Itroduction to cluster algebras and varieties

Lecture 8

Starfish Lemma. Grassmanians

Starfish lemma

- $\mathcal{A}$ - cluster algebra (gener. by all $\mathcal{A}(s')$ for $s' \sim s$

$\bar{\mathcal{A}}$ - upper cluster algebra

$$\bar{\mathcal{A}} = \bigcap_{s' \sim s} \mathbb{Z}[\bar{A}^{\pm 1}(s')] \qquad \bar{\mathcal{A}} = \mathbb{C}[\mathcal{A}\text{-cluster variety}]$$

- Want $\mathcal{A}$ or $\bar{\mathcal{A}}$ to be $R = \mathbb{C}[M]$ (affine) variety
- Assume that we have

$\varphi : \mathcal{A} \to \mathbb{C}(M)$ s.t $\varphi(A_i) \in R = \mathbb{C}[M]$

- Want $\varphi(\bar{\mathcal{A}}) \supset R$ - sufficient to find some generators of R and show that these generators belong to $\varphi(\mathcal{A})$
- Want $\varphi(\bar{\mathcal{A}}) \subset R$

Theorem (Starfish lemma) Let $R = \mathbb{C}[M]$, where M - normal, irreducible, affine variety.

Let $\Sigma = (\bar{A}, b)$ seed

$\varphi : \mathbb{C}(A) \to \mathbb{C}(M)$ s.t

- $\varphi(A_i) \in R$
- $\varphi(A_i)$ pairwise coprime
- $\varphi(A_i') \in R$, $\varphi(A_i')$ coprime to $\varphi(A_i)$

Then $\varphi(\bar{A}) \subset R$

Photo from Wikipedia

Remark ⓐ Usually M is not smooth.

If $\mathbb{C}[M]$ is factorial (unique factorization domain) then M normal.

ⓑ $f, g \in R$ coprime $\iff \{f=0\} \cap \{g=0\}$ has codim ≥ 2.

If R is factorial, then (coprime) $\iff$ ($g.c.d(f,g)$ is invertible in R.

Pf • $Y = \bigcup_{1 \le i < j \le n} \{\varphi(A_i) = \varphi(A_j) = 0\} \cup \bigcup_{1 \le k \le n} \{\varphi(A_k) = \varphi(A_k') = 0\}$

By coprimeness assumption $\operatorname{codim} Y \geq 2$

• Hartogs continuation principle:
If $f \in \mathbb{C}$ is regular outside closed algebraic subset od codim $\geq 2 \Rightarrow f \in \mathbb{C}[M]$

It remains to show that $\varphi(\bar{A})$ is regular on $X \setminus Y$

• $p \in \bar{A}$ $\quad p =$ Laurent polynomial $A_1 \ldots, A_n$

$\Uparrow$

Laurent phenomenon $\quad A_1 \ldots, A_i' \ A_n$

For $pt \in X \setminus Y$ no more the one $\varphi(A_i)(pt)=0$

- If all $\varphi(A_i)(pt) \neq 0 \Rightarrow \varphi(P)(pt) \neq \infty$ since $\varphi(P)$ is Lautent polynomial on $\varphi(A_1), \dots, \varphi(A_n)$

- if $\varphi(A_k)(pt) = 0$, hence $\varphi(A_k')(pt) \neq 0 \Rightarrow$

  $\Rightarrow \varphi(P)(pt) \neq \infty$ since $\varphi(P)$ is Lautent polynomial on $\varphi(A_1), \dots, \varphi(A_n)$ $\square$

Grassmanian

- $Gr(K,N) = \{ V \subset \mathbb{C}^N ; \dim V = K \}$

- Pleucker embedding $Gr(K,N) \hookrightarrow \mathbb{P}^{\binom{N}{K}-1} = \mathbb{P}(\Lambda^K \mathbb{C}^N)$

$$V = \langle v_1, .., v_K \rangle \mapsto [v_1 \wedge .. \wedge v_K]$$

$$\widehat{Gr}(K,N) = \{ v_1 \wedge .. \wedge v_K \} \in \Lambda^K \mathbb{C}^N$$ – affine cone.

