Itroduction to cluster algebras and varieties

Lecture 7

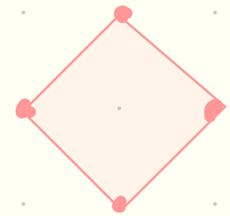
Upper cluster algebras

Newton Polytope

- **Def** $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $P = \sum_{m \in \mathbb{Z}^n} c_m x_1^{m_1} \cdots x_n^{m_n}$

$N(P)$ = convex hull of $m \in \mathbb{Z}^n \subset \mathbb{R}^n$ s.t $c_m \neq 0$

Example $P = x_1 + x_1^{-1} + x_2 + c x_2^{-1}$



- **Prop** $P, Q \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

(i) $N(PQ)$ = Minkowski sum $N(P)$ and $N(Q)$

$N(P) + N(Q) = \{ x + y \in \mathbb{R}^n, \ x \in N(P), \ y \in N(Q) \}$

(ii) $N(P+Q) \subset$ convex hull $N(P) \cup N(Q)$

if P, Q substraction free, then $N(P+Q) = \text{c.h. } N(P) \cup N(Q)$

Cluster algebra

- Fix seed $\Sigma = (B, \bar{A})$ $\quad \mathbb{Z}[A_1^{\pm 1}, \dots, A_m^{\pm 1}]$

Def cluster algebra $\mathcal{A}(\Sigma)$ — subalgebra in $\mathbb{Z}[A_1^{\pm 1}, \dots, A_m^{\pm 1}]$ generated by cluster variables in all seeds Σ' connected to Σ by mutations

Def upper cluster algebra $\bar{\mathcal{A}}(\Sigma) = \bigcap_{\Sigma'} \mathbb{Z}[A_1'^{\pm 1}, \dots, A_n'^{\pm 1}, A_{n+1}, \dots, A_m]$
$\Sigma' = (B', A')$ — seeds connected to Σ

- Remark ⓐ $\mathcal{A}(\Sigma) \subset \bar{\mathcal{A}}(\Sigma)$
ⓑ $\bar{\mathcal{A}}(\Sigma)$ - algebra of regular functions on $\mathcal{A}$ - cluster variety

- $\mathcal{U}(S) = \mathbb{Z}[A_1^{\pm 1}, \dots, A_n^{\pm 1}, \dots, A_m] \cap \bigcap_{j=1}^{n} \mathbb{Z}[A_1^{\pm 1}, \dots, A_{j-1}^{\pm 1}, A_j'^{\pm 1}, \dots, A_n^{\pm 1}, \dots, A_m]$

$S \xrightarrow{j} S_j \qquad A(S_j) = A_1, \dots, A_{j-1}, A_j', \dots A_m$

upper bound

Clearly $\bar{A}(S) \subset \mathcal{U}(S)$. Indeed $\bar{A}(S) = \cap\, \mathcal{U}(S')$

- $S = (\bar{B}, \bar{A}) \qquad A_j' A_j = P_j = \sum_{b_{ij} > 0} A_i^{b_{ij}} + \sum_{b_{ij} < 0} A^{b_{ji}}$

Seed is coprime if all P_j are coprime

- <u>Theorem</u> If $S - S'$ and seeds S, S' are coprime then $\mathcal{U}(S) \simeq \mathcal{U}(S')$

- <u>Lemma</u> Seed is not coprime $\Leftrightarrow$ $\exists i_1, i_2$; $\exists$ a_1, a_2 – odd integers $a_1 b_{i_1 j} = a_2 b_{i_2 j}$ $\forall j$

<u>Pf</u> $\Leftarrow$ w.l.o.g. a_1, a_2 are coprime.
Hence $\forall j$ $a_1^{-1} b_{i_2, j}$, $a_2 \mid b_{i_1, j}$, i.e. $a_1^{-1} b_{i_2, j} = a_2^{-1} b_{i_1, j} \in \mathbb{Z}$
Therefore $P_{i_1} = M_1^{a_2'} + M_2^{a_2}$ and $P_{i_2} = M_1^{a_1} + M_2^{a_2}$ $\Rightarrow$ <u>not coprime</u>

