_{13} P_{24}$ – not cluster

__Def__ T — s/s tableu $\begin{pmatrix} a_1 & \dots & a_k \\ b_1 & \dots & b_k \end{pmatrix}$ $2\times k$

if $a_1 \le a_2 \le \dots \le a_k$ $\quad a_i < b_i$

$b_1 \le b_2 \le \dots \le b_k$

__Notation__ $T \rightsquigarrow P_T = P_{a_1 b_1} \cdot P_{a_2 b_2} \cdot \ldots \cdot P_{a_k b_k}$

__Problem__ (a) $\mathbb{C}[A_n]$ generated by P_T, T — s/s tableu

(b) $\mathbb{C}[A_n]$ is generated by cluster monomials

__Fact__ Both sets above form a basis in $\mathbb{C}[A_n]$

__Remark__ $\mathbb{C}[A_n] = \mathbb{C}[GL_n]^P = \bigoplus V_{k\omega_2}$ $\quad P = \left\{ \begin{array}{cc} 2 & n-2 \\ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right. & \left.\right) \end{array} \right\}$

$= \bigoplus V_{2\underbrace{\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; & \; & \; & \; \\ \hline \end{array}}_{k}}$

Positive Grassmanian

<u>Def</u> $U \subset \mathbb{R}^2$ is positive if all $P_{ab}(U)$ positive (or negative)

<u>Naively</u>: one has to check $\binom{n}{2}$ P-s

<u>Th</u> If for given triang. T all $P_{ae} > 0$, for $ae \in T$ then any $P_{ae} > 0$

<u>Pf</u> Follows from Plucker $\square$

<u>Corollary</u> For given triangulation T, if for $\forall$ $ab \in T$ $P_{ab} > 0$ then U is positive.

Efficient test $\rightsquigarrow$ $2n-3$ checks

Configuration of points in $\mathbb{P}^1$

- $\mathrm{Conf}_n = \{n \text{ distinct points in } \mathbb{P}^1\} / PGL_2$

  $\{(u_1, \dots, u_n)\}$ $\qquad \{1 \dots n\} \to \mathbb{P}^1$

$\mathrm{Conf}_3 = pt$

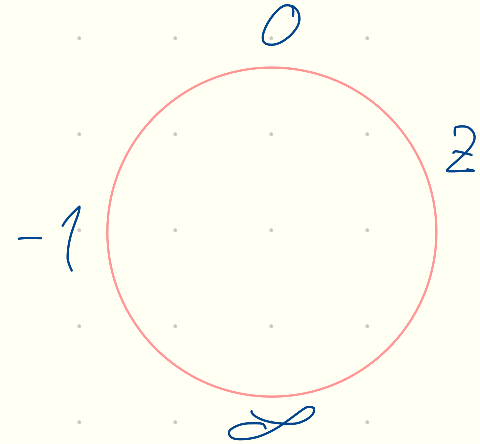
$\mathrm{Conf}_4 = \mathbb{C} \setminus \{0, -1, \infty\}$

$\dim \mathrm{Conf}_n = n-3$

- Coordinates ~ cross ratio

<u>Def</u> $[u_1, u_2, u_3, u_4] = -\left(\frac{(u_1-u_2)\quad(u_3-u_4)}{(u_3-u_2)\quad(u_1-u_4)}\right)$

Equivalently ~ PGL_2 invariant function s.t. $[\infty, -1, 0, z] = z$



- **Remark** $[u_1, u_2, u_3, u_4] = \dfrac{P_{12}\, P_{34}}{P_{23}\, P_{14}}$

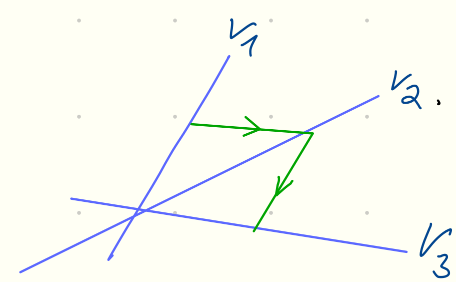
$$\begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ \end{array}$$
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & z \end{pmatrix}$$