- Pleucker coordinates $\mathbb{C}^N = \langle e_1, .., e_N \rangle$ choose a basis

$$v_i = \sum x_{ij} e_j \qquad v_1 \wedge \ldots \wedge v_K = \sum_{i_1 < \ldots < i_K} \Delta_{i_1 \cdots i_K} e_{i_1} \wedge \ldots \wedge e_{i_K}$$

$$\Delta_{i_1, \ldots, i_K} = \operatorname{Det} \begin{pmatrix} x_{1a_1} & & x_{1a_K} \\ & & \\ x_{1a_K} & & x_{Ka_K} \end{pmatrix} \quad \text{minors of} \quad \begin{pmatrix} x_{11} & x_{12} & & x_{1N} \\ & & & \\ x_{K1} & x_{K2} & & x_{KN} \end{pmatrix}$$

$\mathbb{C}[\widehat{Gr}(k,n)] = \mathbb{C}[\text{Plucker coordinates}]/\text{relations}$

- Another description $\mathbb{C}[\widehat{Gr}] = \mathbb{C}[Mat(k\times n)]^{SL_k}$

In particular $\mathbb{C}[Mat(k\times n)]^{SL_k}$ are generated by Δ_I (see Lecture 1 for $k=2$)

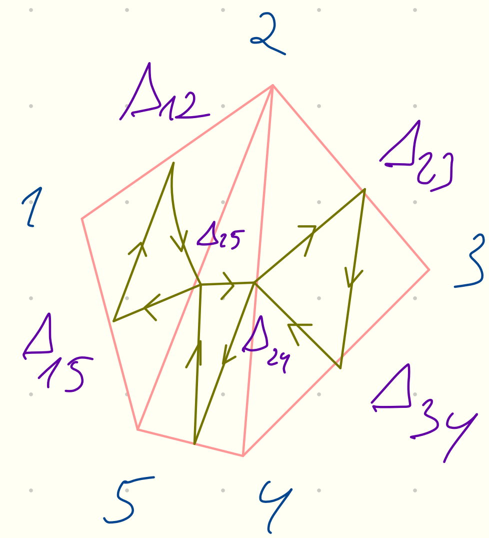
- Th $\mathbb{C}[\widehat{Gr}(k,n)]$ is factorial

Sketch of the proof • $\mathbb{C}[Mat(k\times n)]$ – factorial

• SL_k does not have characters $\Rightarrow$

$\mathbb{C}[Mat(k\times n)]^{SL_k}$ – factorial $\square$

<u>Goal</u> : $\mathbb{C}[\widehat{Gr}(k,n)] \simeq \mathcal{A}$



- <u>Recall</u> $k=2$

We will construct $\varphi : \mathcal{A} \to \mathbb{C}(\widehat{Gr}(k,n))$

- <u>Remark</u> ⓐ $\{$Plucker coordinates$\} \subsetneq \{$cluster variables$\}$

Hence $\mathbb{C}[\widehat{Gr}(k,n)] \subset \varphi(\mathcal{A})$ (for $k=2$ we had equality)