$\Rightarrow$ Newton polygons $N(P_{i_1})$, $N(P_{i_2})$ are segments.
Due to not coprimenes these segments are parallel.
Hence $P_{i_1} = L^{a_1} + M^{a_1}$ $P_{i_2} = L^{a_2} + M^{a_2}$. Note that $t^{a_1}+1$ and $t^{a_2}+1$ are not coprime iff $a_1/a_2 = a_1'/a_2'$ where a_1', a_2'-odd $\square$

- <u>Corollary</u> If B has full rank then seed is coprime.

- <u>Lemma</u> $\operatorname{rk} B' = \operatorname{rk} B$

Proof Consider B as Gram matrix of the form on $\mathbb{Z}^m$, namely $\langle e_i, e_j \rangle = b_{ij}$,

$B \xrightarrow{\mu_k} B'$ equivalent to $e_j' = \begin{cases} -e_k & j=k \\ e_j + \max(b_{ki}, 0) e_k & j \neq k \end{cases}$

Problem Check that $b_{ij}' = \langle e_i', e_j' \rangle$ agrees with mutation of matrix B. $\square$

- **Corollary.** If B has full rank then $\bar{A}(S) = \mathcal{U}(S)$
  In particular, for $\forall S'$ connected to S by mutations $\forall i$ $A_i(S') \in \bar{A}(S') = \bar{A}(S') = \mathcal{U}(S)$ hence
  $A_i(S') \in \mathbb{Z}[A_1^{\pm 1}, \dots, A_n^{\pm 1}, \dots, A_m] \iff$ Laurent phenomenon

- **Remark** If B does not have full rank we can add more frozen vertices $\Rightarrow$ full rank $\Rightarrow$ Laurent phenomenon for new $\tilde{B}$ $\Rightarrow$ Laurent phenomenon for initiae B.

• Proof of the Thm. Idea: everything reduces to vars. say A_1, A_2

Step 1 $\mathcal{U}(t) = \bigcap_{j=1}^{n} \mathbb{Z}[A_1^{\pm1}, \dots, A_{j-1}^{\pm1}, A_j, A_j', A_{j+1}^{\pm1}, \dots, A_n]$

Pf Let $R = \mathbb{Z}[A_2^{\pm}, \dots, A_n^{\pm1}, A_{n+1}, \dots, A_n]$

Enough to show

Problem $R[A_1, A_1^{-1}] \cap R[A_1', A_1'^{-1}] = R[A_1, A_1']$

Step2 If p_1 coprime with p_j $\forall j = 2, \dots, n$

$$\mathcal{U}(t) = \bigcap_{j=2}^{n} \mathbb{Z}[A_1, A_1', A_2^{\pm1}, \dots, A_{j-1}^{\pm1}, A_j, A_j', A_{j+1}^{\pm1}, \dots, A_n^{\pm1}, \dots, A_m]$$

Pf By previous step it is enough to show $R[A_1^{\pm1}, A_2, A_2'] \cap R[A_1, A_1', A_2^{\pm1}] = R[A_1, A_1', A_2, A_2']$

where $R = \mathbb{Z}[A_3^{\pm1}, \dots, A_n^{\pm1}, A_{n+1}, \dots, A_m]$

Embedding "$\supset$" is obvious

Case 1 $b_{ij}=0$. $P_i = A_i A_i' \in R$, $P_j = A_j A_j' \in R$

Let $y \in \sum_{m_1, m_2 \in \mathbb{Z}} c_{m_1, m_2} A_1^{m_1} A_2^{m_2} \in R[A_1^{\pm 1}, A_2, A_2'] \cap R[A_1, A_1', A_2^{\pm 1}]$

for $m_1 < 0$ c_{m_1, m_2} is divisible by $P_1^{-m_1}$

for $m_2 < 0$ c_{m_1, m_2} is divisible by $P_2^{-m_2}$

Since P_1 and P_2 are coprime thic conditions are independent $\Rightarrow$ $y \in R[A_1, A_1', A_2, A_2']$

Case 2 $|b_{12}| = b \neq 0$ Let $A_1 A_1' = P_1 = q_2 A_2^b + r_2$, $A_2 A_2' = P_2 = q_1 A_1^b + r_1$.

