Introduction to cluster algebras and varieties

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Problems for the course Skoltech, fall 2022. There are mistakes here, if you find some please write to mbersht@gmail.com

Definitions and hints are in slides and references.

1 Preliminary examples

Problem 1.1. *a) Show that $\mathbb{C}[\mathcal{A}_n]$ is generated (as a vector space) by P_T, where T is semi-standard tableau.*

b) Show that $\mathbb{C}[\mathcal{A}_n]$ is generated (as a vector space) by cluster monomials.

Problem 1.2 (*). *Show equivalence of two more definitions (through composition of maps and monodromy) of cross-ratios $[u_1, u_2, u_3, u_4]$ to the one given in the lecture.*

Problem 1.3. *Show transformation formula for cross-ration coordinates under the flip of triangulation.*

2 Seeds. Mutations. Laurent phenomenon.

Problem 2.1. *All orientations of tree are mutation equivalent to each other via mutations on sinks or sources.*

Problem 2.2. *a) If $b_{jk} = 0$ then $\mu_j\mu_k = \mu_k\mu_j$.*

b) If $b_{jk} = 1$ then $(\mu_j\mu_k)^5 = \mathrm{id}$.

Problem 2.3 (*). *a) For D_4 quiver find all cluster variables. Establish bijection between them and set $\Phi_+ \sqcup -\Pi$. b) Check the same for another finite root system.*

Problem 2.4. *Find quiver for Somos-5 sequence.*

Problem 2.5. *Show that term in Somos-4 sequence with coefficients $z_{k+4}z_k = az_{k+3}z_{k+1} + bz_{k+2}^2$ are integer, if $z_1 = z_2 = z_= z_4 = 1$ and $a, b \in \mathbb{Z}$.*

3 Total positivity. Networks.

Problem 3.1. *Prove that eigenvalues of totally positive $n \times n$ matrix are real positive and distinct.*

Problem 3.2. *Show that network transformations correspond to cluster mutations.*

Problem 3.3 (*). *Show that any minor appears in seed corresponding to some network.*

Problem 3.4. *a) Show that for $n = 3$ unfrozen part of quiver is equivalent to D_4 quiver. b) Find two cluster variables which are not minors.*

4 Double Bruhat cells. $\mathcal{X}$-varieties

Problem 4.1. *Show that minor conditions that determines Bruhat cell.*

Problem 4.2 (*). *If $l(us_i)l(u) + 1$ then $Bus_iB = BuBBs_iB$*

Problem 4.3. *For given cluster seed $(\mathcal{Q}, \vec{A})$ let $x_j = \prod A_i^{b_{ij}}$. Show that mutation of seed leads to mutation of $\mathcal{X}$-cluster variables.*

Problem 4.4. *Show that braid moves in reduced decomposition of the word in double Weyl group corresponds to cluster transformations of $\mathcal{X}$-cluster variables.*

5 Cluster Poisson structure. Sklyanin bracket

Problem 5.1. *Any totally positive matrix can be represented as $\mathbb{E}$ for the word*

$$(\bar{s}_{n-1} \cdot \dots \cdot \bar{s}_1)(\bar{s}_{n-1} \cdot \dots \cdot \bar{s}_2) \dots (\bar{s}_{n-1}\bar{s}_{n-1})\bar{s}_{n-1}$$

Problem 5.2. *The cluster Poisson bracket is preserved by the mutations.*

Problem 5.3. *a) The cluster closed 2-form is preserved under mutations. b)* This lifts to well defined element of K_2.*

Problem 5.4. *Compute Sklyanin bracket for $GL(2) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\}$. Find two algebraically independent Casimir functions.*

Problem 5.5 (*). *Compute Poisson bracket on the cell $C^{s_i,e}$.*

6 Cluster Poisson structure. Integrable systems

Problem 6.1. *a) Show that Coxeter element depends only on orientation of Dynkin diagram i.e. for any i what is earlier s_i or s_{i+1}.*

b) Show that using transformations preserving quiver $G^{c,c'}/\operatorname{Ad} H$ (i.e. cyclic permutations, $s_i\bar{s}_2 = \bar{s}_2 s_i \quad i \neq j$, $s_i s_j = s_j s_i, \overline{s_i}, \bar{s}_j = \bar{s}_j \bar{s}_i, |i - j| > 1$) any word can be reduced to $\bar{s}_{i_1} \cdot \bar{s}_{i_{n-1}} s_{j_1} \cdot s_{j_{n-1}}$.

c) For $G = GL_{n+1}$ there are no more then 3^{n-1} different quivers for $G^{c,c'}/\operatorname{Ad} H$. All of them are mutation equivalent.