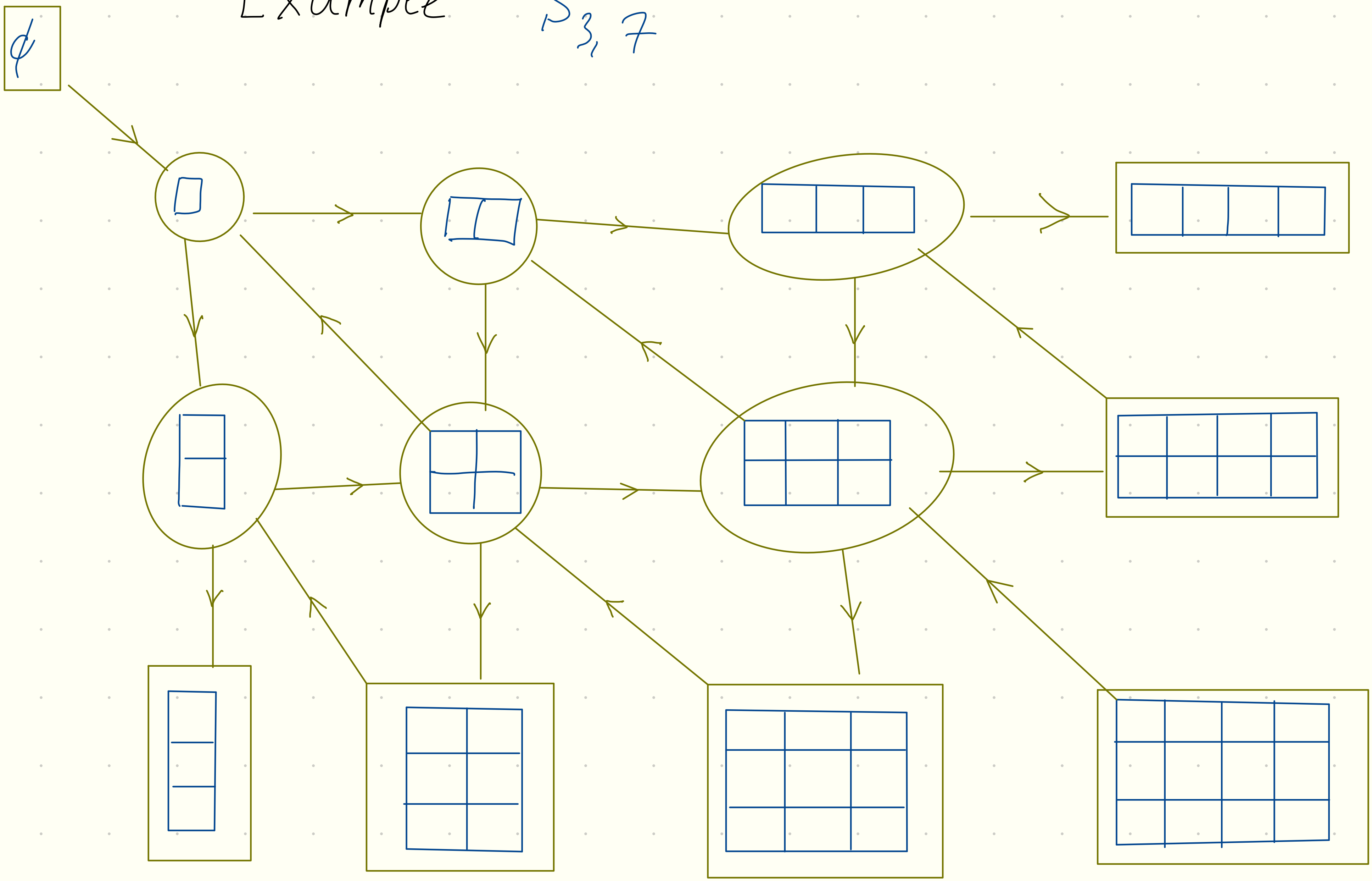
ⓑ In general, cluster algebra of infinite type i.e infinitely many seeds.

ⓒ Cluster monomials are linearly independent but do not form a basis in $\mathbb{C}[\widehat{Gr}(k,n)]$

ⓓ Depends on cyclic order, say $1, 2, \ldots, n$

Special seed $S_{k,N}$ for $\hat{G}_T(k,N)$

Example $S_{3,7}$



$\{\text{vertices}\} \longleftrightarrow \{\text{rectangles in } k \times (N-k)\}$

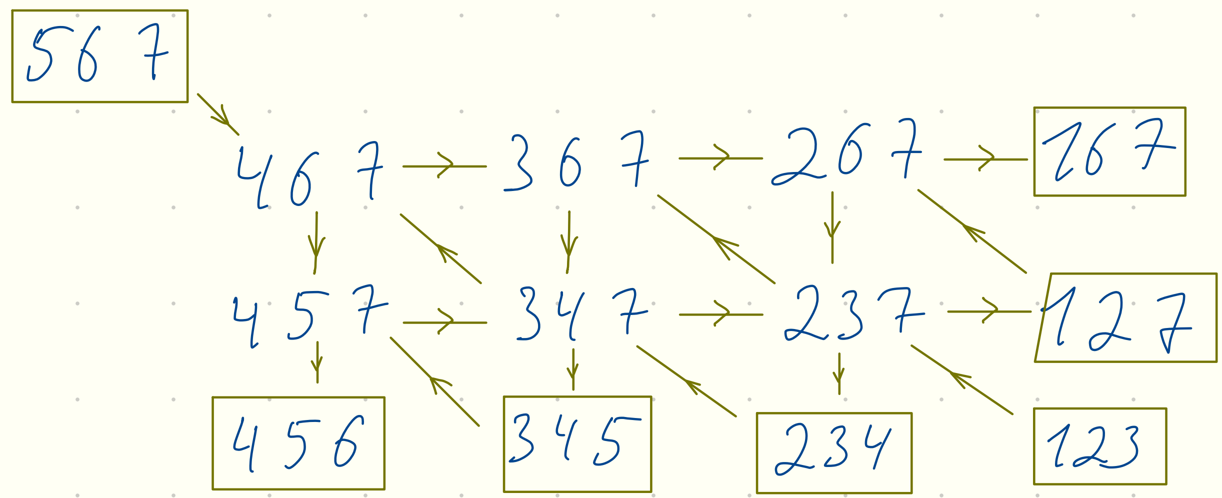
vertices $\quad k(N-k)+1 = \dim \widehat{C_T(k,N)}$