Substep 2.1 $R[A_1, A_2^{\pm 1}] \cap R[A_1^{\pm 1}, A_2, A_2'] = R[A_1, A_2, A_2']$

"$\supset$" obvious

Let $y \in R[A_1^{\pm 1}, A_2, A_2']$, $y = \sum_{m \in \mathbb{Z}} A_1^m (c_m + c_m'(A_2) + c_m''(A_2'))$

where $c_m', c_m'' \in R[x]$ polynomiaes without constant term

Let y also belongs to $R[A_1, A_2^{\pm 1}]$

Substitute $A_2' = A_2^{-1}(q_1 A_1^b + r_1)$, we see that $c_m, c_m', c_m'' = 0$ for $m < 0$.

Substep 2.2 $\quad R[A_1, A_1', A_2^{\pm 1}] = R[A_1, A_1', A_2, A_2'] + R[A_1, A_2^{\pm 1}]$

"$\supset$" obvious

"$\subset$" Enough to show $A_1'^{N} A_2^{-M} \in$ r.h.s $\forall N, M > 0$

Let $p = -q_1/r_1$ - monomial. Then $A_2^{-1} - p A_1^{b} A_2^{-1} = r_1^{-1} A_2'$

Hence $\quad A_2^{-1} \equiv p A_1^{b} A_2^{-1} \equiv p^2 A_1^{2b} A_2^{-1} \equiv \dots \equiv p^N A_1^{Nb} A_2^{-1} \mod R[A_1, A_2']$

Raise into M-th power

$$A_2^{-M} \in R[A_1, A_2'] + A_1^{N} R[A_1, A_2^{-1}]$$

Hence $\quad A_1'^{N} A_2^{-M} \in R[A_1, A_1', A_2'] + R[A_1, A_2^{\pm 1}]$ $\quad \square$

Substep 2.3 $\quad R[A_1, A_1', A_2^{\pm 1}] \cap R[A_1^{\pm 1}, A_2, A_2'] =$

$$= (R[A_1, A_1', A_2, A_2'] + R[A_1, A_2^{\pm 1}]) \cap R[A_1^{\pm 1}, A_2, A_2']$$

$$= R[A_1, A_1', A_2, A_2'] + R[A_1, A_2^{\pm 1}] \cap R[A_1^{\pm 1}, A_2, A_2']$$

$$= R[A_1, A_1', A_2, A_2'] + R[A_1, A_2, A_2'] = R[A_1, A_1', A_2, A_2']$$

<u>Step 3</u> Let $S \xrightarrow{1} S'$. Let $A_2'' = \mu_2 \mu_1 (A_2)$. Then

$$\mathbb{Z}[A_1, A_1', A_2, A_2', A_3^{\pm 1}, \dots, A_n^{\pm 1}, \dots, A_m] = \mathbb{Z}[A_1, A_1', A_2, A_2'', A_3^{\pm 1}, \dots, A_n^{\pm 1}, \dots, A_m]$$

<u>Pf</u> Direct computation $\square$ $\square$

Lower Bound

- <u>Def</u> $L(S) = \mathbb{Z}[A_1, A_1', A_2, A_2', \dots]$

<u>Remark</u> $L(S) \subset A(S)$

- <u>Def</u> Seed is <u>acyclic</u> if no oriented cycles in Q.