Problem 6.2 (*). *Consider cell $G^{c,c}/\operatorname{Ad}H$. Take decomposition*

$$g = H_1(x_1)F_1H_1(y_1)E_1 \cdot \ldots \cdot H_{n-1}(x_{n-1})F_{n-1}H_{n-1}(y_{n-1})E_{n-1}.$$

a) Compute Poisson brackets of $x_1, \ldots x_{n-1}, y_1, \ldots, y_{n-1}$.

b) Compute $I_1 = \operatorname{tr} g$.

c) Let $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ exponential Darboux coordinates

$$\{\eta_i, \xi_j\} = \delta_{ij}\eta_i\xi_j, \quad \{\xi_i, \xi_j\} = \{\eta_i, \eta_j\} = 0$$

Show that $x_i = \eta_i/\eta_{i+1} \quad y_i = \xi_{i+1}/\xi_i$ is a Poisson map.

d) Let $L_i = \begin{pmatrix} \mu\xi_i^{1/2}\eta_i^{-1/2} + \xi_i^{-1/2}\eta_i^{1/2} & \mu\xi_i^{-1/2}\eta_i^{-1/2} \\ \xi_i^{1/2}\eta_i^{1/2} & 0 \end{pmatrix}$ be a Lax matrix. Denote components of the monodromy matrix $L_1 \cdot L_2 \ldots L_n = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let $A = \sum \widetilde{I}_k\mu^{n-k}$. Show that $\tilde{I}_1 \sim I_1$ (proportional by monomial commuting with $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$).

s) Show the same for $\widetilde{I}_k$ and $I_k = \operatorname{Tr}\Lambda^k g$.

7 Upper cluster algebras

Problem 7.1. *Check that $b'_{ij} = \langle e'_i, e'_j\rangle$ agrees with mutation of matrix B*

Problem 7.2. *Show that $R[A_1, A_1^{-1}] \cap R[A'_1, {A'_1}^{-1}] = R[A_1, A'_1]$*

Problem 7.3. *Show for quiver with $B = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$ that*

a) $\bar{\mathcal{A}}(S) = U(S)$,

b) $\mathcal{A}(S) \subsetneq \bar{\mathcal{A}}(S)$.

Problem 7.4. *Show that unfrozen quiver for $\mathbb{C}[N_-]$ for SL_3, SL_3, SL_5 is mutation equivalent to Dynkin diagram A_1, A_3, D_6 correspondingly.*

8 Starfish lemma. Grassmanian

Problem 8.1. *Start from the seed $S_{K,N}$ and mutate at each of the mutable vertices of $S_{K,N}$ exactly once, in the following order: mutate each row from left to right, starting from the bottom row and ending at the top row. Denote the obtained seed by $S^1_{K,N}$*

a) Show that quivers $S_{K,N}$ and $S^1_{K,N}$ are isomorphic.

b) Show that corresponding minors are related as $\Delta_{i_1,\ldots,i_k}$ and $\Delta_{i_1+1,\ldots,i_k+1}$.

Problem 8.2. *a) Consider a map $\operatorname{Gr}(K,N) \to \operatorname{Gr}(N-K,N)$ given $V \mapsto V^\perp$. Compute this map in Plucker coordinates.*

b) Show that formula $\Delta_{J^c} = \Delta_J$ defines well map from $\operatorname{Gr}(K,N)$ to $\operatorname{Gr}(N-K,N)$.

c) Show that this map agrees with cluster structures on $\operatorname{Gr}(K,N)$ and $\operatorname{Gr}(N-K,N)$ up tp change of sign in B.