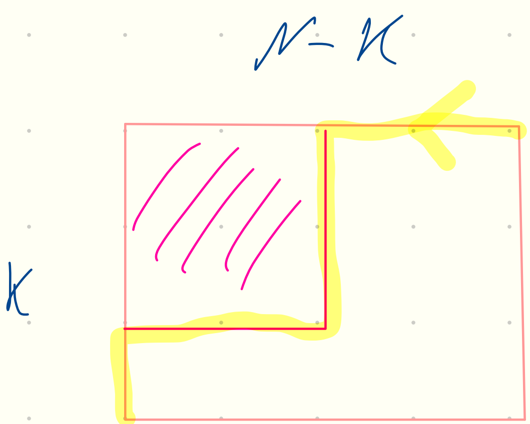
$\square$ vertices – frozen, $\quad N+1$ frozen vertices

Each vertex $i \longleftrightarrow a_1,\dots,a_k$

variable $\quad \varphi(A_i) = \Delta_{a_1\dots a_k}$

Example

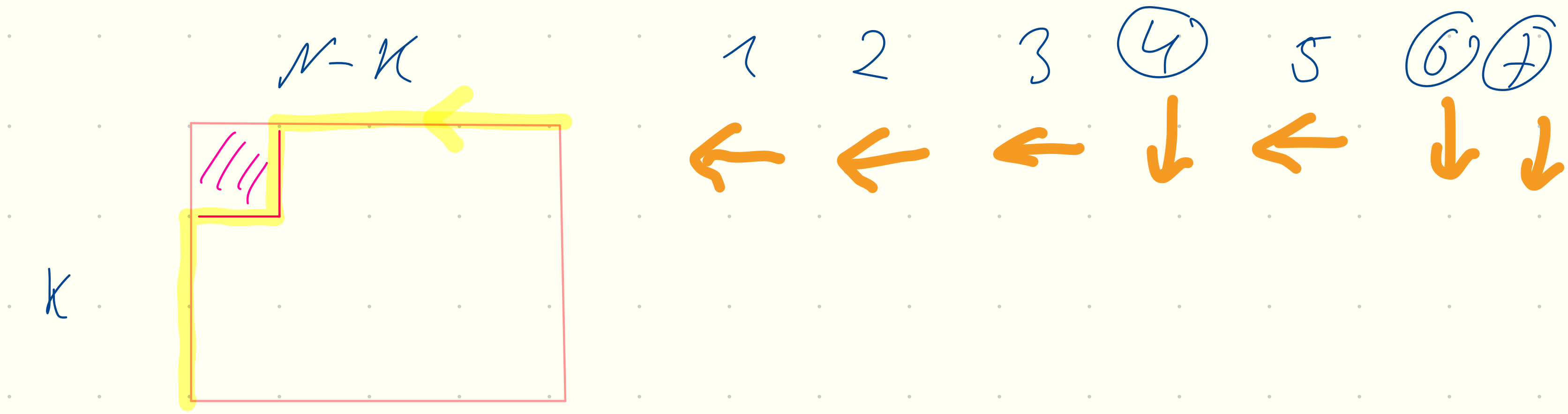




$\leftarrow \ \leftarrow \ \downarrow \ \downarrow \ \leftarrow \ \leftarrow \ \downarrow$

has $N-K$ arrows $\leftarrow$ $\quad K$ arrows $\downarrow$

$a \times b \longmapsto$ numbers of $\downarrow$



- More formally, vertices — $\{(i,j) \mid 1 \le i \le K,\ 1 \le j \le N-K\} \cup \{(0,0)\}$

  Minors $\quad (i,j) \longleftrightarrow \underbrace{N-K+1-j, \dots, N-K+i-j}_{i \text{ elements}}, \underbrace{N-K+i+1, \dots, N}_{K-i \text{ elements}}$

Overall, we have a map $\varphi: A \to \mathbb{C}(\widehat{Gr}(K,N))$

$A_{(i,j)} \to$ corresp minor

Problem Take a seed $S_{k,N}$ and mutate each row from left to right, from bottom to top. Denote the obtained seed by $S^1_{k,N}$