One can renumber vertices in acyclic seed s.t
$b_{ij} > 0 \Rightarrow i > j$

- <u>Theorem</u> If S is acyclic then $L(S) = \mathcal{U}(S)$

<u>Example</u> A_n quiver — $\widehat{Gr}(2, n+3)$

- <u>Corollary</u> Acyclic + full rank $\Rightarrow$ $L(S) = A(S) = \mathcal{U}(S)$

<u>Remark</u> $n=2$ $\Rightarrow$ acyclic

Double Bruhat cells

- <u>Theorem</u> $\mathbb{C}[G^{u,v}] \simeq \overline{\mathcal{A}}(S)$

Equivalently $G^{u,v}$ — $\mathcal{A}$ cluster variety (up to codim 2)

<u>Pf</u> <u>Step 1</u> S – full rank.

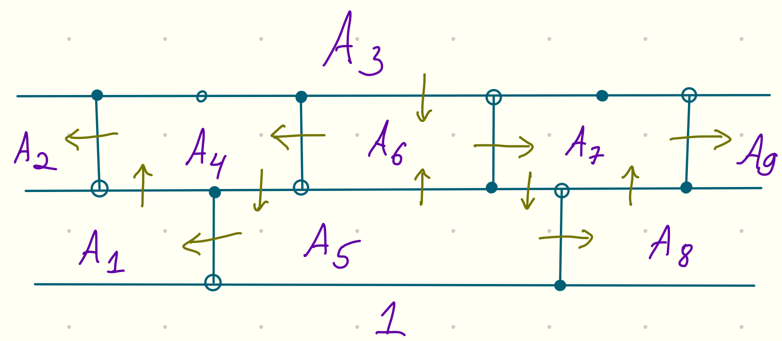
Let N – # of bounded faces = # unfrozen variables

$A_{(1)}, \ldots, A_{(N)}$ – unfrozen variables labelled by their left bourders.

$A_{\{1\}}, \ldots, A_{\{N\}}$ – left neighbours of $A_{(1)}, \ldots, A_{(N)}$

Example

$$A_{(1)} = A_4 \qquad A_{\{1\}} = A_2$$
$$A_{(2)} = A_5 \qquad A_{\{2\}} = A_1$$
$$A_{(3)} = A_6 \qquad A_{\{3\}} = A_4$$
$$A_{(4)} = A_7 \qquad A_{\{4\}} = A_6$$



$$B \quad \begin{array}{c|ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 4 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 \\ 6 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \end{array}$$

$$B^{\{1\}\{2\}\{3\}\{4\}}_{(1)(2)(3)(4)} = \begin{array}{c} \\ (1)=4 \\ (2)=5 \\ (3)=6 \\ (4)=7 \end{array} \begin{array}{c} \begin{array}{cccc} \{1\}=2 & \{2\}=1 & \{3\}=4 & \{4\}=6 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Claim $\quad b_{(i),\{i\}} = \pm 1 \qquad b_{(i),\{j\}} = 0 \quad \text{if} \quad i > j$

Hence $\quad B^{\{1\},\dots,\{N\}}_{(1),\dots,(N)} = \begin{pmatrix} \pm 1 & & * \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix} \Rightarrow \operatorname{rk} B = N \Rightarrow$

$$\mathcal{A}(S') = \mathcal{U}(S')$$

by Theorem

Step 2 Recall, we have a map $\varphi : \bar{A} \to \mathbb{C}(G^{u,v})$

$A_i \mapsto \Delta_i = \Delta$ zig-zags below

zig-zags below

Lemma (i) (a) $\varphi(A_i)$ - regular functions on $G^{u,v}$

(b) Map $G^{u,v} \to \mathbb{C}^{n+\ell(u)+\ell(v)}$ $\quad g \mapsto (\Delta_1(g), \Delta_2(g), \dots, \Delta_{n+\ell(u)+\ell(v)}(g))$

is biregular isomorphism $U = \{g \mid \Delta_i(g) \neq 0 \ \forall i\} \xrightarrow{\sim} \mathbb{C}_{\neq 0}^{n+\ell(u)+\ell(v)}$