- **Problem** ⓐ Composition $u_1 \xrightarrow{u_2} u_3 \xrightarrow{u_4} u_1$ is multiplication by $-[u_1, u_2, u_3, u_4]$

Here u_i - line corresp u_i

Transformation $v_1 \xrightarrow{v_2} v_3$ is in figure

(1 line parallel to v_3, second to v_1)



⑥ Let $U_{123} = \mathrm{Ker}(U_1 \oplus U_2 \oplus U_3) \to \mathbb{C}^2$

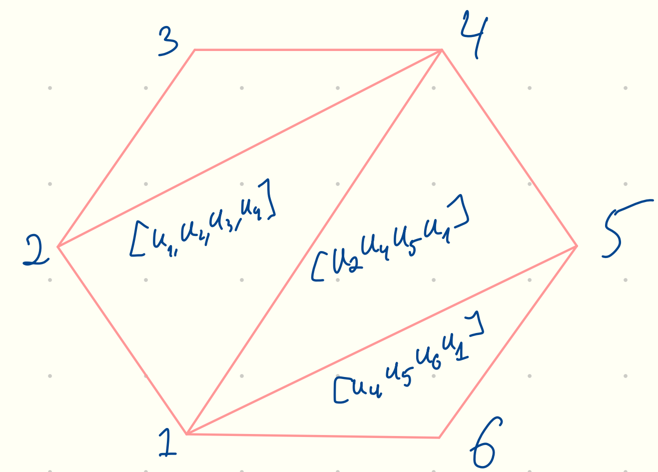
There are natural isomorphisms (in general position) $U_{123} \to U_1, U_2, U_3$

$$\begin{tikzcd}
 & & U_2 & & \\
U_1 & U_{124} \arrow[l, "\sim"'] \arrow[ur, "\sim"] \arrow[dr, "\sim"'] & \arrow[phantom, "\circlearrowleft"] & U_{234} \arrow[ul, "\sim"'] \arrow[r, "\sim"] \arrow[dl, "\sim"] & U_3 \\
 & & U_4 & &
\end{tikzcd}$$

counter-clockwise monodromy is
$-[u_1, u_2, u_3, u_4]$

- Triangulation: coordinates on Conf_n

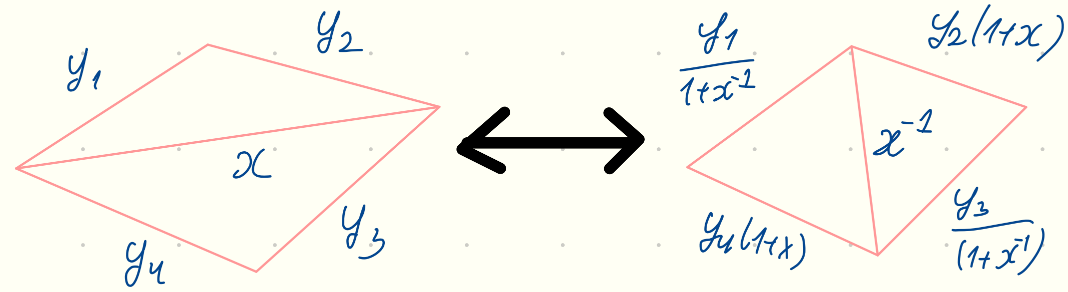
$$(\mathbb{C}^*)^{n-3} \longrightarrow \mathrm{Conf}_n$$



Remark If we assume that points $u_1, \dots, u_n$ are ordered on $\mathbb{RP}^1$, then we get coordinates

- Coordinate transformation

Problem After the flip coordinates transform as:



- Lessons

  - two sets of coordinates A, X coordinates
  - Positivity
  - Laurent phenomenon in A coordinates