ⓐ As quiver $S^1_{k,N}$ is isomorphic to $S_{k,N}$

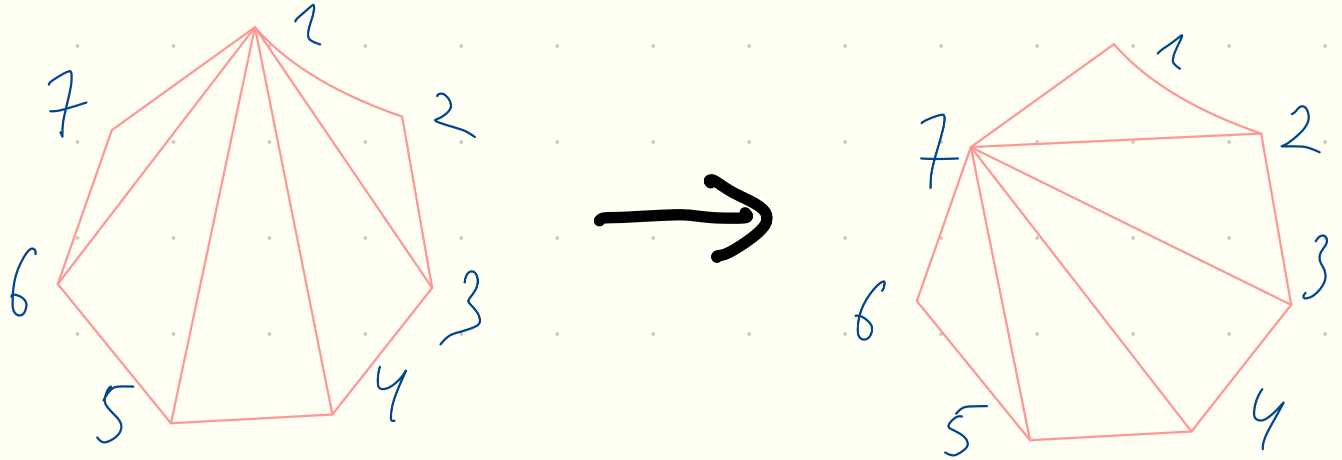
ⓑ Under this isomorphism minors $\Delta_{i_1,\dots,i_k} \to \Delta_{i_1+1,\dots,i_k+1}$ with addition mod N

Hint ⓑ Note that all mutations are performed in vertices of valency 4. Use relation

$$\Delta_{13I}\,\Delta_{24I} = \Delta_{12I}\,\Delta_{34I} + \Delta_{23I}\,\Delta_{14I}$$ for any set $I \subset \{5,\dots,N\}$

(see also next lecture)

Example



Remark Applying this transformation several times we get seed $S^m_{k,N}$ where $\Delta_{i_1,\dots,i_k} \to \Delta_{i_1+m,\dots,i_k+m}$.

Problem ⓐ Consider map $Gr(k,N) \to Gr(N-k,N)$

$$V \subset \mathbb{C}^N \mapsto V^\perp = \{\xi \mid \langle \xi, v\rangle = 0 \ \forall v \in V\} \subset (\mathbb{C}^N)^*$$

compute it in Plucker coordinates.

ⓑ Show that formula $\Delta_{J^c} = \Delta_J$ where $J^c = \{1,\dots,N\} \setminus J$ defines map from $Gr(k,N)$ to $Gr(N-k,N)$

ⓒ Show that this map agrees with cluster structures on $Gr(k,N)$ and $Gr(N-k,N)$ up to change of sign in B.