Lemma (ii) Let $S' \xrightarrow{j} S_j$, $j \in (1), \dots, (N)$. Let $\Delta_j' = \varphi(A_j')$

(a) Δ_j' - is a regular function on $G^{u,v}$

(b) Map $G^{u,v} \to \mathbb{C}^{n+\ell(u)+\ell(v)}$ $\quad g \mapsto (\Delta_1(g), \dots, \Delta_{j-1}(g), \Delta_j'(g), \Delta_{j+1}(g), \dots, \Delta_{n+\ell(u)+\ell(v)}(g))$

is biregular isomorphism $U_j = \{g \mid \Delta_i(g) \neq 0, i \neq j, \Delta_j'(g) = 0\} \xrightarrow{\sim} \mathbb{C}_{\neq 0}^{n+\ell(u)+\ell(v)}$

Remark Δ_j' is not necessary minor of g.

Step 3 Let $V = \mathcal{U} \cup \bigcup_{j=1}^{N} \mathcal{U}_j \subset G^{u,v}$

Lemma Codim $\mathcal{U}$ in $G^{u,v}$ is at least 2.

Hence $\mathbb{C}[G^{u,v}] = \mathbb{C}[G^{u,v}|V] =$

$$= \mathbb{C}[G^{u,v}|\mathcal{U}] \cap \bigcap_{j=1}^{N} \mathbb{C}[G^{u,v}|\mathcal{U}_j]$$

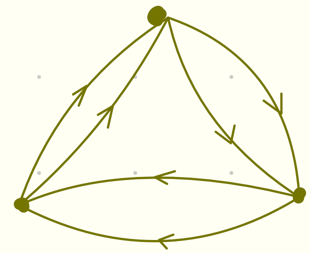
$$= \mathbb{C}[A_1^{\pm 1}, \dots, A_{n+\ell(u)+\ell(v)}^{\pm 1}] \cap \bigcap_{j=1}^{N} \mathbb{C}[A_1^{\pm 1}, \dots, (A_j')^{\pm 1}, \dots, A_{n+\ell(u)+\ell(v)}^{\pm 1}]$$

$$= \mathcal{U}(S') = \bar{A}(S') \qquad \blacksquare$$

- Remark Here in definition of $\mathcal{U}(S')$ and $\bar{A}(S')$ we inverted frozen variables (contrary to discussion above)

- It can be proven that $\mathbb{C}[G^{u,v}] \simeq \bar{A}(S') = \underline{A(S')}$

Problem Show that for quiver



$$B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

(a) $\overline{A}(S') = U(S')$

(b) $A(S') \subsetneq \overline{A}(S')$

Hint (a) Follows from theorem above.

(b) Consider $y = \dfrac{A_1^2 + A_2^2 + A_3^2}{A_1 A_2} = \dfrac{A_1 + A_1'}{A_2} = \dfrac{A_2 + A_2'}{A_1}$

Then $y \in \overline{A}(S')$

On the other hand all $A_i(S')$ have degree 1.

Hence $y \notin A(S')$

Remark $G^{w_0, e} = B w_0 B \cap B_- e B_- = B_-$

B_- has cluster structure

$N_- = H \backslash B_-$ in terms of quiver this is removing of first n frozen variables

<u>Problem</u> Show that

$n=3$ unfrozen quiver for $\mathbb{C}[N_-]$ equivalent to A_1
$n=4$ unfrozen quiver for $\mathbb{C}[N_-]$ equivalent to A_3
$n=5$ unfrozen quiver for $\mathbb{C}[N_-]$ equivalent to D_6

One can show that for $n>5$ quiver is not equivalent to Dynkin graph.

- <u>Remark</u> $G^{w_0, w_0} \subset G$ – cluster structure on open subset of G.

References

- Berenstein Fomin Zelevinsky Cluster algebras III Upper Bounds and Double Bruhat Cells