Problem 8.3. *Show that $\det X$ is irreducible polynomial in $\mathbb{C}[x_{11}, \ldots, x_{NN}]$.*

9 Plabic graphs. Grassmanians

Problem 9.1. *Show that zig-zags starting at different vertices terminate at different vertices.*

Problem 9.2. *Show that* π_G *is invariant under the moves M1,M2,M3.*

Problem 9.3. *a) Moves M2, M3 preserves* Δ_I*. b) Move M1 corresponds to mutation of* Δ_I*.*

Problem 9.4. *a) Find plabic graph corresponding to triangular seed. b) Find plabic graph corresponding to square seed.*

10 Poisson structures on Grassmanians

Problem 10.1 (*)**.** *Show that* $\{y_{i,j}, y_{i',j'}\} = (\mathrm{sgn}(i-i') - \mathrm{sgn}(j-j'))y_{i,j'}y_{i',j}$

Problem 10.2. *Compute* $\{F_{p,q}, F_{p',q'}\}$ *from Sklyanin bracket.*

Problem 10.3 (*)**.** *Compute* $\{F_{p,q}, F_{p',q'}\}$ *from cluster bracket.*

11 Geometric approach to cluster varieties

Problem 11.1 (*)**.** *Find toric varieties for given fans.*

Problem 11.2 (*)**.** *Show that the new difinition agrees with formulas for mutation of cluaster variables introduced above.*

Problem 11.3 (*)**.** *Prove Lemma 1.*

Problem 11.4 (*)**.** *Prove Lemma 2.*

12 Cluster structures on the moduli space of PGL_2 local systems

Problem 12.1. *Find monodromy matrices for the 1 punctured torus. Find the corresponding quiver.*

Problem 12.2. *Show that monodromy around a puncture is conjugated to* $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ *where* $\lambda_1/\lambda_2 = x_1 \cdot \cdots \cdot x_k$*.*

13 Pinnings

Problem 13.1. *Let $G = PGL_m$, B, B' be pair of Borel subgroups in general position, F, F' — corresponding flags. Let $L + i = F_i \cap F_{m+1-i}$. Show is correspondence between 1) pinnings over B, B', 2) projective bases $v_i \in L_i$ 3) Generic lines $< f > \subset \mathbb{C}^m$.*

Problem 13.2. *For any two pinnings p, p' show that there exists and unique $g \in PGL_m$ such that $g \cdot p = p'$.*

Problem 13.3. *Quivers and bipartite graphs for triangulations related by flip are mutation equivalent. (Check for PGL_3, PGL_4).*

14 Cluster structures on the moduli space of PGL_m local systems

Problem 14.1. *Let*

$$F^{(2)} = \{0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots . e_{m-1} \subset \mathbb{C}^m \rangle\},$$
$$F^{(4)} = \{0 \subset \langle e_m \rangle \subset \langle e_{m-1}, e_m \rangle \subset \cdots \subset \langle e_2, \ldots . e_m \subset \mathbb{C}^m \rangle\},$$

and $p = \langle \alpha_1 e_1 + \ldots \alpha_m e_m$. Find p' such that gluing conditions are satisfied.

Problem 14.2. *Show that $v^{A^B C} S = v^{C_B A}$.*

Problem 14.3. *Show some of the relations*
a) $v^{A_p C} H_1(x_1) \cdot \ldots H_{m-1}(x_{m-1}) = v^{A^B C}$.
b) Gluing conditions are satisfied iff $v^{A_p C} = v^{A^{p'} C}$.
c) $v^{A_D C} H_1(x_1) \cdot \ldots H_{m-1}(x_{m-1}) = v^{A^B C}$.

Problem 14.4 (*). *a) For $m = 2$ we have $v^{A_q C} H_1(x_1) E_1 H_1(x_2) = v^{A^r B}$.*
b) For $m = 3$ we have $v^{A_q C} H_1(x_1) H_2(x)_2 E_2 1 E_1 H_2(x_4) E_2 H_1(x_3) H_2(X_6) = v^{A^r B}$.
c) For generic m we have $v^{A_q C} \mathbb{E}_{w_0}(\vec{x}) = v^{A^r B}$.