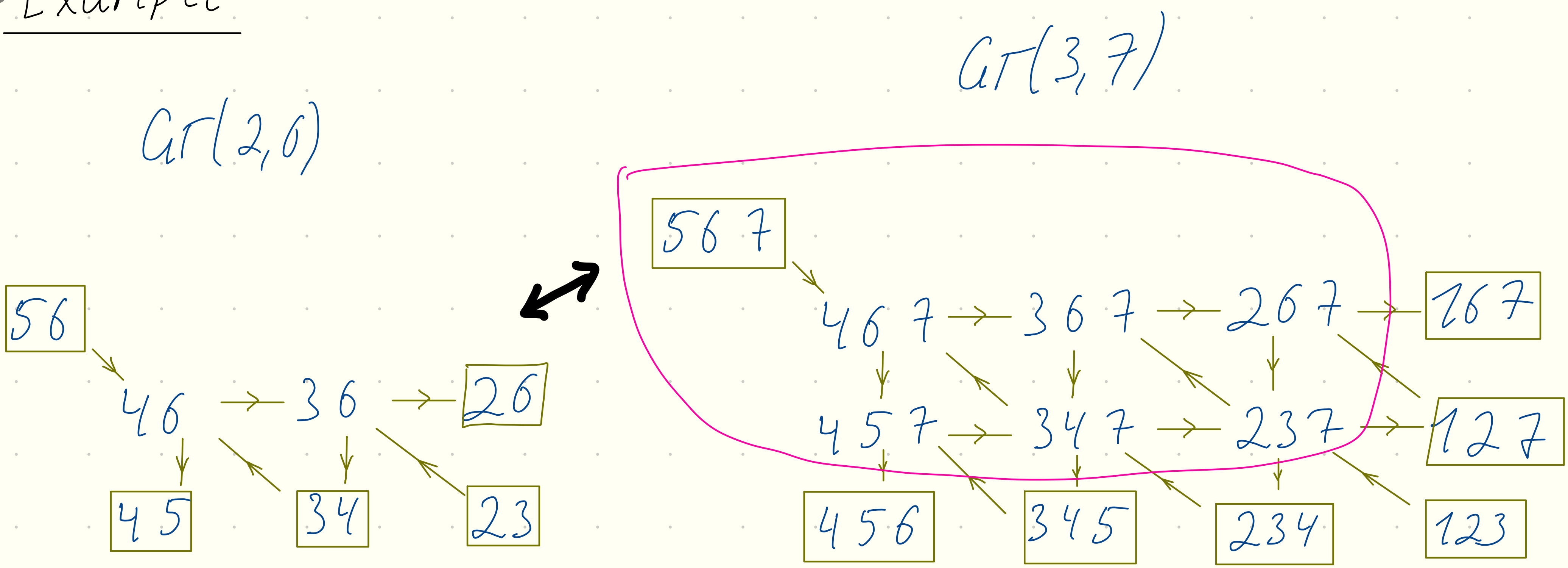
Hint ⓐ Use Jacobi formula for minors of inverse matrix

ⓒ Use plabic graphs interpretation (see next Lecture)

- **Lemma** (a) $\exists$ embedding $\mathbb{C}[Gr(k-1,n-1)] \hookrightarrow \mathbb{C}[Gr(k,n)]$ given by $\Delta_I \mapsto \Delta_{I \cup \{n\}}$

(b) This agrees with cluster structure

- **Example**



all arrows between unfrozen and other vertices agrees.

- **Corollary** Any minor Δ_I is $\varphi(A_k')$ where A_k' is cluster variable for some seed $S' \sim S_{k,n}$

- **Corollary²** $\mathbb{C}[\widehat{Gr}(k,n)] \subset \varphi(\mathcal{A})$

- **Proof of corollary** Consider some Δ_I
  - Applying (if necessary) automorphism shifting by 1 we can assume that $\{n\} \in I$ i.e. $I = I' \cup \{n\}$
  - Use induction $\Delta_{I'}$ -cluster for $\mathbb{C}[Gr(k-1,n-1)]$ Use Lemma $\Rightarrow$ Δ_I is cluster variable $\square$

- **Th** $\varphi: \mathcal{A} \to \mathbb{C}[\widehat{Gr}(k,n)]$ is isomorphism.

Pf • $\dim \widehat{G}(k,n) = k(n-k)+1 = \# \varphi(A_{(i,j)})$. Also, $\varphi(A_{(i,j)})$ generate all Δ_I. Hence $\varphi(A_{(i,j)})$ algebraically independent Hence $\varphi: \mathcal{A} \to \mathbb{C}(\widehat{Gr}(k,n))$ is embedding

- We know $\varphi(A) \supset \mathbb{C}[Gr(k,n)]$. For opposite inclusion use starfish lemma

<u>Problem</u> $\det X$ is irreducible polynomial in $\mathbb{C}[x_{11}, \ldots, x_{nn}]$

Hence Δ_I and Δ_J are coprime for $I \neq J$
Hence $\varphi(A_{(i,j)})$ are coprime.

It remains to show coprimenes $\varphi(A_{(i,j)})$ and $\varphi(A'_{(i,j)})$

- If vertex (i,j) is 4 valent then $\varphi(A'_{(i,j)}) = \Delta_{I'}$ and we are done

- If vertex (i,j) is 4 valent then one can show that $\varphi(A'_{(i,j)}) = \Delta_{I'}\Delta_{I''} - \Delta_{I'''}\Delta_{I''}$ and this is coprime to $\varphi(A_{(i,j)}) = \Delta_I$

$\square$

References

- Fomin Williams Zelevinsky Introduction to Cluster algebras. Chapter 6